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Vector bundles, forcing algebras and local cohomology

Lecture 10

Affineness under deformations

We consider a base scheme B and a morphism

$$Z \longrightarrow B$$

together with an open subscheme $W \subseteq Z$. For every base point $b \in B$ we get the open subset

$$W_b \subseteq Z_b$$

inside the fiber Z_b. It is a natural question to ask how properties of W_b vary with b. In particular we may ask how the cohomological dimension of W_b varies and how the affineness may vary.

In the algebraic setting we have a D-algebra S and an ideal $\mathfrak{a} \subseteq S$ which defines for every prime ideal $\mathfrak{p} \in \operatorname{Spec}(D)$ the extended ideal $\mathfrak{a}_{\mathfrak{p}}$ in $S \otimes_D \kappa(\mathfrak{p})$.

This question is already interesting when B is a one-dimensional integral scheme, in particular in the following two situations.

(1) $B = \operatorname{Spec}(\mathbb{Z})$. Then we talk about an arithmetic deformation and want to know how affineness varies with the characteristic and how the relation is to characteristic zero.
(2) $B = \mathbb{A}^1_K = \operatorname{Spec}(K[t])$, where K is a field. Then we talk about a geometric deformation and want to know how affineness varies with the parameter t, in particular how the behaviour over the special points where the residue class field is algebraic over K is related to the behaviour over the generic point.

It is fairly easy to show that if the open subset in the generic fiber is affine, then also the open subsets are affine for almost all special points.

We deal with this question where W is a torsor over a family of smooth projective curves (or a torsor over a punctured twodimensional spectrum). The arithmetic as well as the geometric variant of this question are directly related to questions in tight closure theory. Because of the above mentioned degree criteria in the strongly semistable case, a weird behavior of the affineness property of torsors is only possible if we have a weird behavior of strong semistability.

Arithmetic deformations

We start with the arithmetic situation, the following example is due to Brenner and Katzman.

EXAMPLE 10.1. Consider $\mathbb{Z}[x,y,z]/(x^7+y^7+z^7)$ and consider the ideal $I=(x^4,y^4,z^4)$ and the element $f=x^3y^3$. Consider reductions $\mathbb{Z}\to\mathbb{Z}/(p)$. Then

$$f\in I^* \text{ holds in } \mathbb{Z}/(p)[x,y,z]/(x^7+y^7+z^7) \text{ for } p=3 \mod 7$$

and

$$f\notin I^* \text{ holds in } \mathbb{Z}/(p)[x,y,z]/(x^7+y^7+z^7) \text{ for } p=2 \mod 7\,.$$

In particular, the bundle $\operatorname{Syz}(x^4,y^4,z^4)$ is semistable in the generic fiber, but not strongly semistable for any reduction $p=2 \mod 7$. The corresponding torsor is an affine scheme for infinitely many prime reductions and not an affine scheme for infinitely many prime reductions.

In terms of affineness (or local cohomology) this example has the following properties: the ideal

$$(x,y,z)\subseteq\mathbb{Z}/(p)[x,y,z,s_1,s_2,s_3]/(x^7+y^7+z^7,s_1x^4+s_2y^4+s_3z^4+x^3y^3)$$

has cohomological dimension 1 if $p=3 \mod 7$ and has cohomological dimension 0 (equivalently, $D(x,y,z)$ is an affine scheme) if $p=2 \mod 7$.

Geometric deformations - A counterexample to the localization problem

Let $S\subseteq R$ be a multiplicative system and I an ideal in R. Then the *localization problem* of tight closure is the question whether the identity

$$(I^*)_S=(IR_S)^*$$

holds.

Here the inclusion $\subseteq$ is always true and $\supseteq$ is the problem. The problem means explicitly:

if $f\in(IR_S)^*$, can we find an $h\in S$ such that $hf\in I^*$ holds in R?

PROPOSITION 10.2. *Let $\mathbb{Z}/(p)\subset D$ be a one-dimensional domain and $D\subseteq R$ of finite type, and I an ideal in R. Suppose that localization holds and that*

$$f\in I^* \text{ holds in } R\otimes_D Q(D)=R_{D^*}=R_{Q(D)}$$

($S=D^*=D-\{0\}$ is the multiplicative system). Then $f\in I^*$ holds in $R\otimes_D\kappa(\mathfrak{p})$ for almost all $\mathfrak{p}$ in Spec D.

Proof. By localization, there exists $h \in D$, $h \neq 0$, such that

$$hf \in I^* \text{ in } R\,.$$

By persistence of tight closure (under a ring homomorphism) we get

$$hf \in I^* \text{ in } R_{\kappa(\mathfrak{p})}\,.$$

The element h does not belong to $\mathfrak{p}$ for almost all $\mathfrak{p}$, so h is a unit in $R_{\kappa(\mathfrak{p})}$ and hence

$$f \in I^* \text{ in } R_{\kappa(\mathfrak{p})}$$

for almost all $\mathfrak{p}$. □

In order to get a counterexample for the localization property we will look now at geometric deformations:

$$D = \mathbb{F}_p[t] \subset \mathbb{F}_p[t][x, y, z]/(g) = S$$

where t has degree 0 and x, y, z have degree 1 and g is homogeneous. Then (for every field $\mathbb{F}_p[t] \to K$)

$$S \otimes_{\mathbb{F}_p[t]} K$$

is a two-dimensional standard-graded ring over K. For residue class fields of points of $\mathbb{A}^1_{\mathbb{F}_p} = \text{Spec } \mathbb{F}_p[t]$ we have basically two possibilities.

- $K = \mathbb{F}_p(t)$, the function field. This is the *generic* or *transcendental* case.
- $K = \mathbb{F}_q$, the *special* or *algebraic* or *finite* case.

How does $f \in I^*$ vary with K? To analyze the behavior of tight closure in such a family we can use what we know in the two-dimensional standard-graded situation.

In order to establish an example where tight closure does not behave uniformly under a geometric deformation we first need a situation where strong semistability does not behave uniformly. Such an example was given, in terms of Hilbert-Kunz theory, by Paul Monsky in 1997.

Example 10.3. Let

$$g = z^4 + z^2xy + z(x^3 + y^3) + (t + t^2)x^2y^2\,.$$

Consider

$$S = \mathbb{F}_2[t, x, y, z]/(g)\,.$$

Then Monsky proved the following results on the *Hilbert-Kunz multiplicity* of the maximal ideal (x, y, z) in $S \otimes_{\mathbb{F}_2[t]} L$, L a field:

$$e_{HK}(S \otimes_{\mathbb{F}_2[t]} L) = \begin{cases} 3 \text{ for } L = \mathbb{F}_2(t) \\ 3 + \frac{1}{4^d} \text{ for } L = \mathbb{F}_q = \mathbb{F}_p(\alpha),\ (t \mapsto \alpha,\ d = \deg(\alpha))\,. \end{cases}$$

By the geometric interpretation of Hilbert-Kunz theory this means that the restricted cotangent bundle

$$\mathrm{Syz}\,(x, y, z) = (\Omega_{\mathbb{P}^2})_C$$

is strongly semistable in the transcendental case, but not strongly semistable in the algebraic case. In fact, for $d = \deg(\alpha)$, $t \mapsto \alpha$, where $K = \mathbb{F}_2(\alpha)$, the d-th Frobenius pull-back destabilizes.

The maximal ideal (x, y, z) can not be used directly. However, we look at the second Frobenius pull-back which is (characteristic two) just

$$I = (x^4, y^4, z^4)\,.$$

By the degree formula we have to look for an element of degree 6. Let's take

$$f = y^3 z^3\,.$$

This is our example (x^3y^3 does not work). First, by strong semistability in the transcendental case we have

$$f \in I^* \text{ in } R \otimes \mathbb{F}_2(t)$$

by the degree formula. If localization would hold, then f would also belong to the tight closure of I for almost all algebraic instances $\mathbb{F}_q = \mathbb{F}_2(\alpha)$, $t \mapsto \alpha$. Contrary to that we show that for all algebraic instances the element f belongs never to the tight closure of I.

LEMMA 10.4. *Let* $\mathbb{F}_q = \mathbb{F}_p(\alpha)$, $t \mapsto \alpha$,
$\deg(\alpha) = d$. *Set* $Q = 2^{d-1}$. *Then*

$$xyf^Q \notin I^{[Q]}\,.$$

Proof. This is an elementary but tedious computation. □

THEOREM 10.5. *Tight closure does not commute with localization.*

Proof. One knows in our situation that xy is a so called test element. Hence the previous Lemma shows that $f \notin I^*$. □

In terms of affineness (or local cohomology) this example has the following properties: the ideal

$$(x, y, z) \subseteq \mathbb{F}_2(t)[x, y, z, s_1, s_2, s_3]/(g, s_1x^4 + s_2y^4 + s_3z^4 + y^3z^3)$$

has cohomological dimension 1 if t is transcendental and has cohomological dimension 0 (equivalently, $D(x, y, z)$ is an affine scheme) if t is algebraic.

COROLLARY 10.6. Tight closure is not plus closure in graded dimension two for fields with transcendental elements.

Proof. Consider

$$R = \mathbb{F}_2(t)[x, y, z]/(g)\,.$$

In this ring $y^3z^3 \in I^*$, but it can not belong to the plus closure. Else there would be a curve mapping $Y \to C_{\mathbb{F}_2(t)}$ which annihilates the cohomology class c and this would extend to a mapping of relative curves almost everywhere. □

Corollary 10.7. There is an example of a smooth projective (relatively over the affine line) variety Z and an effective divisor $D \subset Z$ and a morphism

$$Z \longrightarrow \mathbb{A}^1_{\mathbb{F}_2}$$

such that $(Z - D)_\eta$ is not an affine variety over the generic point, but for every algebraic point x the fiber $(Z - D)_x$ is an affine variety.

Proof. Take $C \to \mathbb{A}^1_{\mathbb{F}_2}$ to be the Monsky quartic and consider the syzygzy bundle

$$\mathcal{S} = \operatorname{Syz}(x^4, y^4, z^4)(6)$$

together with the cohomology class c determined by $f = y^3z^3$. This class defines an extension

$$0 \longrightarrow \mathcal{S} \longrightarrow \mathcal{S}' \longrightarrow \mathcal{O}_C \longrightarrow 0$$

and hence $\mathbb{P}(\mathcal{S}^*) \subset \mathbb{P}(\mathcal{S}'^*)$. Then $\mathbb{P}(\mathcal{S}'^*) - \mathbb{P}(\mathcal{S}^*)$ is an example with the stated properties by the previous results. □

It is an open question whether such an example can exist in characteristic zero.