Prof. Dr. H. Brenner **Osnabrück WS 2011/2012**

Mathematics for natural sciences I

Exercise sheet 4

Warm-up-exercises

Exercise 4.1. Establish for each $n \in \mathbb{N}$ if the function

$$\mathbb{R} \longrightarrow \mathbb{R},\, x \longmapsto x^n,$$

is injective and/or surjective.

Exercise 4.2. Show that there exists a bijection between $\mathbb{N}$ and $\mathbb{Z}$.

Exercise 4.3. Give examples of functions

$$\varphi, \psi : \mathbb{N} \longrightarrow \mathbb{N}$$

such that φ is injective, but not surjective, and ψ is surjective, but not injective.

Exercise 4.4. Let L and M be two sets and let

$$F : L \longrightarrow M$$

be a function. Let

$$G : M \longrightarrow L$$

be another function such that $F \circ G = \mathrm{id}_M$ and $G \circ F = \mathrm{id}_L$. Show that G is the inverse of F.

Exercise 4.5. Determine the composite functions

$$\varphi \circ \psi \text{ and } \psi \circ \varphi$$

for the functions $\varphi, \psi : \mathbb{R} \to \mathbb{R}$, defined by

$$\varphi(x) = x^4 + 3x^2 - 2x + 5 \text{ and } \psi(x) = 2x^3 - x^2 + 6x - 1\,.$$

Exercise 4.6. Let L, M, N and P be sets and let

$$F : L \longrightarrow M,\, x \longmapsto F(x),$$

$$G : M \longrightarrow N,\, y \longmapsto G(y),$$

and

$$H : N \longrightarrow P,\, z \longmapsto H(z),$$

be functions. Show that

$$H \circ (G \circ F) = (H \circ G) \circ F\,.$$

EXERCISE 4.7. Let L, M, N be sets and let

$$f : L \longrightarrow M \text{ and } g : M \longrightarrow N$$

be functions with their composition

$$g \circ f : L \longrightarrow N,\ x \longmapsto g(f(x)).$$

Show that if $g \circ f$ is injective, then also f is injective.

EXERCISE 4.8. Let

$$f_1, \ldots, f_n : \mathbb{R} \longrightarrow \mathbb{R}$$

be functions, which are increasing or decreasing, and let $f = f_n \circ \cdots \circ f_1$ be their composition. Let k be the number of the decreasing functions among the f_i's. Show that if k is even then f is increasing and if k is odd then f is decreasing.

EXERCISE 4.9. Calculate in the polynomial ring $\mathbb{C}[X]$ the product

$$((4+i)X^2 - 3X + 9i) \cdot ((-3+7i)X^2 + (2+2i)X - 1 + 6i)\,.$$

EXERCISE 4.10. Let K be a field and let $K[X]$ be the polynomial ring over K. Prove the following properties concerning the degree of a polynomial:

(1) $\deg(P+Q) \leq \max\{\deg(P), \deg(Q)\}$.
(2) $\deg(P \cdot Q) = \deg(P) + \deg(Q)$.

EXERCISE 4.11. Show that in a polynomial ring over a field K the following statement holds: if $P, Q \in K[X]$ are not zero, then also $PQ \neq 0$.

EXERCISE 4.12. Let K be a field and let $K[X]$ be the polynomial ring over K. Let $a \in K$. Prove that the evaluating function

$$\psi : K[X] \longrightarrow K,\ P \longmapsto P(a),$$

satisfies the following properties (here let $P, Q \in K[X]$).

(1) $(P+Q)(a) = P(a) + Q(a)$.
(2) $(P \cdot Q)(a) = P(a) \cdot Q(a)$.
(3) $1(a) = 1$.

EXERCISE 4.13. Evaluate the polynomial

$$2X^3 - 5X^2 - 4X + 7$$

replacing the variable X by the complex number $2 - 5i$.

EXERCISE 4.14. Perform, in the polynomial ring $\mathbb{Q}[X]$ the division with remainder $\frac{P}{T}$, where $P = 3X^4 + 7X^2 - 2X + 5$, and $T = 2X^2 + 3X - 1$.

EXERCISE 4.15. Let K be a field and let $K[X]$ be the polynomial ring over K. Show that every polynomial $P \in K[X]$, $P \neq 0$, can be decomposed as a product

$$P = (X - \lambda_1)^{\mu_1} \cdots (X - \lambda_k)^{\mu_k} \cdot Q$$

where $\mu_j \geq 1$ and Q is a polynomial with no roots (no zeroes). Moreover the different numbers $\lambda_1, \ldots, \lambda_k$ and the exponents $\mu_1, \ldots, \mu_k$ are uniquely determined apart from the order.

EXERCISE 4.16. Let $F \in \mathbb{C}[X]$ be a non-constant polynomial. Prove that F can be decomposed as a product of linear factors.

EXERCISE 4.17. Determine the smallest real number for which the Bernoulli inequality with exponent $n = 3$ holds.

EXERCISE 4.18. Sketch the graph of the following rational functions

$$f = g/h : U \longrightarrow \mathbb{R},$$

where each time U is the complement set of the set of the zeros of the denominator polynomial h.

(1) $1/x$,
(2) $1/x^2$,
(3) $1/(x^2 + 1)$,
(4) $x/(x^2 + 1)$,
(5) $x^2/(x^2 + 1)$,
(6) $x^3/(x^2 + 1)$,
(7) $(x - 2)(x + 2)(x + 4)/(x - 1)x(x + 1)$.

EXERCISE 4.19. Let $P \in \mathbb{R}[X]$ be a polynomial with real coefficients and let $z \in \mathbb{C}$ be a root of P. Show that also the complex conjugate $\overline{z}$ is a root of P.

Hand-in-exercises

Exercise 4.20. (3 points)

Consider the set $M = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and the function

$$\varphi : M \longrightarrow M,\ x \longmapsto \varphi(x),$$

defined by the following table

x	1	2	3	4	5	6	7	8
$\varphi(x)$	2	5	6	1	4	3	7	7

Compute φ^{1003}, that is the 1003-rd composition (or *iteration*) of φ with itself.

Exercise 4.21. (2 points)

Prove that a strictly increasing function

$$f : \mathbb{R} \longrightarrow \mathbb{R}$$

is injective.

Exercise 4.22. (3 points)

Let L, M, N be sets and let

$$f : L \longrightarrow M \text{ and } g : M \longrightarrow N$$

be functions with their composite

$$g \circ f : L \longrightarrow N,\ x \longmapsto g(f(x)).$$

Show that if $g \circ f$ is surjective, then also g is surjective.

Exercise 4.23. (3 points)

Calculate in the polynomial ring $\mathbb{C}[X]$ the product
$((4+i)X^3 - iX^2 + 2X + 3 + 2i) \cdot ((2-i)X^3 + (3-5i)X^2 + (2+i)X + 1 + 5i)$.

Exercise 4.24. (3 points)

Perform, in the polynomial ring $\mathbb{C}[X]$ the division with remainder $\frac{P}{T}$, where $P = (5+i)X^4 + iX^2 + (3-2i)X - 1$ and $T = X^2 + iX + 3 - i$.

Exercise 4.25. (5 points)

Let $P \in \mathbb{R}[X]$ be a non-constant polynomial with real coefficients. Prove that P can be written as a product of real polynomials of degrees 1 or 2.