In this lecture we will continue with the question when are the torsors given by a forcing algebras over a two-dimensional ring affine? We will look at the graded situation to be able to work on the corresponding projective curve. In particular we want to address the following questions

(1) Is there a procedure to decide whether the torsor is affine?
(2) Is it non-affine if and only if there exists a geometric reason for it not to be affine (because the superheight is too large)?
(3) How does the affineness vary in an arithmetic family, when we vary the prime characteristic?
(4) How does the affineness vary in a geometric family, when we vary the base ring?

In terms of tight closure, these questions are directly related to the tantalizing question of tight closure (is it the same as plus closure), the dependence of tight closure on the characteristic and the localization problem of tight closure.

Geometric interpretation in dimension two

We will restrict now to the two-dimensional homogeneous case in order to work on the corresponding projective curve. We want to find an object over the curve which corresponds to the forcing algebra. Let R be a two-dimensional standard-graded normal domain over an algebraically closed field K. Let $C = \operatorname{Proj} R$ be the corresponding smooth projective curve and let

$$I = (f_1, \ldots, f_n)$$

be an R_+-primary homogeneous ideal with generators of degrees $d_1, \ldots, d_n$. Then we get on C the short exact sequence

$$0 \longrightarrow \operatorname{Syz}(f_1, \ldots, f_n)(m) \longrightarrow \bigoplus_{i=1}^{n} \mathcal{O}_C(m - d_i) \stackrel{f_1,\ldots,f_n}{\longrightarrow} \mathcal{O}_C(m) \longrightarrow 0 \, .$$

Here $\operatorname{Syz}(f_1, \ldots, f_n)(m)$ is a vector bundle, called the *syzygy bundle*, of rank $n-1$ and of degree

$$((n-1)m - \sum_{i=1}^{n} d_i) \deg(C) \, .$$

Thus a homogeneous element f of degree m defines a cohomology class $\delta(f) \in H^1(C, \operatorname{Syz}(f_1, \ldots, f_n)(m)$, so this defiens a torsor over the projective curve. We mention an alternative description of the torsor corresponding to a first cohomology class in a locally free sheaf which is better suited for the projective situation.

REMARK 5.1. Let $\mathcal{S}$ denote a locally free sheaf on a scheme X. For a cohomology class $c \in H^1(X, \mathcal{S})$ one can construct a geometric object: Because of $H^1(X, \mathcal{S}) \cong \operatorname{Ext}^1(\mathcal{O}_X, \mathcal{S})$, the class defines an extension

$$0 \longrightarrow \mathcal{S} \longrightarrow \mathcal{S}' \longrightarrow \mathcal{O}_X \longrightarrow 0\,.$$

This extension is such that under the connecting homomorphism of cohomology, $1 \in \Gamma(X, \mathcal{O}_X)$ is sent to $c \in H^1(X, \mathcal{S})$. The extension yields projective subbundles

$$\mathbb{P}(\mathcal{S}^*) \subset \mathbb{P}(\mathcal{S}'^*)\,.$$

If V is the corresponding vector bundle, one may think of $\mathbb{P}(\mathcal{S}^*)$ as $\mathbb{P}(V)$ which consists for every base point $x \in X$ of all the lines in the fiber V_x running through the zero point. The projective subbundle $\mathbb{P}(V)$ has codimension one inside $\mathbb{P}(V')$, for every point it is a projective space lying (linearly) inside a projective space of one dimension higher. The complement then is over every point then an affine space. One can show that the global complement

$$T = \mathbb{P}(\mathcal{S}'^*) - \mathbb{P}(\mathcal{S}^*)$$

is another model for the torsor given by the cohomology class. The advantage of this viewpoint is that we may work, in particular when X is projective, in an entirely projective setting.

In the situation of a forcing algebra for homogeneous elements, this torsor T can also be obtained as Proj B, where B is the (not necessarily positively) graded forcing algebra. In particular, it follows that the containment $f \in I^*$ is equivalent to the property that T is not an affine variety. For this properties, positivity (ampleness) properties of the syzygy bundle are crucial. We need the concept of semistability.

DEFINITION 5.2. Let $\mathcal{S}$ be a vector bundle on a smooth projective curve C. It is called *semistable*, if $\frac{\deg(\mathcal{T})}{\operatorname{rk}(\mathcal{T})} \leq \frac{\deg(\mathcal{S})}{\operatorname{rk}(\mathcal{S})}$ for all subbundles $\mathcal{T}$. Suppose that the base field has positive characteristic $p > 0$. Then $\mathcal{S}$ is called *strongly semistable*, if all (absolute) Frobenius pull-backs $F^{e*}(\mathcal{S})$ are semistable.

For a strongly semistable vector bundle $\mathcal{S}$ on Y and a cohomology class $c \in H^1(Y, \mathcal{S})$ with corresponding torsor we obtain the following affineness criterion.

THEOREM 5.3. *Let Y denote a smooth projective curve over an algebraically closed field k and let $\mathcal{S}$ be a strongly semistable vector bundle over Y together with a cohomology class $c \in H^1(Y, \mathcal{S})$. Then the torsor $T(c)$ is an affine scheme if and only if* $\deg(\mathcal{S}) < 0$ *and $c \neq 0$ ($F^e(c) \neq 0$ in positive characteristic).*

This implies for a strongly semistable syzygy bundles the following *degree formula* for tight closure.

THEOREM 5.4. *Suppose that* $\mathrm{Syz}\,(f_1,\ldots,f_n)$ *is strongly semistable. Then*

$$R_m \subseteq I^* \text{ for } m \geq \frac{\sum d_i}{n-1} \text{ and } R_m \cap I^* \subseteq I \text{ for } m < \frac{\sum d_i}{n-1}\,.$$

In general, there exists an exact criterion depending on c and the *strong Harder-Narasimhan filtration* of $\mathcal{S}$.

THEOREM 5.5. *Let* Y *denote a smooth projective curve over an algebraically closed field* k *and let* $\mathcal{S}$ *be a vector bundle over* Y *together with a cohomology class* $c \in H^1(Y,\mathcal{S})$. *Let*

$$\mathcal{S}_1 \subset \mathcal{S}_2 \subset \ldots \subset \mathcal{S}_{t-1} \subset \mathcal{S}_t = \mathcal{S}$$

be a strong Harder-Narasimhan filteration, i.e. a Harder-Narasimhan filtration of some Frobenius pull-back such that the quotients $\mathcal{S}_i/\mathcal{S}_{i-1}$ *are strongly semistable (the existence is a theorem of A. Langer). Then the torsor* $T(c)$ *is an affine scheme if and only if the following (inductively defined property starting with* t*) holds: there is an* i *such that* $\deg\,(\mathcal{S}_i/\mathcal{S}_{i-1}) < 0$ *and the image of* c *in this sheaf is non-zero.*

The same criterion holds for plus closure (the existence of projective curves inside the torsor or the trivializing of the cohomology class along a finite curve map). Hence over (the algebraic closure of) a finite field we have that a torsor is not affine if and only if it contains a projective curve. This implies the following theorem.

THEOREM 5.6. *Let* R *be a standard-graded, two-dimensional normal domain over (the algebraic closure of) a finite field. Let* I *be an* R_+*-primary (i.e., the radical of* I *is* R_+*). graded ideal. Then*

$$I^* = I^+\,.$$

Local cohomology under deformations

We consider a base scheme B and a morphism

$$Z \longrightarrow B$$

together with an open subscheme $W \subseteq Z$. For every base point $b \in B$ we get the open subset

$$W_b \subseteq Z_b$$

inside the fiber Z_b. It is a natural question to ask how properties of W_b vary with b. In particular we may ask how the cohomological dimension of W_b varies and how the affineness may vary. In the algebraic setting we have a D-algebra S and an ideal $\mathfrak{a} \subseteq S$ which defines for every prime ideal $\mathfrak{p} \in \mathrm{Spec}\,(D)$ the extended ideal $\mathfrak{a}_{\mathfrak{p}}$ in $S \otimes_D \kappa(\mathfrak{p})$. This question is already interesting when B is a one-dimensional integral scheme, in particular in the following two situations.

(1) $B = \operatorname{Spec}(\mathbb{Z})$. Then we talk about an arithmetic deformation and want to know how affineness varies with the characteristic and how the relation is to characteristic zero.
(2) $B = \mathbb{A}^1_K = \operatorname{Spec}(K[t])$, where K is a field. Then we talk about a geometric deformation and want to know how affineness varies with the parameter t, in particular how the behaviour over the special points where the residue class field is algebraic over K is related to the behaviour over the generic point.

It is fairly easy to show that if the open subset in the generic fiber is affine, then also the open subsets are affine for almost all special points. We deal with this question where W is a torsor over a family of smooth projective curves. The arithmetic as well as the geometric variant of this question are directly related to questions in tight closure theory. Because of the above mentioned degree criteria in the strongly semistable case, a weird behaviour of the affineness property of torsors is only possible if we have a weird behaviour of strong semistability. We start with the arithmetic situation.

EXAMPLE 5.7. Consider $\mathbb{Z}[x, y, z]/(x^7 + y^7 + z^7)$ and consider the ideal $I = (x^4, y^4, z^4)$ and the element $f = x^3y^3$. Consider reductions $\mathbb{Z} \to \mathbb{Z}/(p)$. Then

$$f \in I^* \text{ holds in } \mathbb{Z}/(p)[x, y, z]/(x^7 + y^7 + z^7) \text{ for } p = 3 \mod 7$$

and

$$f \notin I^* \text{ holds in } \mathbb{Z}/(p)[x, y, z]/(x^7 + y^7 + z^7) \text{ for } p = 2 \mod 7 \,.$$

In particular, the bundle $Syz(x^4, y^4, z^4)$ is semistable in the generic fiber, but not for any reduction $p = 2 \mod 7$. The corresponding torsor is an affine scheme for infinitely many prime reductions and not an affine scheme for infinitely many prime reductions.

We will look now at geometric deformations:

$$D = \mathbb{F}_p[t] \subset \mathbb{F}_p[t][x, y, z]/(g) = S$$

where t has degree 0 and x, y, z have degree one and g is homogeneous. Then (for every field $\mathbb{F}_p[t] \to K$)

$$S \otimes_{\mathbb{F}_p[t]} K$$

is a two-dimensional standard-graded ring over K. For residue class fields of points of $\mathbb{A}^1_{\mathbb{F}_p} = \operatorname{Spec} \mathbb{F}_p[t]$ we have basically two possibilities.

- $K = \mathbb{F}_p(t)$, the function field. This is the *generic* or *transcendental* case.
- $K = \mathbb{F}_q$, the *special* or *algebraic* or *finite* case.

How does $f \in I^*$ vary with K? To analyze the behavior of tight closure in such a family we can use what we know in the two-dimensional standard-graded situation.

A counterexample to the localization problem

In order to establish an example where tight closure does not behave uniformly under a geometric deformation we first need a situation where strong semistability does not behave uniformly. Such an example was given by Paul Monsky in 1997 in terms of Hilbert-Kunz multiplicity. We consider the ring

$$\mathbb{F}_2[t][x,y,z]/g)$$

where

$$g = z^4 + z^2xy + z(x^3+y^3) + (t+t^2)x^2y^2 \,,$$

the ideal

$$I = (x^4, y^4, z^4) \,.$$

and the element

$$f = y^3z^3 \,.$$

This is our example (x^3y^3 does not work). First, by strong semistability in the transcendental case we have

$$f \in I^* \text{ in } S \otimes \mathbb{F}_2(t)$$

by the degree formula. If localization would hold, then f would also belong to the tight closure of I for almost all algebraic instances $\mathbb{F}_q = \mathbb{F}_2(\alpha)$, $t \mapsto \alpha$. Contrary to that we can show that for all algebraic instances the element f belongs never to the tight closure of I.

Theorem 5.8. *Tight closure does not commute with localization.*

Corollary 5.9. Tight closure is not plus closure in graded dimension two for fields with transcendental elements.

Proof. Consider

$$R = \mathbb{F}_2(t)[x,y,z]/(g) \,.$$

In this ring $y^3z^3 \in I^*$, but it can not belong to the plus closure. Else there would be a curve mapping $Y \to C_{\mathbb{F}_2(t)}$ which annihilates the cohomology class c and this would extend to a mapping of relative curves almost everywhere. □

Corollary 5.10. There is an example of a smooth projective variety Z and an effective divisor $D \subset Z$ and a morphism

$$Z \longrightarrow \mathbb{A}^1_{\mathbb{F}_2}$$

such that $(Z-D)_\eta$ is not an affine variety over the generic point, but for every algebraic point x the fiber $(Z-D)_x$ is an affine variety.

Proof. Take $C \to \mathbb{A}^1_{\mathbb{F}_2}$ to be the Monsky quartic and consider the syzygzy bundle

$$\mathcal{S} = \mathrm{Syz}\,(x^4, y^4, z^4)(6)$$

together with the cohomology class c determined by $f = y^3z^3$. This class defines an extension

$$0 \longrightarrow \mathcal{S} \longrightarrow \mathcal{S}' \longrightarrow \mathcal{O}_C \longrightarrow 0$$

and hence $\mathbb{P}(\mathcal{S}^*) \subset \mathbb{P}(\mathcal{S}'^*)$. Then $\mathbb{P}(\mathcal{S}'^*) - \mathbb{P}(\mathcal{S}^*)$ is an example with the stated properties by the previous results. $\square$

It is an open question whether such an example can exist in characteristic zero.