1 Formulation of Euler Spiral

1.1 Symbols

R	- Radius of curvature
R_c	- Radius of Circular curve at the end of the spiral
θ	- Angle of curve from begining of spiral (infinite R_c) to a particular point on the spiral
θ_s	- Angle of full spiral curve
s, L	- Length measured along the spiral curve from its initial position
s_o, L_s	- Length of spiral curve position

1.2 Derivation

Euler spiral is defined as a curve whose curvature increases linearly with the distance measured along the curve.

An Euler spiral used is rail track / highway engineering typically connect between a tangent and a circular curve. Thus, the curvature of this Euler spiral starts with zero at one end and increases proportional with the curve distance.

Imagine the tangent extend from –ve x direction to the origin and the spiral starts at the origin in the +ve x direction, turns slowly in anticlockwise direction to meet a circular curve tangentially.

From the definition of the curvature,

$1/R = d\theta/dL \propto L$

i.e. $R.L = \text{constant} = R_cL_s$

$d\theta/dL = L/R_cL_s$

We write in the format

$d\theta/dL = 2a^2.L$

Where $2a^2 = 1/R_cL_s$

Or $a = 1/\sqrt{(2R_cL_s)}$

Thus $\theta = (a.L)^2$

Now

$x = \int dL \cos \theta$

$= \int dL \cos (a.L)^2$

Where $L' = aL$, and $dL = dL'/a$

$x = 1/a.\int dL' \cos (L')^2$

$y = \int dL \sin \theta$

$= \int dL \sin (a.L)^2$

$y = 1/a.\int dL' \sin (L')^2$

1.3 Expansion Of Fresnel Integral

The following integrals are known as Fresnel integrals (or Euler integrals):

$x = \int \cos L^2 \, ds$

$y = \int \sin L^2 \, dL$

$x = \int \cos L^2 \, dL$

$\cos \theta = 1 - \theta^2/2! + \theta^4/4! - \theta^6/6! + \ldots$

$x = \int (1 - L^4/2! + L^8/4! - L^{12}/6! + \ldots) \, dL$

$= L - L^5/(5 \times 2!) + L^9/(9 \times 4!) - L^{13}/(13 \times 6!) + \ldots$

$y = \int \sin L^2 \, dL$

$\text{Sin}\theta = \theta - \theta^3/3! + \theta^5/5! - \theta^7/7! + \ldots$

$y = \int (L^2 - L^6/3! + L^{10}/5! - L^{14}/7! + \ldots) \, dL$

$= L^3/3 - L^7/(7 \times 3!) + L^{11}/(11 \times 5!) - L^{15}/(15 \times 7!) + \ldots$

1.4 Conclusion

For a given Euler curve with:

$$2RL = 2R_cL_s = 1/a^2$$

Or

$$1/R = L/R_cL_s = 2a^2.L$$

Then

$$x = 1/a.\int ds \cos (s)^2$$

$$y = 1/a.\int ds \sin (s)^2$$

Where

$$s = a.L \text{ and } a = 1 / \sqrt{(2 R_cL_s)}$$

The process of solution for obtain (x, y) of an Euler curve thus be viewed as:

- Map L of the original Euler curve to s of a scaled-down Euler curve;
- Find (x', y') from the Fresnel integrals within the limit of L'; and
- Map (x', y') to (x, y) by scaling up with factor 1/a.

In this scaling process,

$$R_c' = R_c / \sqrt{(2R_cL_s)}$$

$$= \sqrt{(R_c / (2L_s)}$$

$$L_s' = L_s / \sqrt{(2R_cL_s)}$$

$$= \sqrt{(L_s / (2R_c)}$$

Then

$$2R_c'L_s' = 2. \sqrt{(R_c / (2L_s)} . \sqrt{(L_s / (2R_c)}$$

$$= 2 / 2$$

$$= 1$$

It is thus proposed that the term **Euler spiral** applies generally where as the term **Cornu spiral** shall only apply to the scaled down version of Euler spiral that has 2RcLs = 1.

Example 1

Given $R_c = 300m$,

$L_s = 100m$,

Then

$\theta_s = L_s / (2R_c)$

$= 100 / (2 \times 300)$

$= 0.1667$ radian, i.e. 9.5493 degrees

$2R_cL_s = 60{,}000$

We scale down the Euler spiral by $\sqrt{60{,}000}$, i.e. $100\sqrt{6}$ to the Cornu spiral that has:

$R_c' = 3/\sqrt{6}m$,

$L_s' = 1/\sqrt{6}m$,

$2R_c'L_s' = 2 \times 3/\sqrt{6} \times 1/\sqrt{6}$

$= 1$

And

$\theta_s = L_s' / (2R_c')$

$= 1/\sqrt{6} / (2 \times 3/\sqrt{6})$

$= 0.1667$ radian, i.e. 9.5493 degrees

The two angles θs are the same. This thus confirms that the big and small Euler spirals are having geometric similarity. The locus of the scale-down curve can be determined from Fresnel Integral, while the locus of the original Euler spiral can be obtained by scaling back (up).

Example 2

Given $R_c = 50m$,

$L_s = 100m$,

Then

$\theta_s = L_s / (2R_c)$

$= 100 / (2 \times 50)$

$= 1$ radian, i.e. 57.296 degrees

$2R_cL_s = 10{,}000$

We scale down by $\sqrt{10{,}000}$, i.e.100 to the transition spiral to the Cornu spiral that has:

$R_c' = 0.5m$,

$L_s' = 1m$,

$2R_c'L_s' = 2 \times 0.5 \times 1$

$= 1$

And

$\theta_s = L_s' / (2R_c')$

$= 1/100 / (2 \times 1/200)$

$= 1$ radian, i.e. 57.296 degrees

The two angles θs are the same. Follow the same process in the previous example to obtain the locus of the original Euler spiral.

Generally the scaling down reduces L' to a small value (<1) and results in good converging characteristics of the Fresnel integral with relative few terms.

1.5 Other Properties Of Cornu Spiral

Cornu Spiral is a special case of the transition spiral / Euler spiral which has $2R_c.L_s = 1$

$\theta_s = L_s / 2R_c \quad = L_s^2$

And

$\theta = \theta_s.(L^2 / L_s^2)$

$= L^2$

$1/R = d\theta/dL$

$= 2L$

Note that $2R_c.L_s = 1$ also means that $1/R_c = 2L_s$, in agreement with the last statement.