M-1.1

Simple Harmonic Motion

All of these systems are simple harmonic oscillators which, when slightly disturbed from their equilibrium or rest position, will oscillate with simple harmonic motion. This is the most fundamental vibration of a single particle or one-dimensional system. A small displacement x from its equilibrium position sets up a restoring force which is proportional to x acting in a direction towards the equilibrium position.

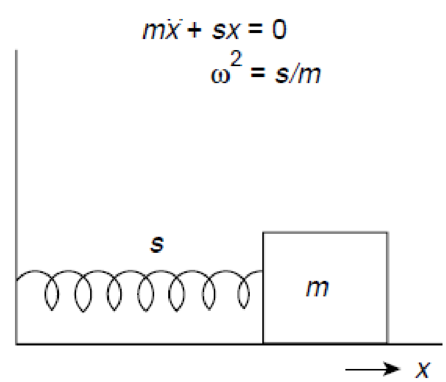
Thus, this restoring force F may be written

$$F = - sx$$

Where s, the constant of proportionality, is called the stiffness and the negative sign shows that the force is acting against the direction of increasing displacement and back towards the equilibrium position. A constant value of the stiffness restricts the displacement x to small values (this is Hooke's Law of Elasticity). The stiffness s is obviously the restoring force per unit distance (or displacement) and has the dimensions

$$\frac{\text{force}}{\text{distance}} \equiv \frac{MLT^{-2}}{L}$$

The equation of motion of such a disturbed system is given by the dynamic balance between the forces acting on the system, which by Newton's Law is



Mass times acceleration = restoring force

Or $m\ddot{x} = -sx$ Where the acceleration $\ddot{x} = \frac{d^2x}{dt^2}$

This gives $m\ddot{x} + sx = 0$ or $\ddot{x} + \frac{s}{m}x = 0$

Where the dimensions of $\frac{s}{m}$ are $\frac{MLT^{-2}}{ML} = T^{-2} = \nu^2$

Here T is a time, or period of oscillation, the reciprocal of v which is the frequency with which the system oscillates.

However, when we solve the equation of motion we shall find that the behavior of x with time has a sinusoidal or cosinusoidal dependence, and it will prove more appropriate to consider, not v, but the angular frequency ω = 2πv so that the period

$$T = \frac{1}{v} = 2\pi\sqrt{\frac{m}{s}}$$

Where s/m is now written as ω^2. Thus the equation of simple harmonic motion becomes

$$\ddot{x} + \frac{s}{m}x = 0$$

$$\ddot{x} + \omega^2 x = 0 \text{ ------------------------ } (1.1)$$

Displacement in Simple Harmonic Motion

The behavior of a simple harmonic oscillator is expressed in terms of its displacement x from equilibrium, its velocity $\dot{x}$, and its acceleration $\ddot{x}$ at any given time. If we try the solution

$$x = A\cos \omega t$$

Where A is constant with the same dimensions as x, we shall find that it satisfies the equation of motion $\ddot{x} + \omega^2 x = 0$ for $\dot{x} = -A\omega \sin\omega t$ and $\ddot{x} = -A\omega^2 \cos\omega t = -\omega^2 x$

Another solution $x = B \sin\omega t$ is equally valid, where B has the same dimensions as A, for then $\dot{x} = B\omega\cos\omega t$ and $\ddot{x} = -B\omega^2 \sin\omega t = -\omega^2 x$

The complete or general solution of equation (1.1) is given by the addition or superposition of both values for x so we have

$$x = A\cos\omega t + B\sin\omega t \text{ ------------------ } (1.2)$$

with

$$\ddot{x} = -\omega^2 \, (A\cos\omega t + B\sin\omega t) = -\omega^2 x$$

Where A and B are determined by the values of x and $\dot{x}$ at a specified time. If we rewrite the constant as $A = a\sin\phi$ and $B = a\cos\phi$

Where ϕ is a constant angle, then

$$A^2 + B^2 = a^2(\sin^2\phi + \cos^2\phi) = a^2$$

so that $a = \sqrt{A^2 + B^2}$

and $x = a\sin\phi \cos\omega t + a\cos\phi \sin\omega t = a\sin(\omega t + \phi)$

The maximum value of $\sin(\omega t + \phi)$ is unity so the constant a is the maximum value of x, known as the amplitude of displacement. The limiting values of $\sin(\omega t + \phi)$ are ± 1 as so the system will oscillate between the values of x = ±a and we shall see the magnitude of a is determined by the total energy of the oscillator.

The angle ϕ is called the 'phase constant' for the following reason. Simple harmonic motion is often introduced by reference to 'circular motion' because each possible value of the displacement x can be represented by the projection of a radius vector of constant length a on the diameter of the circle traced by the tip of the vector as it rotates in a positive

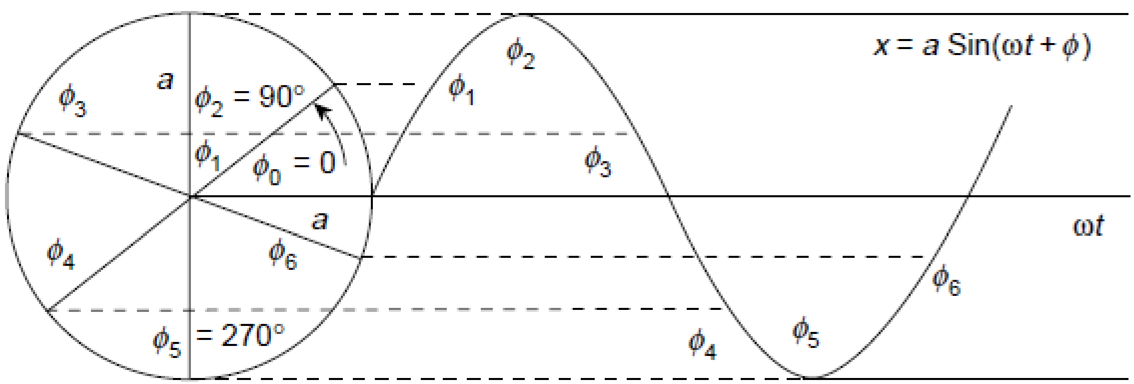


Figure 1.2 Sinusoidal displacement of simple harmonic oscillator with time, showing variation of starting point in cycle in terms of phase angle ϕ

anticlockwise direction with a constant angular velocity ω. Each rotation, as the radius vector sweeps through a phase angle of 2π rad, therefore corresponds to a complete vibration of the oscillator. In the solution

$$x = a \sin(\omega t + \phi)$$

the phase constant ϕ, measured in radians, defines the position in the cycle of oscillation at time t = 0, so that the position in the cycle from which the oscillator started to move is $x = a \sin\phi$.

The solution $x = a \sin\omega t$ defines the displacement only of that system which starts from the origin x = 0 at time t = 0 but the inclusion of ϕ in the solution

$$x = a \sin(\omega t + \phi)$$

Where ϕ may take all values between zero and 2π allows the motion to be defined from any starting point in the cycle. This is illustrated in Figure 1.2 for various values of ϕ.

Velocity and Acceleration in Simple Harmonic Motion

The values of the velocity and acceleration in simple harmonic motion for

$$x = a \sin(\omega t + \phi)$$

are given by $\frac{dx}{dt} = \dot{x} = a\omega\cos(\omega t + \phi)$ and $\frac{d^2x}{dt^2} = \ddot{x} = -a\omega^2\sin(\omega t + \phi)$

The maximum value of the velocity $a\omega$ is called the velocity amplitude and the acceleration amplitude is given by $a\omega^2$.

From figure 1.2 we see that a positive phase angle is $\pi/2$ rad converts a sine into a cosine curve. Thus the velocity

$$\dot{x} = a\omega \cos(\omega t + \phi)$$

leads the displacement $x = a \sin(\omega t + \phi)$

by a phase angle of $\pi/2$ rad and its maxima and minima are always a quarter of a cycle ahead of those of the displacement; the velocity is a maximum when the displacement is zero and is zero at maximum displacement. The acceleration is ‘anti-phase’ (π rad) with respect to the displacement, being maximum positive when the displacement is maximum negative and vice versa. These features are shown in figure 1.3.

Often, the relative displacement or motion between two oscillators having the same frequency and amplitude may be considered in terms of their phase difference $\phi_1 - \phi_2$ which can have any value because one system may have started several cycles before the other and each complete cycle of vibration represents a change in the phase angle of $\phi = 2\pi$. When the motion of the two

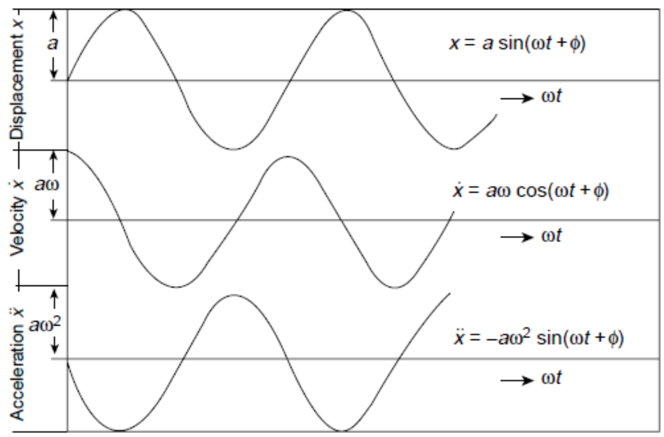


Figure 1.3 Variation with time of displacement, velocity and acceleration in simple harmonic motion. Displacement lags velocity by $\pi/2$ rad and is π rad out of phase with the acceleration. The initial phase constant ϕ is taken as zero

systems are diametrically opposed; that is, one has x = +a whilst the other is at x = -a, the systems are 'anti-phase' and the total phase difference

$$\phi_1 - \phi_2 = n\pi \text{ rad}$$

where n is an odd integer. Identical systems 'in phase' have

$$\phi_1 - \phi_2 = 2n\pi \text{ rad}$$

where n is any integer. They have exactly equal values of displacement, velocity and acceleration at any instant.

Energy of a Simple Harmonic Oscillator

The fact that the velocity is zero at maximum displacement in simple harmonic motion and is a maximum at zero displacement illustrates the important concept of an exchange between kinetic and potential energy. In an ideal case the total energy remains constant but this is never realized in practice. If no energy is dissipated then all the potential energy becomes kinetic energy and vice versa, so that the values of (a) the total energy at any time, (b) the maximum potential energy and (c) the maximum kinetic energy will all be equal; that is

$$E_{total} = KE + PE = KE_{max} = PE_{max}$$

The solution $x = a \sin(\omega t + \phi)$ implies that the total energy remains constant because the amplitude of displacement $x = \pm a$ is regained every half cycle at the position of maximum potential energy; when energy is lost the amplitude gradually decays. The potential energy is found by summing all the small elements of work sx.dx (force sx times distance dx) done by the system against the restoring force over the range zero to x where x = 0 gives zero potential energy.

Thus the potential energy = $\int_0^x sx \times dx = \frac{1}{2}sx^2$

The kinetic energy is give by $\frac{1}{2}m\dot{x}^2$ so that the total energy

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}sx^2$$

Since E is constant we have $\frac{dE}{dt} = (m\ddot{x} + sx)\dot{x} = 0$

giving again the equation of motion

$$m\ddot{x} + sx = 0$$

The maximum potential energy occurs at x = ±a and is therefore

$$PE_{max} = \frac{1}{2}sa^2$$

The maximum kinetic energy is

$$KE_{max} = \left(\frac{1}{2}m\dot{x}^2\right)_{max} = \frac{1}{2}ma^2\omega^2[\cos^2(\omega t + \phi)]_{max} = \frac{1}{2}ma^2\omega^2$$

When the cosine factor is unity.

But $m\omega^2 = s$ so the maximum values of the potential and kinetic energies are equal, showing that the energy exchange is complete.

The total energy at any instant of time or value of x is

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}sx^2 = \frac{1}{2}ma^2\omega^2[\cos^2(\omega t+\phi)+\sin^2(\omega t+\phi)] = \frac{1}{2}ma^2\omega^2 = \frac{1}{2}sa^2$$

as we should expect.

Figure 1.4 shows the distribution of energy versus displacement for simple harmonic motion. Note that the potential energy curve

$$PE = \frac{1}{2}sx^2 = ma^2\omega^2\sin^2(\omega t+\phi)$$

is parabolic with respect to x and is symmetric about x =0, so that energy is stored in the oscillator both when x is positive and when it is negative, e.g. a spring stores energy whether compressed or extended, as does a gas in compression or rarefaction. The kinetic energy curve

$$KE = \frac{1}{2}m\dot{x}^2 = \frac{1}{2}ma^2\omega^2\cos^2(\omega t+\phi)$$

is parabolic with respect to both x and $\dot{x}^2$. The inversion of one curve with respect to the other displays the π/2 phase difference between the displacement (related to the potential energy) and the velocity (related to the kinetic energy).

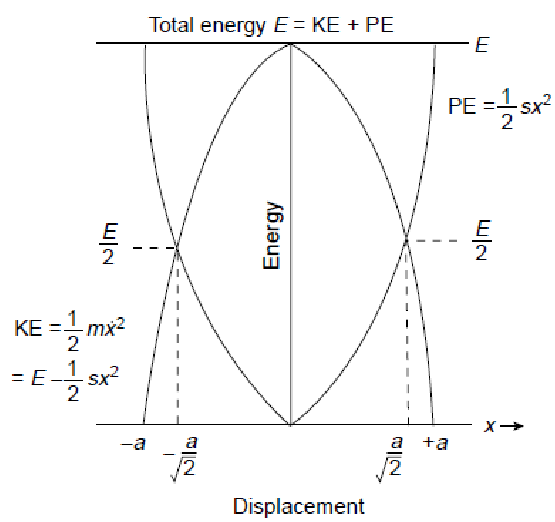


Figure 1.4 Parabolic representation of potential energy and kinetic energy of simple harmonic motion versus displacement. Inversion of one curve with respect to the other shows a 90° phase difference. At any displacement value the sum of the ordinates of the curves equals the total constant energy E

For any

value of the displacement x the sum of the ordinates of both curves equals the total constant energy E.

Simple Harmonic Oscillations in an Electrical system

So far we have discussed the simple harmonic motion of the mechanical and fluid systems, chiefly in terms of the inertial mass stretching the weightless spring of stiffness s. The stiffness s of a spring defines the difficulty of stretching; the reciprocal of the stiffness, the compliance C (where s = 1/C) defines the ease with which the spring is stretched and potential energy stored. This notation of compliance C is useful when discussing the simple harmonic oscillations of the electrical circuit of figure 1.5, where an inductance L is connected across the plates of a capacitance C. The force equation of the mechanical and fluid examples now becomes the voltage equation

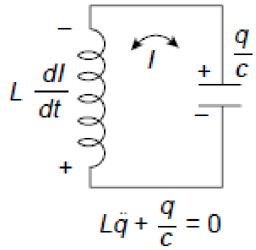


Figure 1.5 Electrical system which oscillates simple harmonically. The sum of the voltages around the circuit is given by Kirchhoff's law as $L\,dI/dt + q/C = 0$

(balance of voltages) of the electrical circuit, but the form and solution of the equations and the oscillatory behaviour of the systems are identical.

In the absence of resistance the energy of the electrical system remains constant and is exchanged between the magnetic field energy stored in the inductance and the electric field energy stored between the plates of the capacitance. At any instant, the voltage across the inductance is

$$V = -L\frac{dI}{dt} = -L\frac{d^2q}{dt^2}$$

Where I is the current flowing and q is the charge on the capacitor, the negative sign showing that the voltage opposes the increase of current. This equals the voltage q/C across the capacitance so that

$$L\ddot{q} + q/C = 0 \quad \text{or} \quad \ddot{q} + \omega^2 q = 0 \quad \text{Where} \quad \omega^2 = \frac{1}{LC}$$

The energy stored in the magnetic field or inductive part of the circuit throughout the cycle, as the current increase from 0 to I, is formed by integrating the power at any instant with respect to time; that is

$$E_L = \int VI \times dt$$

(where V is the magnitude of the voltage across the inductance).

So $$E_L = \int VI\,dt = \int L\frac{dI}{dt}\,I\,dt = \int_0^I LI\,dI = \frac{1}{2}LI^2 = \frac{1}{2}L\dot{q}^2$$

The potential energy stored mechanically by the spring is now stored electrostatically by the capacitance and equals

$$\frac{1}{2}CV^2 = \frac{q^2}{2C}$$

Comparison between the equations for the mechanical and electrical oscillators

Mechanical (force) → $m\ddot{x} + sx = 0$

Electrical (voltage) → $L\ddot{q} + \frac{q}{C} = 0$

Mechanical (energy) → $\frac{1}{2}m\dot{x}^2 + \frac{1}{2}sx^2 = E$

Electrical (energy) → $\frac{1}{2}L\dot{q}^2 + \frac{1}{2}\frac{q^2}{C} = E$

shows that magnetic field inertia (defined by the inductance L) controls the rate of change of current for a given voltage in a circuit in exactly the same way as the inertial mass controls the change of velocity for a given force. Magnetic inertial or inductive behaviour arises from the tendency of the magnetic flux threading a circuit to remain constant and reaction to any change in its value generates a voltage and hence a current which flows to oppose the change of flux. This is the physical basis of Fleming's right-hand rule.