> $restart : assume(n, integer); rounding = simple :$

Solve: $\nabla^2 u = 0$ with $u(0, y) = y - \pi$, $u(1, y) = 0$ and $u_y(x, 0) = x$ and $u_y(x, \pi) = x$

Find the solution $u(x, y) = u_1(x, y) + u_2(x, y)$ as a two part problem.

$\nabla^2 u_1(x, y) = 0$ with $u_1(0, y) = 0$, $u_1(1, y) = 0$ and $u_y(x, 0) = x$ and $u_y(x, \pi) = x$

$\nabla^2 u_2(x, y) = 0$ with $u_2(0, y) = y - \pi$, $u_2(1, y) = 0$ and $u_y(x, 0) = 0$ and $u_y(x, \pi) = 0$

1) Solve $\nabla^2 u_1(x, y) = 0$ with $u_1(0, y) = 0$, $u_1(1, y) = 0$ and $u_y(x, 0) = x$ and $u_y(x, \pi) = x$.

I. $u_1 = X(x) \cdot Y(y)$ substituted into $\nabla^2 u_1(x, y) = 0$ leads to $\frac{X''}{X} + \frac{Y''}{Y} = 0$ and $\frac{X''}{X} = -\frac{Y''}{Y} = constant$ now the constant may be positive , zero or negative.

Ia. For the positive and zero cases the solutions that result are trivial solutions.

For the case of a negative Constant. $X'' + \lambda^2 X = 0$ with $X(0) = 0, X(2) = 0$

>

> $dsolve\left(\left[diff(X(x), x, x) + \lambda^2 \cdot X(x) = 0, X(0) = 0\right], X(x)\right)$

$$X(x) = \_C1 \sin(\lambda x) \tag{1}$$

The only nonzero solution is $\lambda_n = n \cdot \pi$

> lambda $:= n \cdot$Pi

$$\lambda := n\sim \pi \tag{2}$$

$X_n(x) = \sin(\lambda_n \cdot x)$

> X := x→sin(lambda·x)

$$X := x \rightarrow \sin(\lambda x) \tag{3}$$

To solve for the $Y(y)$, solve $Y'' - \lambda_n^2 Y = 0$

> $dsolve\left(diff(Y(y), y, y) - \lambda^2 \cdot Y(y) = 0, Y(y)\right); convert(\%, trigh)$

$$Y(y) = \_C1\, e^{n\sim \pi y} + \_C2\, e^{-n\sim \pi y}$$

$$Y(y) = (\_C1 + \_C2) \cosh(n\sim \pi y) + (\_C1 - \_C2) \sinh(n\sim \pi y) \tag{4}$$

Written as a hyperbolic function $Y_n = a_n \cdot \cosh(\lambda_n y) + b_n \sinh(\lambda_n \cdot y)$

> $Y := y \rightarrow an \cdot \cosh(\text{lambda} \cdot y) + bn \cdot \sinh(\text{lambda} \cdot y)$

$$Y := y \to an\cosh(\lambda y) + bn\sinh(\lambda y) \tag{5}$$

$$u_1(x, y) = \sum_{n=1}^{\infty} X_n(x)\cdot Y_n(y)$$

$$u_1(x, y) = \sum_{n=1}^{\infty} \sin(\lambda_n\cdot x)\cdot(a_n\cdot\cosh(\lambda_n\cdot y) + b_n\cdot\sinh(\lambda_n\cdot y))$$

Use the two boundary conditions $u1_y(x, 0) = x$ and $u1_y(x, \pi) = x$ to find the two constants a_n, b_n.

> $f1 := x$

$$f1 := x \tag{6}$$

$$u1_y(x, 0) = x = \sum_{n=1}^{\infty} \sin(\lambda_n\cdot x)\cdot(a_n\cdot\lambda_n\cdot\sinh(\lambda_n\cdot 0) + b_n\cdot\lambda_n\cosh(\lambda_n\cdot 0))$$

$$x = \sum_{n=1}^{\infty} \sin(\lambda_n\cdot x)\cdot b_n\cdot\lambda_n$$

$$\int_0^1 x\cdot\sin(\lambda_n\cdot x)\,\mathrm{d}x = b_n\cdot\lambda_n\int_0^1 \sin^2(\lambda_n\cdot x)\cdot dx$$

$$b_n = \frac{\int_0^1 x\cdot\sin(\lambda_n\cdot x)\,\mathrm{d}x}{\lambda_n\int_0^1 \sin^2(\lambda_n\cdot x)\cdot dx}$$

> $bn := \frac{1}{\mathrm{lambda}}\cdot\frac{int(f1\cdot X(x), x = 0..1)}{int(X(x)^2, x = 0..1)};$

$$bn := -\frac{2\,(-1)^{n\sim}}{n\sim^2\,\pi^2} \tag{7}$$

Now go after the second coefficients a_n

$$u1_y(x, \pi) = x = \sum_{n=1}^{\infty} \sin(\lambda_n\cdot x)\cdot(a_n\cdot\lambda_n\cdot\sinh(\lambda_n\cdot\pi) + b_n\cdot\lambda_n\cosh(\lambda_n\cdot\pi))$$

$$\frac{1}{\lambda_n}\,\frac{\int_0^1 x\cdot\sin(\lambda_n\cdot x)\,\mathrm{d}x}{\int_0^1 \sin^2(\lambda_n\cdot x)dx} = a_n\cdot\sinh(\lambda_n\cdot\pi) + b_n\cdot\cosh(\lambda_n\cdot\pi)$$

$$a_n = \frac{1}{\sinh(\lambda_n\cdot\pi)\cdot\lambda_n}\,\frac{\int_0^1 x\cdot\sin(\lambda_n\cdot x)\,\mathrm{d}x}{\int_0^1 \sin^2(\lambda_n\cdot x)\cdot dx} - \frac{b_n\cdot\cosh(\lambda_n\cdot\pi)}{\sinh(\lambda_n\cdot\pi)}$$

> $f2 := x$

$$f2 := x \quad \textbf{(8)}$$

> $an := simplify\left(\frac{int(f2 \cdot X(x), x = 0..1)}{\text{lambda} \cdot \sinh(\text{lambda} \cdot \text{Pi}) \cdot int(X(x)^2, x = 0..1)} - \frac{bn \cdot \cosh(\text{lambda} \cdot \text{Pi})}{\sinh(\text{lambda} \cdot \text{Pi})}\right);$

$$an := \frac{2\,(-1)^{n\sim}\left(-1 + \cosh(n\sim \pi^2)\right)}{n\sim^2\, \pi^2 \sinh(n\sim \pi^2)} \quad \textbf{(9)}$$

> $with(plots):$

Only the first two terms of the sum are not zero maple really screws up in calculation the rest of them. These could blow up . They should actually be zero.

> $u1 := (x, y) \rightarrow sum(X(x) \cdot Y(y), n = 1..10)$

$$u1 := (x, y) \rightarrow \sum_{n=1}^{10} X(x)\, Y(y) \quad \textbf{(10)}$$

> $evalf\left(u1\left(\frac{1}{2}, y\right)\right)$

$$\begin{aligned} &-0.2026214058 \cosh(3.141592654\, y) + 0.2026423672 \sinh(3.141592654\, y) \\ &\quad + 0.02251581858 \cosh(9.424777962\, y) - 0.02251581858 \sinh(9.424777962\, y) \\ &\quad - 0.008105694688 \cosh(15.70796327\, y) + 0.008105694688 \sinh(15.70796327\, y) \\ &\quad + 0.004135558514 \cosh(21.99114858\, y) - 0.004135558514 \sinh(21.99114858\, y) \\ &\quad - 0.002501757619 \cosh(28.27433389\, y) + 0.002501757619 \sinh(28.27433389\, y) \end{aligned} \quad \textbf{(11)}$$

So recalculate using only a few terms.

> $u1 := (x, y) \rightarrow sum(X(x) \cdot Y(y), n = 1..2)$

$$u1 := (x, y) \rightarrow \sum_{n=1}^{2} X(x)\, Y(y) \quad \textbf{(12)}$$

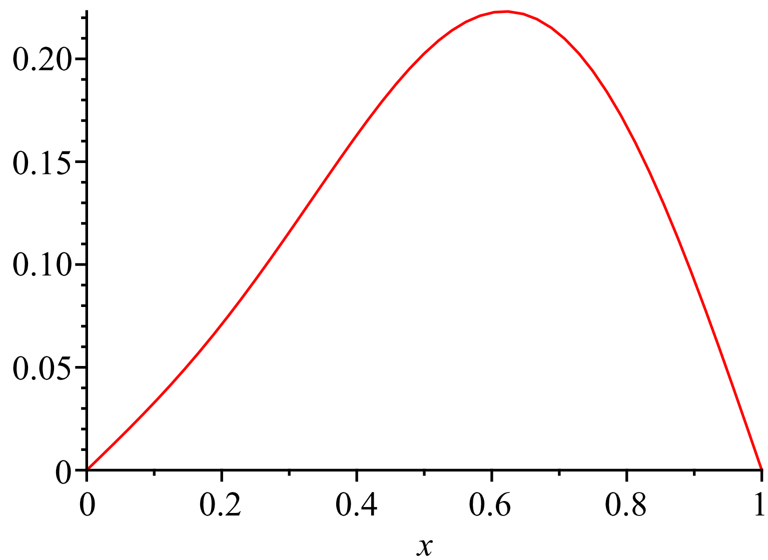
At one of the boundaries $\left.\frac{\partial u_1}{\partial y}\right|_{y = \pi} = x$

> $u1(x, \text{Pi})$

$$\begin{aligned} &\sin(\pi x)\left(-\frac{2\left(-1 + \cosh(\pi^2)\right)\cosh(\pi^2)}{\pi^2 \sinh(\pi^2)} + \frac{2 \sinh(\pi^2)}{\pi^2}\right) \\ &\quad + \sin(2\pi x)\left(\frac{1}{2}\,\frac{\left(-1 + \cosh(2\pi^2)\right)\cosh(2\pi^2)}{\pi^2 \sinh(2\pi^2)} - \frac{1}{2}\,\frac{\sinh(2\pi^2)}{\pi^2}\right) \end{aligned} \quad \textbf{(13)}$$

But it must also meet the zero BCs on the left and right.

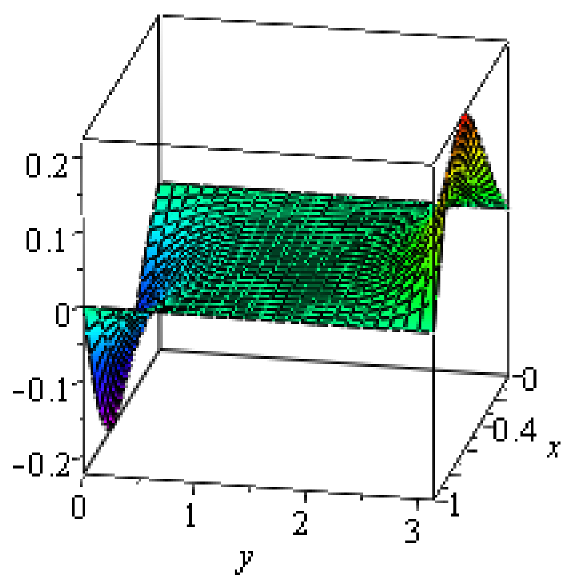
> $plot(u1(x, \text{Pi}), x = 0..1)$



Plot of solution $u_1(x, y)$

> $plot3d(u1(x, y), x = 0 ..1, y = 0 ..Pi, axes = box, shading = zhue, title = \text{"solution for u1"})$

solution for u1



>

Find solution to

2) $\nabla^2 u_2(x, y) = 0$ with $u_2(0, y) = y - \pi$, $u_2(1, y) = 0$ and $u_y(x, 0) = 0$ and $u_y(x, \pi) = 0$.

II. $u_2 = X(x)\cdot Y(y)$ substituted into $\nabla^2 u_2(x,y) = 0$ leads to $\frac{X''}{X} + \frac{Y''}{Y} = 0$ and -

$\frac{X''}{X} = \frac{Y''}{Y} = constant$ now the constant may be positive , zero or negative.

IIa. For the positive case the result is a trivial solution.

IIb. For a zero constant $\mu^2 = 0$, $Y''=0$ $Y(y) = a\cdot y + b$ using BCs $Y'(0) = 0 = a$ and so $Y(y) = b$ other boundary condition does not give any more information.

We need to solve for $X''= 0$ also. $X = c\cdot x + d$.

The solution for $constant = 0$ is $u_2(x,y) = X_0 \cdot Y_0 = b\cdot(c\cdot x + d) = c_0\cdot x + c_1$, these constants will be resolved when we find the series coefficients.

IIc. For the constant < 0 , negative case.

$Y'' + \mu^2\cdot Y = 0$ $Y'(0) = 0$ and $Y'(\pi) = 0$

> $dsolve([diff(Y2(y), y, y) + \mathrm{mu}^2\cdot Y2(y) = 0, \mathrm{D}(Y2)(0) = 0], Y2(y))$

$$Y2(y) = \_C2\cos(\mu y) \qquad (14)$$

Using BCs $Y'(\mathrm{Pi}) = 0$ $Y'(\mathrm{Pi}) = 0 = -c_2\cdot \mu sin(\mu\pi)$ this gives us $\mu_n = n$

The only nonzero solution is $\mu_n = n$

> $\mathrm{mu} := n$

$$\mu := n\sim \qquad (15)$$

So we get $Y_n(y) = \cos(\mu_n\cdot y)$

> $Y2 := y \rightarrow \cos(\mathrm{mu}\cdot y)$

$$Y2 := y \rightarrow \cos(\mu y) \qquad (16)$$

To solve for the $X(x)$, solve $X' - \mu_n^2 X = 0$

> $Y2(1)$

$$\cos(n\sim) \qquad (17)$$

> $dsolve(diff(X2(x), x, x) - \mu^2\cdot X2(x) = 0, X2(x));\ convert(\%, trigh)$

$$X2(x) = \_C1\, \mathrm{e}^{-n\sim x} + \_C2\, \mathrm{e}^{n\sim x}$$

$$X2(x) = (\_C1 + \_C2)\cosh(n\sim x) + (-\_C1 + \_C2)\sinh(n\sim x) \qquad (18)$$

Written as a hyperbolic function $X_n = a_n\cdot\cosh(\mu_n x) + b_n \sinh(\mu_n\cdot x)$

> $X2 := x \rightarrow a2n\cdot\cosh(\mathrm{mu}\cdot x) + b2n\cdot\sinh(\mathrm{mu}\cdot x)$

$$X2 := x \rightarrow a2n\cosh(\mu x) + b2n\sinh(\mu x) \qquad (19)$$

> $X2(1)$

$$a2n\cosh(n\sim)+b2n\sinh(n\sim) \quad (20)$$

$$u_2(x,y)=X_0(x)\cdot Y_0(y)+\sum_{n=1}^{\infty}X_n(x)\cdot Y_n(y)$$

$$u_2(x,y)=c_0\cdot x+c_1+\sum_{n=1}^{\infty}\cos(\mu_n\cdot y)\cdot\left(a_n\cdot\cosh(\mu_n\cdot x)+b_n\cdot\sinh(\mu_n\cdot x)\right)$$

Use the two boundary conditions $u2(0,y)=y-\pi$ and $u1(1,y)=0$ to find the two constants a_n, b_n.

> $f1 := y-\pi$

$$f1 := y-\pi \quad (21)$$

$$u2(0,\mathrm{y})=y-\pi=c_1+\sum_{n=1}^{\infty}\cos(\mu_n\cdot y)\cdot\left(a_n\cdot\cosh(\mu_n\cdot 0)+b_n\cdot\sinh(\mu_n\cdot 0)\right)$$

Multiply each side by orthogonal function for $\cos(\mu_0\cdot y)=1$ where $\mu_0=n=0$ and integrate over interval $(0,\mathrm{Pi})$

$$\int_0^{\pi}(y-\pi)\cdot 1\,\mathrm{d}y=c_1\cdot\int_0^{\pi}1^2\cdot dy$$

$$c_1=\frac{\int_0^{\pi}(y-\pi)\cdot 1\,\mathrm{d}y}{\pi}=-\frac{\pi}{2}$$ this gives the function $c_0\cdot x+c_1=c_0\cdot x-\frac{\pi}{2}$

> $c1 := \frac{int(f1\cdot 1, y=0..\mathrm{Pi})}{\mathrm{Pi}}$

$$c1 := -\frac{1}{2}\pi \quad (22)$$

$$u_2(1,y)=0=c_0\cdot 1-\frac{\pi}{2}+\sum_{n=1}^{\infty}\cos(\mu_n\cdot y)\cdot\left(a_n\cdot\cosh(\mu_n\cdot 1)+b_n\cdot\sinh(\mu_n\cdot 1)\right)$$

Again Multiply both sides by $\cos(\mu_0\cdot y)=1$ where $\mu_0=n=0$ and integrate over interval $(0,\mathrm{Pi})$

$$0=c_0\cdot\int_0^{\pi}1^2\cdot dy-\frac{\pi}{2}\int_0^{\pi}1\,dy$$

$c_0=\frac{\pi}{2}$ this gives the first term in the series as $c_0\cdot x+c_1=\frac{\pi}{2}(x-1)$

> $c0 := \frac{\text{Pi}}{2}; c1 := -\frac{\text{Pi}}{2};$

$$c0 := \frac{1}{2}\,\pi$$

$$c1 := -\frac{1}{2}\,\pi \qquad \textbf{(23)}$$

For the other coefficients

$$y - \pi = \sum_{n=1}^{\infty} \cos(\mu_n \cdot y) \cdot a_n$$

>

$$a_n = \frac{\int_0^{\pi} (y-\pi) \cdot \cos(\mu_n \cdot y) \, \mathrm{d}y}{\int_0^{\pi} \cos^2(\mu_n \cdot y) \cdot dy}$$

> $a2n := \frac{int(\text{f1} \cdot \text{Y2}(\text{y}), y = 0..\text{Pi})}{int(\text{Y2}(\text{y})^2, y = 0..\text{Pi})}; simplify(\%);$

$$a2n := \frac{2\,(-1 + (-1)^{n\sim})}{n\sim^2\,\pi}$$

$$\frac{2\,(-1 + (-1)^{n\sim})}{n\sim^2\,\pi} \qquad \textbf{(24)}$$

Now go after the second coefficients b_n

$$u2(1, y) = 0 = \sum_{n=1}^{\infty} \cos(\mu_n \cdot y) \cdot (a_n \cdot \cosh(\mu_n \cdot 1) + b_n \cdot \sinh(\mu_n \cdot 1))$$

$$0 = a_n \cdot \cosh(\mu_n \cdot 1) + b_n \cdot \sinh(\mu_n \cdot 1)$$

$$b_n = -\frac{a_n \cdot \cosh(\mu_n \cdot 1)}{\sinh(\mu_n \cdot 1)}$$

> $b2n := -\frac{a2n \cdot \cosh(\text{mu})}{\sinh(\text{mu})}$

$$b2n := -\frac{2\,(-1 + (-1)^{n\sim})\cosh(n\sim)}{n\sim^2\,\pi\,\sinh(n\sim)} \qquad \textbf{(25)}$$

> $u2 := (x, y) \rightarrow c1 + c0 \cdot x + sum(Y2(y) \cdot X2(x), n = 1..20)$

$$u2 := (x, y) \rightarrow c1 + c0\,x + \sum_{n=1}^{20} Y2(y)\,X2(x) \qquad \textbf{(26)}$$

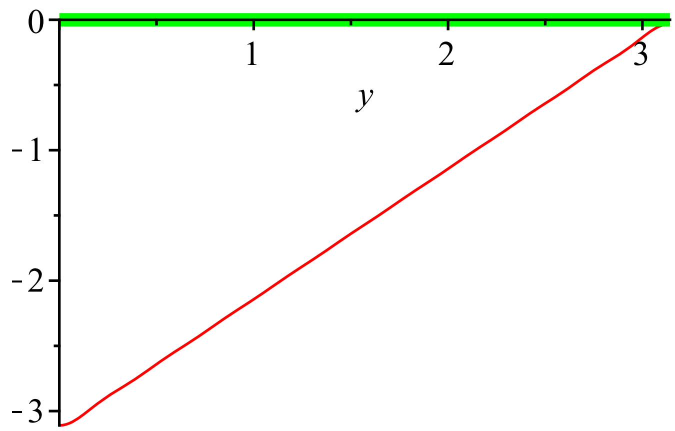
> $\#u2 := (x, y) \rightarrow c1 + c0 \cdot x + sum(\cos(\text{mu} \cdot y) \cdot (a2n \cdot \cosh(\text{mu} \cdot x) + b2n \cdot \sinh(\text{mu} \cdot x)), n = 1..20)$

$u_2(0, y) = y - \pi$ this is the diagonal line below, $u_2(1, y) = 0$ is the horizontal line equal to zero.

> $evalf\left(u2\left(\frac{1}{2}, \frac{1}{2}\right)\right)$

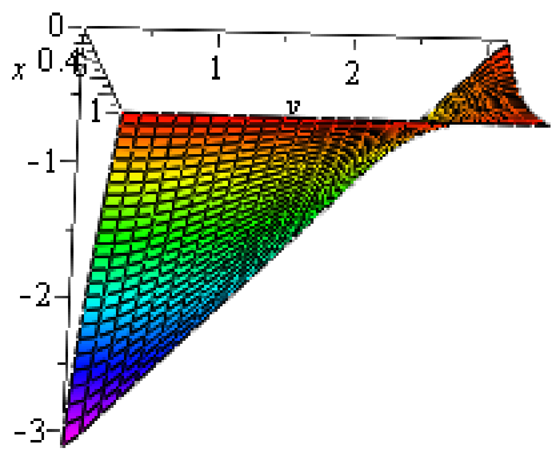
$$-1.278922912 \quad (27)$$

> $plot([u2(0, y), u2(1, y)], y = 0..\text{Pi}, color = [red, green, blue], thickness = [1, 5, 10])$



Solution $u_2(x, y)$

> $plot3d(u2(x, y), x = 0..1, y = 0..\text{Pi}, axes = normal, shading = zhue)$

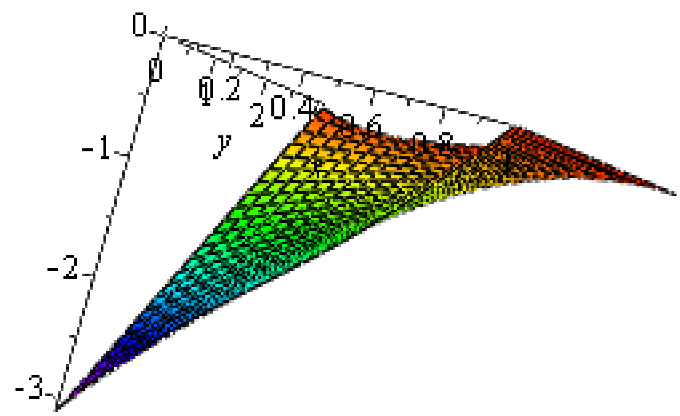


u(x,y) $= u_1 + u_2$

> $u := (x, y) \rightarrow u1(x, y) + u2(x, y);$

$$u := (x, y) \rightarrow u1(x, y) + u2(x, y) \quad (28)$$

> $plot3d(u(x, y), x = 0..1, y = 0..\text{Pi}, axes = normal, shading = zhue)$



Value of the solution at $u(0.5, 0.75)$

> *evalf*$(u(0.5, 0.75))$

$$-1.195815512 \quad (9)$$

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