

Laplace Transform (4B)

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Inverse Laplace Transform

Laplace Transform

$$\begin{aligned} F(s) &= \int_0^{\infty} f(t) e^{-st} dt \\ &= \int_0^{\infty} \{f(t) e^{-xt}\} e^{-iyt} dt \end{aligned}$$

Inverse Laplace Transform

$$f(t) = \frac{1}{2\pi i} \int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} F(s) e^{st} ds$$

Inverse Laplace Transform

Laplace Transform

$$F(s) = \int_0^{\infty} f(t) e^{-st} dt$$

$$= \int_0^{\infty} \{f(t) e^{-xt}\} e^{-iyt} dt$$

→ F(s) converges absolutely
for $\text{Re}(s) = x > \alpha$

f(t) continuous on $[0, \infty)$
 f(t) = 0 for $t < 0$
 f(t) has exponential order α
 f(t) piecewise continuous on $[0, \infty)$

$$\int_0^{\infty} |f(t) e^{-st}| dt = \int_0^{\infty} |f(t)| e^{-xt} dt < \infty$$

$$x > \alpha$$

$$F(x, y) = \int_0^{\infty} \{f(t) e^{-xt}\} e^{-iyt} dt$$

$$F(x, y) = \int_0^{\infty} \underline{g(t)} e^{-iyt} dt$$

Fourier Transform $g(t) = f(t) e^{-xt}$

$$g(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(x, y) e^{iyt} dy$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(x, y) e^{xt} e^{iyt} dy$$

Inverse Fourier Transform

Fourier-Mellin Inversion Formula

$$F(x, y) = \int_0^{\infty} \{ \underline{f(t)} e^{-xt} \} e^{-iyt} dt$$

$$F(x, y) = \int_0^{\infty} \underline{g(t)} e^{-iyt} dt$$

Fourier Transform $g(t) = f(t)e^{-xt}$

$$g(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(x, y) e^{iyt} dy$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(x, y) e^{xt} e^{iyt} dy$$

Inverse Fourier Transform

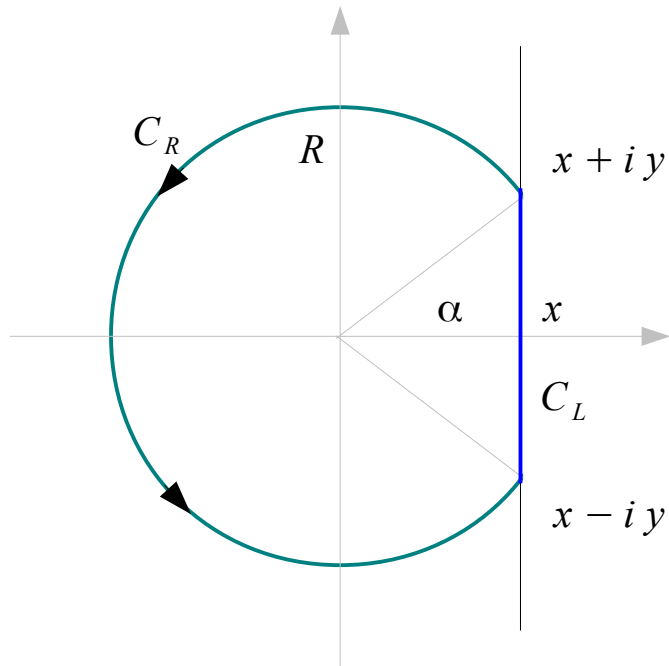
$$s = x + iy \quad ds = i dy \quad x > \alpha \quad (\text{fixed } x)$$

$$f(t) = \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} F(x, y) e^{st} ds = \lim_{y \rightarrow \infty} \frac{1}{2\pi i} \int_{x-iy}^{x+iy} F(s) e^{st} ds$$

Complex Inversion Formula (Fourier-Mellin Inversion Formula)

Vertical line at x : Bromwich line

Contour Integration (1)



$$\frac{1}{2\pi i} \int_C F(s) e^{st} ds$$

$$= \frac{1}{2\pi i} \int_{C_R} F(s) e^{st} ds + \frac{1}{2\pi i} \int_{C_L} F(s) e^{st} ds$$

$F(s)$ is analytic for $\text{Re}(s) = x > \alpha$

➡ $F(s)$ all singularities must lie to the left of Bromwich line

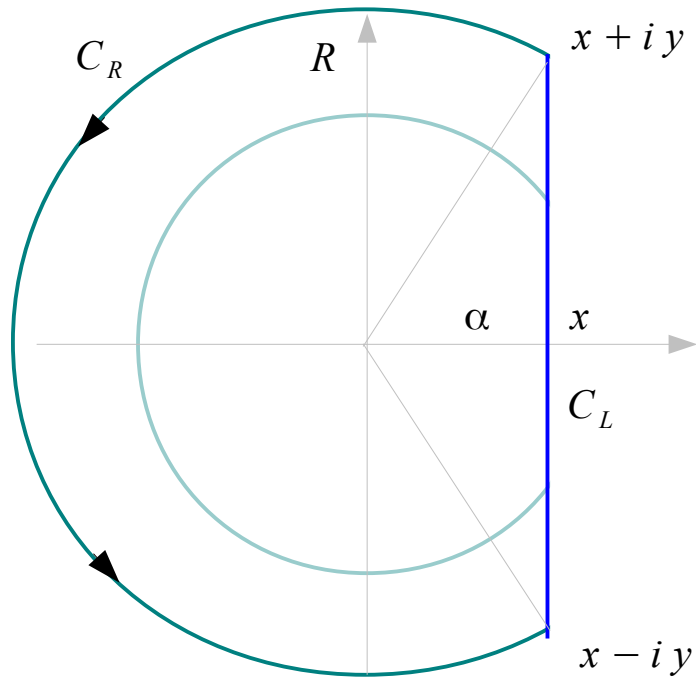
Assume $F(s)$ is analytic for $\text{Re}(s) = x < \alpha$ except for having finitely many poles

$$z_1, z_2, \dots, z_n$$

$$\frac{1}{2\pi i} \int_C F(s) e^{st} ds = \sum_{k=1}^n \text{Res}(z_k)$$

$$= \frac{1}{2\pi i} \int_{C_R} F(s) e^{st} ds + \frac{1}{2\pi i} \int_{x-iy}^{x+iy} F(s) e^{st} ds$$

Contour Integration (2)



$$\frac{1}{2\pi i} \int_C F(s) e^{st} ds = \sum_{k=1}^n \text{Res}(z_k)$$

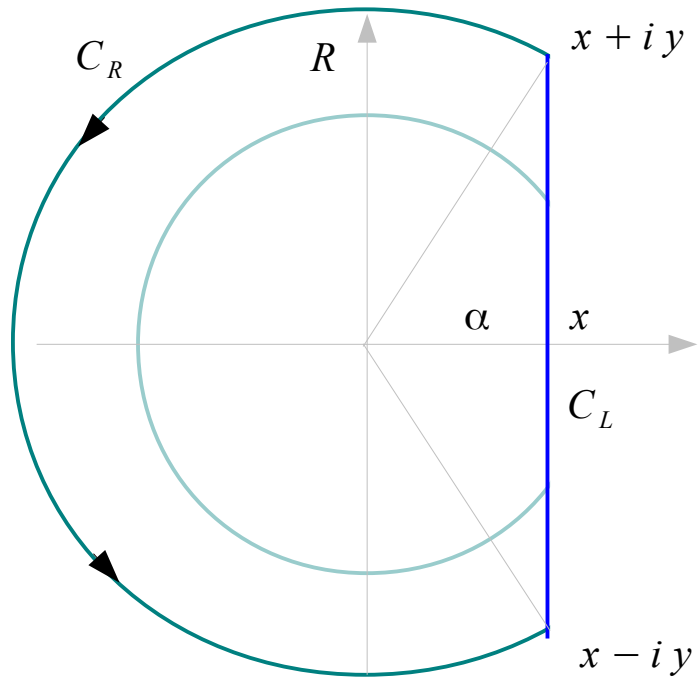
$$= \frac{1}{2\pi i} \int_{C_R} F(s) e^{st} ds + \frac{1}{2\pi i} \int_{x-iy}^{x+iy} F(s) e^{st} ds$$

$$\lim_{R \rightarrow \infty} \int_{C_R} F(s) e^{st} ds = 0 \quad (t > 0)$$



$$f(t) = \lim_{y \rightarrow \infty} \frac{1}{2\pi i} \int_{x-iy}^{x+iy} F(s) e^{st} ds = \sum_{k=1}^n \text{Res}(z_k)$$

Contour Integration (3)



$$s = R e^{i\theta} \quad \theta_1 \leq \theta \leq \theta_2$$

$$ds = i R e^{i\theta} d\theta$$

$$|ds| = R d\theta$$

$$|F(s)| \leq \frac{M}{|s|^p} \quad \text{Growth Restriction}$$

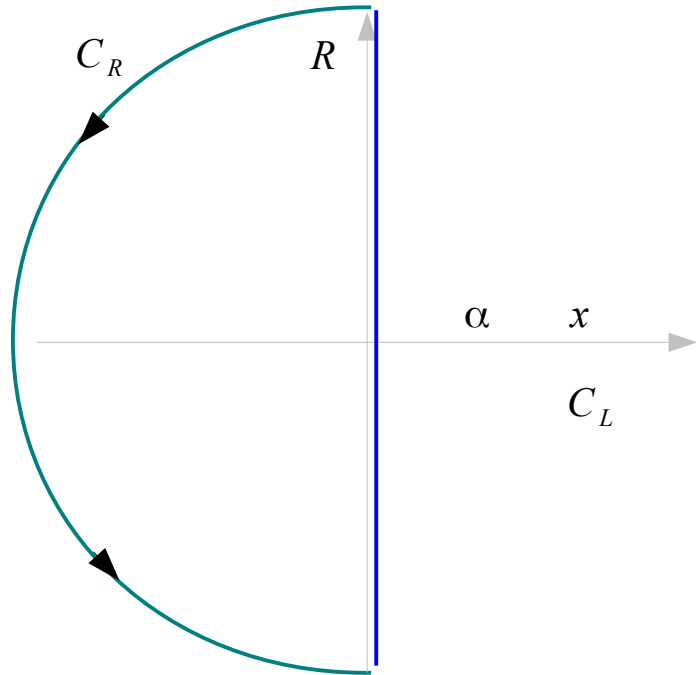
$$\Rightarrow |F(s)| \rightarrow 0 \quad \text{as} \quad |s| \rightarrow \infty$$

for s on C_R

$$|F(s)| \leq \frac{M}{|s|^p} \quad \text{some } p > 0, \text{ all } R > R_0$$

$$\Rightarrow \lim_{R \rightarrow \infty} \int_{C_R} F(s) e^{st} ds = 0 \quad (t > 0)$$

Contour Integration (2)



$$\lim_{R \rightarrow \infty} \int_{C_R} F(s) e^{st} ds = 0 \quad (t > 0)$$

$$s = R e^{i\theta} = R(\cos\theta + i \sin\theta)$$

$$e^{st} = e^{Rt(\cos\theta + i \sin\theta)} = e^{Rt \cos\theta} e^{i \sin\theta}$$

$$|e^{st}| = e^{Rt \cos\theta}$$

$$\begin{aligned} \int_{C_R} F(s) e^{st} ds &\leq \int_{C_R} |F(s)| |e^{st}| |ds| \\ &\leq \frac{M}{R^{p-1}} \int_{\pi/2}^{3\pi/2} e^{Rt \cos\theta} d\theta \end{aligned}$$

References

- [1] <http://en.wikipedia.org/>
- [2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003
- [3] A “graphical interpretation” of the DFT and FFT, by Steve Mann