Fourier Series (2A)

- Fourier Series
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Fourier Series

$$f(x) = a_0 + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{+\pi} f(x)\, dx$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x) \cos kx\, dx$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x) \sin kx\, dx$$

$k = 1, 2, 3, \ldots$ ➡ **one-sided spectrum**

only positive frequencies

Trigonometric Identities

$$\cos\theta\,\cos\varphi = \frac{1}{2}\left(\cos(\theta-\varphi) + \cos(\theta+\varphi)\right)$$

$$\sin\theta\,\sin\varphi = \frac{1}{2}\left(\cos(\theta-\varphi) - \cos(\theta+\varphi)\right)$$

$$\sin\theta\,\cos\varphi = \frac{1}{2}\left(\sin(\theta+\varphi) + \sin(\theta-\varphi)\right)$$

$$\cos\theta\,\sin\varphi = \frac{1}{2}\left(\sin(\theta+\varphi) - \sin(\theta-\varphi)\right)$$

$$\frac{1}{2}\left(1 + \cos(\theta+\varphi)\right) \quad \textit{when} \quad \theta=\varphi$$

$$\frac{1}{2}\left(1 - \cos(\theta+\varphi)\right) \quad \textit{when} \quad \theta=\varphi$$

$$\frac{1}{2}\left(\sin(\theta+\varphi)\right) \quad \textit{when} \quad \theta=\varphi$$

$$\frac{1}{2}\left(\sin(\theta+\varphi)\right) \quad \textit{when} \quad \theta=\varphi$$

$$\int_{-\pi}^{+\pi} \cos \boldsymbol{n} x \cos \boldsymbol{m} x \, dx = 0 \quad (n \neq m)$$

$$\int_{-\pi}^{+\pi} \sin \boldsymbol{n} x \sin \boldsymbol{m} x \, dx = 0 \quad (n \neq m)$$

$$\int_{-\pi}^{+\pi} \sin \boldsymbol{n} x \cos \boldsymbol{m} x \, dx = 0$$

$$\int_{-\pi}^{+\pi} \cos \boldsymbol{n} x \sin \boldsymbol{m} x \, dx = 0$$

$$\int_{-\pi}^{+\pi} \cos \boldsymbol{n} x \cos \boldsymbol{m} x \, dx = \pi \quad (n = m)$$

$$\int_{-\pi}^{+\pi} \sin \boldsymbol{n} x \sin \boldsymbol{m} x \, dx = \pi \quad (n = m)$$

$\boldsymbol{n}, \boldsymbol{m}$: integer

Trigonometric Orthogonality

$$f(x) = a_0 + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{+\pi} f(x)\, dx$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{+\pi} \underline{f(x) \cos kx\, dx}$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{+\pi} \underline{f(x) \sin kx\, dx}$$

$$k = 1, 2, 3, \ldots$$

$$\int_{-\pi}^{+\pi} \cos \boldsymbol{n} x \cos \boldsymbol{m} x\, dx = 0 \quad (n \neq m)$$

$$\int_{-\pi}^{+\pi} \sin \boldsymbol{n} x \sin \boldsymbol{m} x\, dx = 0 \quad (n \neq m)$$

$$\int_{-\pi}^{+\pi} \sin \boldsymbol{n} x \cos \boldsymbol{m} x\, dx = 0$$

$$\int_{-\pi}^{+\pi} \cos \boldsymbol{n} x \sin \boldsymbol{m} x\, dx = 0$$

$$\int_{-\pi}^{+\pi} \underline{\cos \boldsymbol{n} x \cos \boldsymbol{m} x}\, dx = \pi \quad (n = m)$$

$$\int_{-\pi}^{+\pi} \underline{\sin \boldsymbol{n} x \sin \boldsymbol{m} x}\, dx = \pi \quad (n = m)$$

$\boldsymbol{n}, \boldsymbol{m}$: integer

$$a_{\boldsymbol{k}} \leftarrow \underline{f(x) \cdot \cos \boldsymbol{k} x} = a_0 \cdot \cos \boldsymbol{k} x + \sum_{m=1}^{\infty} (a_m \underline{\cos \boldsymbol{m} x \cdot \cos \boldsymbol{k} x} + b_m \sin \boldsymbol{m} x \cdot \cos \boldsymbol{k} x)$$

$$b_{\boldsymbol{k}} \leftarrow \underline{f(x) \cdot \sin \boldsymbol{k} x} = a_0 \cdot \sin \boldsymbol{k} x + \sum_{m=1}^{\infty} (a_m \cos \boldsymbol{m} x \cdot \sin \boldsymbol{n} x + b_m \underline{\sin \boldsymbol{m} x \cdot \sin \boldsymbol{k} x})$$

Any Period $p = 2L$

$$g(v) = a_0 + \sum_{k=1}^{\infty} (a_k \cos kv + b_k \sin kv)$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{+\pi} g(v)\, dv$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{+\pi} g(v) \cos kv\, dv$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{+\pi} g(v) \sin kv\, dv$$

$$k = 1,\ 2,\ \ldots$$

$$v: [-\pi,\ +\pi]$$

$$f(x) = a_0 + \sum_{k=1}^{\infty} \left(a_k \cos \frac{k\pi}{L} x + b_k \sin \frac{k\pi}{L} x \right)$$

$$a_0 = \frac{1}{2L} \int_{-L}^{+L} f(x)\, dx$$

$$a_k = \frac{1}{L} \int_{-L}^{+L} f(x) \cos \frac{k\pi x}{L}\, dx$$

$$b_k = \frac{1}{L} \int_{-L}^{+L} f(x) \sin \frac{k\pi x}{L}\, dx$$

$$k = 1,\ 2,\ 3,\ \ldots$$

$$x: [-L,\ +L]$$

$$v = \frac{\pi}{L} x$$

$$dv = \frac{\pi}{L} dx$$

Time and Frequency

$$f(x) = a_0 + \sum_{k=1}^{\infty} \left(a_k \cos \frac{k\pi}{L} x + b_k \sin \frac{k\pi}{L} x \right)$$

$$a_0 = \frac{1}{2L} \int_{-L}^{+L} f(x)\, dx$$

$$a_k = \frac{1}{L} \int_{-L}^{+L} f(x) \cos \frac{k\pi x}{L}\, dx$$

$$b_k = \frac{1}{L} \int_{-L}^{+L} f(x) \sin \frac{k\pi x}{L}\, dx$$

$$k = 1,\ 2,\ 3,\ \ldots$$

$$x:\ [-L,\ +L]$$

$$x(t) = a_0 + \sum_{k=1}^{\infty} \left(a_k \cos \frac{2\pi k}{T} t + b_k \sin \frac{2\pi k}{T} t \right)$$

$$a_0 = \frac{1}{T} \int_{0}^{T} x(t)\, dt$$

$$a_k = \frac{2}{T} \int_{0}^{T} x(t) \cos \frac{2\pi k t}{T}\, dt$$

$$b_k = \frac{2}{T} \int_{0}^{T} x(t) \sin \frac{2\pi k t}{T}\, dt$$

$$k = 1,\ 2,\ \ldots$$

$$t:\ [0,\ T]$$

$$2L = T$$

Continuous Time Periodic Signal $x(t)$

Harmonic Frquency

$$x(t) = a_0 + \sum_{k=1}^{\infty}\left(a_k \cos\frac{2\pi k}{T}t + b_k \sin\frac{2\pi k}{T}t\right)$$

$$a_0 = \frac{1}{T}\int_0^T x(t)\,dt$$

$$a_k = \frac{2}{T}\int_0^T x(t)\cos\frac{2\pi k t}{T}\,dt$$

$$b_k = \frac{2}{T}\int_0^T x(t)\sin\frac{2\pi k t}{T}\,dt$$

$$k = 1,\ 2,\ \ldots$$

$$t:\ [0,\ T]$$

$$x(t) = a_0 + \sum_{k=1}^{\infty}\left(a_k \cos(2\pi \boldsymbol{k} f_0 t) + b_k \sin(2\pi \boldsymbol{k} f_0 t)\right)$$

$$a_0 = \frac{1}{T}\int_0^T x(t)\,dt$$

$$a_k = \frac{2}{T}\int_0^T x(t)\cos(2\pi \boldsymbol{k} f_0 t)\,dt \qquad k = 1,\ 2,\ \ldots$$

$$b_k = \frac{2}{T}\int_0^T x(t)\sin(2\pi \boldsymbol{k} f_0 t)\,dt \qquad k = 1,\ 2,\ \ldots$$

$$t:\ [0,\ T]$$

resolution frequency $f_0 = \frac{1}{T}$

n-th harmonic frequency $f_n = \boldsymbol{n} f_0 = \boldsymbol{n}\frac{1}{T}$

Radial Frequency

$$x(t) = a_0 + \sum_{n=1}^{\infty} \left(a_k \cos(k 2\pi f_0 t) + b_k \sin(k 2\pi f_0 t) \right)$$

$$a_0 = \frac{1}{T} \int_0^T x(t)\, dt$$

$$a_k = \frac{2}{T} \int_0^T x(t) \cos(k 2\pi f_0 t)\, dt \qquad k = 1, 2, \dots$$

$$b_k = \frac{2}{T} \int_0^T x(t) \sin(k 2\pi f_0 t)\, dt \qquad k = 1, 2, \dots$$

$$t : [0, T]$$

$$x(t) = a_0 + \sum_{n=1}^{\infty} \left(a_k \cos(\boldsymbol{k} \omega_0 t) + b_k \sin(\boldsymbol{k} \omega_0 t) \right)$$

$$a_0 = \frac{1}{T} \int_0^T x(t)\, dt$$

$$a_k = \frac{2}{T} \int_0^T x(t) \cos(\boldsymbol{k} \omega_0 t)\, dt$$

$$b_k = \frac{2}{T} \int_0^T x(t) \sin(\boldsymbol{k} \omega_0 t)\, dt$$

$$k = 1, 2, \dots$$

$$t : [0, T]$$

linear frequency f

angular (radial) frequency $\omega = 2\pi f$

Complex Fourier Series Coefficients

$$x(t) = a_0 + \sum_{k=1}^{\infty} \left(a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t) \right)$$

$$a_0 = \frac{1}{T} \int_0^T x(t)\, dt$$

$$a_k = \frac{2}{T} \int_0^T x(t) \cos(k\omega_0 t)\, dt$$

$$b_k = \frac{2}{T} \int_0^T x(t) \sin(k\omega_0 t)\, dt$$

$$k = 1,\ 2,\ \ldots$$

$$t : [0,\ T]$$

$$x(t) = A_0 + \sum_{k=1}^{\infty} \left(A_k e^{jk\omega_0 t} + B_k e^{-jk\omega_0 t} \right)$$

$$A_0 = \frac{1}{T} \int_0^T x(t)\, dt$$

$$A_k = \frac{1}{T} \int_0^T x(t)\, e^{-jk\omega_0 t}\, dt$$

$$B_k = \frac{1}{T} \int_0^T x(t)\, e^{+jk\omega_0 t}\, dt$$

$$t : [0,\ T]$$

Real coefficients

$$a_0,\ a_k,\ b_k,\quad k = 1,\ 2,\ \ldots$$

→ **one-sided spectrum**
only positive frequencies

Complex coefficients

$$A_0,\ A_k,\ B_k,\quad k = 1,\ 2,\ \ldots$$

→ **two-sided spectrum**
Both pos and neg frequencies

Euler Equation (1)

$$x(t) = a_0 + \sum_{k=1}^{\infty} \left(a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t) \right)$$

$$a_0 = \frac{1}{T} \int_0^T x(t)\, dt$$

$$a_k = \frac{2}{T} \int_0^T x(t) \cos(k\omega_0 t)\, dt$$

$$b_k = \frac{2}{T} \int_0^T x(t) \sin(k\omega_0 t)\, dt$$

$$k = 1,\ 2,\ \ldots$$

$$e^{+j\omega t} = \cos\omega t + j\sin\omega t$$

$$e^{-j\omega t} = \cos\omega t - j\sin\omega t$$

$$\cos\omega t = \frac{e^{j\omega t} + e^{-j\omega t}}{2}$$

$$\sin\omega t = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}$$

$$a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t)$$

$$= a_k \frac{1}{2}\left(e^{jk\omega_0 t} + e^{-jk\omega_0 t}\right) + b_k \frac{1}{2j}\left(e^{jk\omega_0 t} - e^{-jk\omega_0 t}\right)$$

$$= \frac{(a_k - jb_k)}{2} e^{jk\omega_0 t} + \frac{(a_k + jb_k)}{2} e^{-jk\omega_0 t}$$

$$= A_k e^{jk\omega_0 t} + B_k e^{-jk\omega_0 t}$$

$$x(t) = A_0 + \sum_{k=1}^{\infty} \left(A_k e^{jk\omega_0 t} + B_k e^{-jk\omega_0 t} \right)$$

zero freq → $A_0 = a_0$

pos freq → $A_k = \frac{1}{2}(a_k - jb_k)$

neg freq → $B_k = \frac{1}{2}(a_k + jb_k)$

only positive frequencies

Euler Equation (2)

$$x(t) = a_0 + \sum_{k=1}^{\infty} \left(a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t) \right)$$

$$a_0 = \frac{1}{T}\int_0^T x(t)\, dt$$

$$a_k = \frac{2}{T}\int_0^T x(t) \cos(k\omega_0 t)\, dt$$

$$b_k = \frac{2}{T}\int_0^T x(t) \sin(k\omega_0 t)\, dt$$

$$k = 1,\ 2,\ \ldots$$

$$x(t) = A_0 + \sum_{k=1}^{\infty} \left(A_k e^{jk\omega_0 t} + B_k e^{-jk\omega_0 t} \right)$$

zero freq → $A_0 = a_0$

pos freq → $A_k = \frac{1}{2}(a_k - j b_k)$

neg freq → $B_k = \frac{1}{2}(a_k + j b_k)$

only positive frequencies

$$A_k = \frac{1}{T}\int_0^T x(t) \left(\cos(k\omega_0 t) - j \sin(k\omega_0 t) \right) dt$$

$$B_k = \frac{1}{T}\int_0^T x(t) \left(\cos(k\omega_0 t) + j \sin(k\omega_0 t) \right) dt$$

$$A_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt$$

$$B_k = \frac{1}{T}\int_0^T x(t)\, e^{+jk\omega_0 t}\, dt$$

$$x(t) = A_0 + \sum_{k=1}^{\infty} \left(A_k e^{+jk\omega_0 t} + B_k e^{-jk\omega_0 t} \right)$$

$$x(t) = \sum_{k=0}^{\infty} \left(A_k e^{+jk\omega_0 t} + B_k e^{-jk\omega_0 t} \right)$$

Complex Fourier Series

$$x(t) = a_0 + \sum_{k=1}^{\infty}\left(a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t)\right)$$

$$a_0 = \frac{1}{T}\int_0^T x(t)\, dt$$

$$a_k = \frac{2}{T}\int_0^T x(t)\cos(k\omega_0 t)\, dt$$

$$b_k = \frac{2}{T}\int_0^T x(t)\sin(k\omega_0 t)\, dt$$

$$k = 1, 2, \ldots$$

$$x(t) = \sum_{k=0}^{\infty}\left(A_k e^{+jk\omega_0 t} + B_k e^{-jk\omega_0 t}\right)$$

$$A_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt$$

$$k = 0, 1, 2, \ldots$$

$$B_k = \frac{1}{T}\int_0^T x(t)\, e^{+jk\omega_0 t}\, dt$$

$$k = 1, 2, \ldots$$

$$x(t) = A_0 + \sum_{k=1}^{\infty}\left(A_k e^{jk\omega_0 t} + B_k e^{-jk\omega_0 t}\right)$$

$$A_0 = a_0$$

$$A_k = \frac{1}{2}(a_k - jb_k)$$

$$B_k = \frac{1}{2}(a_k + jb_k)$$

$$k = 1, 2, \ldots$$

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k e^{+jk\omega_0 t}$$

$$C_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt$$

$$k = -2, -1, 0, +1, +2, \ldots$$

$$C_k = \begin{cases} A_0 & (k = 0) \\ A_k & (k > 0) \\ B_k & (k < 0) \end{cases}$$

Single-Sided Spectrum

$$x(t) = a_0 + \sum_{k=1}^{\infty} \left(a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t) \right)$$

$$a_0 = \frac{1}{T}\int_0^T x(t)\,dt$$

$$a_k = \frac{2}{T}\int_0^T x(t)\cos(k\omega_0 t)\,dt$$

$$b_k = \frac{2}{T}\int_0^T x(t)\sin(k\omega_0 t)\,dt$$

$\boldsymbol{k} = +1, +2, \ldots$

$$x(t) = g_0 + \sum_{k=1}^{\infty} g_k \cos(k\omega_0 t + \varphi_k)$$

$$g_0 = a_0$$

$$g_k = \sqrt{a_k^2 + b_k^2}$$

$$\varphi_k = \tan^{-1}\left(-\frac{b_k}{a_k}\right)$$

$\boldsymbol{k} = +1, +2, \ldots$

$$\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

$$g_k \cos(k\omega_0 t + \varphi_k) = g_k \cos(\varphi_k)\cos(k\omega_0 t) - g_k \sin(\varphi_k)\sin(k\omega_0 t)$$

$$a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t)$$

$$a_k = g_k \cos(\varphi_k)$$

$$-b_k = g_k \sin(\varphi_k)$$

$$a_k^2 + b_k^2 = g_k^2$$

$$-\frac{b_k}{a_k} = \tan(\varphi_k)$$

Phasor Representation (1)

$$x(t) = a_0 + \sum_{k=1}^{\infty} \left(a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t) \right)$$

$$a_0 = \frac{1}{T}\int_0^T x(t)\,dt$$

$$a_k = \frac{2}{T}\int_0^T x(t)\cos(k\omega_0 t)\,dt$$

$$b_k = \frac{2}{T}\int_0^T x(t)\sin(k\omega_0 t)\,dt$$

$$k = 1,\ 2,\ \ldots$$

$$x(t) = g_0 + \sum_{k=1}^{\infty} g_k \cos(k\omega_0 t + \varphi_k)$$

$$g_0 = a_0$$

$$g_k = \sqrt{a_k^2 + b_k^2}$$

$$\varphi_k = \tan^{-1}\left(-\frac{b_k}{a_k}\right)$$

$$k = 1,\ 2,\ \ldots$$

$$x(t) = g_0 + \sum_{k=1}^{\infty} g_k \cos(k\omega_0 t + \varphi_k)$$

$$x(t) = g_0 + \sum_{k=1}^{\infty} g_k\, \Re\left\{e^{+j(k\omega_0 t + \varphi_k)}\right\}$$

$$x(t) = g_0 + \sum_{k=1}^{\infty} \Re\left\{g_k \cdot e^{+j\varphi_k} \cdot e^{+jk\omega_0 t}\right\}$$

$$x(t) = X_0 + \sum_{k=1}^{\infty} \Re\left\{X_k e^{+jk\omega_0 t}\right\}$$

$$X_0 = g_0$$

$$X_k = g_k \cdot e^{+j\varphi_k}$$

$$k = 1,\ 2,\ \ldots$$

Phasor Representation (2)

$$x(t) = g_0 + \sum_{k=1}^{\infty} \frac{g_k}{2} \cdot \left(e^{+j(k\omega_0 t + \varphi_k)} + e^{-j(k\omega_0 t + \varphi_k)}\right)$$

$$x(t) = g_0 + \sum_{k=1}^{\infty} \left(\frac{g_k}{2} e^{+j\varphi_k} e^{+jk\omega_0 t} + \frac{g_k}{2} e^{-j\varphi_k} e^{-jk\omega_0 t} \right)$$

$$x(t) = g_0 + \sum_{k=1}^{\infty} \left(\frac{g_k e^{+j\varphi_k}}{2} e^{+jk\omega_0 t} + \frac{g_k e^{-j\varphi_k}}{2} e^{-jk\omega_0 t} \right)$$

$$x(t) = g_0 + \sum_{k=1}^{\infty} g_k \cos(k\omega_0 t + \varphi_k)$$

$$g_0 = a_0$$

$$g_k = \sqrt{a_k^2 + b_k^2}$$

$$\varphi_k = \tan^{-1}\left(-\frac{b_k}{a_k}\right)$$

$$k = 1,\ 2,\ \ldots$$

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k\, e^{+jk\omega_0 t}$$

$$C_k = \frac{g_k e^{+j\varphi_k}}{2} \qquad (k > 0)$$

$$C_{-k} = \frac{g_k e^{-j\varphi_k}}{2} \qquad (k < 0)$$

$$k = -2,\ -1,\ 0,\ +1,\ +2,\ \ldots$$

$$x(t) = X_0 + \sum_{k=1}^{\infty} \mathfrak{R}\{X_k e^{+jk\omega_0 t}\}$$

$$X_0 = g_0$$

$$X_k = g_k \cdot e^{+j\varphi_k}$$

$$k = 1,\ 2,\ \ldots$$

Two-Sided Spectrum

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k \, e^{+jk\omega_0 t}$$

$$C_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt$$

$$k = \ldots, -2, -1, 0, +1, +2, \ldots$$

$$C_k = \begin{cases} a_0 & (k = 0) \\ \frac{1}{2}(a_k - jb_k) & (k > 0) \\ \frac{1}{2}(a_k + jb_k) & (k < 0) \end{cases}$$

$$|C_k| = \begin{cases} a_0 & (k = 0) \\ \frac{1}{2}\sqrt{a_k^2 + b_k^2} & (k \neq 0) \end{cases}$$

$$Arg(C_k) = \begin{cases} \tan^{-1}(-b_k/a_k) & (k > 0) \\ \tan^{-1}(+b_k/a_k) & (k < 0) \end{cases}$$

Power Spectrum *Two-Sided*

$$|C_k|^2 + |C_{-k}|^2 = \frac{1}{2}|g_k|^2 = \frac{1}{2}(a_k^2 + b_k^2)$$

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k \, e^{+jk\omega_0 t}$$

$$C_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt$$

$$k = \ldots, -2, -1, 0, +1, +2, \ldots$$

$$C_k = \begin{cases} a_0 & (k = 0) \\ \frac{1}{2} g_k \, e^{+j\varphi_k} & (k > 0) \\ \frac{1}{2} g_k \, e^{-j\varphi_k} & (k < 0) \end{cases}$$

$$|C_k| = \begin{cases} a_0 & (k = 0) \\ \frac{1}{2}|g_k| & (k \neq 0) \end{cases}$$

$$Arg(C_k) = \begin{cases} +\varphi_k & (k > 0) \\ -\varphi_k & (k < 0) \end{cases}$$

Periodogram *One-Sided*

$$2 \cdot |C_k| = |g_k| = \sqrt{a_k^2 + b_k^2}$$

Square Wave CTFS (1)

Continuous Time Fourier Series

$$C_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt \quad \Longleftrightarrow \quad x(t) = \sum_{n=0}^{\infty} C_k\, e^{+jk\omega_0 t}$$

$$C_k = \frac{1}{T_0}\int_{-T_0/2}^{+T_0/2} x_{T_0}(t)\, e^{-jk\omega_0 t}\, dt$$

$$C_k T_0 = \int_{-T_0/2}^{+T_0/2} x_{T_0}(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \int_{-T_0/2}^{+T_0/2} e^{-jk\omega_0 t}\, dt = \left[\frac{-1}{jk\omega_0}\, e^{-jk\omega_0 t}\right]_{-T/2}^{+T/2}$$

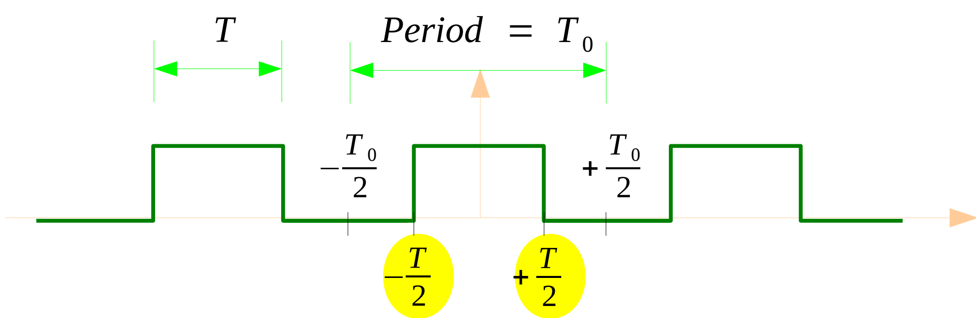
$$= -\frac{e^{-jk\omega_0 T/2} - e^{+jk\omega_0 T/2}}{jk\omega_0} = \frac{\sin(k\omega_0 T/2)}{k\omega_0/2}$$

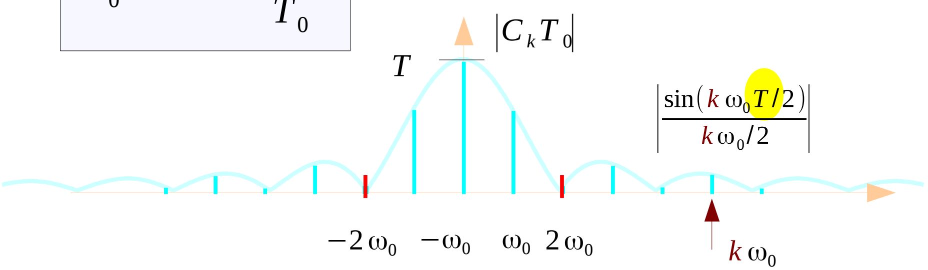
Fundamental Frequency

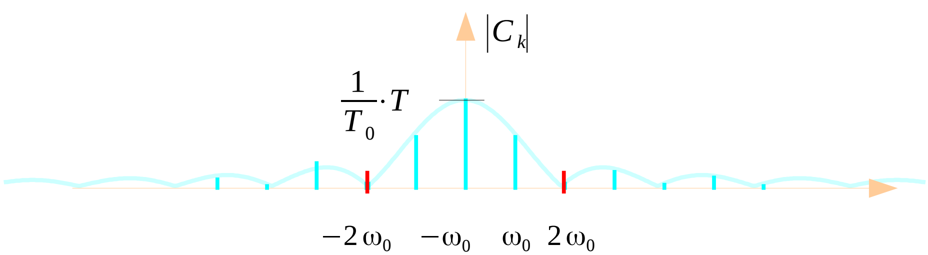
$$\omega_0 = \frac{2\pi}{T_0}$$

$$\omega_0 T_0 = 2\pi \qquad \omega_0 T = \pi \qquad \Leftarrow \qquad \frac{T_0}{T} = \frac{2}{1}$$

$$\omega_0 T = 2\pi \frac{T}{T_0}$$







Square Wave CTFS (2)

$$C_k = \frac{1}{T_0}\cdot\frac{\sin(k\omega_0 T/2)}{k\omega_0/2} = \frac{1}{T_0}\cdot\frac{\sin(T k\omega_0/2)}{k\omega_0/2}$$

$$\omega_0 = \frac{2\pi}{T_0}$$

$$\omega_0\frac{T_0}{T} = \frac{2\pi}{T}$$

$$\omega_0 T = 2\pi\frac{T}{T_0}$$

$$C_k = 0 \quad \Rightarrow \quad \sin(k\omega_0 T/2) = 0$$

$$\sin\left(k\frac{2\pi}{T_0}\frac{T}{2}\right) = 0 \quad \Rightarrow \quad \sin(\pm n\pi) = 0$$

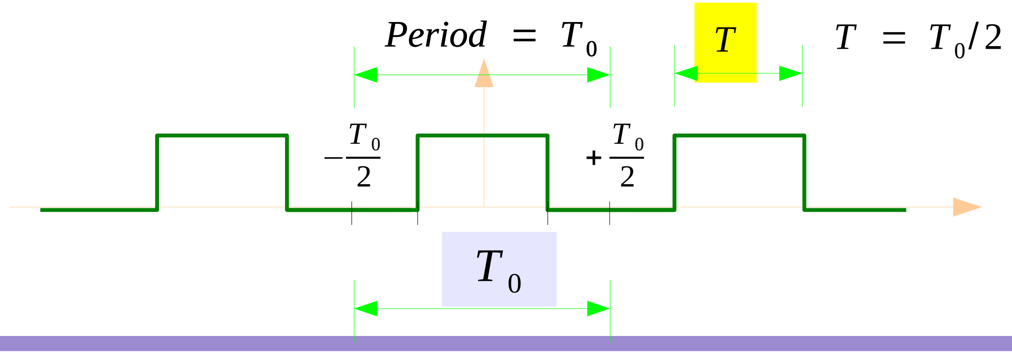
$$k = \pm n\frac{T_0}{T} \quad \Rightarrow \quad \omega = k\omega_0 = \pm n\frac{T_0}{T}\omega_0$$

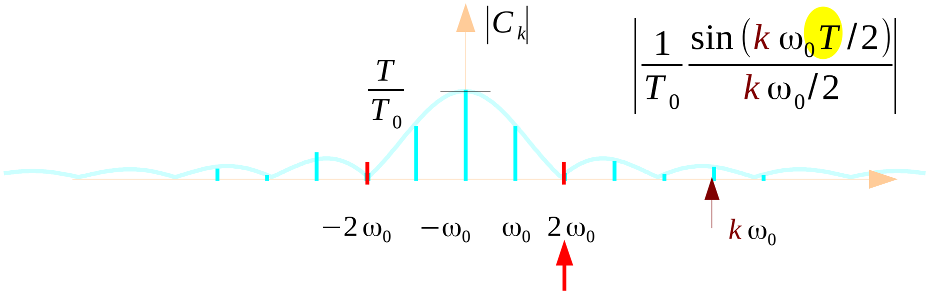
$$C_k = \frac{1}{T_0}\cdot\frac{\sin(k\omega_0 T/2)}{k\omega_0/2}$$

$$C_0 = \frac{T}{T_0}$$

$$C_0 = \lim_{k\to 0}\frac{1}{T_0}\cdot\frac{\sin(T k\omega_0/2)}{k\omega_0/2}$$

$$C_0 = \lim_{k\to 0}\frac{1}{T_0}\cdot\frac{(T\omega_0/2)\cos(T k\omega_0/2)}{\omega_0/2} = \frac{T}{T_0}$$





Square Wave CTFS (3)

$$C_k = \frac{1}{T_0} \cdot \frac{\sin(k\omega_0 T/2)}{k\omega_0/2}$$

$$C_0 = \frac{T}{T_0}$$

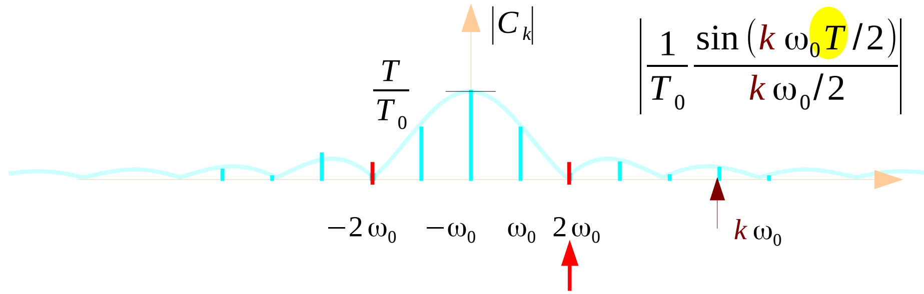
$$\omega_0 = \frac{2\pi}{T_0}$$

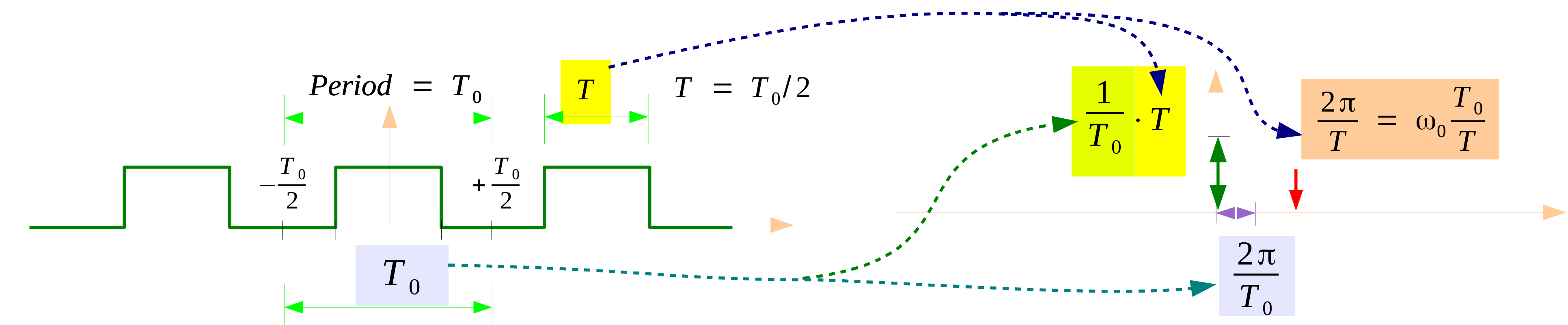
$$\omega_0 \frac{T_0}{T} = \frac{2\pi}{T}$$

$$\omega_0 T = 2\pi \frac{T}{T_0}$$

$$\omega = k\omega_0 = \pm n \frac{T_0}{T} \omega_0$$

$$C_k = 0 \quad \left(k = \pm n \frac{T_0}{T} \right)$$





Square Wave CTFS (4)

Continuous Time Fourier Series

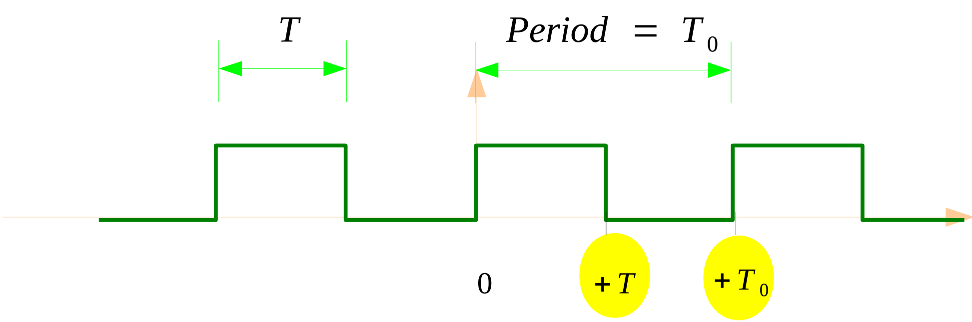
$$C_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt \quad \Leftrightarrow \quad x(t) = \sum_{n=0}^{\infty} C_k\, e^{+jk\omega_0 t}$$

$$C_k = \frac{1}{T_0}\int_0^{+T_0} x_{T_0}(t)\, e^{-jk\omega_0 t}\, dt$$

$$C_k T_0 = \int_0^{+T_0} x_{T_0}(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \int_0^{+T} e^{-jk\omega_0 t}\, dt = \left[\frac{-1}{jk\omega_0} e^{-jk\omega_0 t}\right]_0^{+T}$$

$$= -\frac{e^{-jk\omega_0 T} - e^0}{jk\omega_0} = \frac{1 - e^{-jk\omega_0 T}}{jk\omega_0} = \frac{1 - e^{-jk2\pi T/T_0}}{jk2\pi/T_0}$$



Fundamental Frequency

$$\omega_0 = \frac{2\pi}{T_0}$$

$$\omega_0 T_0 = 2\pi$$
$$\omega_0 T = \pi$$

$$\frac{T_0}{T} = \frac{2}{1}$$

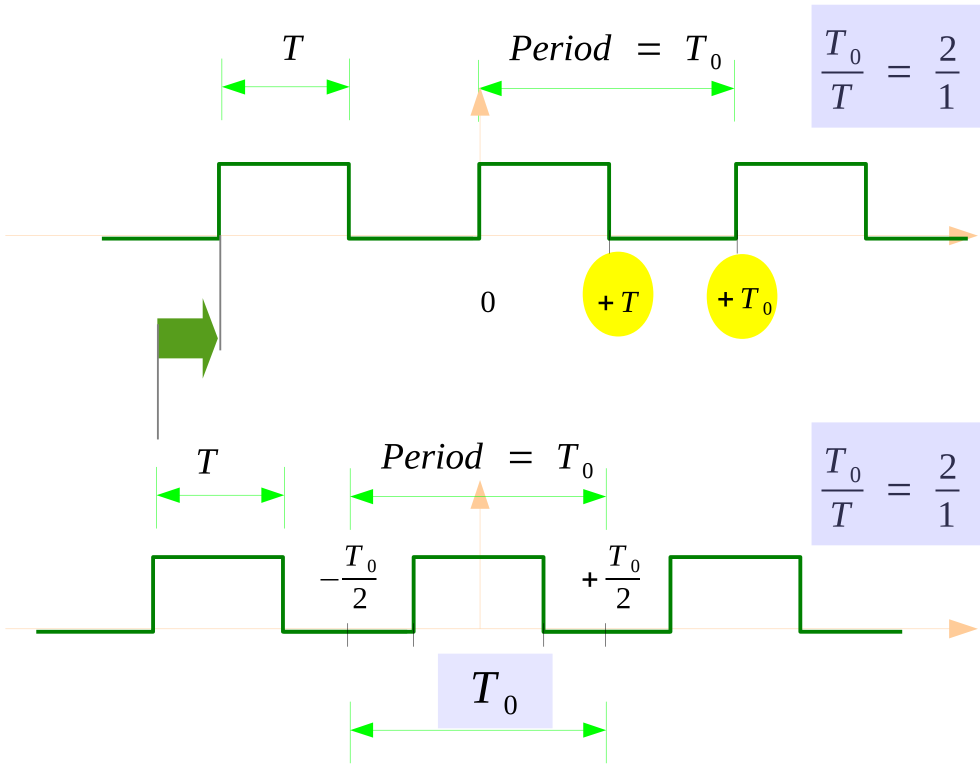
$$\omega_0 T = 2\pi \frac{T}{T_0}$$

$$C_k T_0 = \frac{1 - e^{-jk2\pi T/T_0}}{jk2\pi/T_0} \Rightarrow C_k = \frac{1-(-1)^k}{j2\pi k}$$

$$C_0 T_0 = \int_0^{+T} e^{-j0\omega_0 t}\, dt = T \Rightarrow C_0 = \frac{1}{2}$$

C_{-4}	C_{-3}	C_{-2}	C_{-1}	C_0	C_1	C_2	C_3	C_4
0	$\frac{-1}{j3\pi}$	0	$\frac{-1}{j\pi}$	$\frac{1}{2}$	$\frac{1}{j\pi}$	0	$\frac{1}{j3\pi}$	0

Square Wave CTFS (5)



$$C_k = \frac{1 - e^{-jk2\pi T/T_0}}{jk2\pi}$$

$$C_0 = \frac{T}{T_0}$$

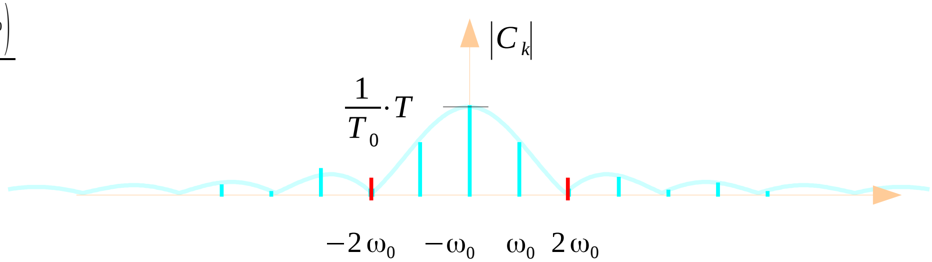
$\Rightarrow$ $e^{+jk\omega_0 T/2}$

$$C_k = \frac{1}{T_0}\cdot\frac{\sin(k\omega_0 T/2)}{k\omega_0/2}$$

$$C_0 = \frac{T}{T_0}$$

$$\frac{1}{T_0}\cdot\frac{\sin(k\omega_0 T/2)}{k\omega_0/2} = \frac{\sin(k\pi T/T_0)}{k\pi} = \frac{\left(e^{+jk\pi T/T_0} - e^{-jk\pi T/T_0}\right)}{jk2\pi}$$

$$= e^{+jk\pi T/T_0}\left(\frac{1 - e^{-jk2\pi T/T_0}}{jk2\pi}\right) = e^{+jk\omega_0 T/2}\left(\frac{1 - e^{-jk2\pi T/T_0}}{jk2\pi}\right)$$



Inner Product Space

Hilbert Space real / complex inner product space

$$\langle f , g \rangle = \int_a^b f(t)\overline{g(t)}\,dt$$

complex conjugate

$$\langle y , x \rangle = \overline{\langle x , y \rangle}$$

linear

$$\langle a x_1 + b x_2 , y \rangle = a \langle x_1 , y \rangle + b \langle x_2 , y \rangle$$

positive semidefinite

$$\langle x , x \rangle \geq 0$$

Norm

$$\|x\| = \sqrt{\langle x , x \rangle}$$

Cauchy-Schwartz Inequality

$$|\langle x , y \rangle| \leq \|x\| \, \|y\|$$

Orthogonality

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k \, e^{+jk\omega_0 t}$$

$$C_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt$$

$k = \ldots, -2, -1, 0, +1, +2, \ldots$

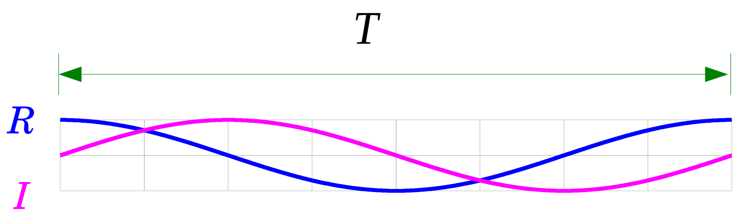
fundamental frequency $f_0 = \frac{1}{T}$ $\quad \omega_0 = 2\pi f_0 = \frac{2\pi}{T}$

n-th harmonic frequency $f_n = n\, f_0$ $\quad \omega_n = 2\pi f_n = \frac{2\pi n}{T}$

$$\langle e^{jm\omega_0 t}, \; e^{jn\omega_0 t}\rangle = \int_0^T e^{+j(m-n)\omega_0 t}\, dt = \begin{cases} 0 & (m \neq n) \\ T & (m = n) \end{cases}$$

m, n : integer

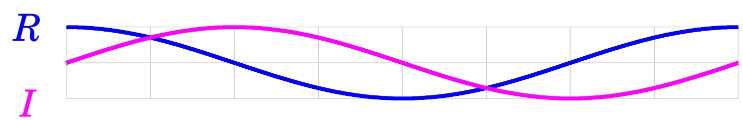
Inner Product Examples



$$f_0 = 1/T \qquad \omega_0 = 2\pi/T$$

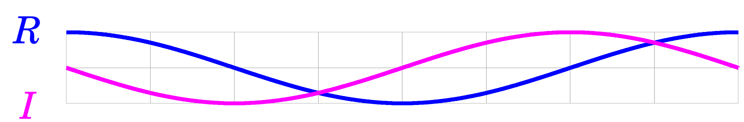
$e^{j1\omega_0 t}$

$$\langle e^{j1\omega_0 t},\ e^{j1\omega_0 t}\rangle = \int_0^T e^{+j(1-1)\omega_0 t}\,dt = T$$



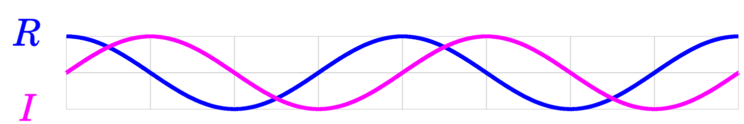
$e^{j1\omega_0 t}$

$$\langle e^{j1\omega_0 t},\ e^{j-1\omega_0 t}\rangle = \int_0^T e^{+j(1+1)\omega_0 t}\,dt = 0$$



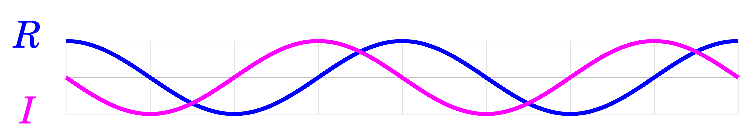
$e^{j-1\omega_0 t}$

$$\langle e^{j1\omega_0 t},\ e^{j2\omega_0 t}\rangle = \int_0^T e^{+j(1-2)\omega_0 t}\,dt = 0$$



$e^{j2\omega_0 t}$

$$\langle e^{j1\omega_0 t},\ e^{j-2\omega_0 t}\rangle = \int_0^T e^{+j(1+2)\omega_0 t}\,dt = 0$$



$e^{j-2\omega_0 t}$

References

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[3] M.L. Boas, “Mathematical Methods in the Physical Sciences”
[4] D.G. Zill, “Advanced Engineering Mathematics”