Prof. Dr. H. Brenner **Medellín 2012**

Vector bundles, forcing algebras and local cohomology

Lecture 9

Plus closure in dimension two

Let K be a field and let R be a normal two-dimensional standard-graded domain over K with corresponding smooth projective curve C. A homogeneous $\mathfrak{m}$-primary ideal with homogeneous ideal generators $f_1, \ldots, f_n$ and another homogeneous element f of degree m yield a cohomology class

$$c = \delta(f) \in H^1(C, \operatorname{Syz}(f_1, \ldots, f_n)(m)).$$

Let $T(c)$ be the corresponding torsor.

We have seen in the last lecture that the affineness of this torsor over C is equivalent to the affineness of the corresponding torsor over $D(\mathfrak{m}) \subseteq \operatorname{Spec}(R)$. Now we want to understand what the property $f \in I^+$ means for c and for $T(c)$. Instead of the plus closure we will work with the graded plus closure $I^{+\mathrm{gr}}$, where $f \in I^{+\mathrm{gr}}$ holds if and only if there exists a finite graded extension $R \subseteq S$ such that $f \in IS$. The existence of such an S translates into the existence of a finite morphism

$$\varphi : C' = \operatorname{Proj}(S) \longrightarrow \operatorname{Proj}(R) = C$$

such that $\varphi^*(c) = 0$. Here we may assume that C' is also smooth. Therefore we discuss the more general question when a cohomology class $c \in H^1(C, \mathcal{S})$, where $\mathcal{S}$ is a locally free sheaf on C, can be annihilated by a finite morphism

$$C' \longrightarrow C$$

of smooth projective curves. The advantage of this more general approach is that we may work with short exact sequences (in particular, the sequences coming from the Harder-Narasimhan filtration) in order to reduce the problem to semistable bundles which do not necessarily come from an ideal situation.

Lemma 9.1. *Let C denote a smooth projective curve over an algebraically closed field K, let $\mathcal{S}$ be a locally free sheaf on C and let $c \in H^1(C, \mathcal{S})$ be a cohomology class with corresponding torsor $T \to C$. Then the following conditions are equivalent.*

(1) *There exists a finite morphism*

$$\varphi : C' \longrightarrow C$$

from a smooth projective curve C' such that $\varphi^*(c) = 0$.

(2) *There exists a projective curve $Z \subseteq T$.*

Proof. If (1) holds, then the pull-back $\varphi^*(T) = T \times_C C'$ is trivial (as a torsor), as it equals the torsor given by $\varphi^*(c) = 0$. Hence $\varphi^*(T)$ is isomorphic to a vector bundle and contains in particular a copy of C'. The image Z of this copy is a projective curve inside T.

If (2) holds, then let C' be the normalization of Z. Since Z dominates C, the resulting morphism

$$\varphi : C' \longrightarrow C$$

is finite. Since this morphism factors through T and since T annihilates the cohomology class by which it is defined, it follows that $\varphi^*(c) = 0$. □

We want to show that the cohomological criterion for (non)-affineness of a torsor along the Harder-Narasimhan filtration of the vector bundle also holds for the existence of projective curves inside the torsor, under the condition that the projective curve is defined over a finite field. This implies that tight closure is (graded) plus closure for graded $\mathfrak{m}$-primary ideals in a two-dimensional graded domain over a finite field.

Annihilation of cohomology classes of strongly semistable sheaves

We deal first with the situation of a strongly semistable sheaf $\mathcal{S}$ of degree 0. The following two results are due to Lange and Stuhler. We say that a locally free sheaf is étale trivializable if there exists a finite étale morphism $\varphi : C' \to C$ such that $\varphi^*(\mathcal{S}) \cong \mathcal{O}_{C'}^r$. Such bundles are directly related to linear representations of the étale fundamental group.

Lemma 9.2. *Let K denote a finite field (or the algebraic closure of a finite field) and let C be a smooth projective curve over K. Let $\mathcal{S}$ be a locally free sheaf over C. Then $\mathcal{S}$ is étale trivializable if and only if there exists some n such that $F^{n*}\mathcal{S} \cong \mathcal{S}$.*

Theorem 9.3. *Let K denote a finite field (or the algebraic closure of a finite field) and let C be a smooth projective curve over K. Let $\mathcal{S}$ be a strongly semistable locally free sheaf over C of degree 0. Then there exists a finite mapping*

$$\varphi : C' \longrightarrow C$$

such that $\varphi^*(\mathcal{S})$ is trivial.

Proof. We consider the family of locally free sheaves $F^{e*}(\mathcal{S})$, $e \in \mathbb{N}$. Because these are all semistable of degree 0, and defined over the same finite field, we must have (by the existence of the moduli space for vector bundles) a repetition, i.e. $F^{e*}(\mathcal{S}) \cong F^{e'*}(\mathcal{S})$ for some $e' > e$. By Lemma 9.2 the bundle $F^{e*}(\mathcal{S})$ admits an étale trivialization $\varphi : C' \to C$. Hence the finite map $F^e \circ \varphi$ trivializes the bundle. □

Theorem 9.4. *Let K denote a finite field (or the algebraic closure of a finite field) and let C be a smooth projective curve over K. Let $\mathcal{S}$ be a strongly semistable locally free sheaf over C of nonnegative degree and let $c \in H^1(C, \mathcal{S})$ denote a cohomology class. Then there exists a finite mapping*

$$\varphi : C' \longrightarrow C$$

such that $\varphi^*(c)$ is trivial.

Proof. If the degree of $\mathcal{S}$ is positive, then a Frobenius pull-back $F^{e*}(\mathcal{S})$ has arbitrary large degree and is still semistable. By Serre dualtiy we get that $H^1(C, F^{e*}(\mathcal{S})) = 0$. So in this case we can annihilate the class by an iteration of the Frobenius alone.

So suppose that the degree is 0. Then there exists by Theorem 9.3 a finite morphism which trivializes the bundle. So we may assume that $\mathcal{S} \cong \mathcal{O}_C^r$. Then the cohomology class has several components $c_i \in H^1(C, \mathcal{O}_C)$ and it is enough to annihilate them separately by finite morphisms. But this is possible by the parameter theorem of K. Smith (or directly using Frobenius and Artin-Schreier extensions). □

The general case

We look now at an arbitrary locally free sheaf $\mathcal{S}$ on C, a smooth projective curve over a finite field. We want to show that the same numerical criterion (formulated in terms of the Harder-Narasimhan filtration) for non-affineness of a torsor holds also for the finite annihilation of the corresponding cohomomology class (or the existence of a projective curve inside the torsor).

Theorem 9.5. *Let K denote a finite field (or the algebraic closure of a finite field) and let C be a smooth projective curve over K. Let $\mathcal{S}$ be a locally free sheaf over C and let $c \in H^1(C, \mathcal{S})$ denote a cohomology class. Let $\mathcal{S}_1 \subset \ldots \subset \mathcal{S}_t$ be a strong Harder-Narasimhan filtration of $F^{e*}(\mathcal{S})$. We choose i such that $\mathcal{S}_i/\mathcal{S}_{i-1}$ has degree ≥ 0 and that $\mathcal{S}_{i+1}/\mathcal{S}_i$ has degree < 0. We set $\mathcal{Q} = F^{e*}(\mathcal{S})/\mathcal{S}_i$. Then the following are equivalent.*

1. *The class c can be annihilated by a finite morphism.*
2. *Some Frobenius power of the image of $F^{e*}(c)$ inside $H^1(C, \mathcal{Q})$ is 0.*

Proof. Suppose that (1) holds. Then the torsor is not affine and hence by Theorem 8.7 also (2) holds.

So suppose that (2) is true. By applying a certain power of the Frobenius we may assume that the image of the cohomology class in $\mathcal{Q}$ is 0. Hence the class stems from a cohomology class $c_i \in H^1(C, \mathcal{S}_i)$. We look at the short exact sequence

$$0 \longrightarrow \mathcal{S}_{i-1} \longrightarrow \mathcal{S}_i \longrightarrow \mathcal{S}_i/\mathcal{S}_{i-1} \longrightarrow 0 \,,$$

where the sheaf of the right hand side has a nonnegative degree. Therefore the image of c_i in $H^1(C, \mathcal{S}_i/\mathcal{S}_{i-1})$ can ba annihilated by a finite morphism due to Theorem 9.4. Hence after applying a finite morphism we may assume that c_i stems from a cohomology class $c_{i-1} \in H^1(C, \mathcal{S}_{i-1})$. Going on inductively we see c can be annihilated by a finite morphism. □

Theorem 9.6. *Let C denote a smooth projective curve over the algebraic closure of a finite field field K, let $\mathcal{S}$ be a locally free sheaf on C and let $c \in H^1(C, \mathcal{S})$ be a cohomology class with corresponding torsor $T \to C$. Then T is affine if and only if it does not contain any projective curve.*

Proof. Due to Theorem 8.7 and Theorem 9.5, for both properties the same numerical criterion does hold. □

These results imply the following theorem in the setting of a twodimensional graded ring.

Theorem 9.7. *Let R be a standard-graded, two-dimensional normal domain over (the algebraic closure of) a finite field. Let I be an R_+-primary graded ideal. Then*

$$I^* = I^+ \, .$$

This is also true for non-primary graded ideals and also for submodules in finitely generated graded submodules. Moreover, G. Dietz has shown that one can get rid also of the graded assumption (of the ideal or module, but not of the ring).