Prof. Dr. H. Brenner **Osnabrück WS 2011/2012**

Mathematics for natural sciences I

Exercise sheet 10

Warm-up-exercises

Exercise 10.1. The telephone companies A, B and C compete for a market, where the market customers in a year j are given by the customers-tuple $K_j = (a_j, b_j, c_j)$ (where a_j is the number of customers of A in the year j etc.). There are customers passing from one provider to another one during a year.

(1) The customers of A remain for 80% with A while 10% of them goes to B and the same percentage goes to C.
(2) The customers of B remain for 70% with B while 10% of them goes to A and 20% goes to C.
(3) The customers of C remain for 50% with C while 20% of them goes to A and 30% goes to B.

a) Determine the linear map (i.e. the matrix), which expresses the customers-tuple K_{j+1} with respect to K_j.

b) Which customers-tuple arises from the customers-tuple $(12000, 10000, 8000)$ within one year?

c) Which customers-tuple arises from the customers-tuple $(10000, 0, 0)$ in four years?

Exercise 10.2. Let K be a field and let V and W be vector spaces over K of dimensions n and m. Let

$$\varphi : V \longrightarrow W$$

be a linear map, described by the matrix $M \in \operatorname{Mat}_{m \times n}(K)$ with respect to two bases. Prove that φ is surjective if and only if the columns of the matrix form a system of generators for K^m.

Exercise 10.3. Let K be a field and let V and W be two K-vector spaces. Let

$$\varphi : V \longrightarrow W$$

be a bijective linear map. Prove that also the inverse map

$$\varphi^{-1} : W \longrightarrow V$$

is linear.

Exercise 10.4. Determine the inverse matrix of
$$M = \begin{pmatrix} 2 & 7 \\ -4 & 9 \end{pmatrix}.$$

Exercise 10.5. Determine the inverse matrix of
$$M = \begin{pmatrix} 1 & 2 & 3 \\ 6 & -1 & -2 \\ 0 & 3 & 7 \end{pmatrix}.$$

Exercise 10.6. Determine the inverse matrix of the complex matrix
$$M = \begin{pmatrix} 2+3i & 1-i \\ 5-4i & 6-2i \end{pmatrix}.$$

Exercise 10.7. a) Determine if the complex matrix
$$M = \begin{pmatrix} 2+5i & 1-2i \\ 3-4i & 6-2i \end{pmatrix}$$
is invertible.

b) Find a solution to the inhomogeneous linear system of equations
$$M \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 54+72i \\ 0 \end{pmatrix}.$$

Exercise 10.8. Prove that the matrix
$$\begin{pmatrix} 0 & 0 & k+2 & k+1 \\ 0 & 0 & k+1 & k \\ -k & k+1 & 0 & 0 \\ k+1 & -(k+2) & 0 & 0 \end{pmatrix}$$
for all k is the inverse of itself.

Exercise 10.9. We consider the linear map
$$\varphi : K^3 \longrightarrow K^2, \begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto \begin{pmatrix} 1 & 2 & 5 \\ 4 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$
Let $U \subseteq K^3$ be the subspace of K^3 defined by the linear equation $2x+3y+4z=0$, and let ψ be the restriction of φ on U. On U are given vectors of the form
$$u = (0,1,a),\ v = (1,0,b) \text{ and } w = (1,c,0).$$
Compute the change of basismatrix between the bases
$$\mathfrak{b}_1 = v, w,\ \mathfrak{b}_2 = u, w \text{ and } \mathfrak{b}_3 = u, v$$

of U and the transformation matrix of ψ with respect to these three bases (and the standard basis of K^2).

EXERCISE 10.10. Prove that the elementary matrices are invertible. What are the inverse matrices of the elementary matrices?

EXERCISE 10.11. Let K be a field and M a $n \times n$-matrix with entries in K. Prove that the multiplication by the elementary matrices from the left with M has the following effects.

(1) $V_{ij} \circ M =$ exchange of the i-th and the j-th row of M.
(2) $(S_k(s)) \circ M =$ multiplication of the k-th row of M by s.
(3) $(A_{ij}(a)) \circ M =$ addition of a-times the j-th row of M to the i-th row $(i \neq j)$.

EXERCISE 10.12. Describe what happens when a matrix is multiplied from the right by an elementary matrix.

Hand-in-exercises

EXERCISE 10.13. (3 points)

Compute the inverse matrix of

$$M = \begin{pmatrix} 2 & 3 & 2 \\ 5 & 0 & 4 \\ 1 & -2 & 3 \end{pmatrix} .$$

EXERCISE 10.14. (3 points)

Perform the procedure to find the inverse matrix of the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

under the assumption that $ad - bc \neq 0$.

EXERCISE 10.15. (6 (3+1+2) points)

An animal population consists of babies (first year), freshers (second year), Halbstarke (third year), mature ones (fourth year) and veterans (fifth year), these animals can not become older. The total stock of these animals in a given year j is given by a 5-tuple $B_j = (b_{1,j}, b_{2,j}, b_{3,j}, b_{4,j}, b_{5,j})$.

During a year 7/8 of the babies become freshers, 9/10 of the freshers become Halbstarke, 5/6 of the Halbstarken become mature ones and 2/3 of the mature ones reach the fifth year.

Babies and freshes can not reproduce yet, then they reach the sexual maturity and 10 Halbstarke generate 5 new pets and 10 of the mature ones generate 8 new babies, and the babies are born one year later.

a) Determine the linear map (i.e. the matrix), which expresses the total stock B_{j+1} with respect to the stock B_j.

b) What will happen to the stock $(200, 150, 100, 100, 50)$ in the next year?

c) What will happen to the stock $(0, 0, 100, 0, 0)$ in five years?

EXERCISE 10.16. (3 points)

Let $z \in \mathbb{C}$ be a complex number and let

$$\mathbb{C} \longrightarrow \mathbb{C},\ w \longmapsto zw,$$

be the multiplication map, which is a $\mathbb{C}$-linear map. How does the matrix related to this map with respect to the real basis 1 and i look like? Let z_1 and z_2 be two complex numbers with corresponding real matrices M_1 and M_2. Prove that the matrix product $M_2 \circ M_1$ is the real matrix corresponding to $z_1 z_2$.