In this lecture we deal with closure operations which depend only on the torsor which the forcing algebra defines, so they only depend on the cohomology class of the forcing data inside the syzygy bundle. Our main example is tight closure, a theory developed by Hochster and Huneke, and related closure operations like solid closure and plus closure.

Tight closure and solid closure

Let R be a noetherian domain of positive characteristic, let

$$F : R \longrightarrow R,\ f \longmapsto f^p,$$

be the *Frobenius homomorphism*, and

$$F^e : R \longrightarrow R,\ f \longmapsto f^q,\ q = p^e\,,$$

its eth iteration. Let I be an ideal and set

$$I^{[q]} = \text{ extended ideal of } I \text{ under } F^e$$

Then define the *tight closure* of I to be the ideal

$$I^* := \{f \in R : \text{ there exists } z \neq 0 \text{ such that } zf^q \in I^{[q]} \text{ for all } q = p^e\}\,.$$

The element f defines the cohomology class $c \in H^1(D(I), \mathrm{Syz}\,(f_1, \ldots, f_n))$. Suppose that R is normal and that I has height at least 2 (think of a local normal domain of dimension at least 2 and an $\mathfrak{m}$-primary ideal I). Then the eth Frobenius pull-back of the cohomology class is

$$F^{e*}(c) \in H^1(D(I), F^{e*}(\mathrm{Syz}\,(f_1, \ldots, f_n)) \cong H^1(D(I), \mathrm{Syz}\,(f_1^q, \ldots, f_n^q))$$

($q = p^e$) and this is the cohomology class corresponding to f^q. By the height assumption, $zF^e(c) = 0$ if and only if $zf^q \in (f_1^q, \ldots, f_n^q)$, and if this holds for all e then $f \in I^*$ by definition. This shows already that tight closure under the given conditions does only depend on the cohomology class.

This is also a consequence of the following theorem of Hochster which gives a characterization of tight closure in terms of forcing algebra and local cohomology.

Theorem 4.1. *Let R be a normal excellent local domain with maximal ideal $\mathfrak{m}$ over a field of positive characteristic. Let $f_1, \ldots, f_n$ generate an $\mathfrak{m}$-primary ideal I and let f be another element in R. Then $f \in I^*$ if and only if*

$$H^{\dim(R)}_{\mathfrak{m}}(B) \neq 0\,,$$

where $B = R[T_1, \ldots, T_n]/(f_1T_1 + \ldots + f_nT_n + f)$ denotes the forcing algebra of these elements.

If the dimension d is at least two, then

$$H^d_{\mathfrak{m}}(R) \longrightarrow H^d_{\mathfrak{m}}(B) \cong H^d_{\mathfrak{m}B}(B) \cong H^{d-1}(D(\mathfrak{m}B), \mathcal{O}_B)\,.$$

This means that we have to look at the cohomological properties of the complement of the exceptional fiber over the closed point, i.e. the torsor given by these data. If the dimension is two, then we have to look whether the first cohomology of the structure sheaf vanishes. This is true (by Serre's cohomological criterion for affineness) if and only if the open subset $D(\mathfrak{m}B)$ is an *affine scheme* (the spectrum of a ring).

The right hand side of this equivalence - the non-vanishing of the top-dimensional local cohomology - is independent of any characteristic assumption, and can be taken as the basis for the definition of another closure operation, called *solid closure*. So the theorem above says that in positive characteristic tight closure and solid closure coincide. There is also a definition of tight closure for algebras over a field of characteristic 0 by reduction to positive characteristic.

An important property of tight closure is that it is trivial for regular rings, i.e. $I^* = I$ for every ideal I. This rests upon Kunz's theorem saying that the Frobenius homomorphism for regular rings is flat. This property implies the following cohomological property of torsors.

Corollary 4.2. Let $(R, \mathfrak{m})$ denote a regular local ring of dimension d and of positive characteristic, let $I = (f_1, \ldots, f_n)$ be an $\mathfrak{m}$-primary ideal and $f \in R$ an element with $f \notin I$. Let $B = R[T_1, \ldots, T_n]/(f_1T_1 + \ldots + f_nT_n + f)$ be the corresponding forcing algebra. Then for the extended ideal $\mathfrak{m}B$ we have

$$H^d_{\mathfrak{m}B}(B) \; = \; H^{d-1}(D(\mathfrak{m}B), \mathcal{O}_B) \; = \; 0.$$

Proof. This follows from Theorem 4.1 and $f \notin I^*$. □

In dimension two this is true in every (even mixed) characteristic.

Theorem 4.3. *Let $(R, \mathfrak{m})$ denote a two-dimensional regular local ring, let $I = (f_1, \ldots, f_n)$ be an $\mathfrak{m}$-primary ideal and $f \in R$ an element with $f \notin I$. Let $B = R[T_1, \ldots, T_n]/(f_1T_1 + \ldots + f_nT_n + f)$ be the corresponding forcing algebra. Then for the extended ideal $\mathfrak{m}B$ we have*

$$H^2_{\mathfrak{m}B}(B) \; = \; H^1(D(\mathfrak{m}B), \mathcal{O}_B) \; = \; 0.$$

In particular, the open subset $T = D(\mathfrak{m}B)$ is an affine scheme if and only if $f \notin I$.

The main point for the proof of this result is that for $f \notin I$, the natural mapping

$$H^1(U, \mathcal{O}_X) \longrightarrow H^1(T, \mathcal{O}_T)$$

is not injective by a Matlis duality argument. Since the local cohomology of a regular ring is explicitely known, this map annihilates some cohomology class of the form $\frac{1}{fg}$ where f, g are parameters. But then it annihilates the complete local cohomology module and then T is an affine scheme.

For non-regular two-dimensional rings it is a difficult question in general to decide whether a torsor is affine or not. A satisfactory answer is only known in the normal twodimensional graded case over a field, which we will deal with in the final lecture.

In higher dimension in characteristic zero it is not true that a regular ring is solidly closed (meaning that every ideal equals its solid closure), as was shown by the following example of Paul Roberts.

Example 4.4. Let K be a field of characteristic 0 and let

$$B = K[X, Y, Z][U, V, W]/(X^2U + Y^2V + Z^2W - X^3Y^3Z^3)\,.$$

Then the ideal $\mathfrak{a} = (X, Y, Z)B$ has the property that $H^3_{\mathfrak{a}}(B) \neq 0$. This means that in $R = K[X, Y, Z]$ the element $X^3Y^3Z^3$ belongs to the solid closure of the ideal (X^2, Y^2, Z^2), and hence the threedimensional polynomial ring is not solidly closed.

This example was the motivation for the introduction of parasolid closure, which has all the good properties of solid closure but which is also trivial for regular rings.

Plus closure

For an ideal $I \subseteq R$ in a domain R define

$I^+ = \{f \in R :\ \text{there exists a finite domain extension } R \subseteq T \text{ such that } f \in IT\}\,.$

Equivalent: let R^+ be the *absolute integral closure* of R. This is the integral closure of R in an algebraic closure of the quotient field $Q(R)$ (first considered by Artin). Then

$$f \in I^+ \text{ if and only if } f \in IR^+\,.$$

The plus closure commutes with localization.

We also have the inclusion $I^+ \subseteq I^*$. Here the question arises:

Question: Is $I^+ = I^*$?

This question is known as the *tantalizing question* in tight closure theory.

In terms of forcing algebras and their torsors, the containment inside the plus closure means that there exists a d-dimensional closed subscheme inside the torsor which meets the exceptional fiber (the fiber over the maximal ideal) in one point, and this means that the superheight of the extended ideal is d. In this case the local cohomological dimension of the torsor must be d as well, since it contains a closed subscheme with this cohomological dimension. So also the plus closure depends only on the torsor.

In characteristic zero, the plus closure behaves very differently compared with positive characteristic. If R is a normal domain of characteristic 0, then the trace map shows that the plus closure is trivial, $I^+ = I$ for every ideal I.