Bandpass Sampling (2B)

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3/21/12

Please send corrections (or suggestions) to youngwlim@hotmail.com.

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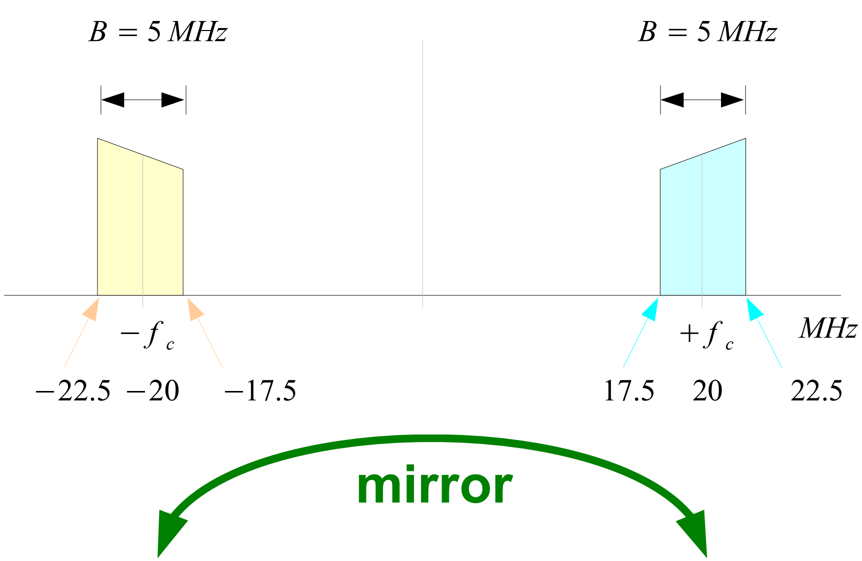
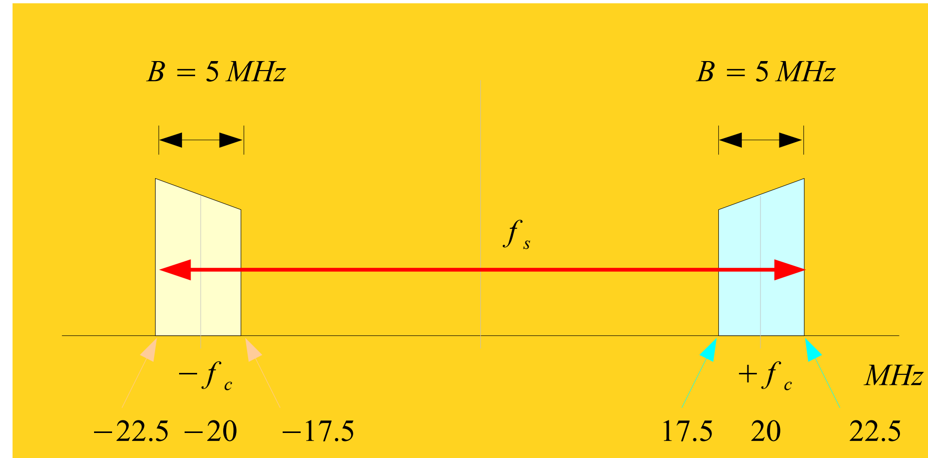
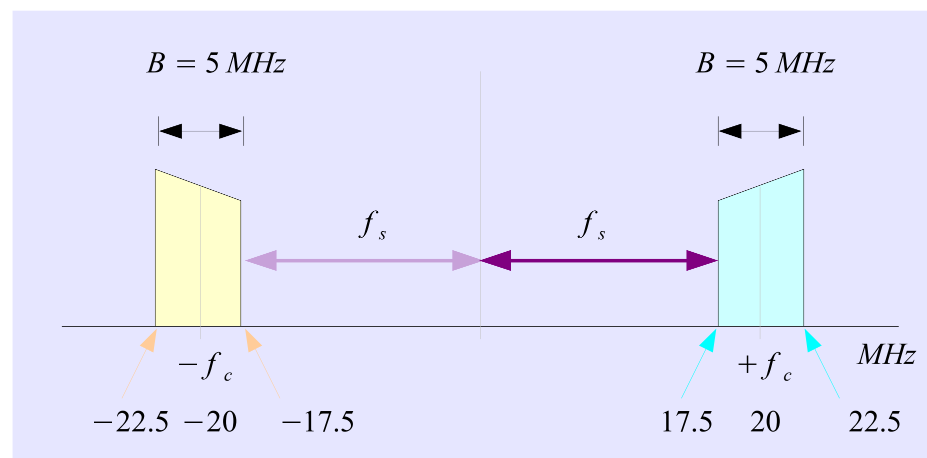
Band-limited Signal



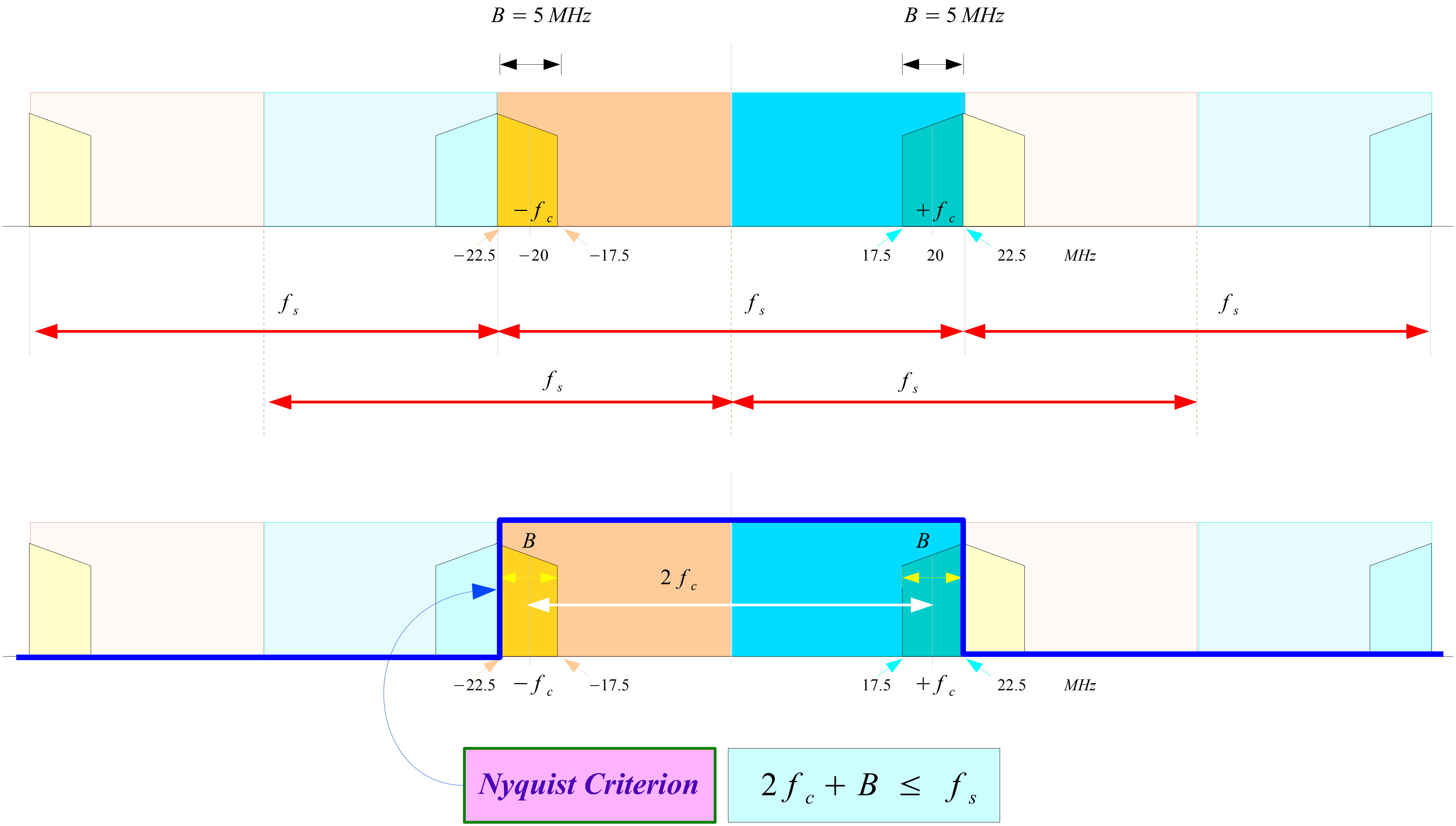


- **Lowpass Sampling**

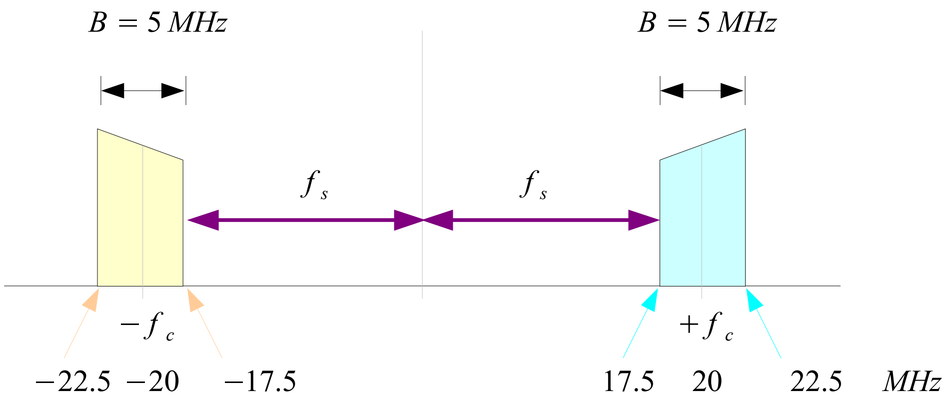
- **Bandpass Sampling**
- **IF filtering**
- **Harmonic Sampling**
- **Sub-Nyquist Sampling**



Low-pass Signal Sampling

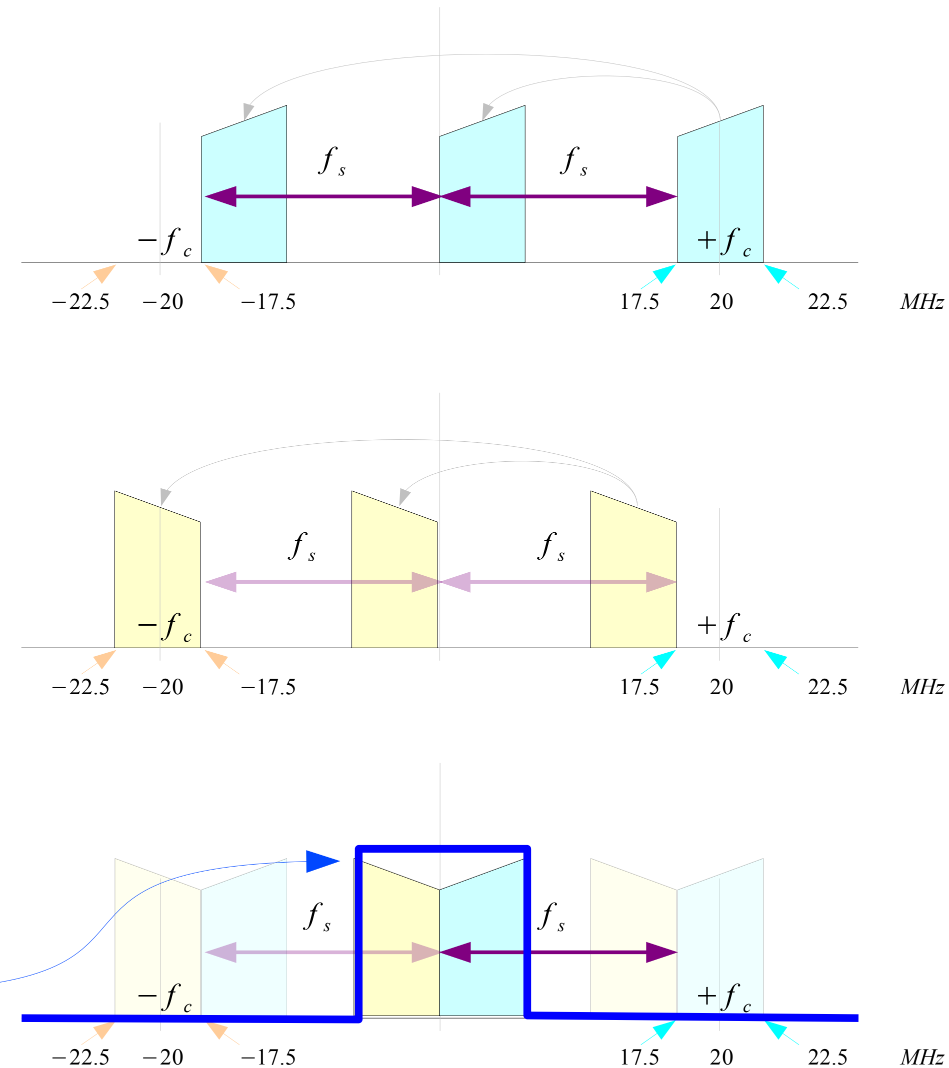


Band-pass Signal Sampling



- **Bandpass Sampling**
- **IF filtering**
- **Harmonic Sampling**
- **Sub-Nyquist Sampling**

Nyquist Criterion	$2B \leq f_s$



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Sampling Frequency f_s (1)

Assume there are m multiples of f_s

$$2f_c - B = m \cdot f_s$$

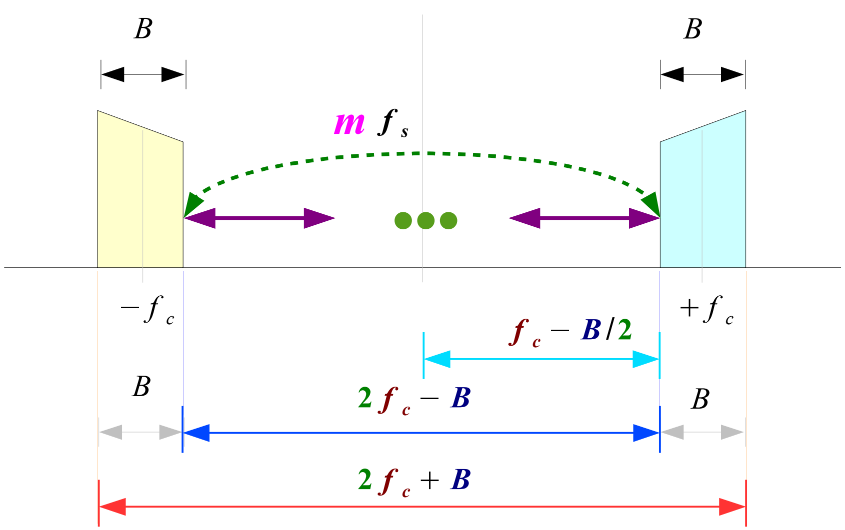
← **Max f_s condition**

Given an integer m

f_s *can be decreased according to the following condition without introducing aliasing problems*

$$2f_c + B = (m+1) \cdot f_s$$

← **Min f_s condition**



Given Band-pass Signal is characterized by

- ***Bandwidth B***
- ***Carrier Frequency f_c***

$$\frac{2f_c + B}{m+1} \le f_s \le \frac{2f_c - B}{m}$$

Sampling Frequency f_s (2)

$$\frac{2f_c + B}{m+1} \le f_s \le \frac{2f_c - B}{m}$$

Given Band-pass Signal is characterized by

- ***Bandwidth B***
- ***Carrier Frequency f_c***

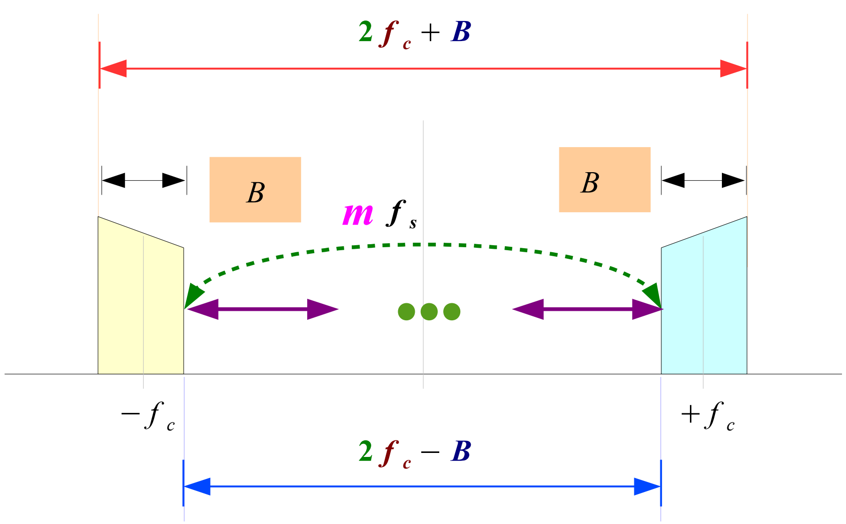
➡ ***Normalization by B***

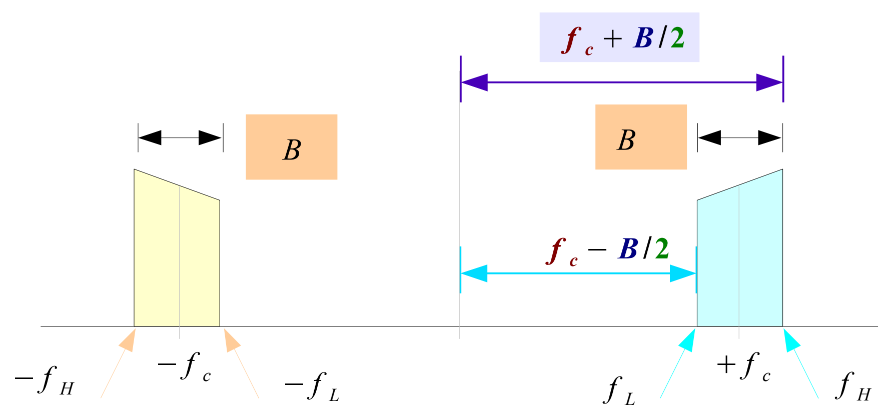
$$\frac{2f_c + B}{(m+1)B} \le \frac{f_s}{B} \le \frac{2f_c - B}{mB}$$

$$\frac{2f_H}{(m+1)B} \le \frac{f_s}{B} \le \frac{2f_L}{mB}$$

$$f_H = f_c + B/2$$ ***Highest frequency***

$$f_L = f_c - B/2$$ ***Lowest frequency***





Sampling Frequency f_s (3)

$$\frac{2 f_c + B}{m + 1} \le f_s \le \frac{2 f_c - B}{m}$$

$$\frac{f_c + B/2}{B} = R \;\rightarrow\; \mathbf{X}$$

$\Rightarrow$ **highest signal frequency / bandwidth B**

$$\frac{2 f_c + B}{(m + 1) B} = \frac{f_{s,min}}{B} \;\rightarrow\; \mathbf{Y}$$

$\Rightarrow$ **minimum sampling rate / bandwidth B**

X-Y Plot

Y $\frac{f_{s,min}}{B}$

X

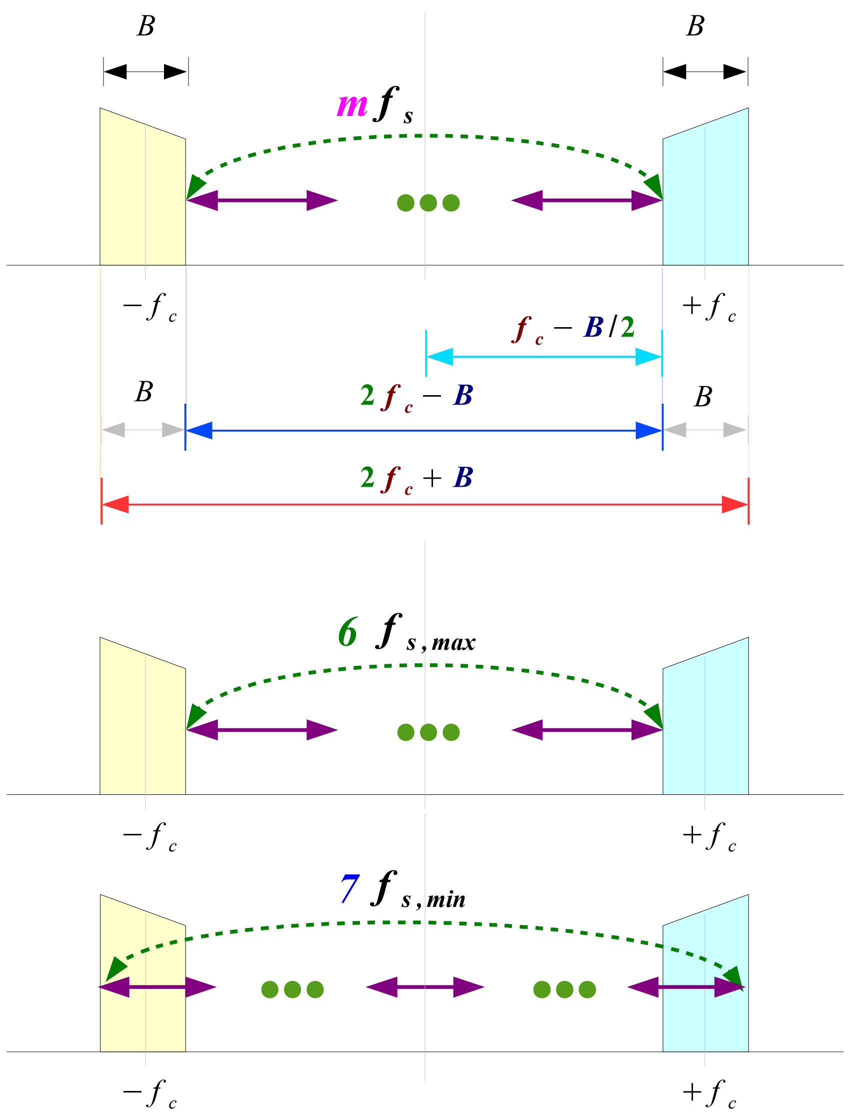
*This plot shows **min f_s** normalized by **B**, for the given bandpass signal that is characterized by **R** and the given parameter **m***

Characterized by

- ***Bandwidth B***
- ***Carrier Frequency f_c***

$$R = \frac{f_H}{B} = \frac{f_c + B/2}{B}$$

Example m=6 (1)



$$\frac{2f_c + B}{m+1} \le f_s \le \frac{2f_c - B}{m}$$

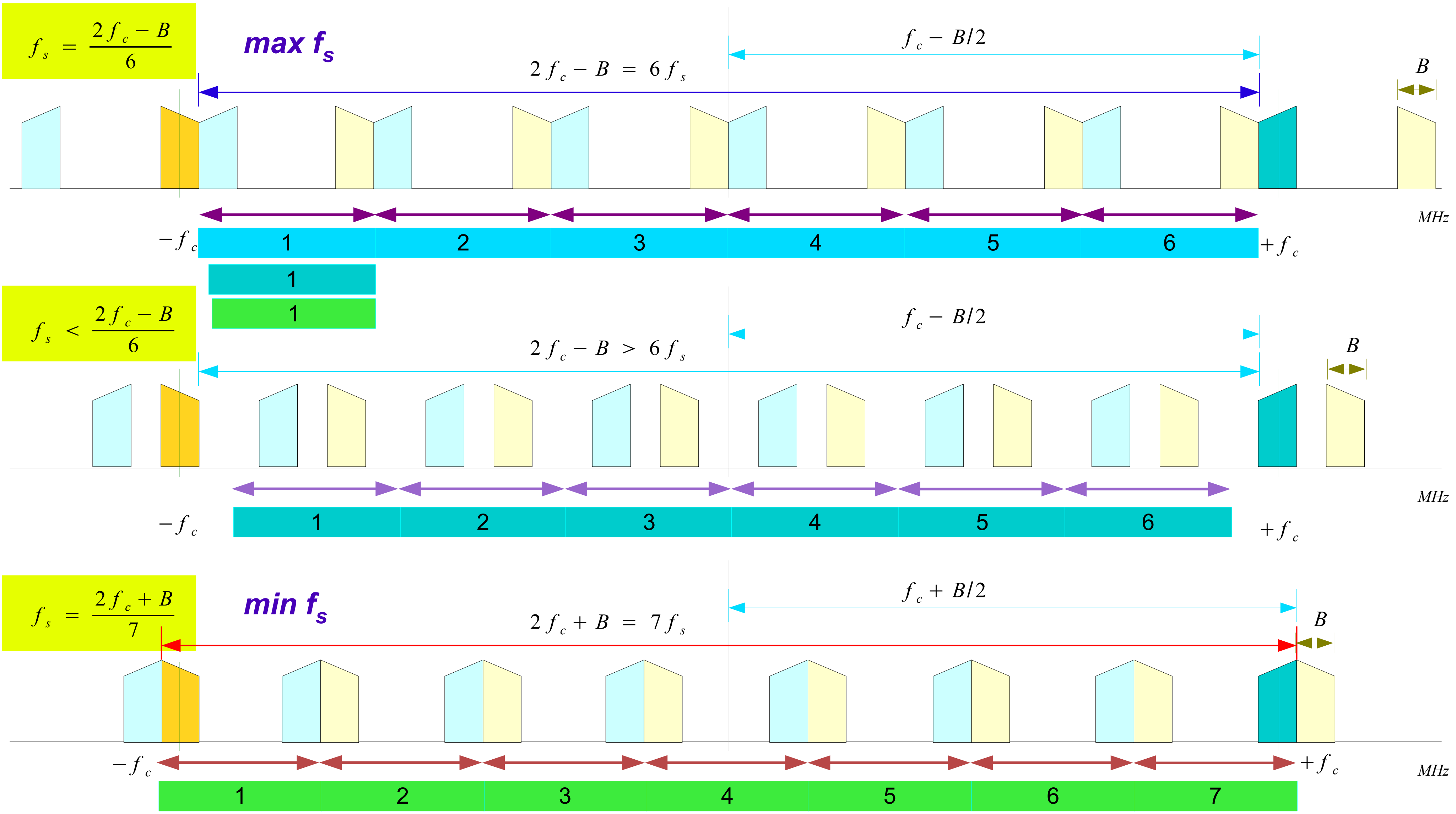
When m = 6

$$\text{min } f_s \quad \frac{2f_c + B}{7} \le f_s \le \frac{2f_c - B}{6} \quad \text{max } f_s$$

$$\text{max } f_s = \frac{2f_c - B}{6}$$

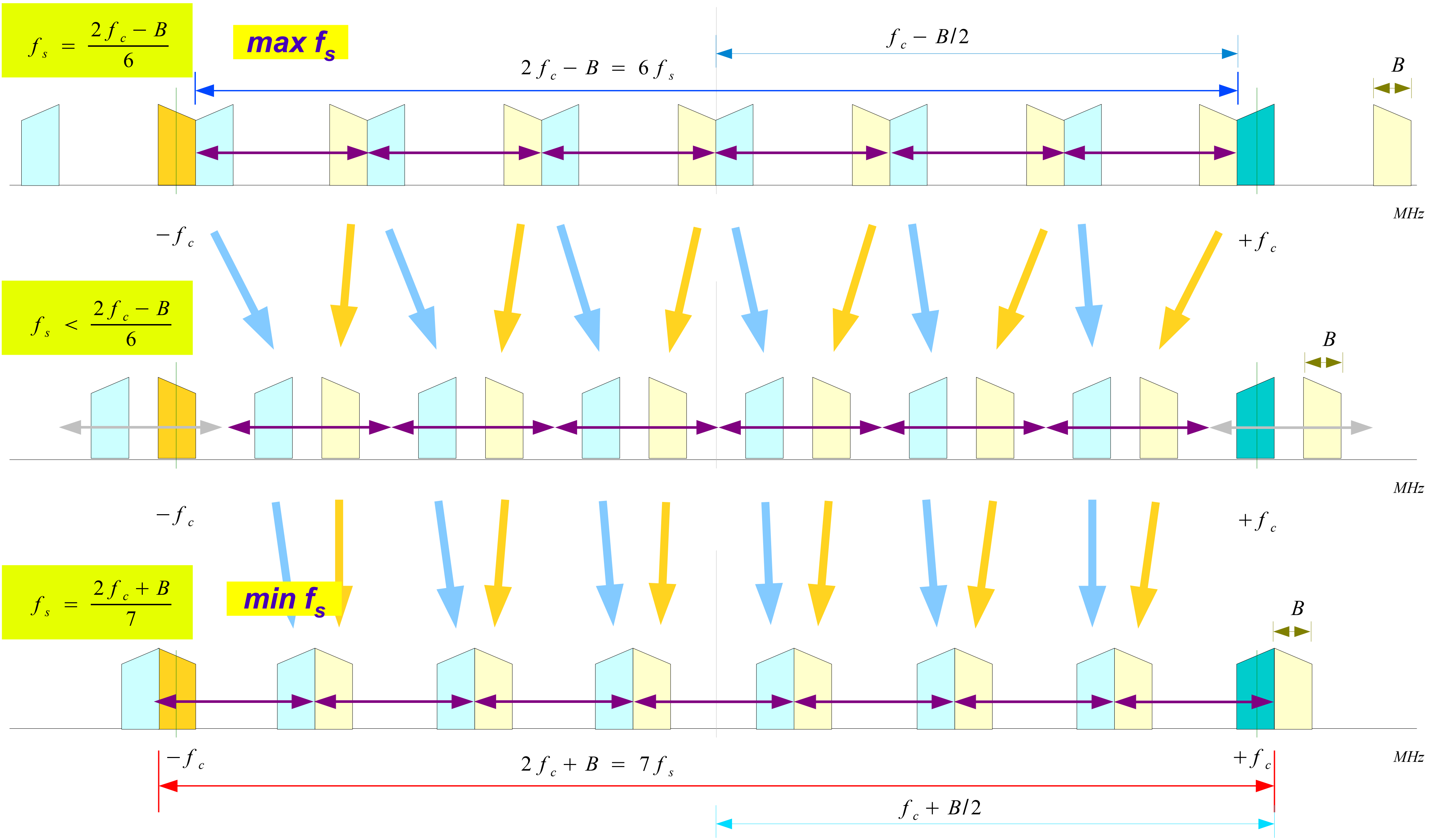
$$\text{min } f_s = \frac{2f_c + B}{7}$$

Example m=6 (2)



$f_s = \frac{2f_c - B}{6}$
max f_s
$f_c - B/2$
$2f_c - B = 6f_s$
B
MHz
$-f_c$
1 2 3 4 5 6
$+f_c$
1
1
$f_s < \frac{2f_c - B}{6}$
$f_c - B/2$
$2f_c - B > 6f_s$
B
MHz
$-f_c$
1 2 3 4 5 6
$+f_c$
$f_s = \frac{2f_c + B}{7}$
min f_s
$f_c + B/2$
$2f_c + B = 7f_s$
B
$-f_c$
$+f_c$
MHz
1 2 3 4 5 6 7

Example m=6 (3)



$f_s = \frac{2f_c - B}{6}$
max f_s
$f_c - B/2$
$2f_c - B = 6f_s$
B
MHz
$-f_c$
$+f_c$
$f_s < \frac{2f_c - B}{6}$
B
MHz
$-f_c$
$+f_c$
$f_s = \frac{2f_c + B}{7}$
min f_s
B
MHz
$-f_c$
$2f_c + B = 7f_s$
$+f_c$
$f_c + B/2$

Range of f_s (1)

For a given m

$$\frac{2 f_c + B}{m+1} \le f_s \le \frac{2 f_c - B}{m}$$

Nyquist Criterion

$$2B \le f_s$$

$f_c = 20\ MHz$

$B = 5\ MHz$

min f_s ***max f_s***

Optimum Sampling Frequency

$m = 1$: $\frac{2 \cdot 20 + 5}{1 + 1} = 22.5 \le f_s \le \frac{2 \cdot 20 - 5}{1} = 35$ → $f_s = 22.5\ MHz \quad (10 \le f_s)$

$m = 2$: $\frac{2 \cdot 20 + 5}{2 + 1} = 15 \le f_s \le \frac{2 \cdot 20 - 5}{2} = 17.5$ → $f_s = 17.5\ MHz \quad (10 \le f_s)$

$m = 3$: $\frac{2 \cdot 20 + 5}{3 + 1} = 11.25 \le f_s \le \frac{2 \cdot 20 - 5}{3} = 11.67$ → $f_s = 11.25\ MHz \quad (10 \le f_s)$

$m = 4$: $\frac{2 \cdot 20 + 5}{4 + 1} = 9 \ge \frac{2 \cdot 20 - 5}{4} = 8.75$ → X

$m = 5$: $\frac{2 \cdot 20 + 5}{5 + 1} = 7.5 \ge \frac{2 \cdot 20 - 5}{5} = 7.0$ → X

Range of f_s (2)

$$\frac{2f_c + B}{m+1} \le f_s \le \frac{2f_c - B}{m}$$

$$\frac{f_c + B/2}{B} = R$$

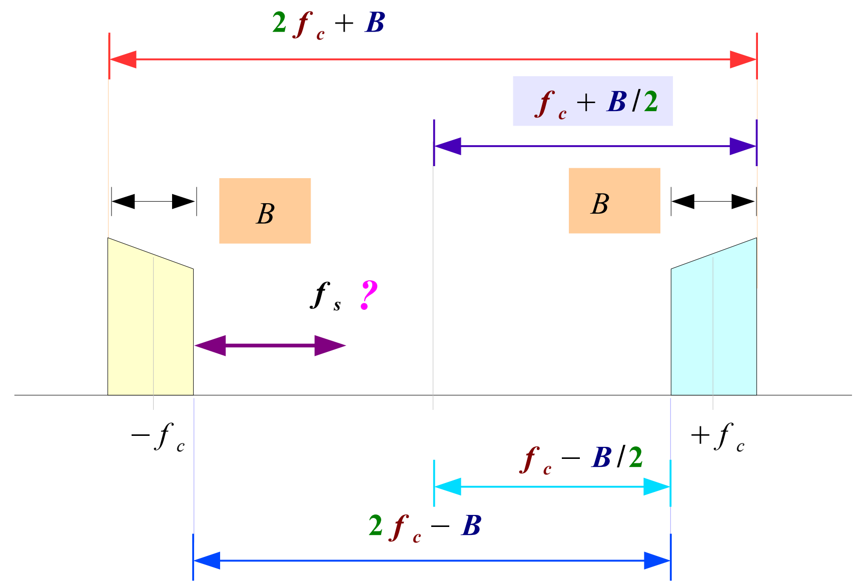
$\Rightarrow$ highest signal frequency / bandwidth B

$$\frac{2f_c + B}{(m+1)B} = \frac{f_{s,min}}{B} = g(m, R)$$

$\Rightarrow$ minimum sampling rate / bandwidth B

$$\frac{2(f_c + B/2)}{(m+1)B} = \frac{2R}{m+1} = g(m, R)$$

$m = 1$	$g(1, R) = R$	$m = 5$	$g(5, R) = \frac{1}{3}R$
$m = 2$	$g(2, R) = \frac{2}{3}R$	$m = 6$	$g(6, R) = \frac{2}{7}R$
$m = 3$	$g(3, R) = \frac{1}{2}R$	$m = 7$	$g(7, R) = \frac{1}{4}R$
$m = 4$	$g(4, R) = \frac{2}{5}R$	$m = 8$	$g(8, R) = \frac{2}{9}R$



Range of f_s (3)

$$\frac{2f_c + B}{m+1} \le f_s \le \frac{2f_c - B}{m}$$

$$\frac{f_c + B/2}{B} = R \;\Rightarrow\; \frac{\text{highest signal frequency}}{\text{bandwidth } B}$$

$$f_H = f_c + B/2$$

$$R = f_H / B$$

$$\frac{2f_c + B}{(m+1)B} = g(m,R) \;\Rightarrow\; \frac{\text{minimum sampling rate}}{\text{bandwidth } B}$$

$$f_{s,min} = \frac{2f_c + B}{m+1} = \frac{2f_H}{k}$$

$$\frac{2f_c + B}{(m+1)B} = g(m,R) = \frac{2R}{m+1} = \frac{2R}{k}$$

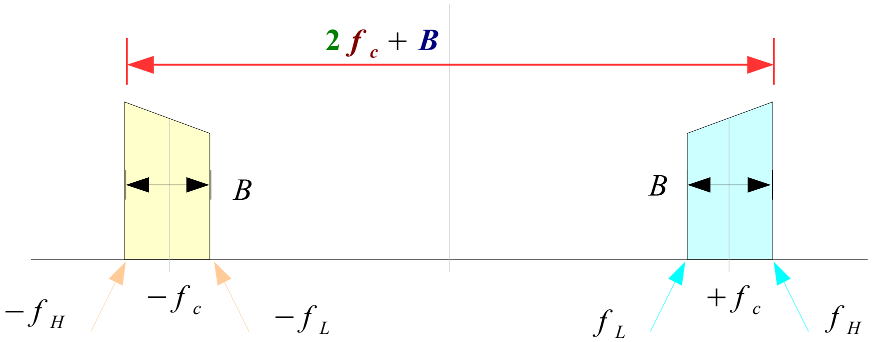
$$m + 1 = k$$

$$g(m,R) = \frac{2f_H}{kB} = \frac{2R}{k}$$

k represents how many f_s are in $2f_c + B$ in

Min f_s condition

$$2f_c + B = (m+1)\cdot f_s = k\cdot f_s$$



Range of f_s (4)

$$\frac{2f_c+B}{m+1} \le f_s \le \frac{2f_c-B}{m}$$

$$\frac{2f_c+B}{k} \le f_s \le \frac{2f_c-B}{k-1}$$

$$\frac{2(f_c+B/2)}{k} \le f_s \le \frac{2(f_c+B/2)-2B}{k-1}$$

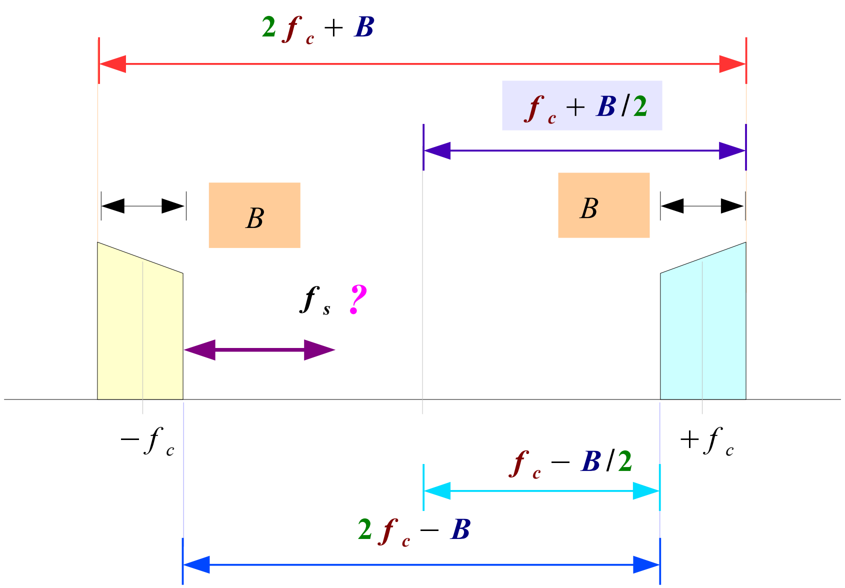
$$\frac{2f_H}{k} \qquad \frac{2(f_H-B)}{k-1}$$

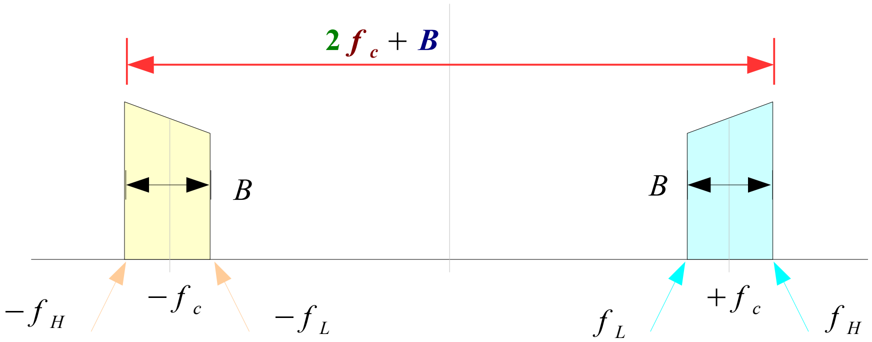
$$\frac{2f_H}{k} \le f_s \le \frac{2(f_H-B)}{k-1}$$

$k=2 \qquad f_H \le f_s \le 2f_H - 2B$

$k=3 \qquad \frac{2f_H}{3} \le f_s \le f_H - B$

$k=4 \qquad \frac{f_H}{2} \le f_s \le \frac{2f_H}{3} - \frac{3B}{3}$

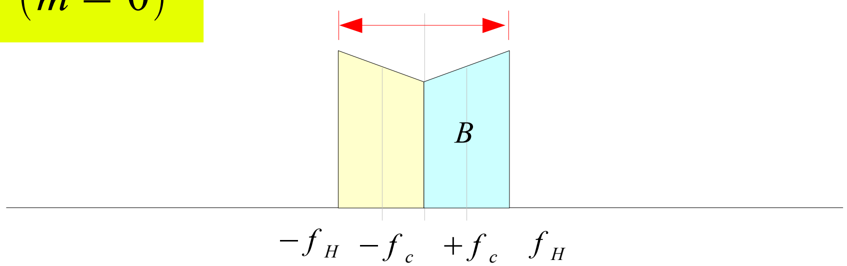




Example k=1

$k = 1$
$(m = 0)$

$f_H = f_c + B/2 = 1B$



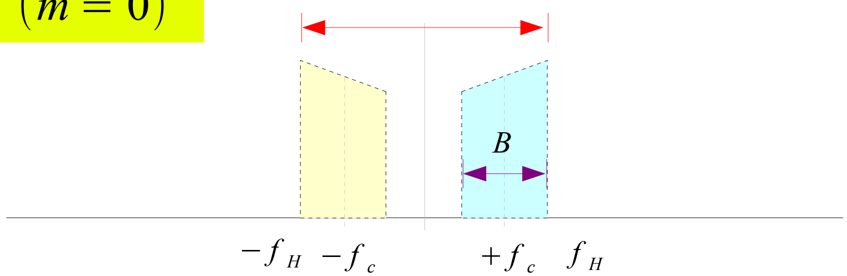
$R = f_H / B = 1$

$$\frac{f_{s,min}}{B} = \frac{2f_H}{kB} = \mathbf{2}$$

$$\frac{f_{s,max}}{B} = \frac{2(f_H - B)}{(k-1)B} = +\infty$$

$k = 1$
$(m = 0)$

$f_H = f_c + B/2 = 1.5B$



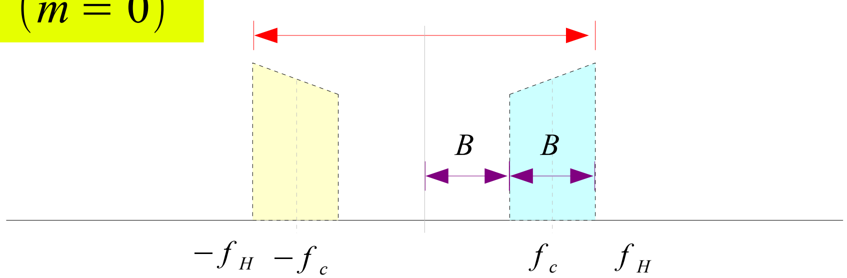
$R = f_H / B = 1.5$

$$\frac{f_{s,min}}{B} = \frac{2f_H}{kB} = \mathbf{3}$$

$$\frac{f_{s,max}}{B} = \frac{2(f_H - B)}{(k-1)B} = +\infty$$

$k = 1$
$(m = 0)$

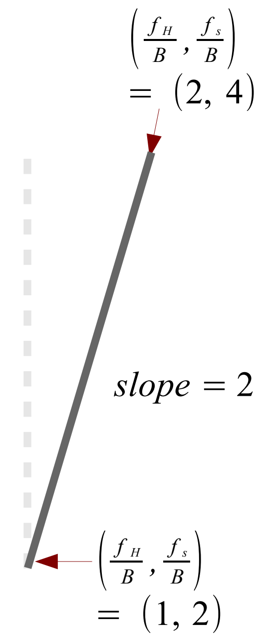
$f_H = f_c + B/2 = 2B$



$R = f_H / B = 2$

$$\frac{f_{s,min}}{B} = \frac{2f_H}{kB} = \mathbf{4}$$

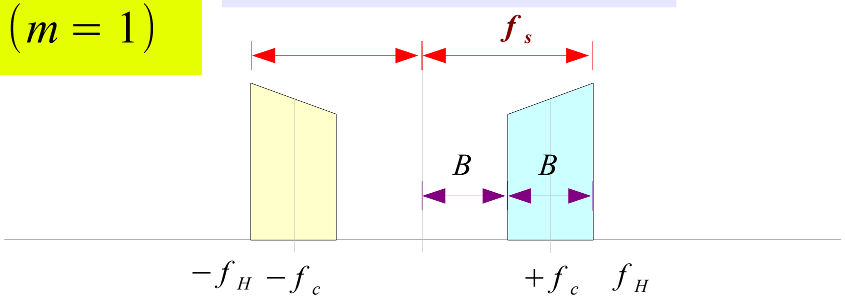
$$\frac{f_{s,max}}{B} = \frac{2(f_H - B)}{(k-1)B} = +\infty$$



Example k=2

$k = 2$ $(m = 1)$

$f_H = f_c + B/2 = 2B$



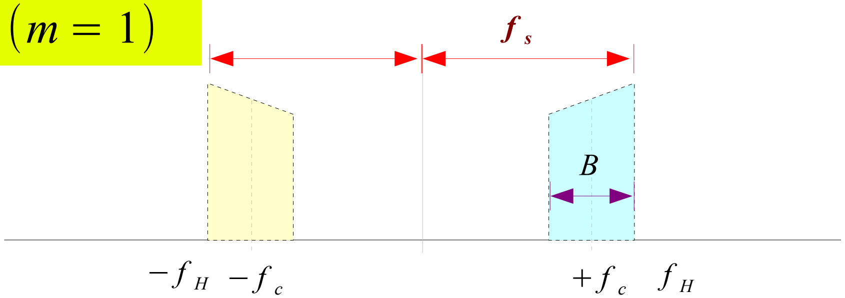
$R = f_H / B = 2$

$$\frac{f_{s,min}}{B} = \frac{2 f_H}{k B} = 2$$

$$\frac{f_{s,max}}{B} = \frac{2(f_H - B)}{(k-1) B} = 2$$

$k = 2$ $(m = 1)$

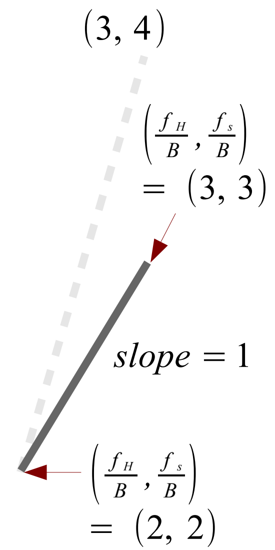
$f_H = f_c + B/2 = 2.5B$



$R = f_H / B = 2.5$

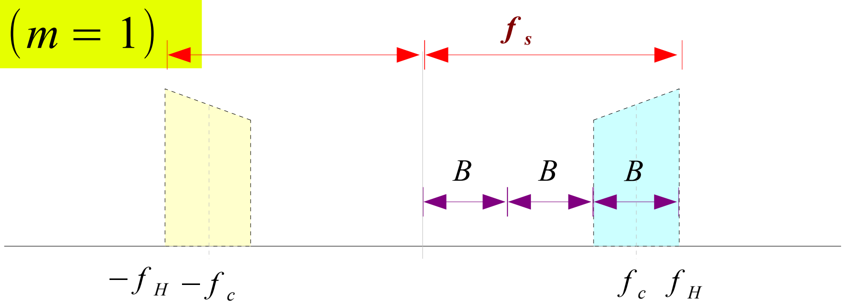
$$\frac{f_{s,min}}{B} = \frac{2 f_H}{k B} = 2.5$$

$$\frac{f_{s,max}}{B} = \frac{2(f_H - B)}{(k-1) B} = 3$$



$k = 2$ $(m = 1)$

$f_H = f_c + B/2 = 3B$



$R = f_H / B = 3$

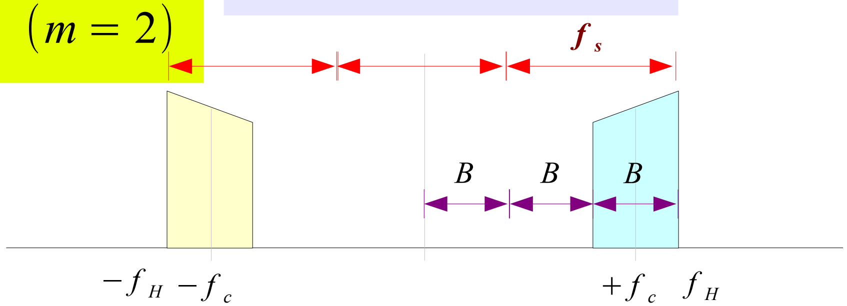
$$\frac{f_{s,min}}{B} = \frac{2 f_H}{k B} = 3$$

$$\frac{f_{s,max}}{B} = \frac{2(f_H - B)}{(k-1) B} = 4$$

Example k=3

$k = 3$ $(m = 2)$

$f_H = f_c + B/2 = 3B$



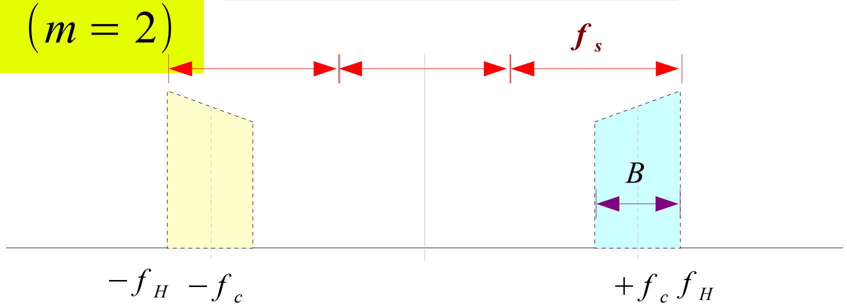
$$R = f_H / B = 3$$

$$\frac{f_{s,min}}{B} = \frac{2 f_H}{k B} = \mathbf{2}$$

$$\frac{f_{s,max}}{B} = \frac{2(f_H - B)}{(k-1)B} = \mathbf{2}$$

$k = 3$ $(m = 2)$

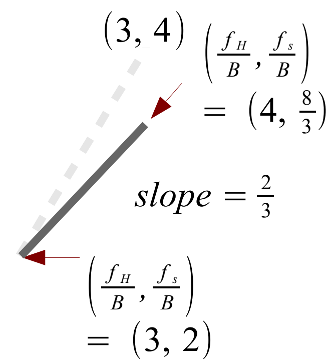
$f_H = f_c + B/2 = 3.5B$



$$R = f_H / B = 3.5$$

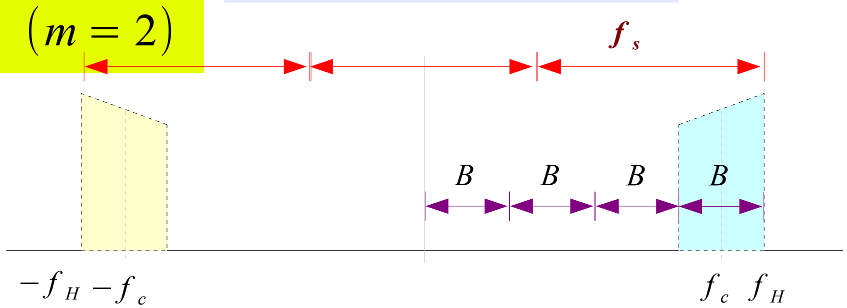
$$\frac{f_{s,min}}{B} = \frac{2 f_H}{k B} = \frac{\mathbf{7}}{\mathbf{3}}$$

$$\frac{f_{s,max}}{B} = \frac{2(f_H - B)}{(k-1)B} = \mathbf{3}$$



$k = 3$ $(m = 2)$

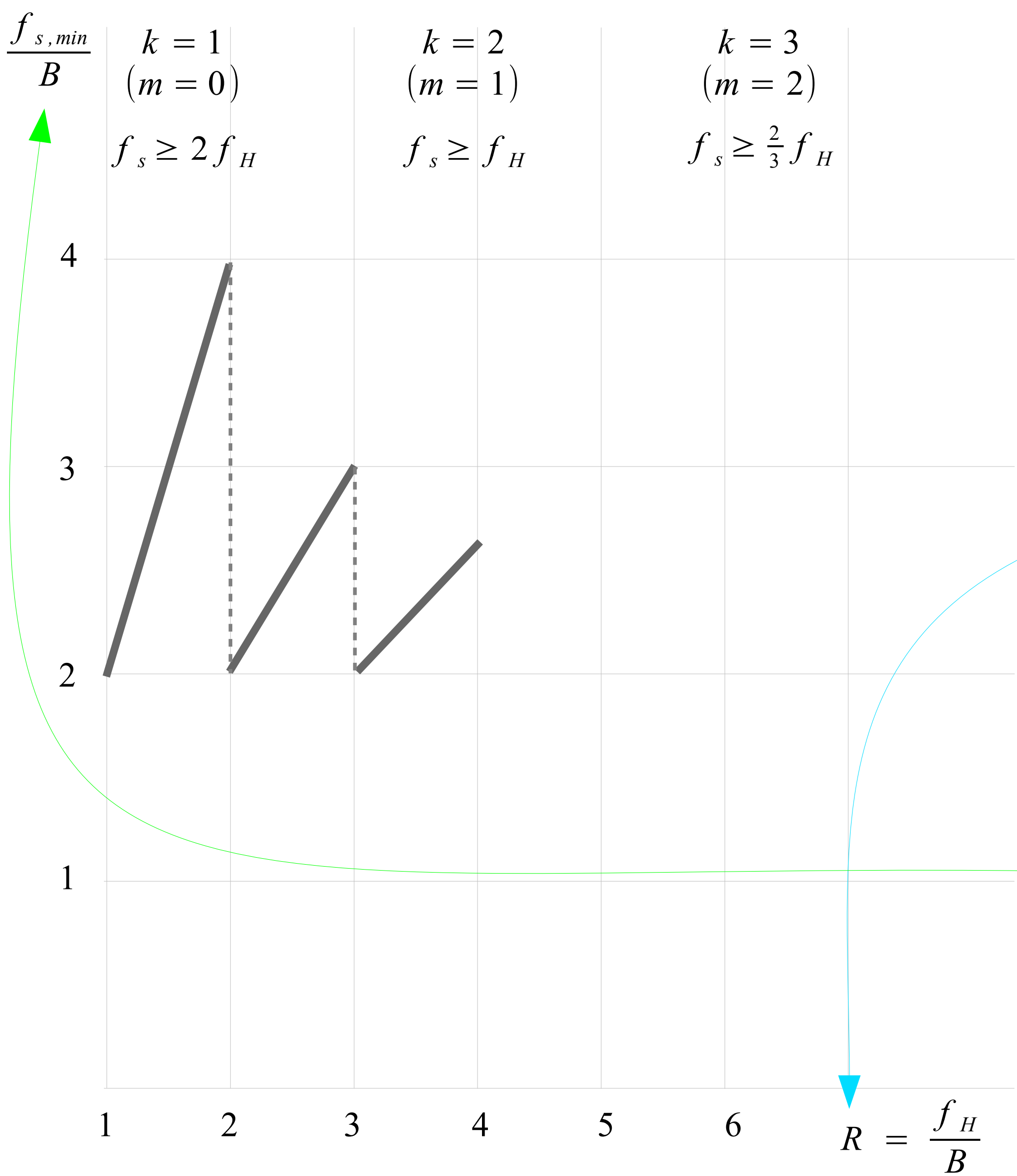
$f_H = f_c + B/2 = 4B$



$$R = f_H / B = 4$$

$$\frac{f_{s,min}}{B} = \frac{2 f_H}{k B} = \frac{\mathbf{8}}{\mathbf{3}}$$

$$\frac{f_{s,max}}{B} = \frac{2(f_H - B)}{(k-1)B} = \mathbf{4}$$



$$\frac{2 f_c + B}{m + 1} \leq f_s \leq \frac{2 f_c - B}{m}$$

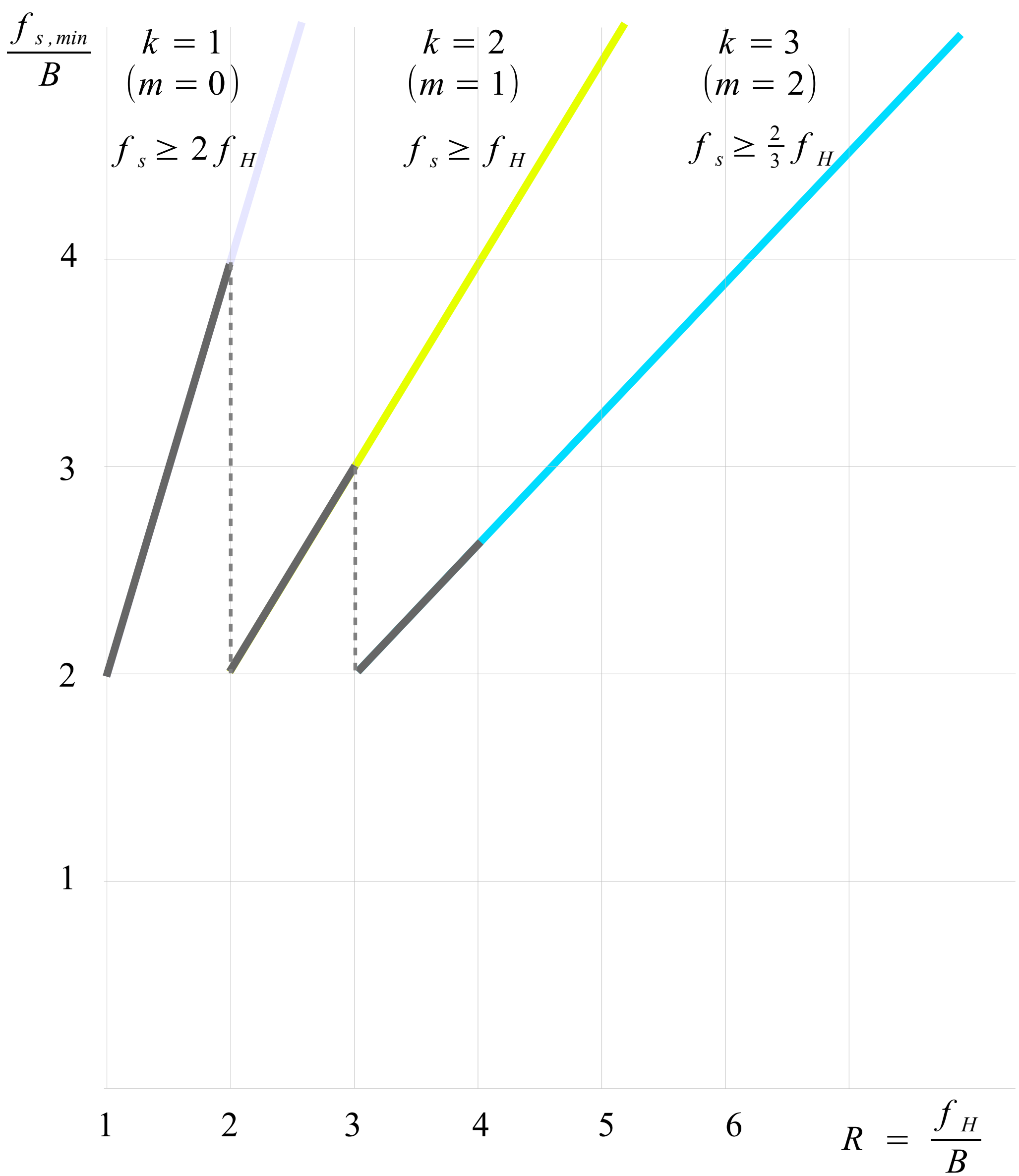
$$\frac{2 f_c + B}{k} \leq f_s \leq \frac{2 f_c - B}{k - 1}$$

$$\frac{f_c + B/2}{B} = R$$

→ $\dfrac{\text{highest signal frequency}}{\text{bandwidth } B}$

$$\frac{2 f_c + B}{(m + 1) B} = \frac{f_{s,min}}{B} = g(m, R)$$

→ $\dfrac{\text{minimum sampling rate}}{\text{bandwidth } B}$



$$\frac{2 f_c + B}{m + 1} \leq f_s \leq \frac{2 f_c - B}{m}$$

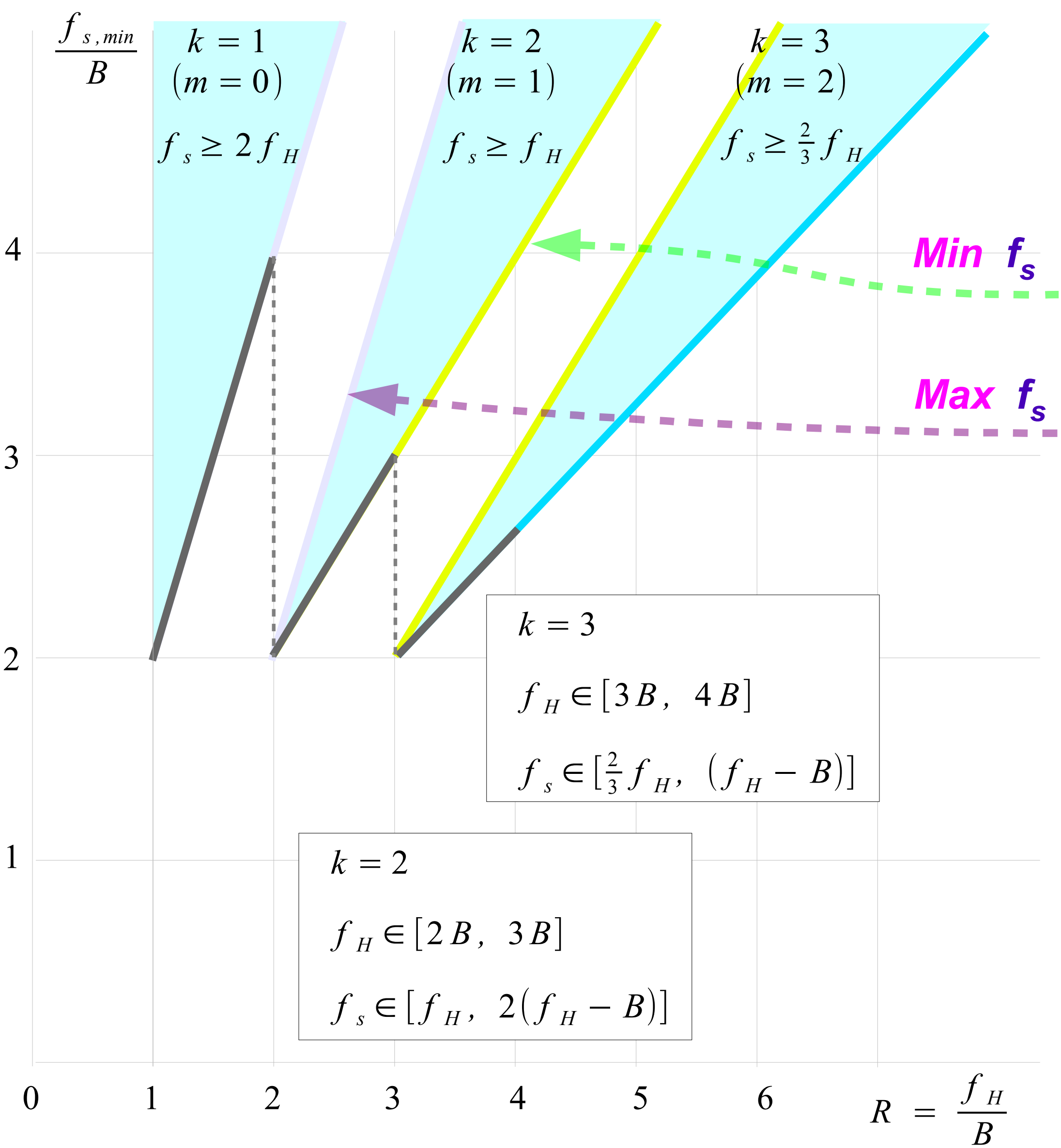
$$\frac{2 f_c + B}{k} \leq f_s \leq \frac{2 f_c - B}{k - 1}$$

$$\frac{f_c + B/2}{B} = R$$

➡ $\dfrac{\text{highest signal frequency}}{\text{bandwidth } B}$

$$\frac{2 f_c + B}{(m + 1) B} = \frac{f_{s,min}}{B} = g(m, R)$$

➡ $\dfrac{\text{minimum sampling rate}}{\text{bandwidth } B}$



$$\frac{2 f_c + B}{m + 1} \leq f_s \leq \frac{2 f_c - B}{m}$$

$$\frac{2 f_c + B}{k} \leq f_s \leq \frac{2 f_c - B}{k - 1}$$

Min f_s

Max f_s

	Max f_s	**Min f_s**
	$y = 2(x-2)+2$	$y = 1(x-2)+2$
$k = 2$	$y = 2x-2$	$y = x$
	$y = 1(x-3)+2$	$y = \frac{2}{3}(x-3)+2$
$k = 2$	$y = x-1$	$y = \frac{2}{3}x$

References

[1] http://en.wikipedia.org/
[2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003
[3] A “graphical interpretation” of the DFT and FFT, by Steve Mann
[4] R. G. Lyons, Understanding Digital Signal Processing, 1997