Divergence and Curl (2A)

- Divergence
- Curl
- Green's Theorem

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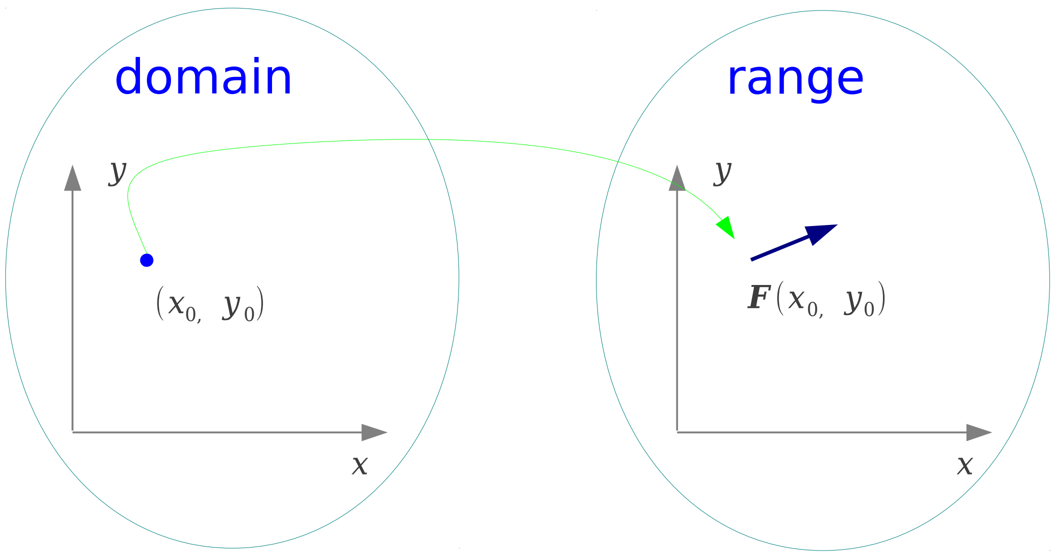
2-D Vector Field

a given point in a 2-d space ➡ A vector

$(x_{0,}\ \ y_0)$ $\langle\, M(x_{0,} y_0),\ \ N(x_{0,} y_0)\,\rangle$



2 functions

$(x_{0,}\ \ y_0) \longrightarrow M(x_{0,} y_0)$

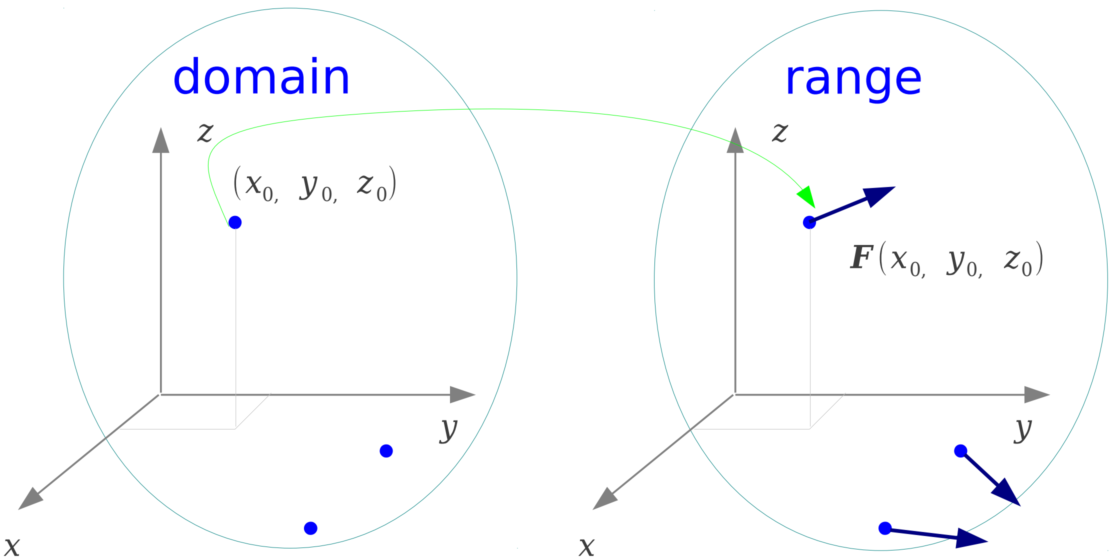
$(x_{0,}\ \ y_0) \longrightarrow N(x_{0,} y_0)$

$$(x_{0,}\ \ y_0) \longrightarrow \boldsymbol{F}(x_{0,}\ \ y_0) = M(x_{0,} y_0)\boldsymbol{i} + N(x_{0,} y_0)\boldsymbol{j}$$

2-D Vector Field

A given point in a 3-d space ➡ A vector

$(x_{0,}\ \ y_{0,}\ \ z_0)$ $\langle\, M(x_{0,}y_0),\ N(x_{0,}y_0),\ P(x_{0,}y_0)\,\rangle$



3 functions

$(x_{0,}\ \ y_{0,}\ \ z_0) \longrightarrow M(x_{0,}y_{0,}Z_0)$

$(x_{0,}\ \ y_{0,}\ \ z_0) \longrightarrow N(x_{0,}y_{0,}z_0)$

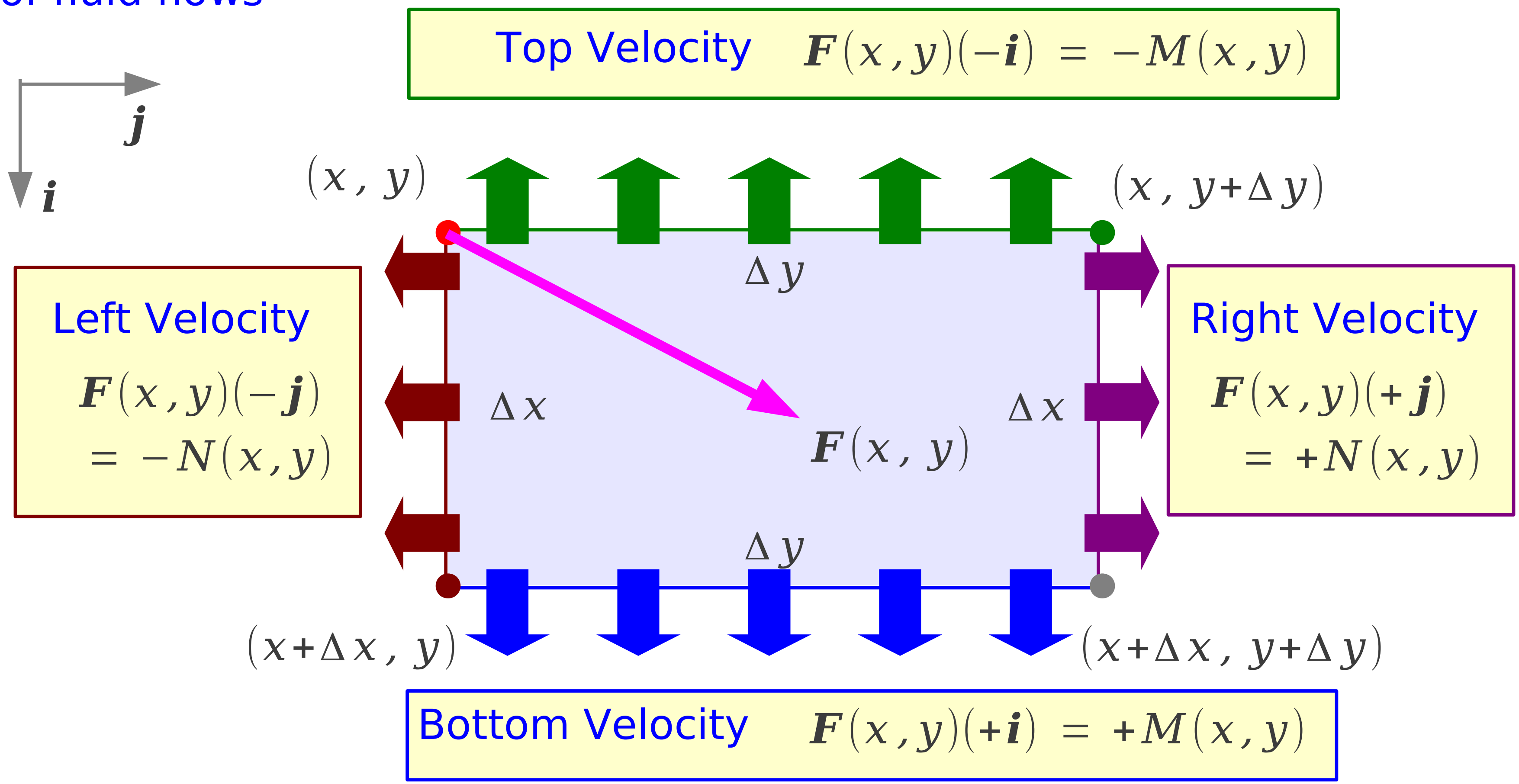
$(x_{0,}\ \ y_{0,}\ \ z_0) \longrightarrow P(x_{0,}y_{0,}z_0)$

$$(x_{0,}\ \ y_{0,}\ \ z_0) \longrightarrow \boldsymbol{F}(x_{0,}\ \ y_{0,}\ \ z_0) = M(x_{0,}y_{0,}z_0)\boldsymbol{i} + N(x_{0,}y_{0,}z_0)\boldsymbol{j} + P(x_{0,}y_{0,}z_0)\boldsymbol{k}$$

2-D Divergence (1)

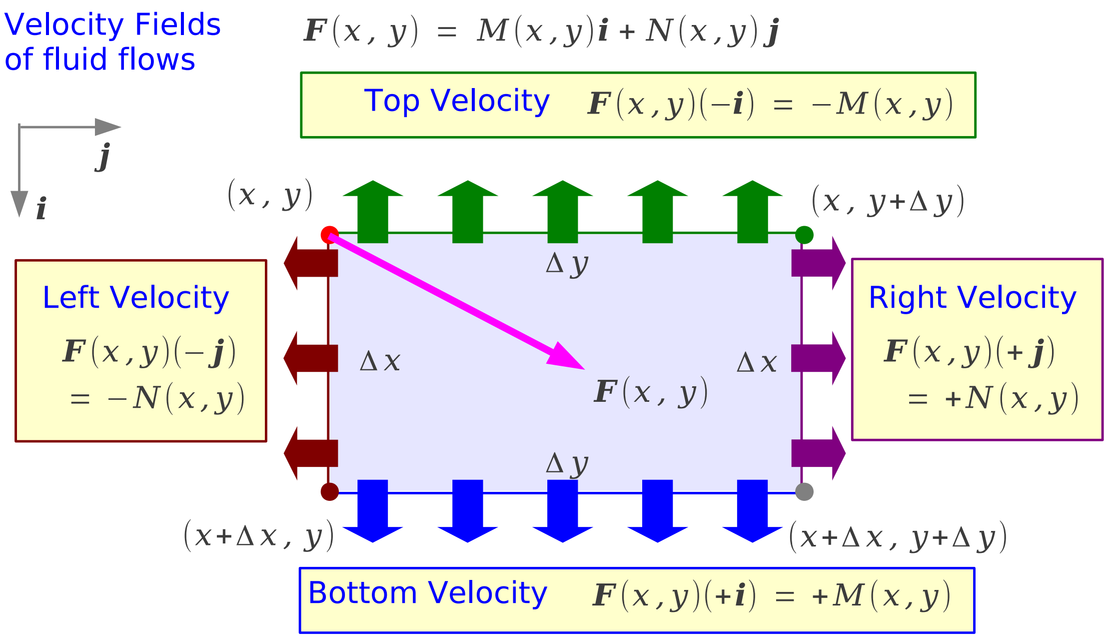
Velocity Fields
of fluid flows

$\boldsymbol{F}(x, y) = M(x,y)\boldsymbol{i} + N(x,y)\boldsymbol{j}$



Flow rate of outward bound fluid

2-D Divergence (2)



Flow rate of outward bound fluid

The rate at which fluid leave the rectangle

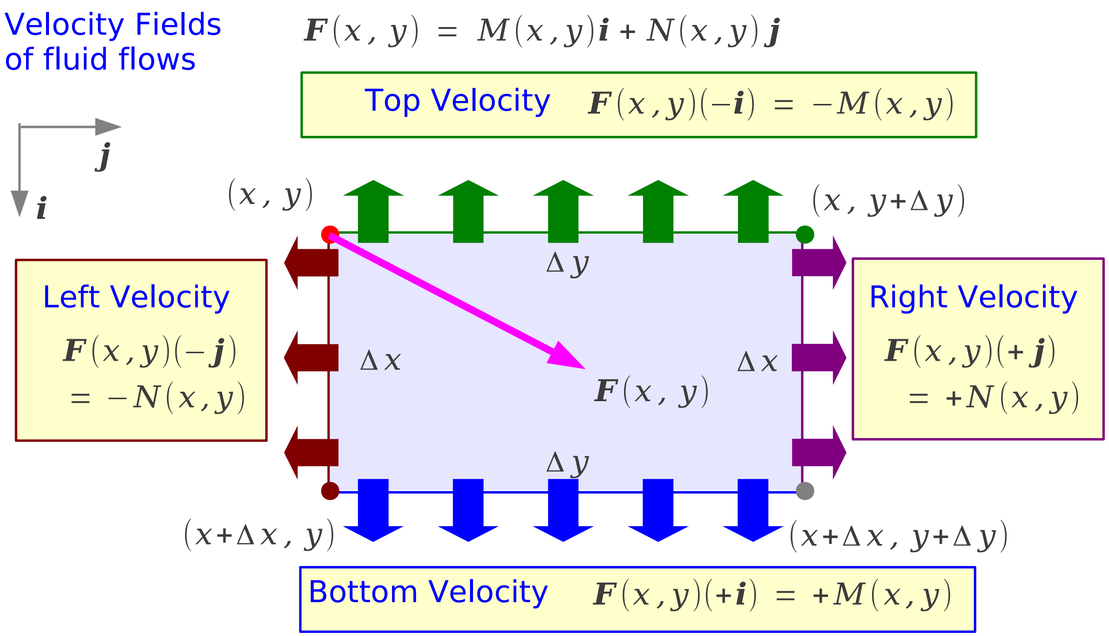
Across top $\quad \boldsymbol{F}(x, y)\cdot(-\boldsymbol{i})\Delta y = -M(x, y)\Delta y$

Across bottom $\quad \boldsymbol{F}(x+\Delta x, y)\cdot(+\boldsymbol{i})\Delta y = M(x+\Delta x, y)\Delta y$

Across left $\quad \boldsymbol{F}(x, y)\cdot(-\boldsymbol{j})\Delta x = -N(x, y)\Delta x$

Across right $\quad \boldsymbol{F}(x, y+\Delta y)\cdot(+\boldsymbol{j})\Delta x = N(x, y+\Delta y)\Delta x$

2-D Divergence (3)



$$\boldsymbol{F}(x,\ y)\cdot(-\boldsymbol{i})\,\Delta y \;=\; -M(x,\ y)\,\Delta y$$
$$\boldsymbol{F}(x+\Delta x,\ y)\cdot(+\boldsymbol{i})\,\Delta y \;=\; M(x+\Delta x,\ y)\,\Delta y$$
$$\boldsymbol{F}(x,\ y)\cdot(-\boldsymbol{j})\,\Delta x \;=\; -N(x,\ y)\,\Delta x$$
$$\boldsymbol{F}(x,\ y+\Delta y)\cdot(+\boldsymbol{j})\,\Delta x \;=\; N(x,\ y+\Delta y)\,\Delta x$$

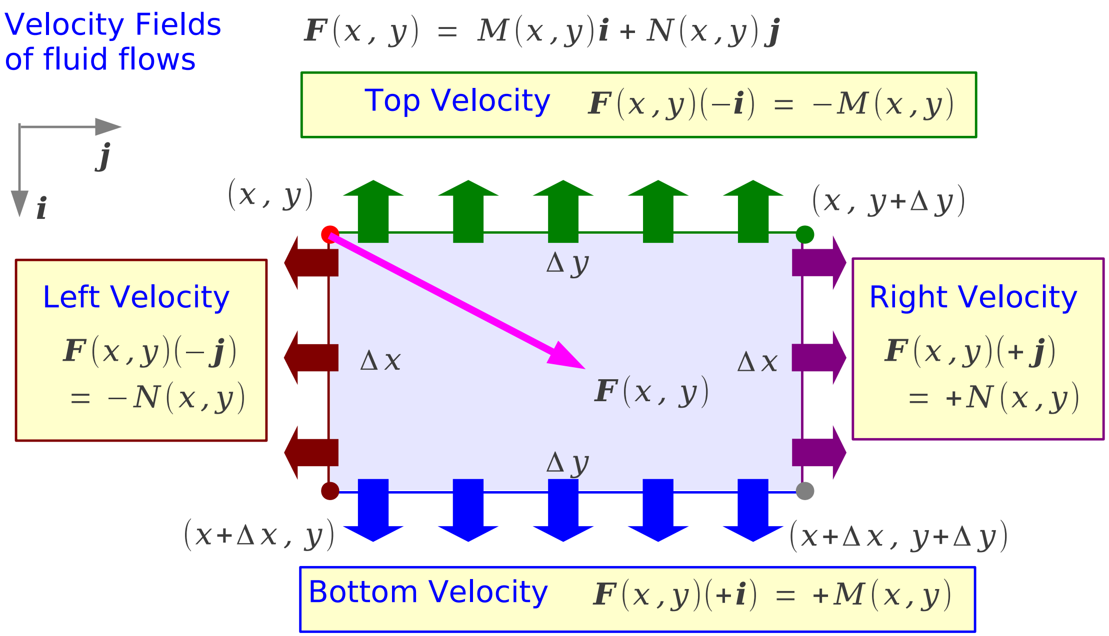
Flow rate of outward bound fluid

The rate at which fluid leave the rectangle

Across top + bottom $\quad \{M(x+\Delta x, y) - M(x,\ y)\}\,\Delta y \quad = \left(\dfrac{\partial M}{\partial x}\Delta x\right)\Delta y$

Across left + right $\quad \{N(x,\ y+\Delta y) - N(x,\ y)\}\,\Delta x \quad = \left(\dfrac{\partial N}{\partial y}\Delta y\right)\Delta x$

2-D Divergence (4)



$$\{M(x+\Delta x, y) - M(x, y)\}\Delta y = \left(\frac{\partial M}{\partial x}\Delta x\right)\Delta y$$

$$\{N(x, y+\Delta y) - N(x, y)\}\Delta x = \left(\frac{\partial N}{\partial y}\Delta y\right)\Delta x$$

Flow rate of outward bound fluid

Flux across rectangle boundary

$$\approx \left(\frac{\partial M}{\partial x}\Delta x\right)\Delta y + \left(\frac{\partial N}{\partial y}\Delta y\right)\Delta x = \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)\Delta x \Delta y$$

Flux density

$$= \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)$$

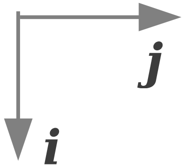
Divergence of **F**   Flux Density

$\boldsymbol{i}$

Velocity Fields
of fluid flows

$$\boldsymbol{F}(x, y) = M(x,y)\,\boldsymbol{i} + N(x,y)\,\boldsymbol{j}$$

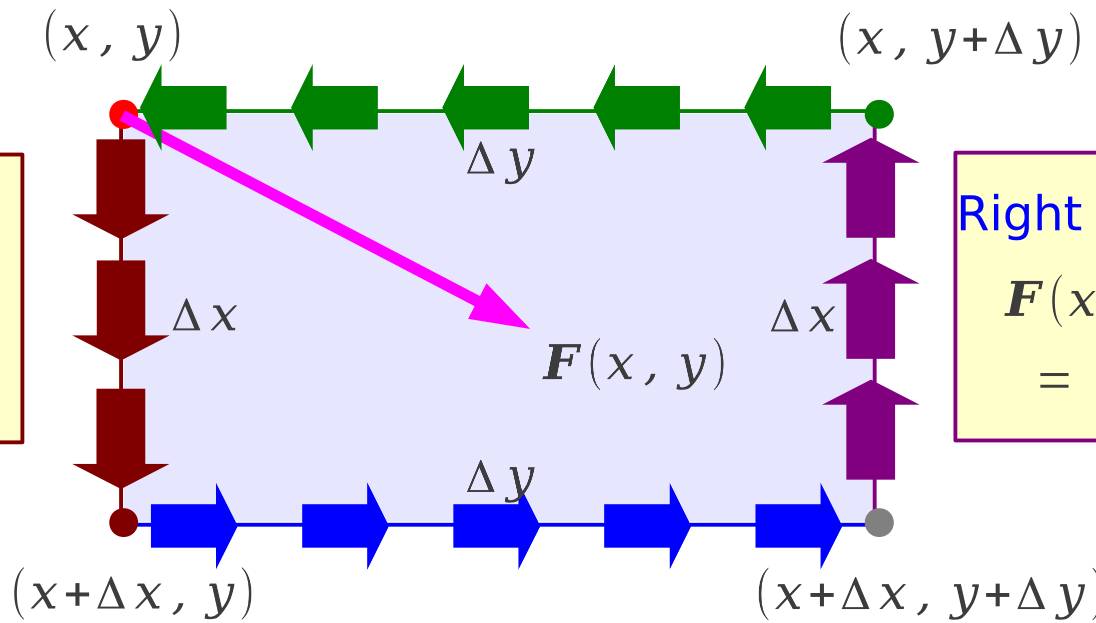
Top Velocity $\boldsymbol{F}(x,y)(-\boldsymbol{j}) = -N(x,y)$



Left Velocity

$\boldsymbol{F}(x,y)(+\boldsymbol{i})$

$= +M(x,y)$



Right Velocity

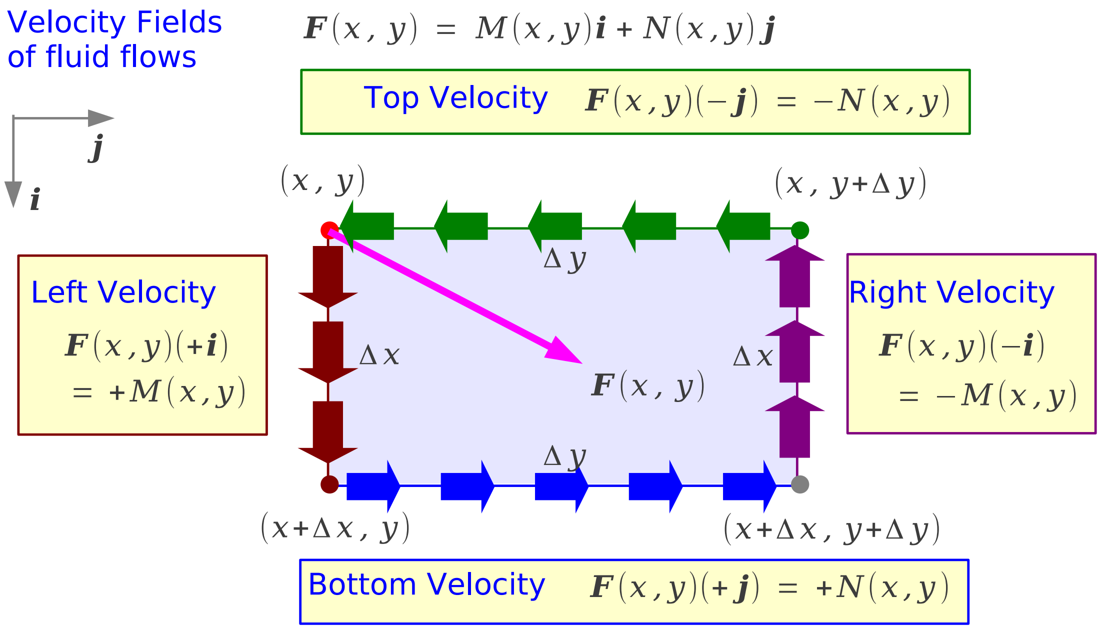
$\boldsymbol{F}(x,y)(-\boldsymbol{i})$

$= -M(x,y)$

Bottom Velocity $\boldsymbol{F}(x,y)(+\boldsymbol{j}) = +N(x,y)$

Flow rate of counter clock wise circulating fluid

2-D Curl (2)



Flow rate of counter clock wise circulating fluid

The flow rate of counter clock wise circulation

Across top $\quad \boldsymbol{F}(x, y)\cdot(-\boldsymbol{j})\Delta y = -N(x, y)\Delta y$

Across bottom $\quad \boldsymbol{F}(x+\Delta x, y)\cdot(+\boldsymbol{j})\Delta y = N(x+\Delta x, y)\Delta y$

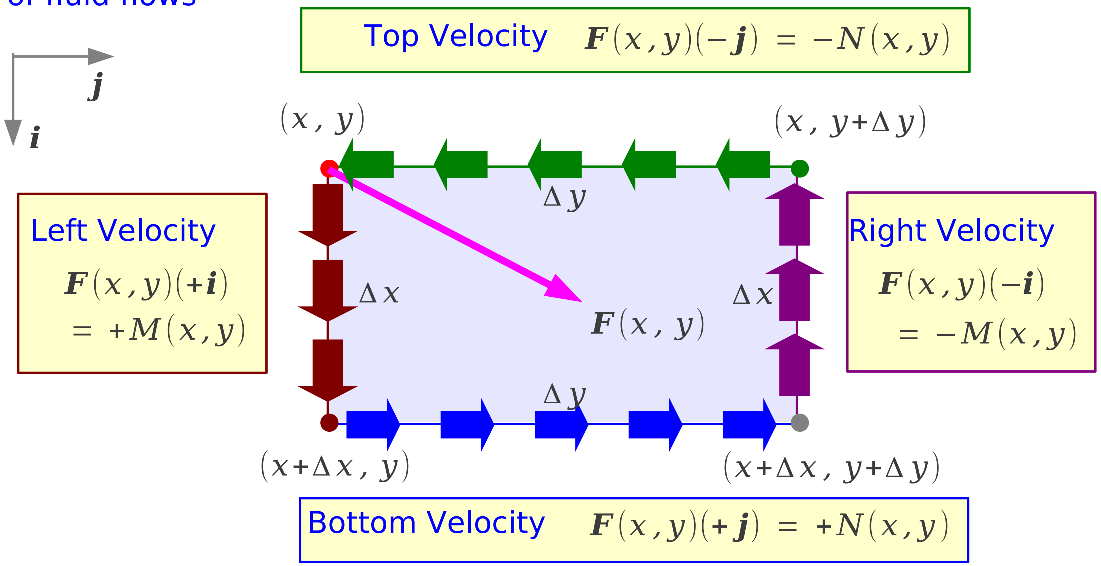
Across left $\quad \boldsymbol{F}(x, y)\cdot(+\boldsymbol{i})\Delta x = M(x, y)\Delta x$

Across right $\quad \boldsymbol{F}(x, y+\Delta y)\cdot(-\boldsymbol{i})\Delta x = -M(x, y+\Delta y)\Delta x$

2-D Curl (3)

Velocity Fields
of fluid flows

$\boldsymbol{F}(x, y) = M(x,y)\boldsymbol{i} + N(x,y)\boldsymbol{j}$



$$\boldsymbol{F}(x, y)\cdot(-\boldsymbol{j})\Delta y = -N(x, y)\Delta y$$
$$\boldsymbol{F}(x+\Delta x, y)\cdot(+\boldsymbol{j})\Delta y = N(x+\Delta x, y)\Delta y$$
$$\boldsymbol{F}(x, y)\cdot(+\boldsymbol{i})\Delta x = M(x, y)\Delta x$$
$$\boldsymbol{F}(x, y+\Delta y)\cdot(-\boldsymbol{i})\Delta x = -M(x, y+\Delta y)\Delta x$$

Flow rate of counter clock wise circulating fluid

The flow rate of counter clock wise circulation

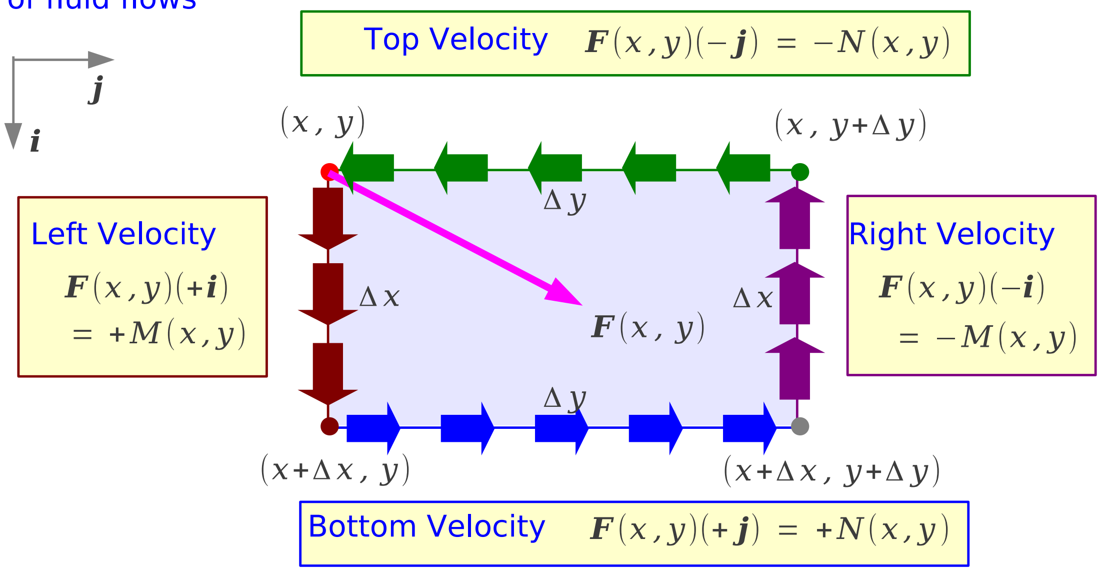
Across top + bottom $\quad \{N(x+\Delta x, y) - N(x, y)\}\Delta y \quad = \left(\dfrac{\partial N}{\partial x}\Delta x\right)\Delta y$

Across left + right $\quad -\{M(x, y+\Delta y) - M(x, y)\}\Delta x \quad = -\left(\dfrac{\partial M}{\partial y}\Delta y\right)\Delta x$

2-D Curl (4)

Velocity Fields
of fluid flows

$\boldsymbol{F}(x, y) = M(x,y)\boldsymbol{i} + N(x,y)\boldsymbol{j}$



$$\{N(x+\Delta x, y) - N(x, y)\}\Delta y = \left(\frac{\partial N}{\partial x}\Delta x\right)\Delta y$$

$$-\{M(x, y+\Delta y) - M(x, y)\}\Delta x = -\left(\frac{\partial M}{\partial y}\Delta y\right)\Delta x$$

Flow rate of counter clock wise circulating fluid

Flux across rectangle boundary

$$\approx \left(\frac{\partial N}{\partial x}\Delta x\right)\Delta y - \left(\frac{\partial M}{\partial y}\Delta y\right)\Delta x = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\Delta x \Delta y$$

Circulation density $= \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$

k-component
Curl of **F**

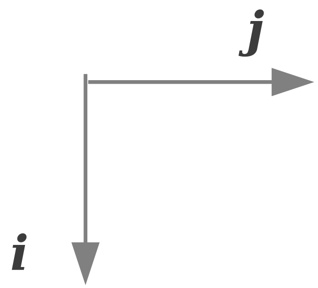
Circulation Density

i

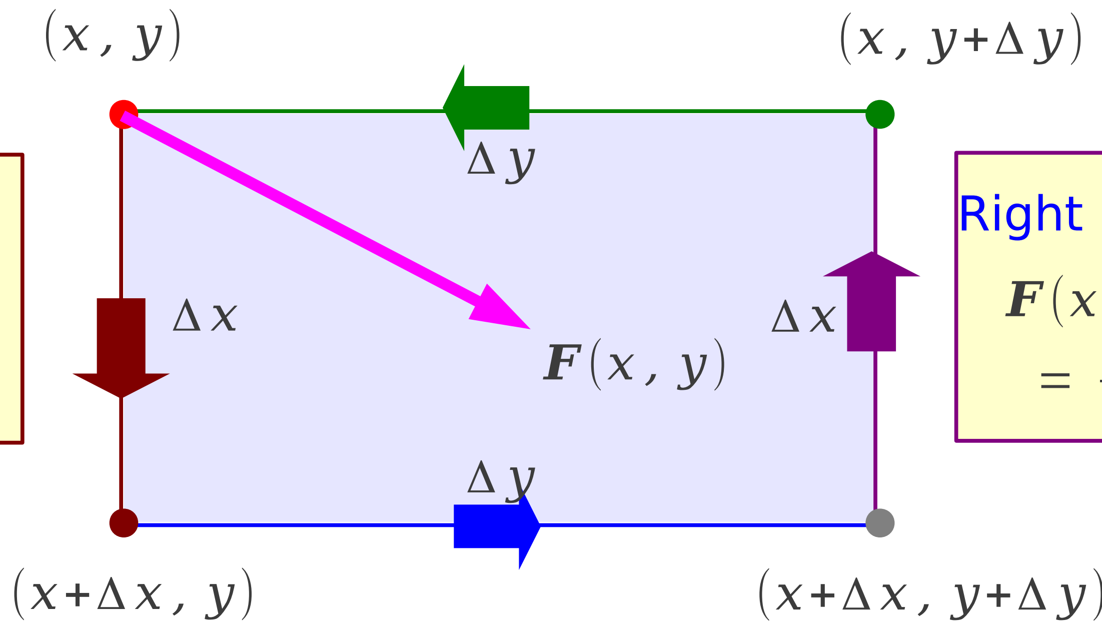
2-D Curl (2)

Velocity Fields
of fluid flows

$$\boldsymbol{F}(x, y) = M(x, y)\,\boldsymbol{i} + N(x, y)\,\boldsymbol{j}$$



Top Flow Rate $\boldsymbol{F}(x,y)(-\boldsymbol{j}) = -N(x,y)$



Left Flow Rate

$$\boldsymbol{F}(x,y)(+\boldsymbol{i}) = +M(x,y)$$

Right Flow Rate

$$\boldsymbol{F}(x,y)(-\boldsymbol{i}) = -M(x,y)$$

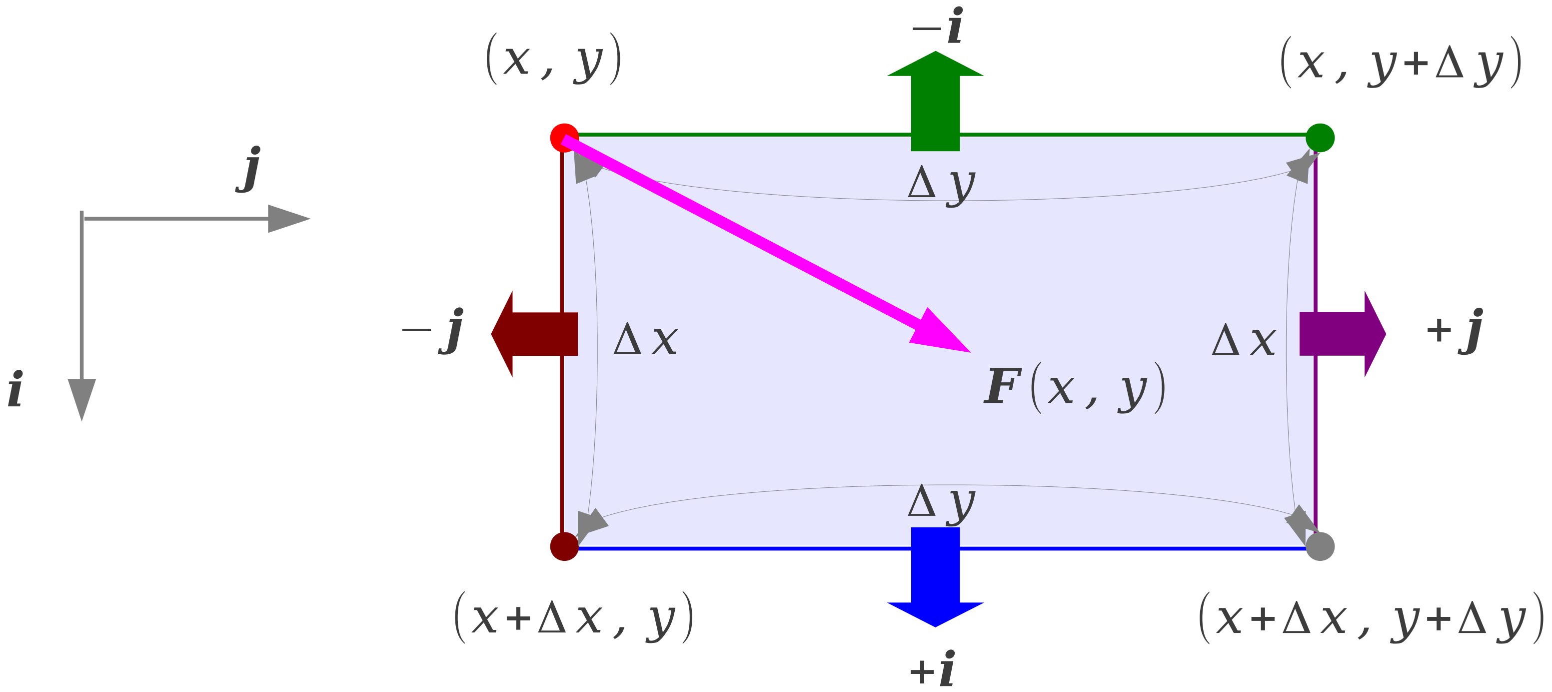
Bottom Flow Rate $\boldsymbol{F}(x,y)(+\boldsymbol{j}) = +N(x,y)$

CounterClockWise Circulation

2-D Curl

$$\boldsymbol{F}(x, y) \cdot (-\boldsymbol{j}) \Delta x = -N(x, y) \Delta x$$

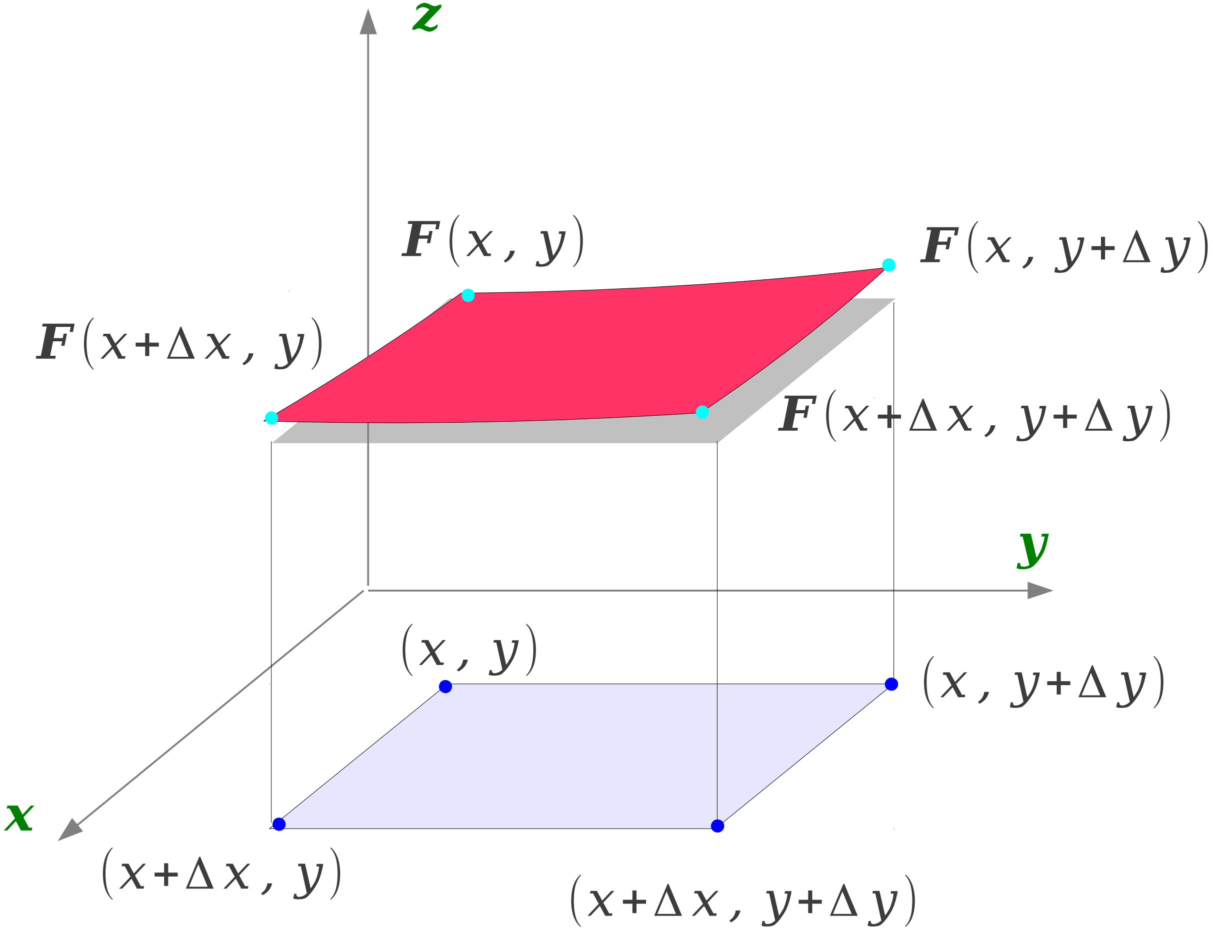
$$\boldsymbol{F}(x, y) \cdot (-\boldsymbol{i}) \Delta y = -M(x, y) \Delta y$$



$$\boldsymbol{F}(x+\Delta x, y) \cdot (+\boldsymbol{i}) \Delta y = M(x+\Delta x, y) \Delta y$$

$$\boldsymbol{F}(x, y+\Delta y) \cdot (+\boldsymbol{j}) \Delta x = N(x, y+\Delta y) \Delta x$$

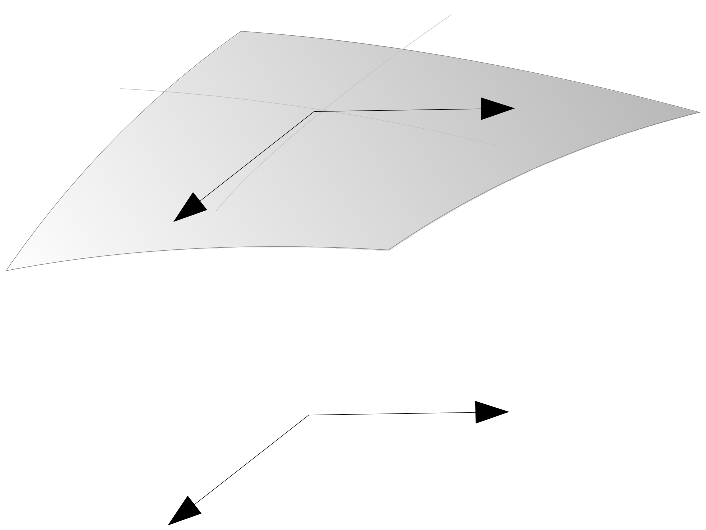
2-D Divergence

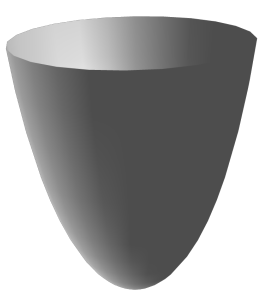


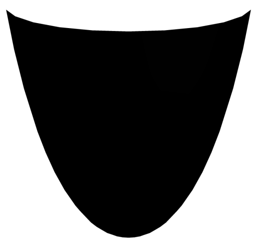
Chain Rule

Function of two variable $y = f(u, v)$

$u = g(x, y)$ $v = h(x, y)$







References

[1] http://en.wikipedia.org/
[2] http://planetmath.org/
[3] M.L. Boas, “Mathematical Methods in the Physical Sciences”
[4] D.G. Zill, “Advanced Engineering Mathematics”