Bandpass Sampling (2B)

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3/31/12

Please send corrections (or suggestions) to youngwlim@hotmail.com.

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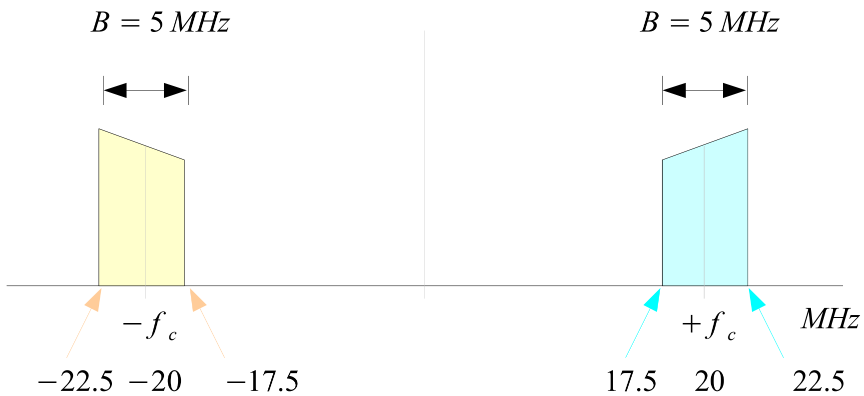
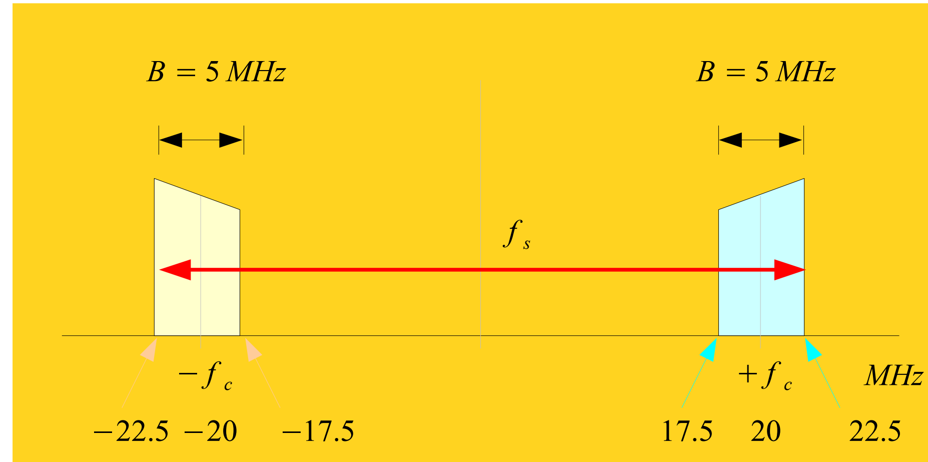
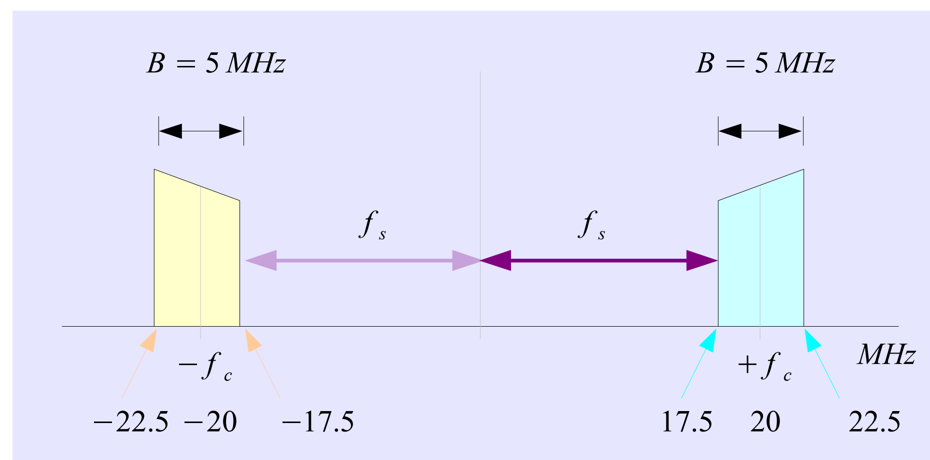
Band-limited Signal



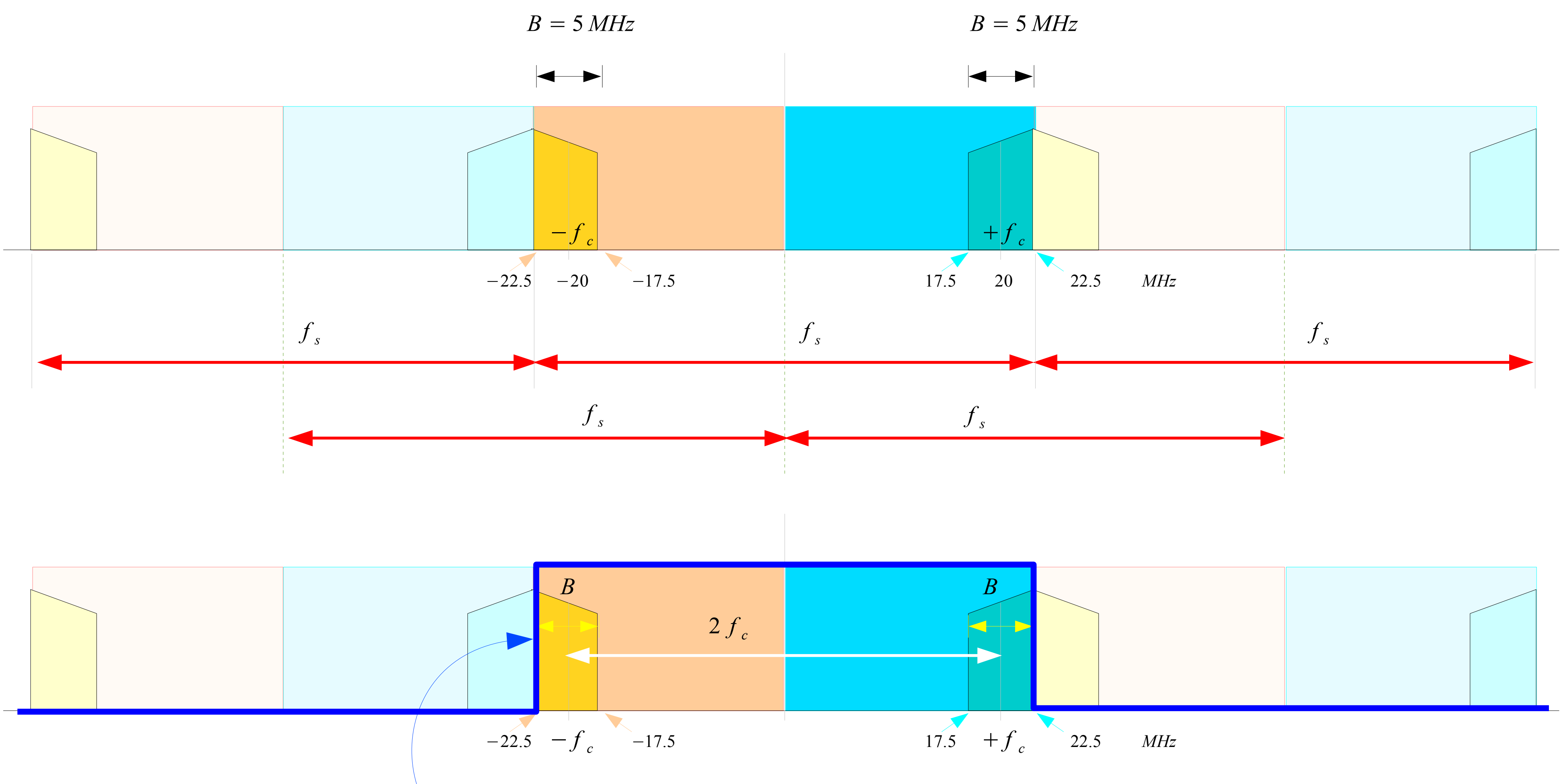


- **Lowpass Sampling**

- **Bandpass Sampling**
- **IF filtering**
- **Harmonic Sampling**
- **Sub-Nyquist Sampling**



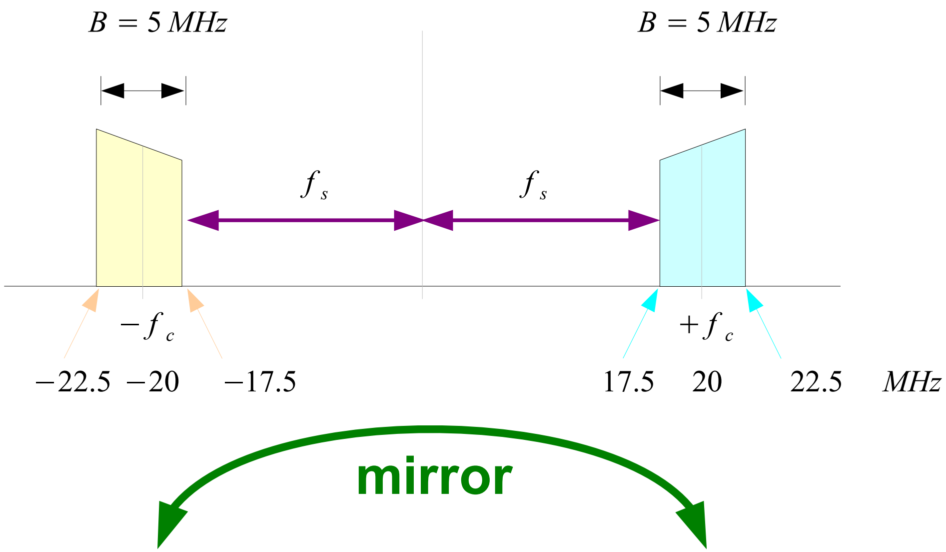
Low-pass Signal Sampling



Nyquist Criterion

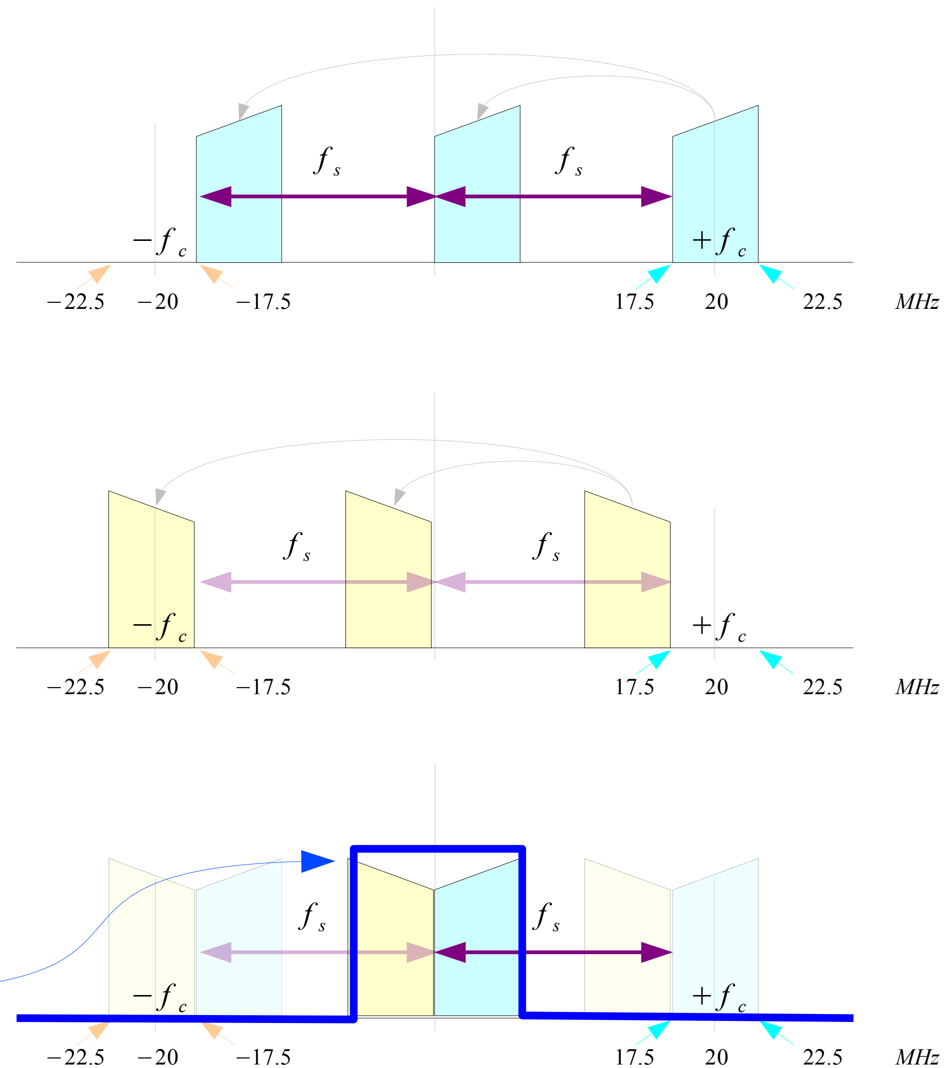
$$2f_c + B \leq f_s$$

Band-pass Signal Sampling



- **Bandpass Sampling**
- **IF filtering**
- **Harmonic Sampling**
- **Sub-Nyquist Sampling**

Nyquist Criterion	$2B \leq f_s$



Sampling Frequency f_s (1)

Assume there are m multiples of f_s — *Given an integer m*

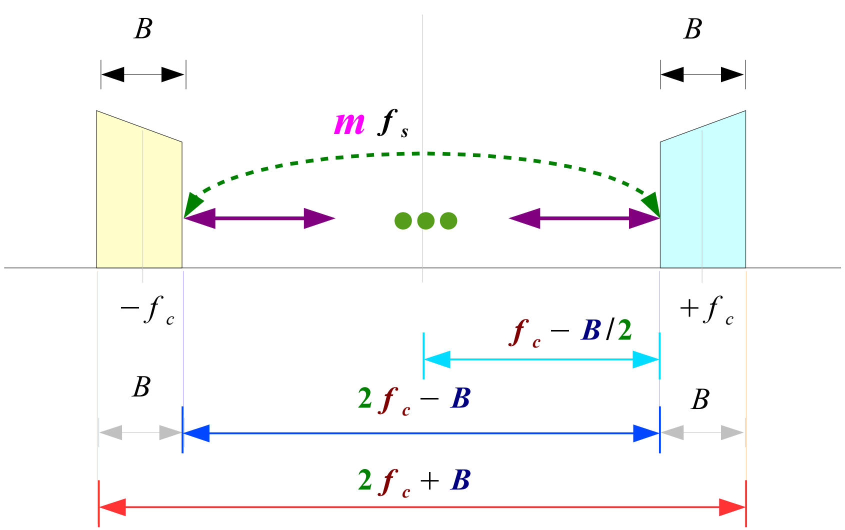
$$2f_c - B = m \cdot f_s$$

Max f_s condition

f_s *can be decreased according to the following condition without introducing aliasing problems*

$$2f_c + B = (m+1) \cdot f_s$$

Min f_s condition



Given Band-pass Signal is characterized by

- *Bandwidth B*
- *Carrier Frequency* f_c

$$\frac{2f_c + B}{m+1} \le f_s \le \frac{2f_c - B}{m}$$

Sampling Frequency f_s (2)

$$\frac{2f_c + B}{m+1} \le f_s \le \frac{2f_c - B}{m}$$

Given Band-pass Signal is characterized by

- ***Bandwidth B***
- ***Carrier Frequency f_c***

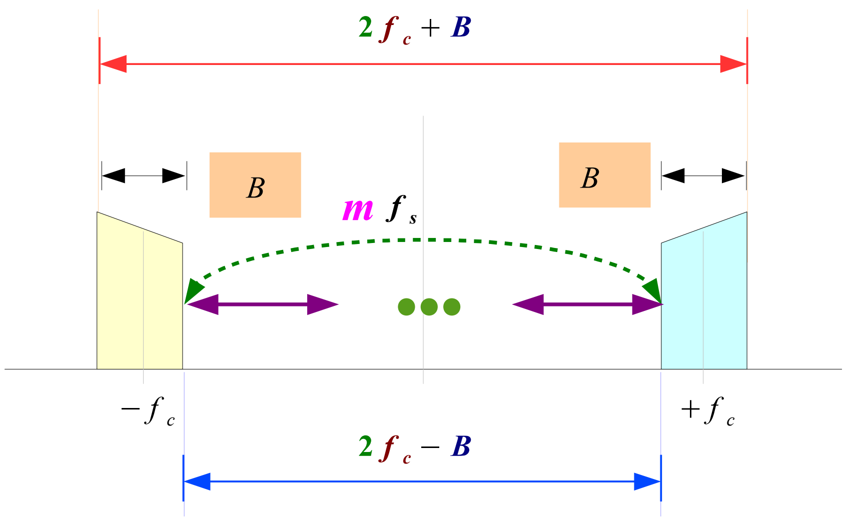
Normalization by B

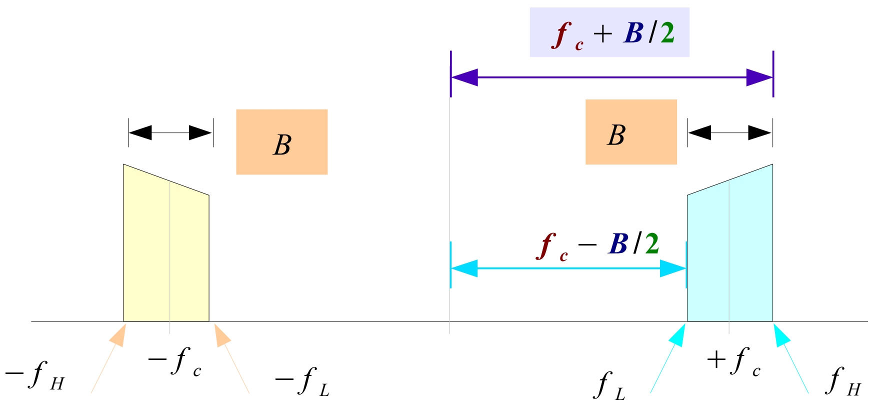
$$\frac{2f_c + B}{(m+1)B} \le \frac{f_s}{B} \le \frac{2f_c - B}{mB}$$

$$\frac{2f_H}{(m+1)B} \le \frac{f_s}{B} \le \frac{2f_L}{mB}$$

$f_H = f_c + B/2$ ***Highest frequency***

$f_L = f_c - B/2$ ***Lowest frequency***





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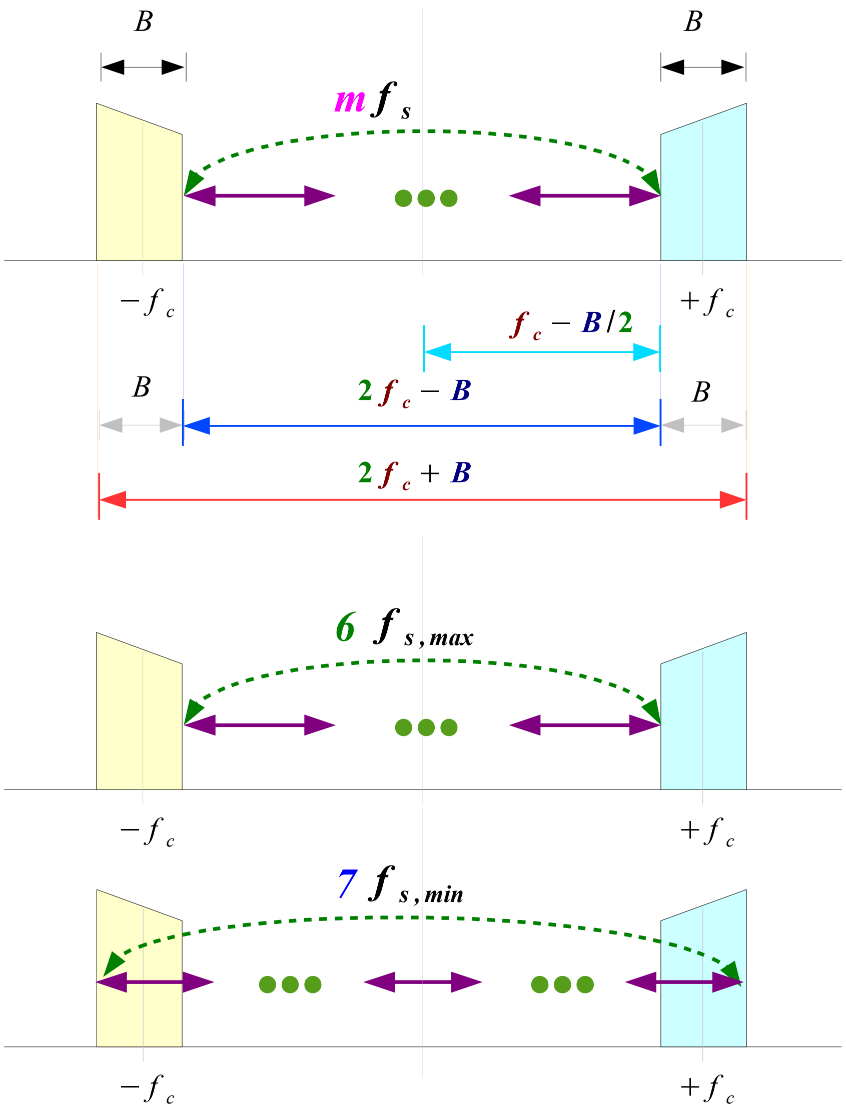
Example m=6 (1)



$$\frac{2f_c + B}{m+1} \le f_s \le \frac{2f_c - B}{m}$$

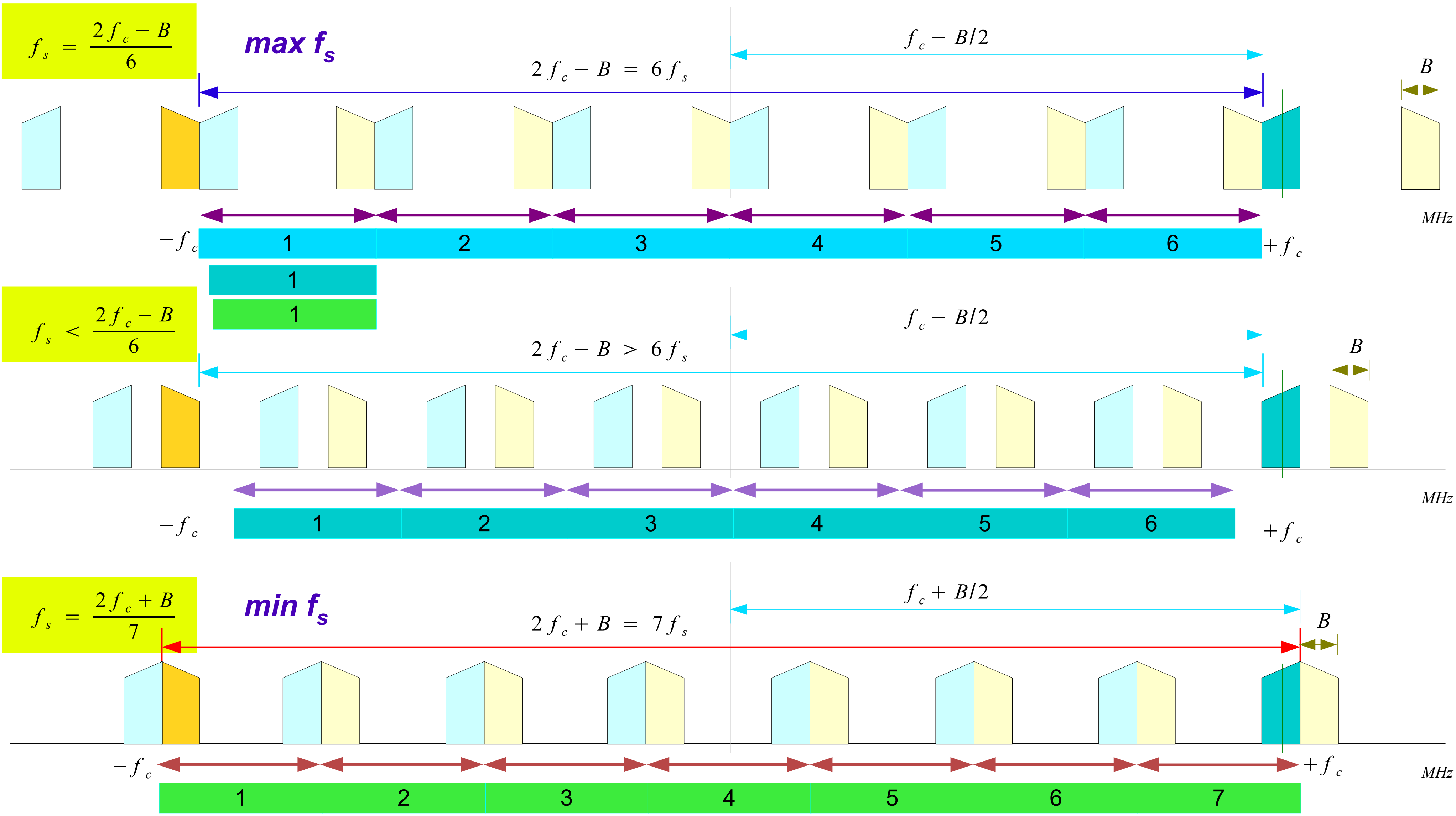
When $m = 6$

min f_s $\quad \frac{2f_c + B}{7} \le f_s \le \frac{2f_c - B}{6}$ $\quad$ max f_s

$$\text{max } f_s = \frac{2f_c - B}{6}$$

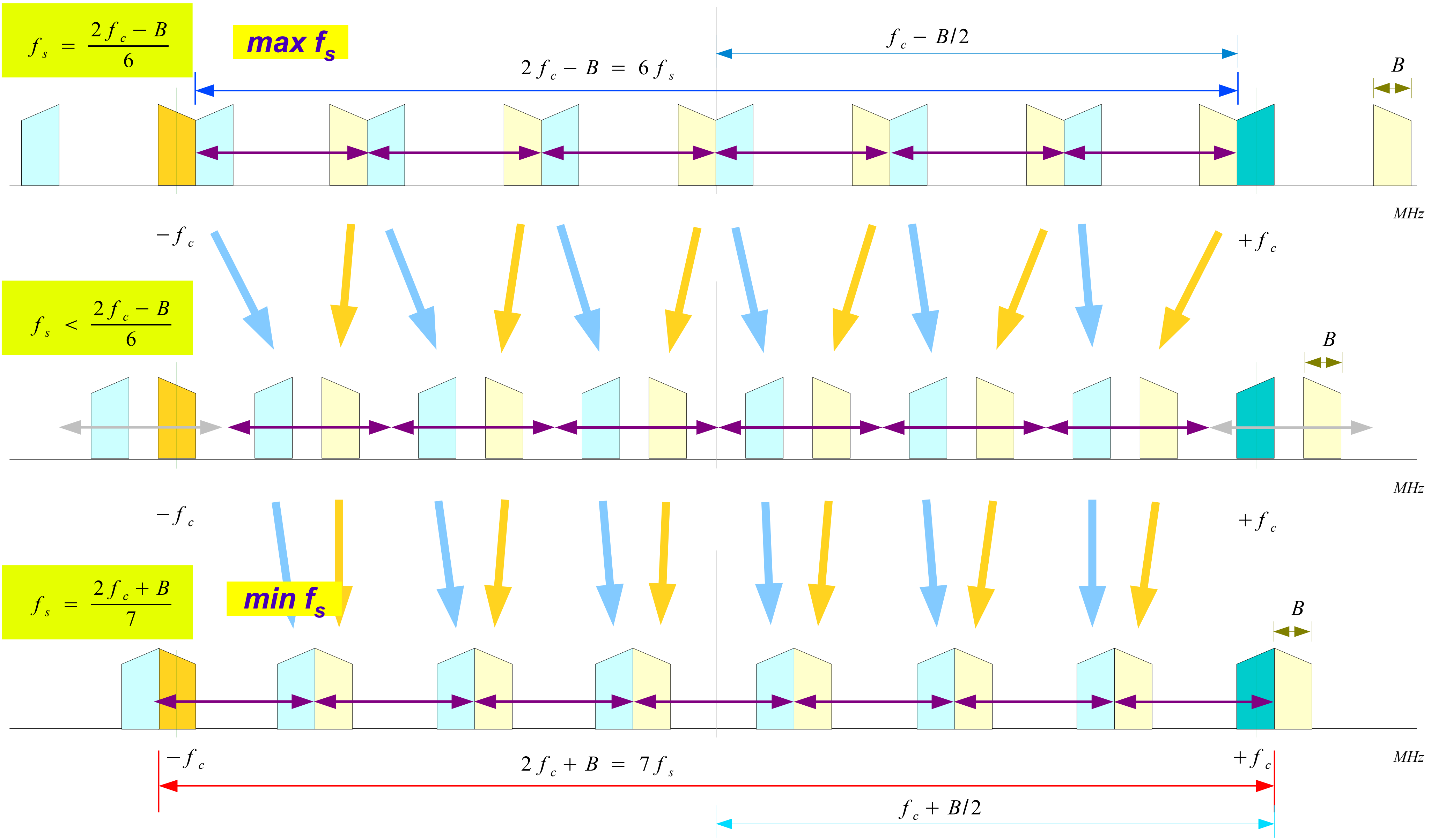
$$\text{min } f_s = \frac{2f_c + B}{7}$$

Example m=6 (2)



$f_s = \frac{2f_c - B}{6}$
max f_s
$f_c - B/2$
$2f_c - B = 6f_s$
B
MHz
$-f_c$
1 2 3 4 5 6
$+f_c$
1
1
$f_s < \frac{2f_c - B}{6}$
$f_c - B/2$
$2f_c - B > 6f_s$
B
MHz
$-f_c$
1 2 3 4 5 6
$+f_c$
$f_s = \frac{2f_c + B}{7}$
min f_s
$f_c + B/2$
$2f_c + B = 7f_s$
B
$-f_c$
$+f_c$
MHz
1 2 3 4 5 6 7

Example m=6 (3)



$f_s = \frac{2f_c - B}{6}$
max f_s
$f_c - B/2$
$2f_c - B = 6f_s$
B
MHz
$-f_c$
$+f_c$
$f_s < \frac{2f_c - B}{6}$
B
MHz
$-f_c$
$+f_c$
$f_s = \frac{2f_c + B}{7}$
min f_s
B
$-f_c$
$2f_c + B = 7f_s$
$+f_c$
MHz
$f_c + B/2$

Minimum f_s Plot (1)

$$\frac{2f_c + B}{m+1} \le f_s \le \frac{2f_c - B}{m}$$

$$\frac{f_c + B/2}{B} = R \rightarrow X$$

$\rightarrow$ $\dfrac{\text{highest signal frequency}}{\text{bandwidth } B}$

$$\frac{2f_c + B}{(m+1)B} = \frac{f_{s,min}}{B} \rightarrow Y$$

$\rightarrow$ $\dfrac{\text{minimum sampling rate}}{\text{bandwidth } B}$

X-Y Plot

Y: $\dfrac{f_{s,min}}{B}$

X

This plot shows min f_s
normalized by B,
for the given bandpass signal
that is characterized by R
and the given parameter m

Characterized by

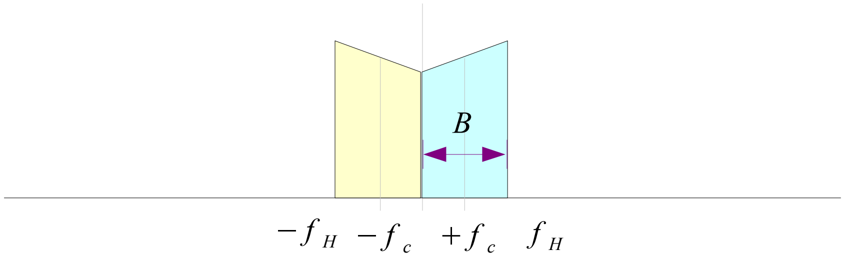
- ***Bandwidth B***
- ***Carrier Frequency f_c***

$$R = \frac{f_H}{B} = \frac{f_c + B/2}{B}$$

Minimum f_s Plot (2)

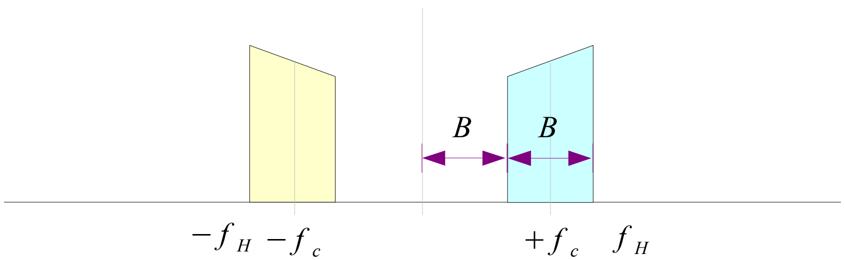
$f_H = f_c + B/2 = 1B$

$R = f_H / B = 1$



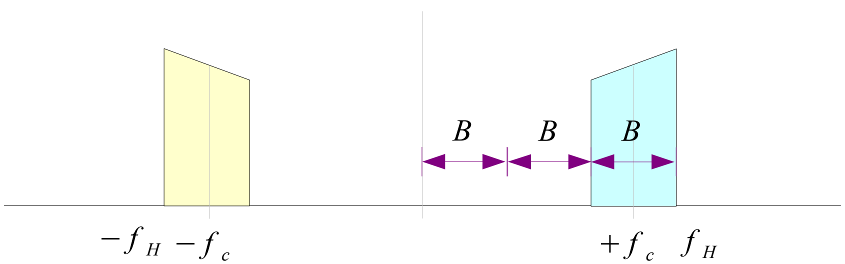
$f_H = f_c + B/2 = 2B$

$R = f_H / B = 2$

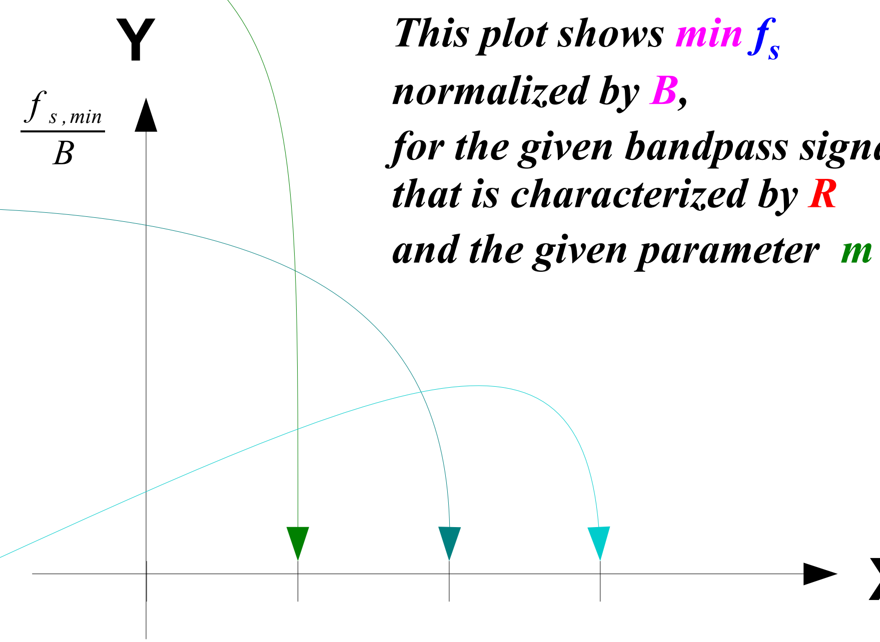


$f_H = f_c + B/2 = 3B$

$R = f_H / B = 3$



X-Y Plot



This plot shows min f_s normalized by B, for the given bandpass signal that is characterized by R and the given parameter m

Characterized by

- ***Bandwidth B***
- ***Carrier Frequency f_c***

$$R = \frac{f_H}{B} = \frac{f_c + B/2}{B}$$

Minimum f_s Plot (3)

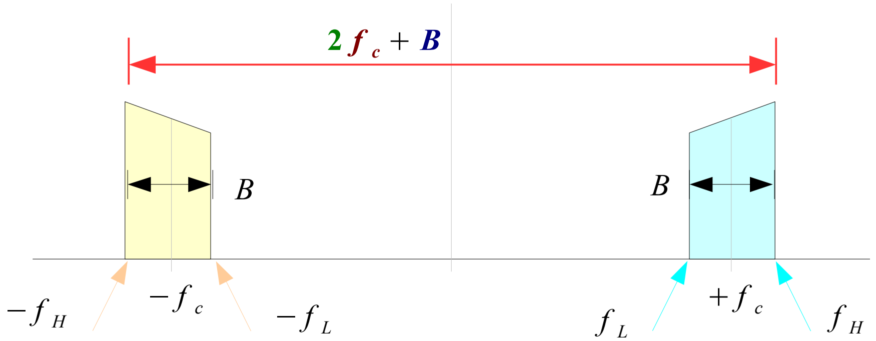
$$\frac{2f_c + B}{m+1} \le f_s \le \frac{2f_c - B}{m}$$

$$g(m, R) = \frac{2}{(m+1)}\frac{f_H}{B} = \frac{2}{(m+1)}R$$

$$\frac{f_H}{B} = \mathbf{X} \Rightarrow \frac{f_c + B/2}{B} = R$$

$$\frac{f_{s,min}}{B} = \mathbf{Y} \Rightarrow \frac{2f_c + B}{(m+1)B} = \frac{2f_H}{(m+1)B}$$

$$\Rightarrow g(m, R)$$



$m = 0$	$g(0, R) = 2R$	$slope = 2$
$m = 1$	$g(1, R) = R$	$slope = 1$
$m = 2$	$g(2, R) = \frac{2}{3}R$	$slope = 2/3$
$m = 3$	$g(3, R) = \frac{1}{2}R$	$slope = 1/2$
$m = 4$	$g(4, R) = \frac{2}{5}R$	$slope = 2/5$
$m = 5$	$g(5, R) = \frac{1}{3}R$	$slope = 1/3$
$m = 6$	$g(6, R) = \frac{2}{7}R$	$slope = 2/7$
$m = 7$	$g(7, R) = \frac{1}{4}R$	$slope = 1/4$
$m = 8$	$g(8, R) = \frac{2}{9}R$	$slope = 2/9$

Minimum f_s Plot (4)

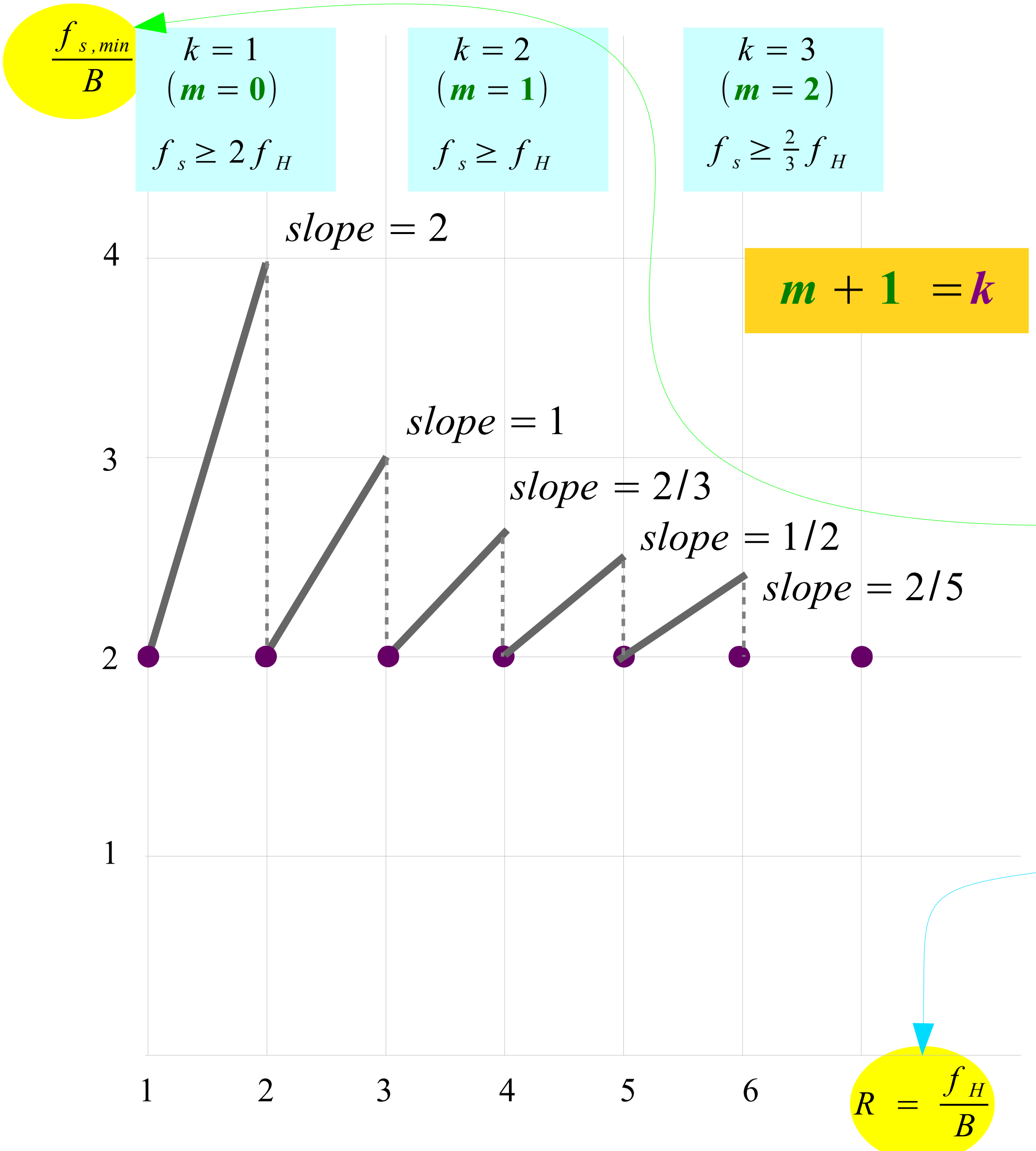
$$g(m,R) = \frac{2}{(m+1)}\frac{f_H}{B} = \frac{2}{(m+1)}R$$

$$R = m+1 \Rightarrow g(m, m+1) = 2$$

$m = 0$	$g(0,R) = 2R$	$slope = 2$
$m = 1$	$g(1,R) = R$	$slope = 1$
$m = 2$	$g(2,R) = \frac{2}{3}R$	$slope = 2/3$
$m = 3$	$g(3,R) = \frac{1}{2}R$	$slope = 1/2$
$m = 4$	$g(4,R) = \frac{2}{5}R$	$slope = 2/5$
$m = 5$	$g(5,R) = \frac{1}{3}R$	$slope = 1/3$
$m = 6$	$g(6,R) = \frac{2}{7}R$	$slope = 2/7$
$m = 7$	$g(7,R) = \frac{1}{4}R$	$slope = 1/4$
$m = 8$	$g(8,R) = \frac{2}{9}R$	$slope = 2/9$

$m = 0$	$R = 1$	➡	$g(0,1) = 2$
$m = 1$	$R = 2$	➡	$g(1,2) = 2$
$m = 2$	$R = 3$	➡	$g(2,3) = 2$
$m = 3$	$R = 4$	➡	$g(3,4) = 2$
$m = 4$	$R = 5$	➡	$g(4,5) = 2$
$m = 5$	$R = 6$	➡	$g(5,6) = 2$
$m = 6$	$R = 7$	➡	$g(6,7) = 2$
$m = 7$	$R = 8$	➡	$g(7,8) = 2$
$m = 8$	$R = 9$	➡	$g(8,9) = 2$

Minimum f_s Plot (5)



$$\frac{2f_c + B}{m+1} \leq f_s \leq \frac{2f_c - B}{m}$$

$$\frac{2f_c + B}{k} \leq f_s \leq \frac{2f_c - B}{k-1}$$

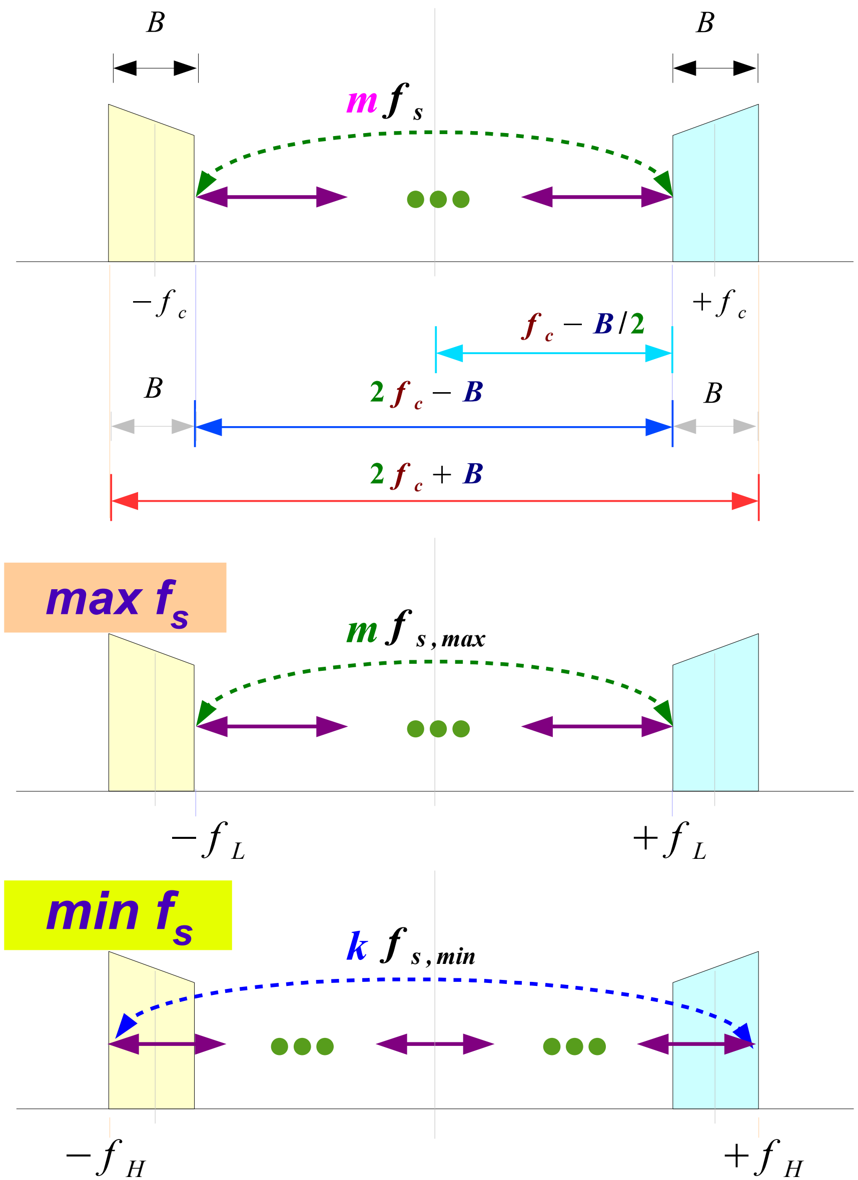
$$\frac{2f_c + B}{(m+1)B} = \frac{f_{s,min}}{B} = g(m, R)$$

➡ $\frac{\text{minimum sampling rate}}{\text{bandwidth B}}$

$$\frac{f_c + B/2}{B} = R$$

➡ $\frac{\text{highest signal frequency}}{\text{bandwidth B}}$

Min, Max Condition on f_s (1)



$$\frac{2f_c + B}{m+1} \le f_s \le \frac{2f_c - B}{m}$$

$$\frac{2f_c + B}{k} \le f_s \le \frac{2f_c - B}{k-1}$$

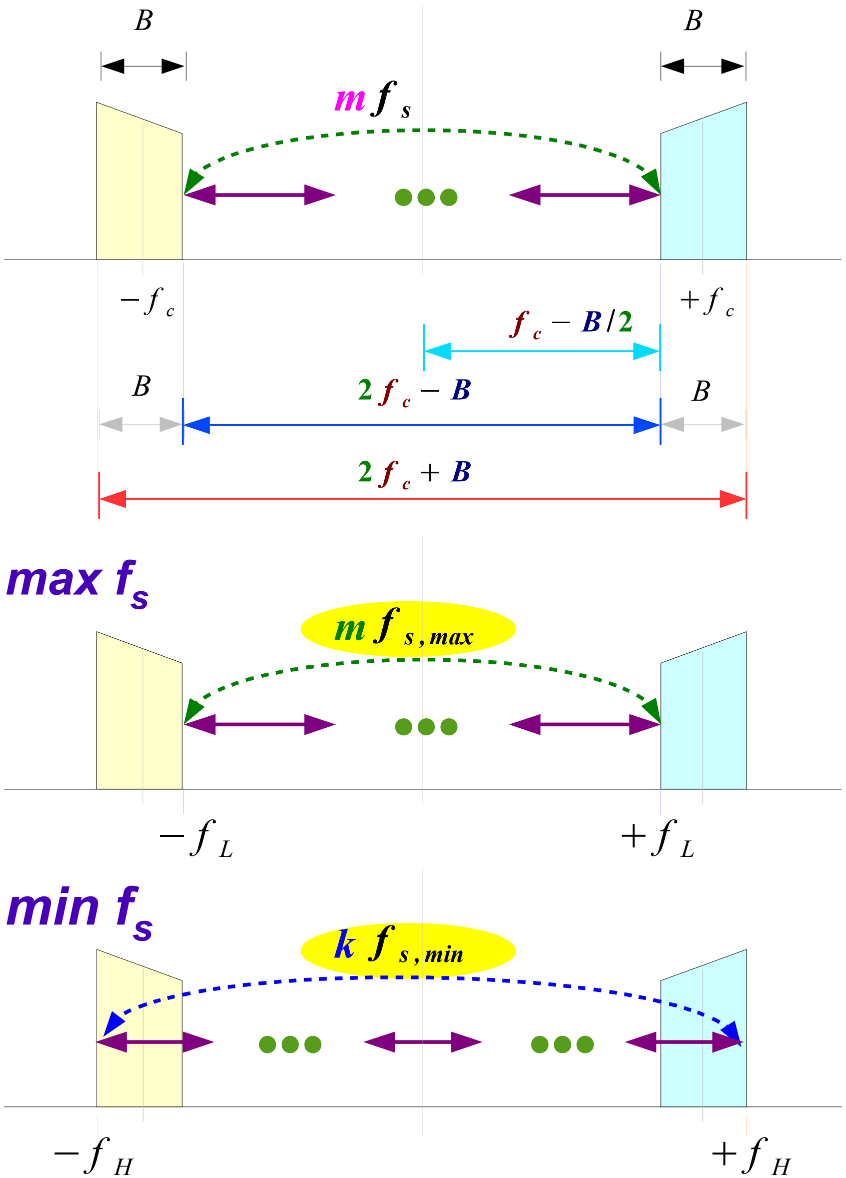
$$m + 1 = k$$

min f_s **max f_s**

$$\frac{2f_H}{k} \le f_s \le \frac{2f_L}{m}$$

$k = 2$	$f_H \le f_s \le 2f_L$	$m = 1$
$k = 3$	$\frac{2}{3}f_H \le f_s \le f_L$	$m = 2$
$k = 4$	$\frac{1}{2}f_H \le f_s \le \frac{2}{3}f_L$	$m = 3$

Min, Max Condition on f_s (2)



min f_s		max f_s
$\frac{2f_H}{k}$	$\le f_s \le$	$\frac{2f_L}{m}$

$$k = m + 1$$

m represents how many f_s are in $2f_c - B$ in max f_s

$$\text{max } f_s = \frac{2f_c - B}{m} = \frac{2f_L}{m}$$

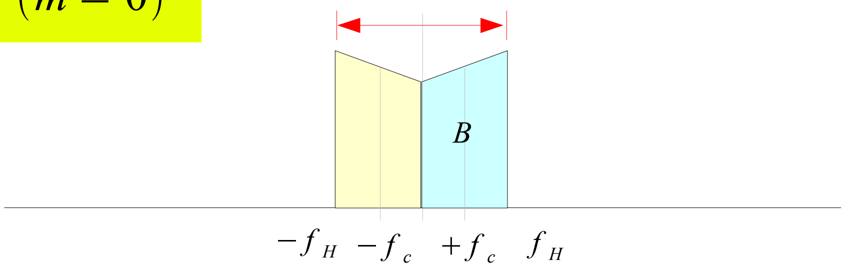
k represents how many f_s are in $2f_c + B$ in min f_s

$$\text{min } f_s = \frac{2f_c + B}{k} = \frac{2f_H}{k}$$

Example k=1 (m=0)

$k = 1$
$(m = 0)$

$f_H = f_c + B/2 = 1B$

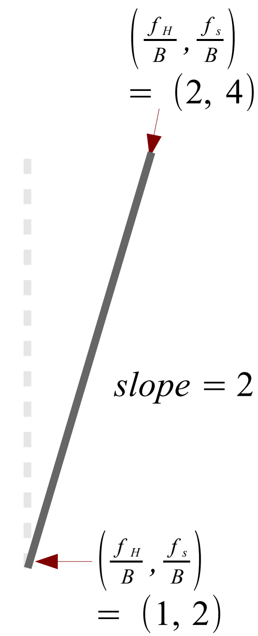


$R = f_H / B = 1$

$R \in [1, 2]$

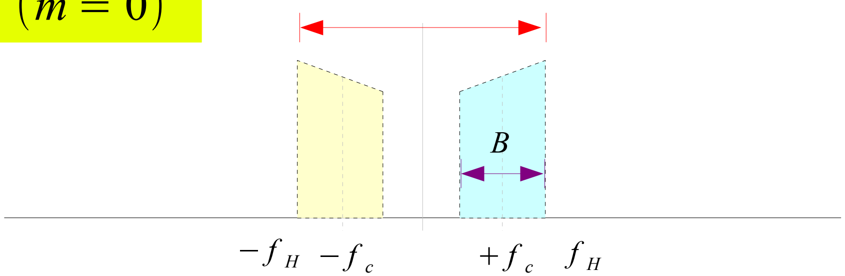
$$\frac{f_{s,min}}{B} = \frac{2 f_H}{k B} = \mathbf{2}$$

$$\frac{f_{s,max}}{B} = \frac{2(f_H - B)}{(k-1) B} = +\infty$$



$k = 1$
$(m = 0)$

$f_H = f_c + B/2 = 1.5B$



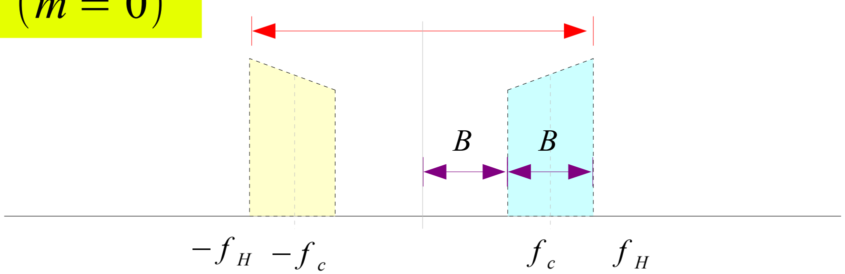
$R = f_H / B = 1.5$

$$\frac{f_{s,min}}{B} = \frac{2 f_H}{k B} = \mathbf{3}$$

$$\frac{f_{s,max}}{B} = \frac{2(f_H - B)}{(k-1) B} = +\infty$$

$k = 1$
$(m = 0)$

$f_H = f_c + B/2 = 2B$



$R = f_H / B = 2$

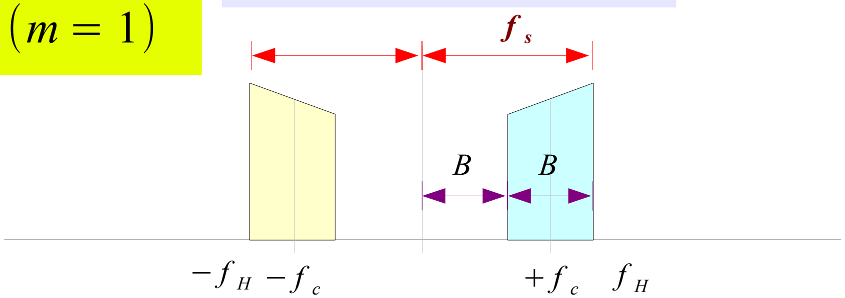
$$\frac{f_{s,min}}{B} = \frac{2 f_H}{k B} = \mathbf{4}$$

$$\frac{f_{s,max}}{B} = \frac{2(f_H - B)}{(k-1) B} = +\infty$$

Example k=2 (m=1)

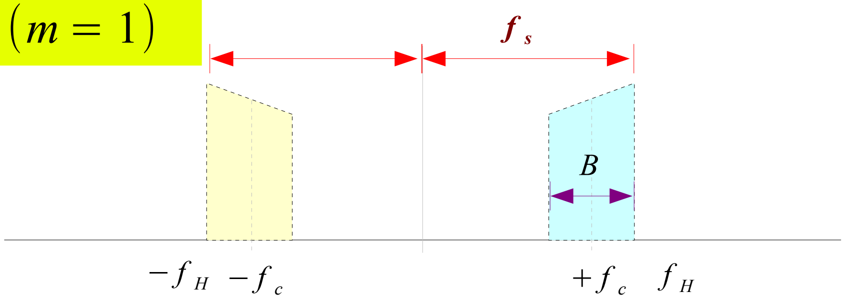
$k = 2$
$(m = 1)$

$f_H = f_c + B/2 = 2B$



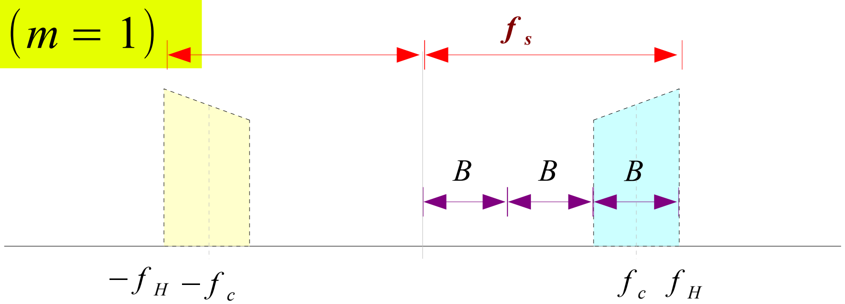
$k = 2$
$(m = 1)$

$f_H = f_c + B/2 = 2.5B$



$k = 2$
$(m = 1)$

$f_H = f_c + B/2 = 3B$



$R = f_H / B = 2$

$$\frac{f_{s,min}}{B} = \frac{2f_H}{kB} = \mathbf{2}$$

$$\frac{f_{s,max}}{B} = \frac{2(f_H - B)}{(k-1)B} = \mathbf{2}$$

$R = f_H / B = 2.5$

$$\frac{f_{s,min}}{B} = \frac{2f_H}{kB} = \mathbf{2.5}$$

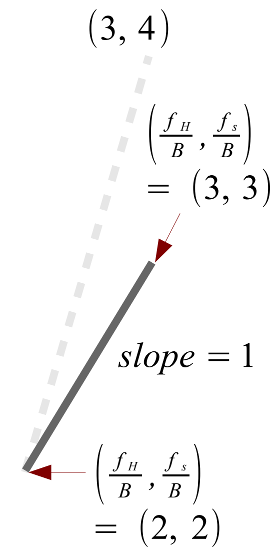
$$\frac{f_{s,max}}{B} = \frac{2(f_H - B)}{(k-1)B} = \mathbf{3}$$

$R = f_H / B = 3$

$$\frac{f_{s,min}}{B} = \frac{2f_H}{kB} = \mathbf{3}$$

$$\frac{f_{s,max}}{B} = \frac{2(f_H - B)}{(k-1)B} = \mathbf{4}$$

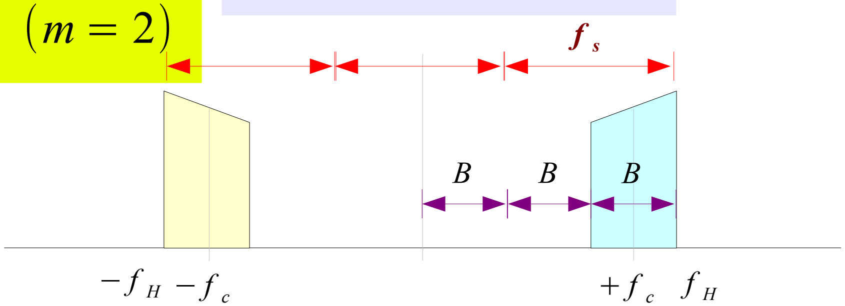
$R \in [2,\ 3]$



Example k=3 (m=2)

$k = 3$
$(m = 2)$

$f_H = f_c + B/2 = 3B$



$R = f_H / B = 3$

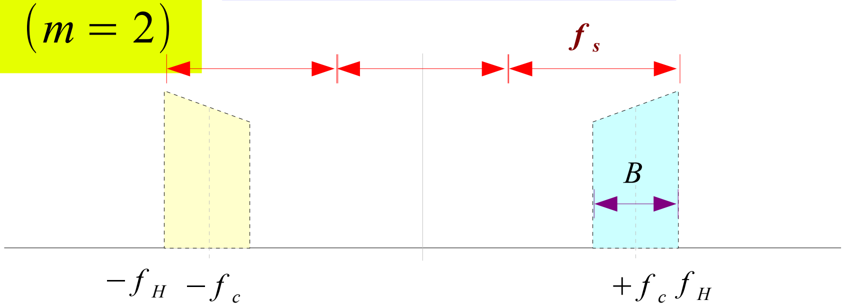
$$\frac{f_{s,min}}{B} = \frac{2 f_H}{k B} = 2$$

$$\frac{f_{s,max}}{B} = \frac{2(f_H - B)}{(k-1) B} = 2$$

$R \in [3,\ 4]$

$k = 3$
$(m = 2)$

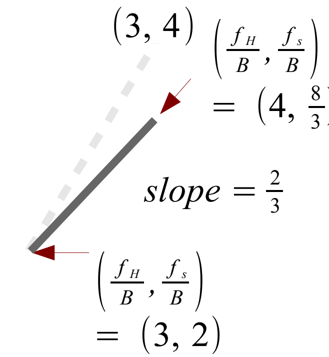
$f_H = f_c + B/2 = 3.5B$



$R = f_H / B = 3.5$

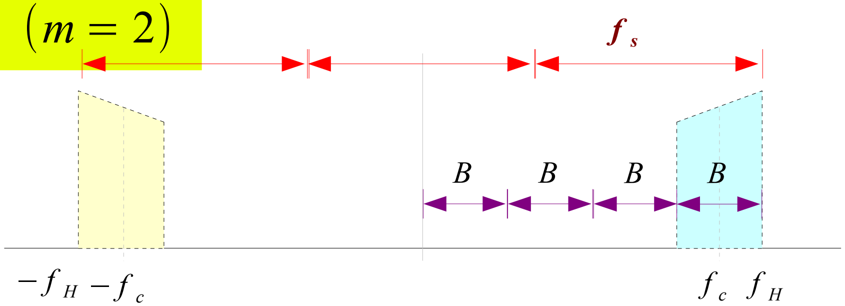
$$\frac{f_{s,min}}{B} = \frac{2 f_H}{k B} = \frac{7}{3}$$

$$\frac{f_{s,max}}{B} = \frac{2(f_H - B)}{(k-1) B} = 3$$



$k = 3$
$(m = 2)$

$f_H = f_c + B/2 = 4B$

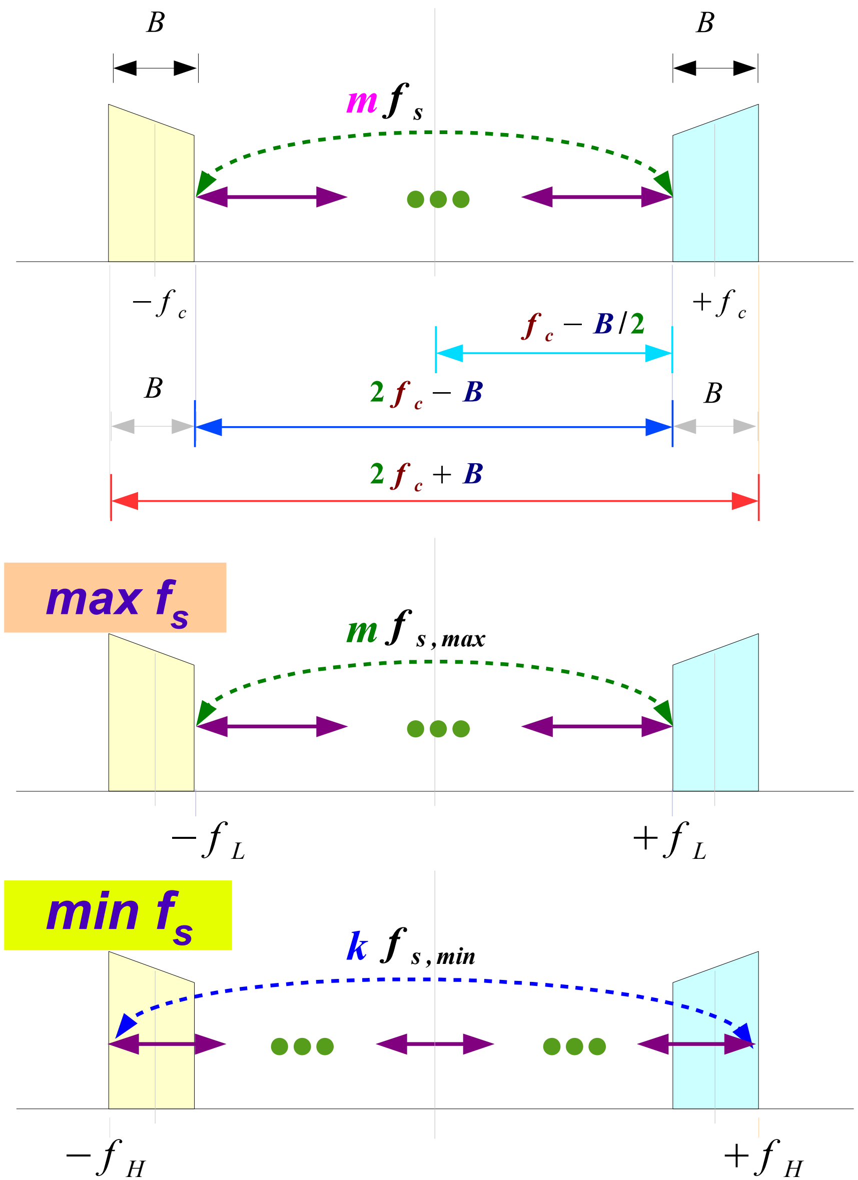


$R = f_H / B = 4$

$$\frac{f_{s,min}}{B} = \frac{2 f_H}{k B} = \frac{8}{3}$$

$$\frac{f_{s,max}}{B} = \frac{2(f_H - B)}{(k-1) B} = 4$$

Min, Max Condition on f_s (2)



$$\frac{2f_c + B}{m+1} \le f_s \le \frac{2f_c - B}{m}$$

$$\frac{2f_c + B}{k} \le f_s \le \frac{2f_c - B}{k-1}$$

$$m + 1 = k$$

min f_s **max f_s**

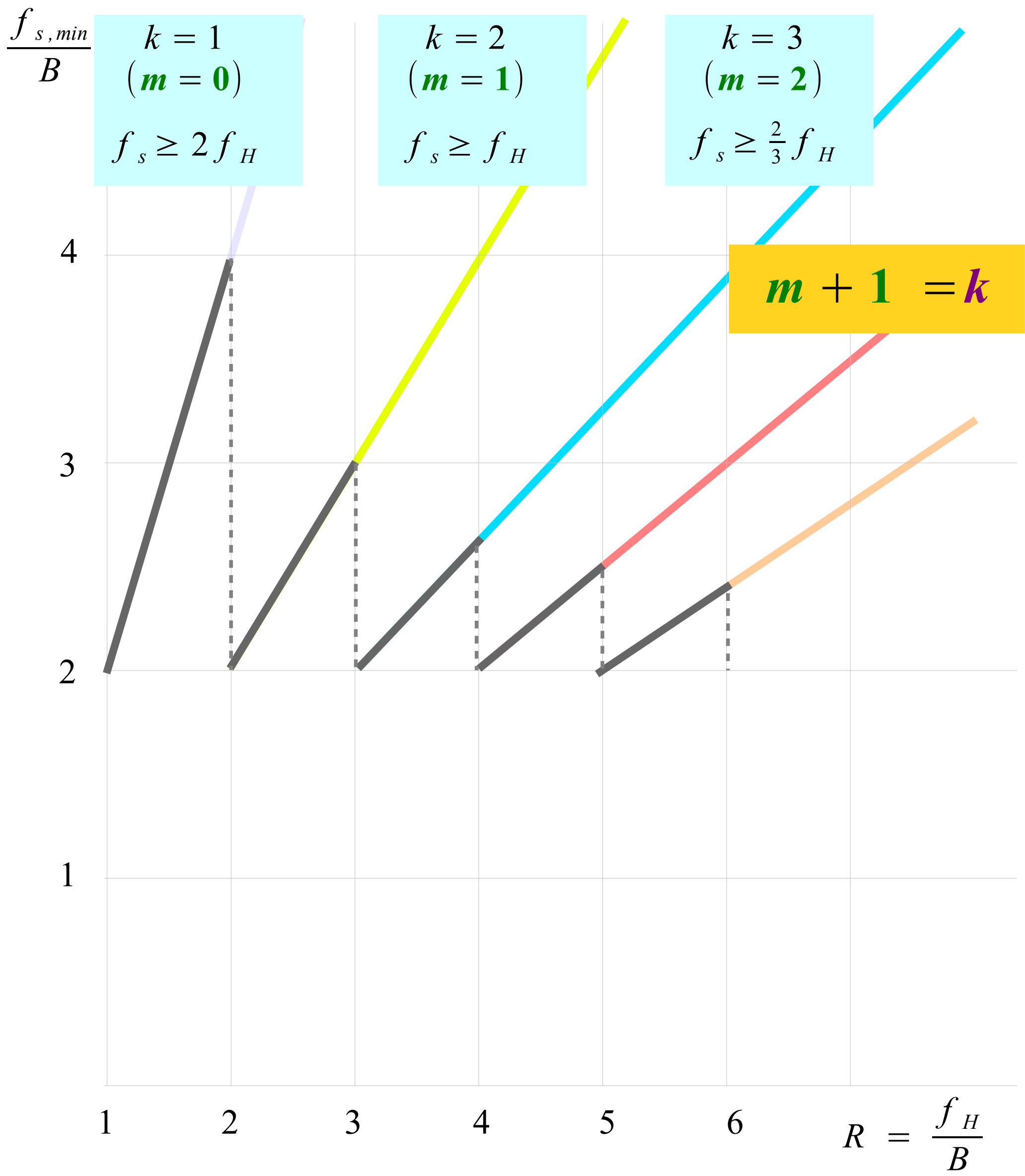
$$\frac{2f_H}{k} \le f_s \le \frac{2f_L}{m}$$

$k = 2 \quad f_H \le f_s \le 2f_L \quad m = 1$

$k = 3 \quad \frac{2}{3}f_H \le f_s \le f_L \quad m = 2$

$k = 4 \quad \frac{1}{2}f_H \le f_s \le \frac{2}{3}f_L \quad m = 3$

Min Max f_s Plot (1)

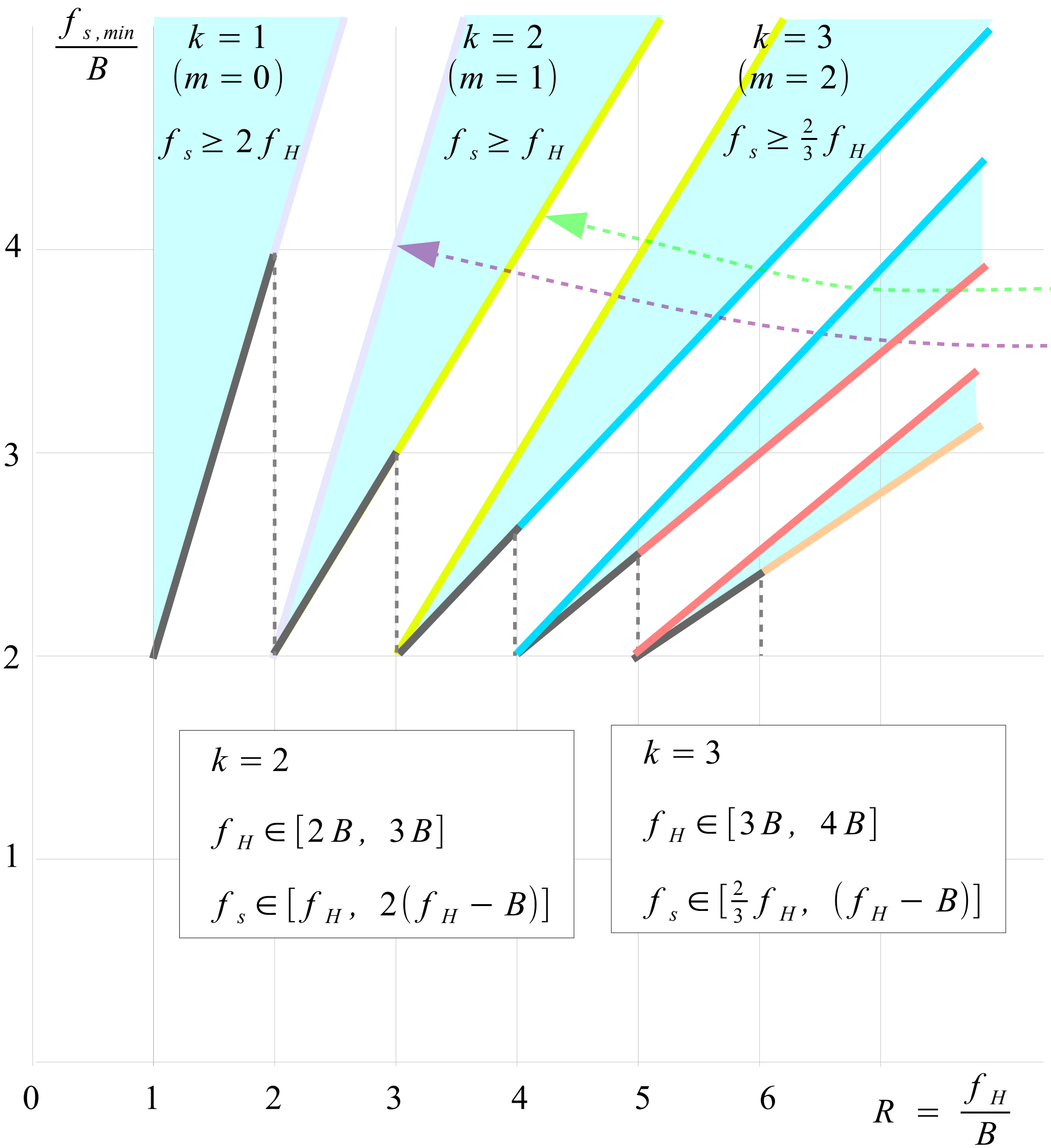


$$\frac{2f_c + B}{m+1} \leq f_s \leq \frac{2f_c - B}{m}$$

$$\frac{2f_c + B}{k} \leq f_s \leq \frac{2f_c - B}{k-1}$$

$$\frac{2f_H}{k} \leq f_s \leq \frac{2f_L}{m}$$

Min Max f_s Plot (2)



$$\frac{2 f_H}{k} \leq f_s \leq \frac{2 f_L}{m}$$

Min f_s **Max f_s**

$k = 2$	$f_H \leq f_s \leq 2 f_L$
$k = 3$	$\frac{2}{3} f_H \leq f_s \leq f_L$

Min f_s		Max f_s
$y = 1(x-2)+2$ $y = x$	$k = 2$	$y = 2(x-2)+2$ $y = 2x-2$
$y = \frac{2}{3}(x-3)+2$ $y = \frac{2}{3}x$	$k = 3$	$y = 1(x-3)+2$ $y = x-1$

Range of f_s (1)

For a given m

$$\frac{2f_c + B}{m+1} \le f_s \le \frac{2f_c - B}{m}$$

Nyquist Criterion

$$2B \le f_s$$

$f_c = 20\ MHz$

$B = 5\ MHz$

	min f_s		max f_s	Optimum Sampling Frequency
$m = 1$	$\frac{2\cdot 20 + 5}{1+1} = 22.5$	$\le f_s \le$	$\frac{2\cdot 20 - 5}{1} = 35$	$f_s = 22.5\ MHz \quad (10 \le f_s)$
$m = 2$	$\frac{2\cdot 20 + 5}{2+1} = 15$	$\le f_s \le$	$\frac{2\cdot 20 - 5}{2} = 17.5$	$f_s = 17.5\ MHz \quad (10 \le f_s)$
$m = 3$	$\frac{2\cdot 20 + 5}{3+1} = 11.25$	$\le f_s \le$	$\frac{2\cdot 20 - 5}{3} = 11.67$	$f_s = 11.25\ MHz \quad (10 \le f_s)$
$m = 4$	$\frac{2\cdot 20 + 5}{4+1} = 9$	$\ge$	$\frac{2\cdot 20 - 5}{4} = 8.75$	X
$m = 5$	$\frac{2\cdot 20 + 5}{5+1} = 7.5$	$\ge$	$\frac{2\cdot 20 - 5}{5} = 7.0$	X

Range of f_s (2)

For a given m

$$\frac{2f_c + B}{m+1} \le f_s \le \frac{2f_c - B}{m}$$

Nyquist Criterion

$$2B \le f_s$$

$f_c = 20\ MHz$

$B = 5\ MHz$

$$\frac{2f_H}{k} \le f_s \le \frac{2f_L}{m}$$

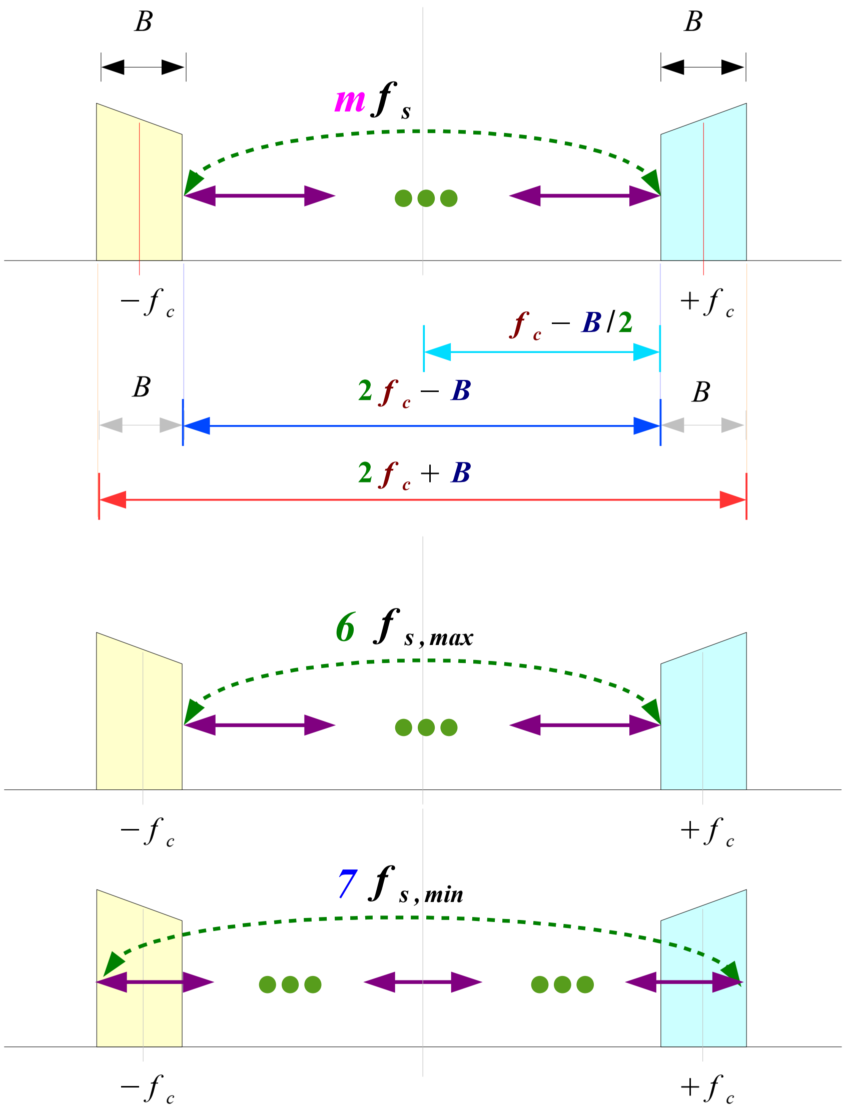
min f_s — max f_s

$f_H = f_c + B/2 = 22.5\ MHz$

$f_L = f_c - B/2 = 17.5\ MHz$

$k=2$	$m=1$	➡	$f_H \le f_s \le 2f_L$	➡ $22.5 \le f_s \le 35$
$k=3$	$m=2$	➡	$\frac{2}{3}f_H \le f_s \le f_L$	➡ $15.0 \le f_s \le 17.5$
$k=4$	$m=3$	➡	$\frac{1}{2}f_H \le f_s \le \frac{2}{3}f_L$	➡ $11.2 \le f_s \le 11.67$
$k=5$	$m=4$	➡	$\frac{2}{5}f_H \le f_s \le \frac{1}{2}f_L$	➡ $9.0 \le f_s \le 8.75$ (X)
$k=6$	$m=5$	➡	$\frac{1}{3}f_H \le f_s \le \frac{2}{5}f_L$	➡ $7.5 \le f_s \le 7.0$ (X)

Example m=6 (1)



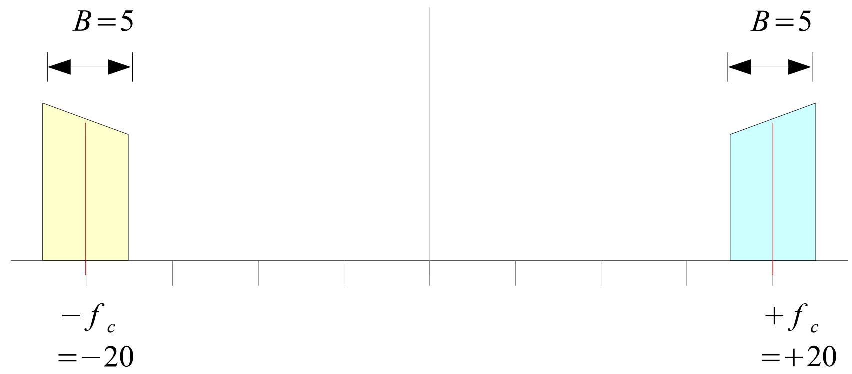
$$\frac{2f_c + B}{m+1} \le f_s \le \frac{2f_c - B}{m}$$

When m = 6

$$\text{min } f_s \quad \frac{2f_c + B}{7} \le f_s \le \frac{2f_c - B}{6} \quad \text{max } f_s$$

$$\text{max } f_s = \frac{2f_c - B}{6}$$

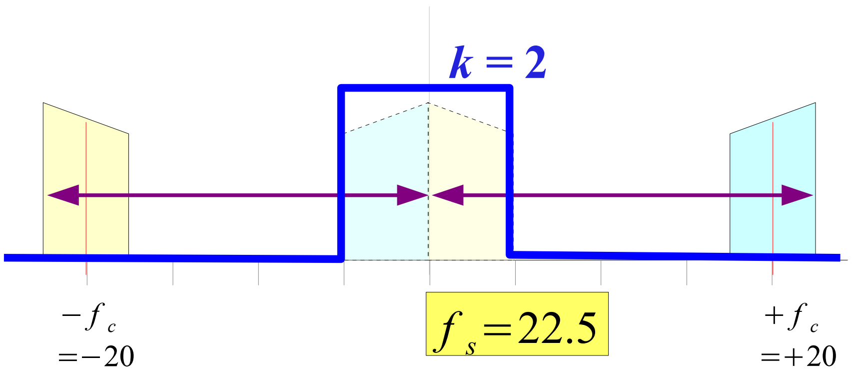
$$\text{min } f_s = \frac{2f_c + B}{7}$$

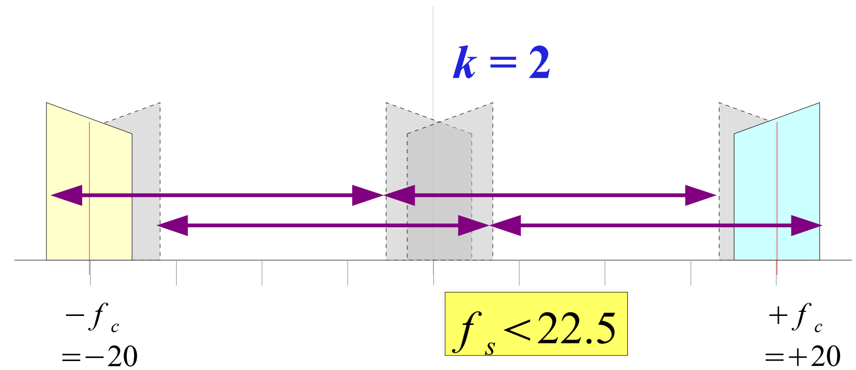


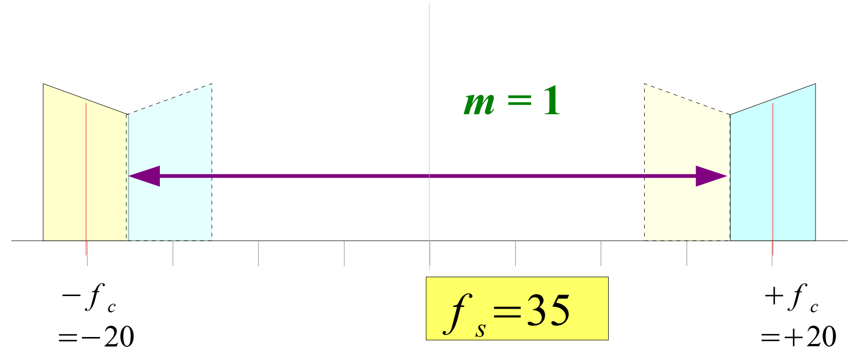
$$\frac{2f_H}{k} \le f_s \le \frac{2f_L}{m}$$

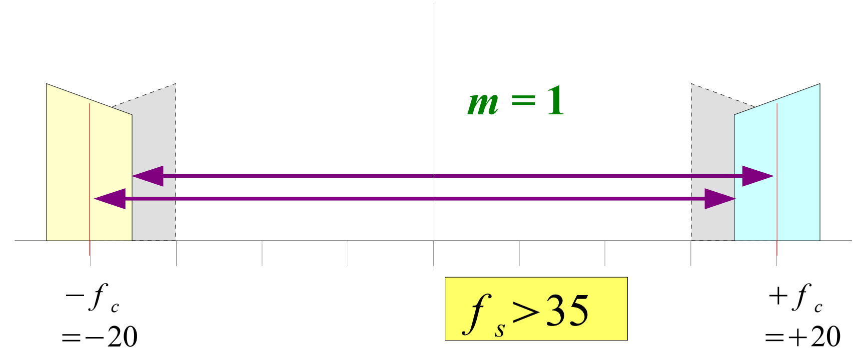
min f_s | $k = 2$ $m = 1$ | max f_s

$$22.5 = f_H \le f_s \le 2f_L = 35$$





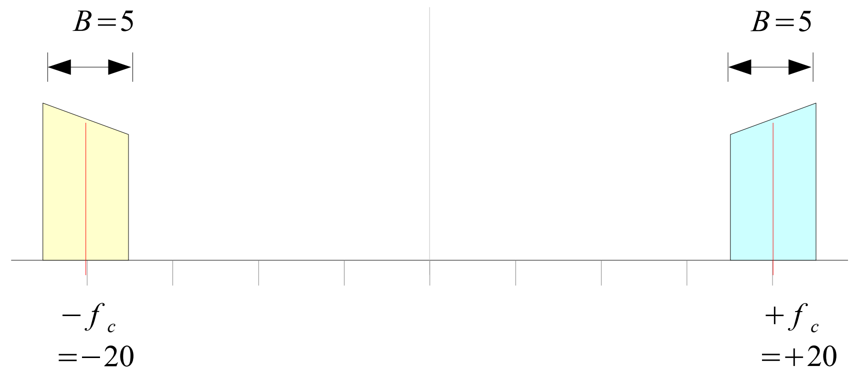


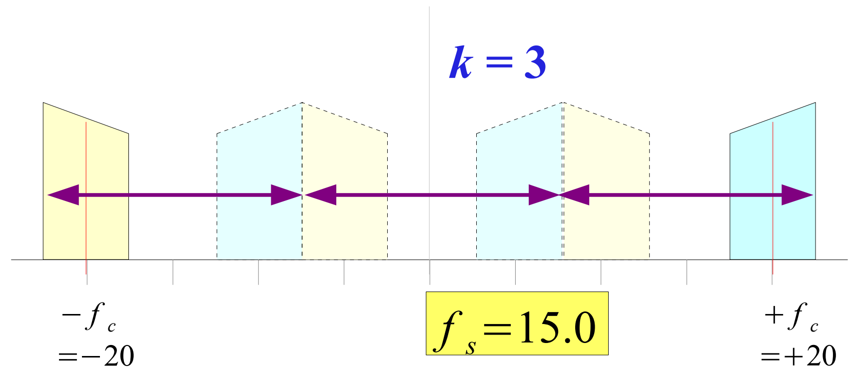


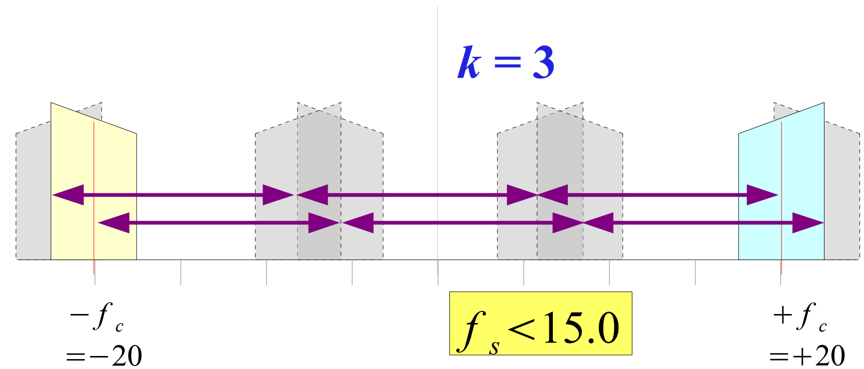
$$\frac{2f_H}{k} \le f_s \le \frac{2f_L}{m}$$

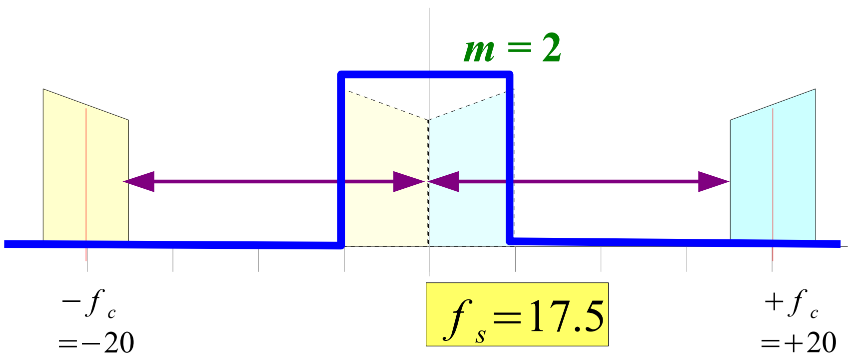
min f_s $k = 3$ $m = 2$ **max f_s**

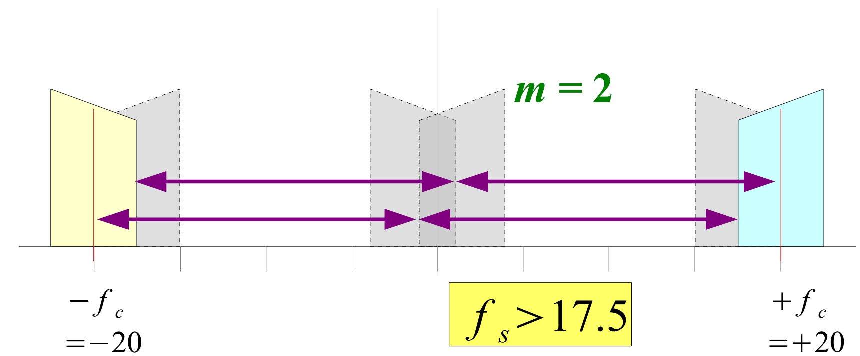
$$15.0 = \frac{2}{3} f_H \le f_s \le f_L = 17.5$$







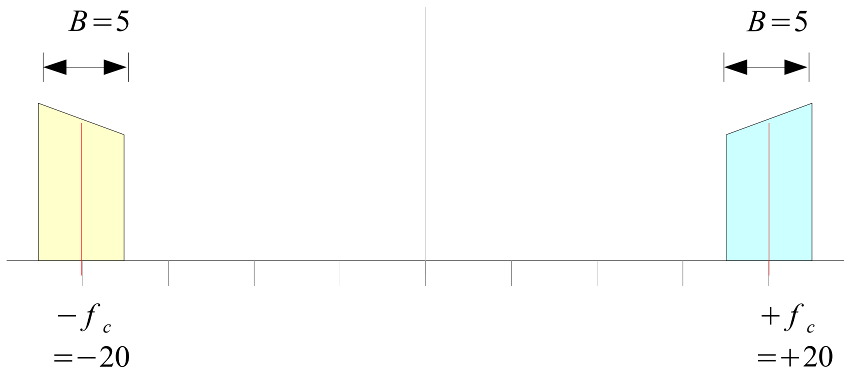


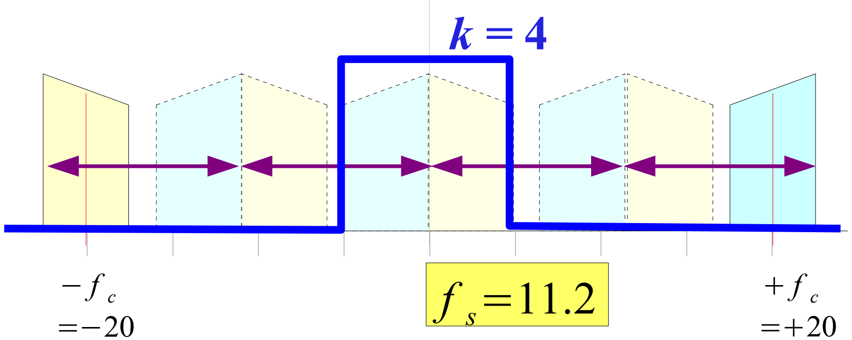


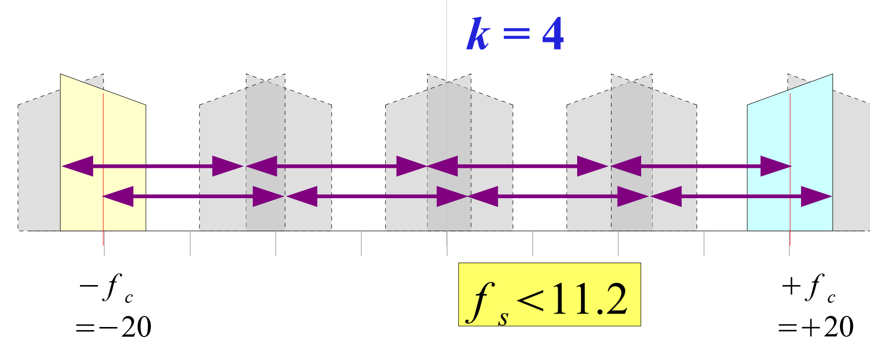
$$\frac{2 f_H}{k} \le f_s \le \frac{2 f_L}{m}$$

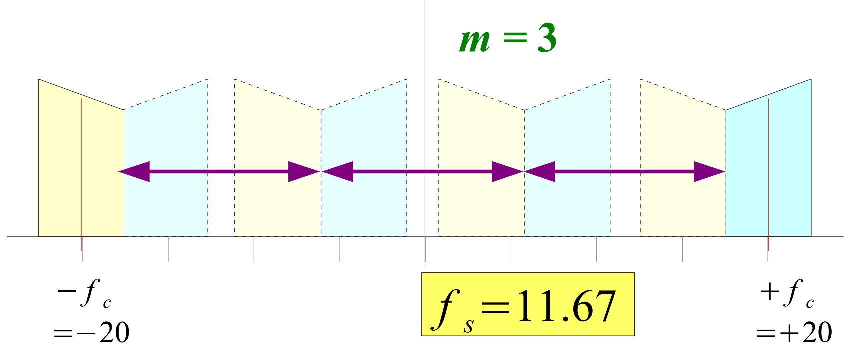
min f_s $k = 4 \quad m = 3$ **max f_s**

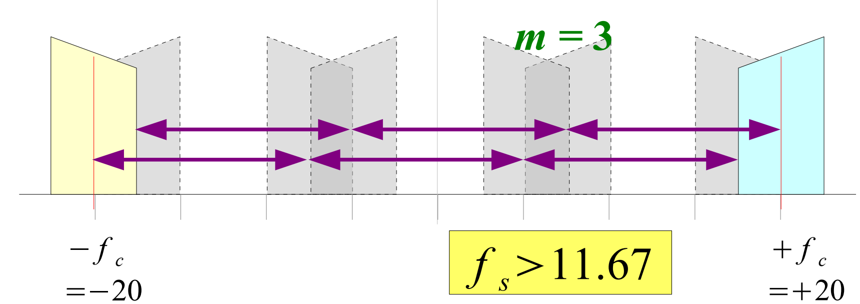
$$11.2 = \frac{1}{2} f_H \le f_s \le \frac{2}{3} f_L = 11.67$$

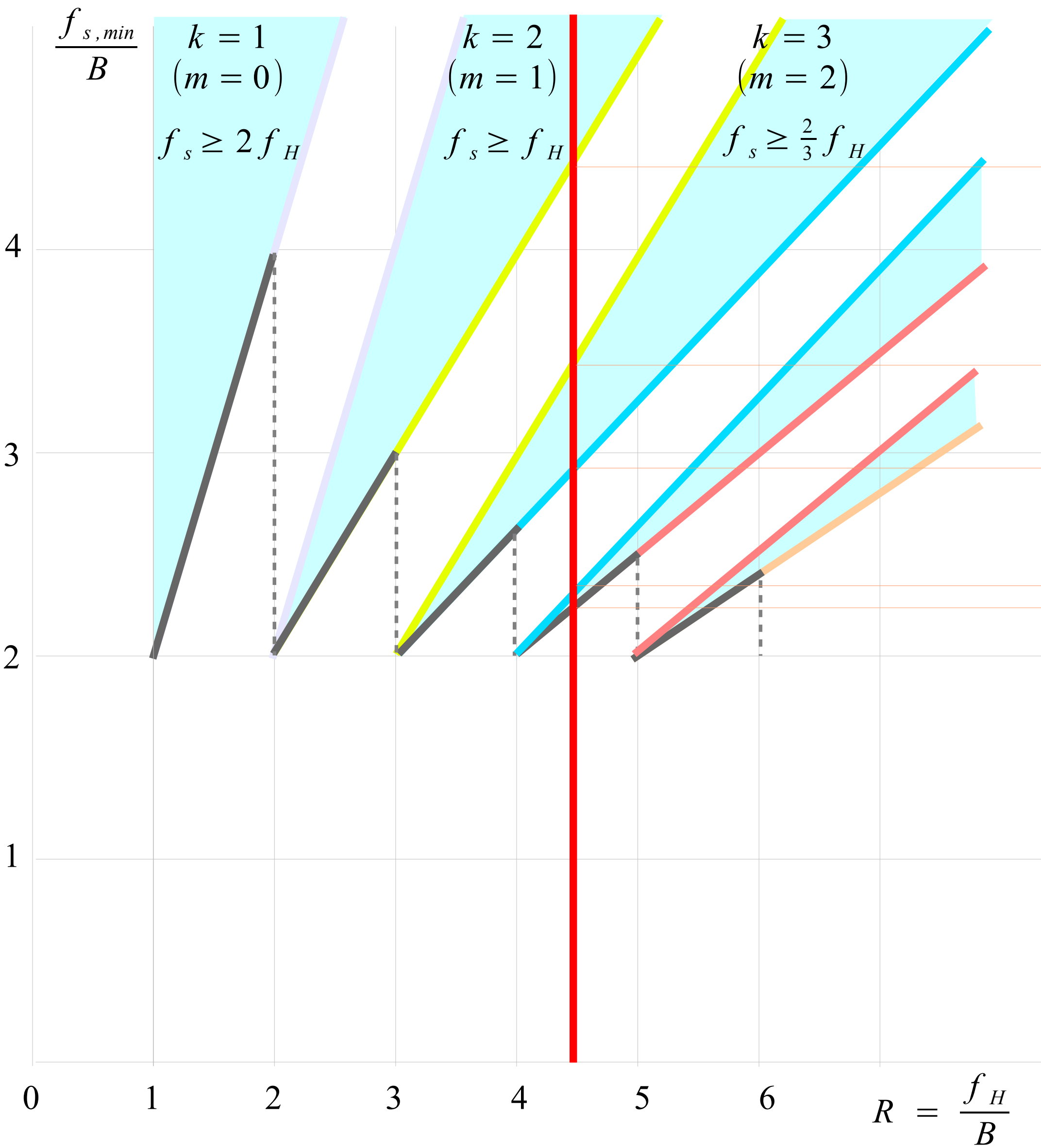












$22.5 \leq f_s$

$f_s \leq 17.5$

$15.0 \leq f_s$

$f_s \leq 11.67$

$11.2 \leq f_s$

$k = 2$ $m = 1$ ➡ $22.5 \leq f_s \leq 35$

$k = 3$ $m = 2$ ➡ $15.0 \leq f_s \leq 17.5$

$k = 4$ $m = 3$ ➡ $11.2 \leq f_s \leq 11.67$

References

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[4] R. G. Lyons, Understanding Digital Signal Processing, 1997