Prof. Dr. H. Brenner **Osnabrück WS 2011/2012**

Mathematics for natural sciences I

Exercise sheet 2

Warm-up-exercises

EXERCISE 2.1. Let x, y, z, w be elements in a field and suppose that z and w are not zero. Prove the following fraction rules.

(1)
$$\frac{x}{1} = x\,,$$
(2)
$$\frac{1}{-1} = -1\,,$$
(3)
$$\frac{0}{z} = 0\,,$$
(4)
$$\frac{z}{z} = 1\,,$$
(5)
$$\frac{x}{z} = \frac{xw}{zw}\,,$$
(6)
$$\frac{x}{z} \cdot \frac{y}{w} = \frac{xy}{zw}\,,$$
(7)
$$\frac{x}{z} + \frac{y}{w} = \frac{xw + yz}{zw}\,.$$

Does there exist an analogue of formula (7), which arises when one replaces addition by multiplication (and subtraction by division), that is

$$(x - z) \cdot (y - w) = (x + w) \cdot (y + z) - (z + w)\,?$$

Show that the "popular formula"

$$\frac{x}{z} + \frac{y}{w} = \frac{x + y}{z + w}$$

does not hold.

EXERCISE 2.2. Determine which of the two rational numbers p and q is larger:

$$p = \frac{573}{-1234} \text{ und } q = \frac{-2007}{4322}\,.$$

EXERCISE 2.3. a) Give an example of rational numbers $a, b, c \in]0,1[$ such that

$$a^2 + b^2 = c^2\,.$$

b) Give an example of rational numbers $a, b, c \in]0,1[$ such that

$$a^2 + b^2 \neq c^2\,.$$

c) Give an example of irrational numbers $a, b \in]0,1[$ and a rational number $c \in]0,1[$ such that

$$a^2 + b^2 = c^2\,.$$

The following exercises should only be made with reference to the ordering axioms of the real numbers.

EXERCISE 2.4. Prove the following properties of real numbers.

(1) $1 > 0$.
(2) From $a \geq b$ and $c \geq 0$ it follows $ac \geq bc$.
(3) From $a \geq b$ and $c \leq 0$ it follows $ac \leq bc$.
(4) $a^2 \geq 0$ holds.
(5) From $a \geq b \geq 0$ it follows $a^n \geq b^n$ for all $n \in \mathbb{N}$.
(6) From $a \geq 1$ it follows $a^n \geq a^m$ für ganze Zahlen $n \geq m$.
(7) From $a > 0$ it follows $\frac{1}{a} > 0$.
(8) From $a > b > 0$ it follows $\frac{1}{a} < \frac{1}{b}$.

EXERCISE 2.5. Show that for a real number $x \geq 3$ the inequality

$$x^2 + (x+1)^2 \geq (x+2)^2$$

holds.

EXERCISE 2.6. Let $x < y$ be two real numbers. Show that for the arithmetic mean $\frac{x+y}{2}$ the inequalities

$$x < \frac{x+y}{2} < y$$

hold.

EXERCISE 2.7. Prove the following properties for the absolute value function

$$\mathbb{R} \longrightarrow \mathbb{R},\ x \longmapsto |x|,$$

(here let x, y be arbitrary real numbers).

(1) $|x| \geq 0$.
(2) $|x| = 0$ if and only if $x = 0$.
(3) $|x| = |y|$ if and only if $x = y$ or $x = -y$.
(4) $|y - x| = |x - y|$.
(5) $|xy| = |x||y|$.
(6) For $x \neq 0$ we have $|x^{-1}| = |x|^{-1}$.
(7) We have $|x + y| \leq |x| + |y|$ (*Triangle inequality for the absolute value*).

EXERCISE 2.8. Sketch the following subsets of $\mathbb{R}^2$.

(1) $\{(x, y) |\, x = 5\}$,
(2) $\{(x, y) |\, x \geq 4 \text{ und } y = 3\}$,
(3) $\{(x, y) |\, y^2 \geq 2\}$,
(4) $\{(x, y) |\, |x| = 3 \text{ und } |y| \leq 2\}$,
(5) $\{(x, y) |\, 3x \geq y \text{ und } 5x \leq 2y\}$,
(6) $\{(x, y) |\, xy = 0\}$,
(7) $\{(x, y) |\, xy = 1\}$,
(8) $\{(x, y) |\, xy \geq 1 \text{ und } y \geq x^3\}$,
(9) $\{(x, y) |\, 0 = 0\}$,
(10) $\{(x, y) |\, 0 = 1\}$.

Hand-in-exercises

EXERCISE 2.9. (2 points)

Let $x_1, \ldots, x_n$ be real numbers. Show by induction the following inequality

$$|\sum_{i=1}^{n} x_i| \leq \sum_{i=1}^{n} |x_i|\,.$$

EXERCISE 2.10. (5 points)

Prove the general distributive property for a field.

EXERCISE 2.11. (3 points)

Sketch the following subsets of $\mathbb{R}^2$.

(1) $\{(x, y) |\, x + y = 3\}$,

(2) $\{(x,y)|\, x+y \leq 3\}$,
(3) $\{(x,y)|\, (x+y)^2 \geq 4\}$,
(4) $\{(x,y)|\, |x+2| \geq 5$ and $|y-2| \leq 3\}$,
(5) $\{(x,y)|\, |x| = 0$ and $|y^4 - 2y^3 + 7y - 5| \geq -1\}$,
(6) $\{(x,y)|\, -1 \leq x \leq 3$ and $0 \leq y \leq x^3\}$.

Exercise 2.12. (5 points)

A page has been ripped from a book. The sum of the numbers of the remaining pages is 65000. How many pages did the book have?

Hint: Show that it cannot be the last page. From the two statements "A page is missing" and "The last page is not missing" two inequalities can be set up to deliver the (reasonable) upper and lower bound for the number of pages.