Prof. Dr. H. Brenner **Medellin 2012**

Vector bundles, forcing algebras and local cohomology

Lecture 1

Systems of linear equations

We start with some linear algebra. Let K be a field. We consider a system of linear homogeneous equations over K,

$$f_{11}t_1 + \ldots + f_{1n}t_n = 0\,,$$

$$f_{21}t_1 + \ldots + f_{2n}t_n = 0\,,$$

$$\vdots$$

$$f_{m1}t_1 + \ldots + f_{mn}t_n = 0\,,$$

where the f_{ij} are elements in K. The solution set to this system of homogeneous equations is a vector space V over K, its dimension is $n - \operatorname{rk}(A)$, where $A = (f_{ij})_{ij}$ is the matrix given by these elements. Additional elements $f_1, \ldots, f_m \in K$ give rise to the system of inhomogeneous linear equations,

$$f_{11}t_1 + \ldots + f_{1n}t_n = f_1\,,$$

$$f_{21}t_1 + \ldots + f_{2n}t_n = f_2\,,$$

$$\vdots$$

$$f_{m1}t_1 + \ldots + f_{mn}t_n = f_m\,.$$

The solution set T of this inhomogeneous system may be empty, but nevertheless it is tightly related to the solution space of the homogeneous system. First of all, there exists an action

$$V \times T \longrightarrow T,\ (v, t) \longmapsto v + t,$$

because the sum of a solution of the homogeneous system and a solution of the inhomogeneous system is again a solution of the inhomogeneous system. This action is a group action of the group $(V, +, 0)$ on the set T. Moreover, if we fix one solution $t_0 \in T$ (supposing that at least one solution exists), then there exists a bijection

$$V \longrightarrow T,\ v \longmapsto v + t_0.$$

So T can be identified with the vector space V, however not in a canonical way. The group V acts simply transitive on T.

Suppose now that X is a geometric object (a topological space, a manifold, a variety, the spectrum of a ring) and that instead of elements in the field K we have functions

$$f_{ij} : X \longrightarrow K$$

on X (which are continuous, or differentiable, or algebraic). We form the Matrix of functions $A = (f_{ij})_{ij}$, which yields for every point $P \in X$ a matrix $A(P)$ over K. Then we get from these data the space

$$V = \left\{ (P, t_1, \ldots, t_n) |\, A(P) \begin{pmatrix} t_1 \\ \\ t_n \end{pmatrix} = 0 \right\} \subseteq X \times K^n$$

together with the projection to X. For a fixed point $P \in X$, the fiber of V over P is the solution space to the corresponding homogeneous system of linear equations given by inserting P. In particular, all fibers of the map

$$V \longrightarrow X,$$

are vector spaces (maybe of non-constant dimension). This vector space structures yield an addition

$$V \times_X V \longrightarrow V,\ (P; t_1, \ldots, t_n; s_1, \ldots, s_n) \longmapsto (P; t_1 + s_1, \ldots, t_n + s_n)$$

(only points in the same fiber can be added). The mapping

$$X \longrightarrow V,\ P \longmapsto (P; 0, \ldots, 0)$$

is called the zero-section.

Suppose now that there are additionally functions

$$f_1, \ldots, f_m : X \longrightarrow K$$

given. Then we can form the set

$$T = \left\{ (P, t_1, \ldots, t_n) |\, A(P) \begin{pmatrix} t_1 \\ \\ t_n \end{pmatrix} = \begin{pmatrix} f_1(P) \\ \\ f_n(P) \end{pmatrix} \right\} \subseteq X \times K^n$$

with the mapping to X. Again, every fiber of T over a point $P \in X$ is the solution set to the system of inhomogeneous linear equations which arises by inserting P. The actions of the fibers V_P on T_P (coming from linear algebra) extend to an action

$$V \times_X T \longrightarrow T,\ (P; t_1, \ldots, t_n; s_1, \ldots, s_n) \longmapsto (P; t_1 + s_1, \ldots, t_n + s_n).$$

Also, if a (continuous, differentiable, algebraic) map

$$s : X \longrightarrow T$$

with $s(P) \in T_P$ exists, then we can construct an (continuous, differentiable, algebraic) isomorphism between V and T. However, different from the situation in linear algebra (which corresponds to the situation where X is just one point), such a section does rarely exist.

These objects T have new and sometimes difficult global properties which we try to understand in these lectures. We will work mainly in an algebraic setting and restrict to the situation where just one equation

$$f_1 T_1 + \ldots + f_n T_n = f$$

is given. Then in the homogeneous case ($f = 0$) the fibers are vector spaces of dimension $n-1$ or n, and the later holds exactly for the points $P \in X$ where $f_1(P) = \ldots = f_n(P) = 0$. In the inhomogeneous case the fibers are either empty or of dimension $n-1$ or n. We give some typical examples.

Example 1.1. We consider the line ($X = \mathbb{A}^1_K$) (or $X = K, \mathbb{R}, \mathbb{C}$ etc.) with the (identical) function x. For $f_1 = x$ and $f = 0$, i.e. for the equation $xt = 0$, the geometric object V consists of a horizontal line (corresponding to the zero-solution) and a vertical line over $x = 0$. So all fibers except one are zero-dimensional vector spaces. For the equation $0t = x$, V consists of one vertical line, almost all fibers are empty. For the equation $xt = 1$, V is a hyperbola, and all fibers are zero-dimensional with the exception that the fiber over $x = 0$ is empty.

Example 1.2. Let X denote a plane ($K^2, \mathbb{R}^2, \mathbb{A}^2_K$) with coordinate functions x and y. We consider a linear equation of type

$$x^a t_1 + y^b t_2 = x^c y^d \, .$$

The fiber of the solution set T over a point $\neq (0,0)$ is onedimensional, whereas the fiber over $(0,0)$ has dimension two (for $a, b, c, d \geq 1$). Many properties of T depend on these four exponents.

In (most of) these example we can observe the following behavior. On an open subset, the dimension of the fibers is constant and equals $n-1$, wheres the fiber over some special points degenerates to an n-dimensional solution set (or becomes empty).

Forcing algebras

We describe now the algebraic setting of systems of linear equations depending on a base space. For a commutative ring R, its spectrum $X = \mathrm{Spec}\,(R)$ is a topological space on which the ring elements can be considered as functions. The value of $f \in R$ at a prime ideal $P \in \mathrm{Spec}\,(R)$ is just the image of f under the morphism $R \to R/P \to \kappa(P) = Q(R/P)$. In this interpretation, a ring element is a function with values in different fields. Suppose that R contains a field K. Then an element $f \in R$ gives rise to the ring homomorphism

$$K[Y] \longrightarrow R,\, Y \longmapsto f,$$

which itself gives rise to a scheme morphism

$$\mathrm{Spec}\,(R) \longrightarrow \mathrm{Spec}\,(K[Y]) \cong \mathbb{A}^1_K.$$

This is another way to consider f as a function on $\mathrm{Spec}\,(R)$.

Definition 1.3. Let R be a commutative ring and let $f_1, \ldots, f_n$ and f be elements in R. Then the R-algebra

$$R[T_1, \ldots, T_n]/(f_1T_1 + \ldots + f_nT_n - f)$$

is called the *forcing algebra* of these elements (or these data).

The forcing algebra B forces f to lie inside the extended ideal $(f_1, \ldots, f_n)B$ (hence the name) For every R-algebra S such that $f \in (f_1, \ldots, f_n)S$ there exists a (non unique) ring homomorphism $B \to S$ by sending T_i to the coefficient $s_i \in S$ in an expression $f = s_1f_1 + \ldots + s_nf_n$.

The forcing algebra induces the spectrum morphism

$$\mathrm{Spec}\,(B) \longrightarrow \mathrm{Spec}\,(R).$$

Over a point $P \in X = \mathrm{Spec}\,(R)$, the fiber of this morphism is given by

$$\mathrm{Spec}\,(B \otimes_R \kappa(P))\,,$$

and we can write

$$B \otimes_R \kappa(P) = \kappa(P)[T_1, \ldots, T_n]/(f_1(P)T_1 + \ldots + f_n(P)T_n - f(P))\,,$$

where $f_i(P)$ means the evaluation of the f_i in the residue class field. Hence the $\kappa(P)$-points in the fiber are exactly the solution to the inhomogeneous linear equation $f_1(P)T_1 + \ldots + f_n(P)T_n = f(P)$. In particular, all the fibers are (empty or) affine spaces.