

DFT (4B)

- CTFS and DFT
- CTFT and DFT
- DTFT and DFT
- CTFT and DFT
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CTFS with Complex Coefficients

$$x(t) = a_0 + \sum_{k=1}^{\infty} (a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t))$$

$$a_0 = \frac{1}{T} \int_0^T x(t) dt$$

$$a_k = \frac{2}{T} \int_0^T x(t) \cos(k\omega_0 t) dt$$

$$b_k = \frac{2}{T} \int_0^T x(t) \sin(k\omega_0 t) dt$$
$$k = 1, 2, \dots$$

$$x(t) = A_0 + \sum_{k=1}^{\infty} (A_k e^{jk\omega_0 t} + B_k e^{-jk\omega_0 t})$$

$$A_0 = a_0$$

$$A_k = \frac{1}{2} (a_k - jb_k)$$

$$B_k = \frac{1}{2} (a_k + jb_k)$$

$$k = 1, 2, \dots$$

$$x(t) = \sum_{k=0}^{\infty} (A_k e^{jk\omega_0 t} + B_k e^{-jk\omega_0 t})$$

$$A_k = \frac{1}{T} \int_0^T x(t) e^{-jk\omega_0 t} dt$$
$$k = 0, 1, 2, \dots$$

$$B_k = \frac{1}{T} \int_0^T x(t) e^{jk\omega_0 t} dt$$
$$k = 1, 2, \dots$$

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k e^{jk\omega_0 t}$$

$$C_k = \frac{1}{T} \int_0^T x(t) e^{-jk\omega_0 t} dt$$
$$k = -2, -1, 0, +1, +2, \dots$$

$$C_k = \begin{cases} A_0 = a_0 & (k = 0) \\ A_k = \frac{1}{2} (a_k - jb_k) & (k > 0) \\ B_k = \frac{1}{2} (a_k + jb_k) & (k < 0) \end{cases}$$

CTFS and DTFS

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k e^{+jk\omega_0 t} \quad \text{CTFS}$$

$$C_k = \frac{1}{T} \int_0^T x(t) e^{-jk\omega_0 t} dt$$

$$k = -2, -1, 0, +1, +2, \dots$$

$$x(t) \approx \sum_{k=-M}^{+M} C_k e^{+jk\omega_0 t} \quad N = 2M + 1$$

$$jk\omega_0 t \rightarrow k \left(\frac{2\pi}{T} \right) n \left(\frac{T}{N} \right) = \left(\frac{2\pi}{T} \right) nk$$

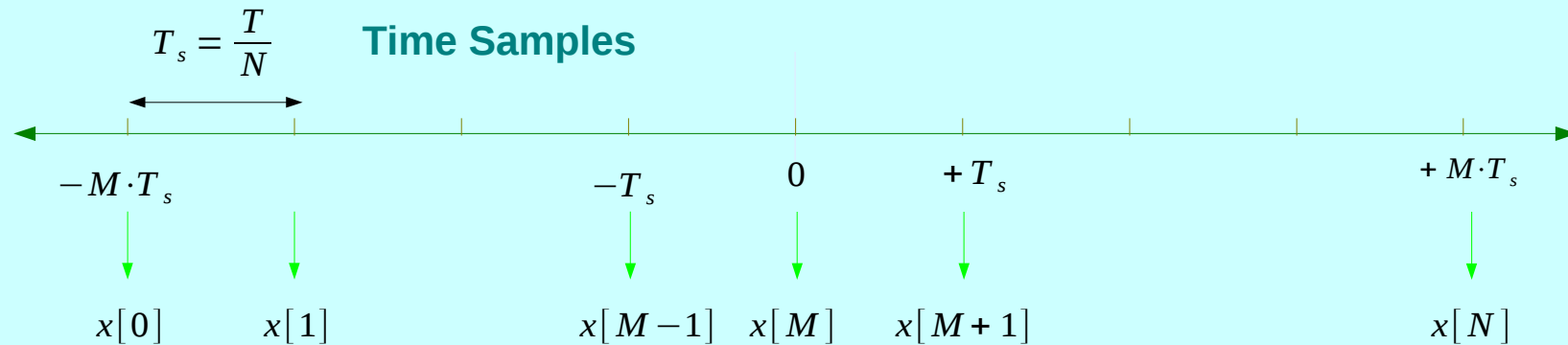
$$x[n] = \sum_{k=-M}^{+M} \alpha_k e^{+j \left(\frac{2\pi}{N} \right) nk}$$

$$n = 0, 1, 2, \dots, N-1,$$

$$\alpha_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j \left(\frac{2\pi}{N} \right) nk\omega_0 t}$$

$$k = -M, \dots, 0, \dots, +M$$

$$x_{FS}(t) = \sum_{k=-\infty}^{+\infty} \alpha_k e^{+jk\omega_0 t} \quad \text{DTFS}$$



CTFT and DFT

Continuous Time Fourier Transform

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt \quad \longleftrightarrow \quad x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{+j\omega t} d\omega$$

Discrete Fourier Transform

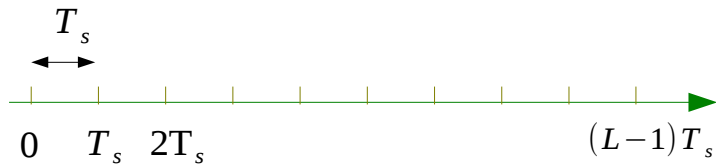
$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j(2\pi/N)kn} \quad \longleftrightarrow \quad x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{+j(2\pi/N)kn}$$

From CTFT to DFT (1)

Continuous Time Fourier Transform

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt \quad \longleftrightarrow \quad x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{+j\omega t} d\omega$$

Time Samples

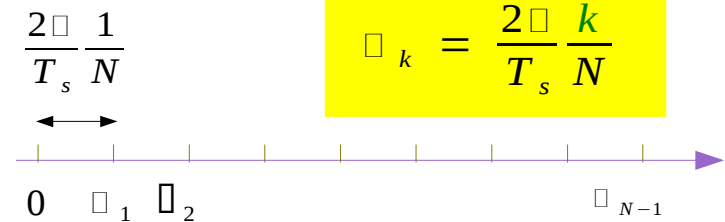


$$t \rightarrow nT_s \quad dt \rightarrow T_s \quad \int \rightarrow \sum$$

$$0 \leq n < L$$

$$T_s \rightarrow 0$$

Frequency Samples



$$\omega \rightarrow \omega_k \quad d\omega \rightarrow \frac{2\pi}{T_s} \frac{1}{N} \quad \int \rightarrow \sum$$

$$0 \leq k < N$$

$$0 \leq \omega_k < \frac{2\pi}{T_s}$$

From CTFT to DFT (2)

Continuous Time Fourier Transform

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt \quad \longleftrightarrow \quad x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{+j\omega t} d\omega$$

$$t \rightarrow nT_s \quad dt \rightarrow T_s \quad \int \rightarrow \sum \quad 0 \leq n < L$$

$$\hat{X}(j\omega) = \sum_{n=-\infty}^{+\infty} x(nT_s) e^{-j\omega nT_s} \cdot T_s \quad \longleftrightarrow \quad x(nT_s) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{X}(j\omega) e^{+j\omega nT_s} d\omega$$

$$\omega \rightarrow \omega_k \quad d\omega \rightarrow \frac{2\pi}{T_s} \frac{1}{N} \quad \int \rightarrow \sum \quad 0 \leq k < N$$

$$\omega_k = \frac{2\pi}{T_s} \frac{k}{N}$$

$$\hat{X}(j\omega_k) = T_s \sum_{n=0}^{L-1} x[n] e^{-j\omega_k nT_s} \quad \longleftrightarrow \quad x[n] = \frac{1}{2\pi} \sum_{k=0}^{N-1} \hat{X}(j\omega_k) e^{+j\omega_k nT_s} \frac{2\pi}{T_s} \frac{1}{N}$$

From CTFT to DFT (3)

Continuous Time Fourier Transform

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt \quad \longleftrightarrow \quad x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{+j\omega t} d\omega$$

Time Samples

$$t \rightarrow nT_s \quad dt \rightarrow T_s \quad \int \rightarrow \sum$$

Frequency Samples

$$\omega \rightarrow \omega_k \quad d\omega \rightarrow \frac{2\pi}{T_s} \frac{1}{N} \quad \int \rightarrow \sum$$

$$\hat{X}(j\omega_k) = T_s \sum_{n=0}^{L-1} x[n] e^{-j\omega_k n T_s} \quad \longleftrightarrow \quad x[n] = \frac{1}{2\pi} \sum_{k=0}^{N-1} \hat{X}(j\omega_k) e^{+j\omega_k n T_s} \frac{2\pi}{T_s} \frac{1}{N}$$

$$\omega_k T_s \rightarrow \frac{2\pi}{N} k \quad \longrightarrow \quad \omega_k n T_s \rightarrow \frac{2\pi}{N} k n$$

$$\omega_k = \frac{2\pi}{T_s} \frac{k}{N}$$

$$\frac{1}{T_s} \hat{X}(j\omega_k) = \sum_{n=0}^{L-1} x[n] e^{-j\left(\frac{2\pi}{N}\right)kn} \quad \longleftrightarrow \quad x[n] = \frac{1}{N} \sum_{k=0}^{N-1} \frac{1}{T_s} \hat{X}(j\omega_k) e^{+j\left(\frac{2\pi}{N}\right)kn}$$

From CTFT to DFT (4)

Continuous Time Fourier Transform

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt \quad \longleftrightarrow \quad x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{+j\omega t} d\omega$$

Time Samples

$$t \rightarrow nT_s \quad dt \rightarrow T_s \quad \int \rightarrow \sum$$

Frequency Samples

$$\omega \rightarrow \omega_k \quad d\omega \rightarrow \frac{2\pi}{T_s} \frac{1}{N} \quad \int \rightarrow \sum$$

$$\omega_k T_s \rightarrow \frac{2\pi}{N} k$$



$$\omega_k nT_s \rightarrow \frac{2\pi}{N} kn$$

$$\omega_k = \frac{2\pi}{T_s} \frac{k}{N}$$

$$\frac{1}{T_s} X(j\omega_k) = \sum_{n=0}^{L-1} x[n] e^{-j\left(\frac{2\pi}{N}\right)kn}$$



$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} \frac{1}{T_s} X(j\omega_k) e^{+j\left(\frac{2\pi}{N}\right)kn}$$

From CTFT to DFT (5)

Continuous Time Fourier Transform

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt \quad \longleftrightarrow \quad x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{+j\omega t} d\omega$$

$$\frac{1}{T_s} X(j\omega_k) = \sum_{n=0}^{L-1} x[n] e^{-j\left(\frac{2\pi}{N}\right)kn} \quad \longleftrightarrow \quad x[n] = \frac{1}{N} \sum_{k=0}^{N-1} \frac{1}{T_s} X(j\omega_k) e^{+j\left(\frac{2\pi}{N}\right)kn}$$

Discrete Fourier Transform

$$L = N$$

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j(2\pi/N)kn} \quad \longleftrightarrow \quad x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{+j(2\pi/N)kn}$$

CTFT of a Sampled Signal

Continuous Time Fourier Transform

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{+j\omega t} d\omega$$



$$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$$

$$x_s(t) = x_c(t) \sum_{n=-\infty}^{+\infty} \delta(t - nT_s)$$

CTFT



$$X_s(j\Omega) = \sum_{n=-\infty}^{+\infty} x_c(nT_s) e^{-j\Omega nT_s}$$

$$x_s(t) = \sum_{n=-\infty}^{+\infty} x_c(nT_s) \delta(t - nT_s)$$

$$X_s(j\Omega) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\Omega nT_s}$$

CTFS



DTFT



$$x_s(t) = x_c(t) \sum_{k=-\infty}^{+\infty} \frac{1}{T_s} e^{jk\omega_s t}$$

CTFT



$$X_s(j\Omega) = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} X_c(j(\Omega - k\Omega_s))$$

$$\omega_s = \frac{2\pi}{T_s}$$

$$\Omega_s = \frac{2\pi}{T_s}$$

DTFT and CTFT

Continuous Time Fourier Transform

CTFT

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt \quad \longleftrightarrow \quad x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{+j\omega t} d\omega$$

Discrete Time Fourier Transform

DTFT

$$X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\hat{\omega}n} \quad \longleftrightarrow \quad x[n] = \frac{1}{2\pi} \int_{-\pi}^{+\pi} X(e^{j\hat{\omega}}) e^{+j\hat{\omega}n} d\hat{\omega}$$

$$X_s(j\hat{\omega}) = \sum_{n=-\infty}^{+\infty} x_c(nT_s) e^{-j\hat{\omega}nT_s}$$

$$X_s(j\hat{\omega}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\hat{\omega}nT_s}$$



$$X_s(j\hat{\omega}) = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} X_c(j(\hat{\omega} - k\hat{\omega}_s))$$

$$\hat{\omega}_s = \frac{2\pi}{T_s}$$

DTFT and CTFT

$$X_s(j\omega) = \sum_{n=-\infty}^{+\infty} x_c(nT_s) e^{-j\omega nT_s}$$

$$X_s(j\omega) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega nT_s}$$



$$X_s(j\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} X_c(j(\omega - k\omega_s))$$

$$\omega_s = \frac{2\pi}{T_s}$$

CTFT of a sampled signal

$$X(e^{j\hat{\omega}})$$

$$\hat{\omega} = \omega T_s$$

$$= X(e^{j\omega T_s}) = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} X_c(j(\omega - k\omega_s))$$

$$= \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} X_c(j(\omega - \frac{2\pi k}{T_s}))$$

$$X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\hat{\omega}n}$$

DTFT of a sampled signal

DTFT and DFT

DTFT of a sampled signal

$$X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\hat{\omega}n}$$

$$\hat{\omega} = \omega T_s$$

Frequency Samples

$$\hat{\omega} \rightarrow \hat{\omega}_k \quad 0 \leq \hat{\omega}_k < 2\pi \quad 0 \leq k < N \quad 0 \leq n < L$$

$$X(e^{j\hat{\omega}_k}) = \sum_{n=0}^{L-1} x[n] e^{-j\hat{\omega}_k n}$$

$$\hat{\omega}_k = \left(\frac{2\pi}{N} \right) k$$

DFT of a sampled signal

$$X[k] =$$

$$X(e^{j(2\pi/N)k}) = \sum_{n=0}^{L-1} x[n] e^{-j(2\pi/N)kn}$$

DTFT sampled in frequency

CTFT and DFT

DFT of a sampled signal

$$X[k]$$

$$= X(e^{j(2\pi/N)k}) = \sum_{n=0}^{L-1} x[n] e^{-j(2\pi/N)kn}$$

DTFT sampled in frequency

$$X(e^{j\omega T_s}) \Big|_{\omega = \frac{2\pi k}{NT_s}}$$

CTFT evaluated at $\omega = \frac{2\pi k}{NT_s}$

$$= \frac{1}{T_s} \sum_{l=-\infty}^{+\infty} X_c(j(\omega - l\omega_s)) \Big|_{\omega = \frac{2\pi k}{NT_s}}$$

$$= \frac{1}{T_s} \sum_{l=-\infty}^{+\infty} X_c(j(\omega - l\frac{2\pi}{T_s})) \Big|_{\omega = \frac{2\pi k}{NT_s}}$$

From DTFT to DFT (1)

DTFT of a sampled signal

$$X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\hat{\omega}n}$$

$$\hat{\omega} = \omega T_s$$

Frequency Samples

$$\hat{\omega} \rightarrow \hat{\omega}_k \quad 0 \leq \hat{\omega}_k < 2\pi \quad 0 \leq k < N \quad 0 \leq n < L$$

$$X(e^{j\hat{\omega}_k}) = \sum_{n=0}^{L-1} x[n] e^{-j\hat{\omega}_k n}$$

$$\hat{\omega}_k = \left(\frac{2\pi}{N} \right) k$$

DFT of a sampled signal

$$X[k] =$$

$$X(e^{j(2\pi/N)k}) = \sum_{n=0}^{L-1} x[n] e^{-j(2\pi/N)kn}$$

DTFT sampled in frequency

From FT to DFT (1)

$$f(t)$$

$$x(k\tau) = f(t)\delta(t - k\tau)w(t)$$

sampling time τ

window function $0 \leq t \leq T$

$$X(\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$$

Fourier Transform

discrete time $t \Rightarrow k\tau$
discrete freq $\omega \Rightarrow \omega_k = 2\pi f_k$

$$X(\omega_k) = \sum_{n=0}^{N-1} x(n\tau) e^{-j2\pi f_k n\tau}$$

$$e^{-j2\pi f_k n\tau} = e^{-j2\pi k \left(\frac{1}{N\tau}\right) n\tau}$$

$$X(\omega_k) = \sum_{n=0}^{N-1} x(n\tau) e^{-j\frac{2\pi}{N}kn}$$

fundamental frequency

$$f_0 = \frac{1}{T} = \frac{1}{N\tau} = \frac{f_s}{N}$$

k-th harmonics

$$f_k = k \cdot \frac{1}{T} = k \cdot \frac{1}{N\tau} = k \cdot \frac{f_s}{N}$$

From FT to DFT (2)

$$f(t)$$

$$x(k\tau) = f(t)\delta(t - k\tau)w(t)$$

sampling time τ

window function $0 \leq t \leq T$

$$X(\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$$

discrete time $t \Rightarrow k\tau$
discrete freq $\omega \Rightarrow \omega_k = 2\pi f_k$

$$X(\omega_k) = \sum_{n=0}^{N-1} x(n\tau) e^{-j\frac{2\pi}{N}kn}$$

$$f_k = k \cdot \frac{f_s}{N}$$

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}$$

Fourier Transform

fundamental frequency

$$f_0 = \frac{1}{T} = \frac{1}{N\tau} = \frac{f_s}{N}$$

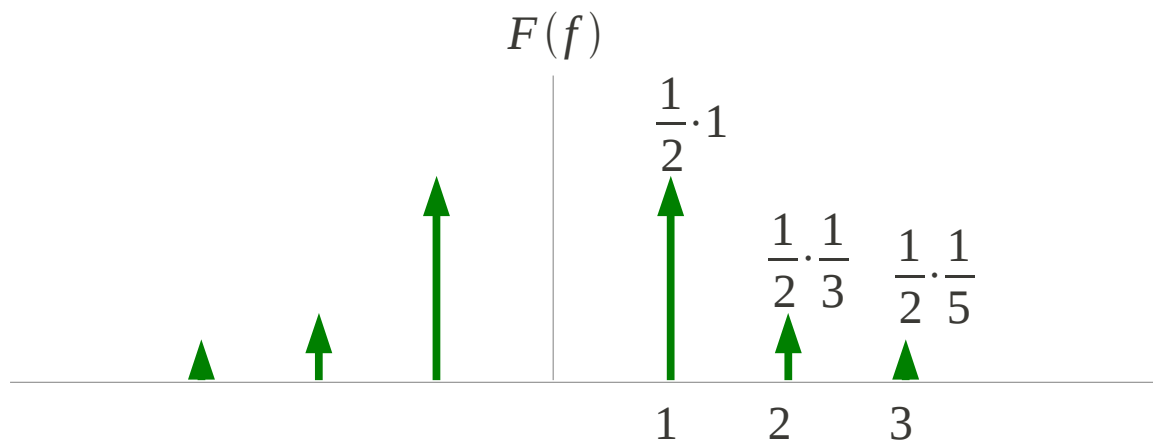
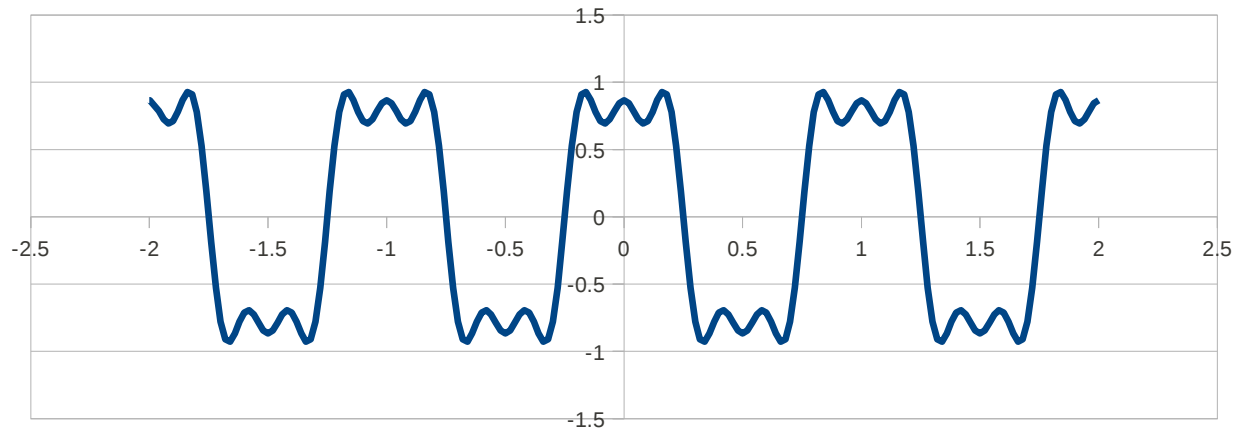
k-th harmonics

$$f_k = k \cdot \frac{1}{T} = k \cdot \frac{1}{N\tau} = k \cdot \frac{f_s}{N}$$

Discrete Fourier Transform

FT → DFT Example (1)

$$f(t) = \cos(2\pi \cdot 1 \cdot t) - \frac{1}{3} \cos(2\pi \cdot 3 \cdot t) + \frac{1}{5} \cos(2\pi \cdot 5 \cdot t)$$



FT → DFT Example (2)

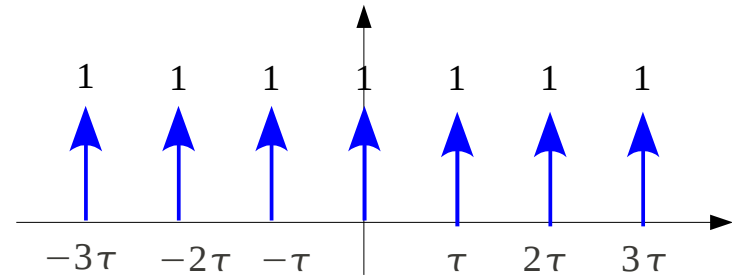
$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - n\tau)$$

Fourier Series Expansion of Impulse Train

$$p(t) = \sum_{k=-\infty}^{+\infty} a_k e^{+jk\omega_s t}$$

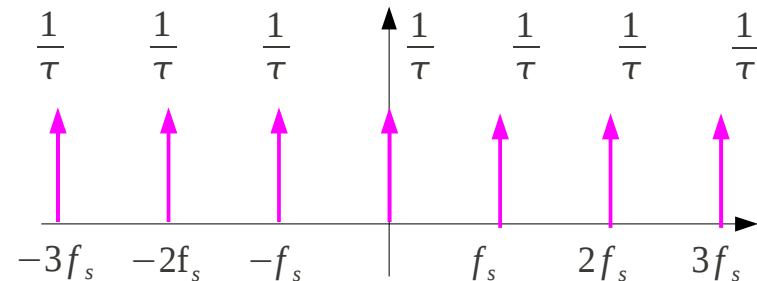
Fourier Series Coefficients

$$\begin{aligned} a_k &= \frac{1}{\tau} \int_{-\tau/2}^{+\tau/2} \delta(t) e^{-jk\omega_s t} dt \\ &= \frac{1}{\tau} \int_{-\tau/2}^{+\tau/2} \delta(t) e^{-jk\omega_s 0} dt \\ &= \frac{1}{\tau} \int_{-\tau/2}^{+\tau/2} \delta(t) dt = \frac{1}{T_s} \end{aligned}$$



$$\omega_s = \frac{2\pi}{\tau}$$

$$f_s = \frac{1}{\tau}$$



FT → DFT Example (3)

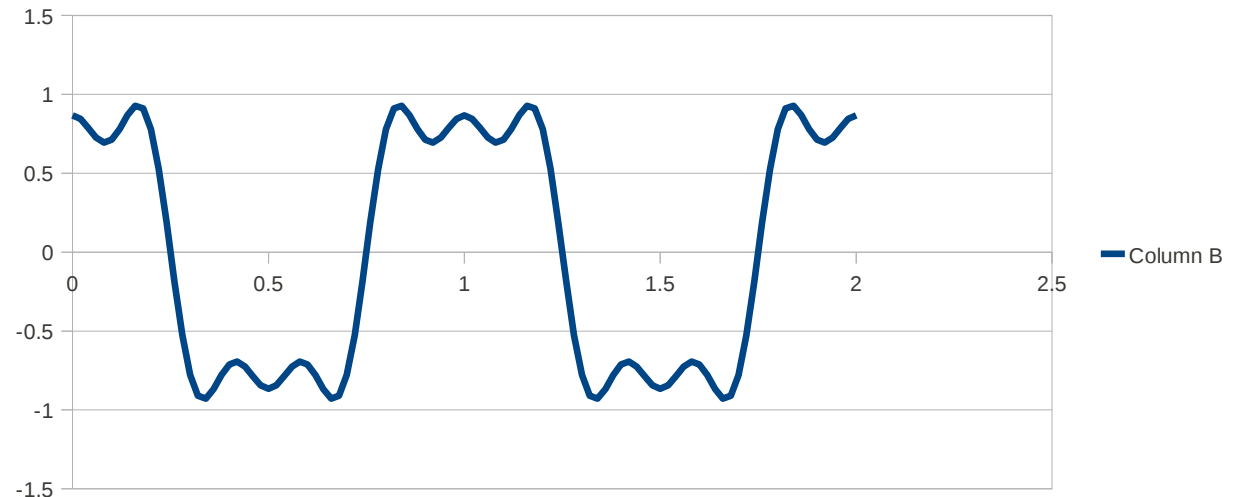
$$\begin{aligned}W(\omega) &= \int_0^T e^{-j\omega t} dt \\&= \left[\frac{e^{-j\omega t}}{-j\omega} \right]_0^T \\&= \frac{e^{-j\omega T} - 1}{-j\omega} \\&= \frac{e^{-j\omega T/2} (e^{-j\omega T/2} - e^{+j\omega T/2})}{-j\omega} \\&= \frac{\sin(\omega T/2)}{\omega T/2} \cdot e^{-j\omega T/2}\end{aligned}$$

From DTFT to DFT (3)

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}kn}$$

$$X\left(\frac{k}{N}\right) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}kn}$$

$$X\left(k\frac{f_s}{N}\right) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}kn}$$



References

- [1] <http://en.wikipedia.org/>
- [2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003
- [3] <http://www.complextoreal.com/fft3.htm>