Prof. Dr. H. Brenner **Osnabrück WS 2011/2012**

Mathematics for natural sciences I

Exercise sheet 20

Warm-up-exercises

Exercise 20.1. Prove that the function

$$\mathbb{R} \longrightarrow \mathbb{R},\, x \longmapsto x|x|,$$

is differentiable but not twice differentiable.

Exercise 20.2. Consider the function

$$f : \mathbb{R} \longrightarrow \mathbb{R},$$

defined by

$$f(x) = \begin{cases} x - \lfloor x \rfloor, \text{ if } \lfloor x \rfloor \text{ is even,} \\ \lfloor x \rfloor - x + 1, \text{ if } \lfloor x \rfloor \text{ is odd.} \end{cases}$$

Examine f in terms of continuity, differentiability and extremes.

Exercise 20.3. Determine local and global extrema of the function

$$f : [-2, 5] \longrightarrow \mathbb{R},\, x \longmapsto f(x) = 2x^3 - 5x^2 + 4x - 1.$$

Exercise 20.4. Determine local and global extrema of the function

$$f : [-4, 4] \longrightarrow \mathbb{R},\, x \longmapsto f(x) = 3x^3 - 7x^2 + 6x - 3.$$

Exercise 20.5. Consider the function

$$f : \mathbb{R} \longrightarrow \mathbb{R},\, x \longmapsto f(x) = 4x^3 + 3x^2 - x + 2.$$

Find the point $a \in [-3, 3]$ such that the tangent of the function at a is parallel to the secant between -3 and 3.

Exercise 20.6. Prove that a real polynomial function

$$f : \mathbb{R} \longrightarrow \mathbb{R}$$

of degree $d \geq 1$ has at most $d-1$ extrema, and moreover the real numbers can be divided into at most d sections, where f is strictly increasing or strictly decreasing.

Exercise 20.7. Determine the limit

$$\lim_{x\to 2} \frac{3x^2 - 5x - 2}{x^3 - 4x^2 + x + 6}$$

by polynomial division (see Example 20.11).

Exercise 20.8. Determine the limit of the rational function

$$\frac{x^3 - 2x^2 + x + 4}{x^2 + x}$$

at point $a = -1$.

Exercise 20.9. Next to a rectilinear river we want to fence a rectangular area of $1000m^2$, one side of the area is the river itself. For the other three sides, we need a fence. Which is the minimal length of the fence we need?

Exercise 20.10. Discuss the following properties of the rational function

$$f : D \longrightarrow \mathbb{R},\ x \longmapsto f(x) = \frac{2x - 3}{5x^2 - 3x + 4},$$

domain, zeros, growth behavior, (local) extrema. Sketch the graph of the function.

Exercise 20.11. Consider

$$f(x) = x^3 + x - 1\,.$$

a) Prove that the function f has in the real interval $[0, 1]$ exactly one zero.

b) Compute the first decimal digit in the decimal system of this zero point.

c) Find a rational number $q \in [0, 1]$ such that $|f(q)| \leq \frac{1}{10}$.

Exercise 20.12. Determine the limit of

$$\frac{x^2 - 3x + 2}{x^3 - 2x + 1}$$

at point $x = 1$, and specifically

a) by polynomial division.

b) by the rule of l'Hospital.

Exercise 20.13. Let $P \in \mathbb{R}[X]$ be a polynomial $a \in \mathbb{R}$ and $n \in \mathbb{N}$. Prove that P is a multiple of $(X - a)^n$ if and only if a is a zero of all the derivatives $P, P', P'', \ldots, P^{(n-1)}$.

Hand-in-exercises

Exercise 20.14. (5 points)

From a sheet of paper with side lengths of 20 cm and 30 cm we want to realize a box (without cover) with the greatest possible volume. We do it in this way. We remove from each corner a square of the same size, then we lift up the sides and we glue them. Which box height do we need to realize the maximum volume?

Exercise 20.15. (4 points)

Discuss the following properties of the rational function

$$f : D \longrightarrow \mathbb{R},\ x \longmapsto f(x) = \frac{3x^2 - 2x + 1}{x - 4},$$

domain, zeros, growth behavior, (local) extrema. Sketch the graph of the function.

Exercise 20.16. (5 points)

Prove that a non-costant rational function of the shape

$$f(x) = \frac{ax + b}{cx + d}$$

(with $a, b, c, d \in \mathbb{R}, a, c \neq 0$), has no local extrema.

Exercise 20.17. (3 points)

Determine the limit of the rational function

$$\frac{x^4 + 2x^3 - 3x^2 - 4x + 4}{2x^3 - x^2 - 4x + 3}$$

at point $a = 1$.

Exercise 20.18. (5 points)

Let $D \subseteq \mathbb{R}$ and

$$F : D \longrightarrow \mathbb{R}$$

be a rational function. Prove that F is a polynomial if and only if there is a higher derivative such that$F^{(n)} = 0$.