1 Formulation Of Euler spiral

All angular measurements are in radians.

1.1 Symbols

R	- Radius of curvature
R_c	- Radius of Circular curve at the end of the spiral
θ	- Angle of curve from begining of spiral (infinite R_c) to a particular point on the spiral
θ_s	- Angle of full spiral curve
L	- Length measured along the spiral curve from its initial position
L_s	- Length of spiral curve
$\boldsymbol{v}$	- velocity vector
V	- speed or amplitude of $\boldsymbol{v}$
$\boldsymbol{T}$	- unit tangential vector
$\boldsymbol{X}$	- unit vector in x-direction, or Northing
$\boldsymbol{Y}$	- unit vector in y-direction, or Easting
t	- time, measured from the instant a vehicle, which travels towards increasing curvature of the spiral, is at the beginning of the spiral
t_s	- time required to travel the length L_s
$\boldsymbol{z}$	- position vector whose coordinate is (x, y). Origin is at the position when t = 0. The initial direction is aligned with x-axis.

1.2 Original Derivation

Euler spiral is defined as a curve whose curvature increases linearly with the distance measured along the curve.

An Euler spiral used is rail track / highway engineering typically connect between a tangent and a circular curve. Thus, the curvature of this Euler spiral starts with zero at one end and increases proportional with the curve distance.

Imagine a vehicle travel at constant speed V on the spiral, which starts at the origin. Let the initial tangent be parallel to x-axis and the initial direction of travel to be in +ve x direction.

For constant speed,

$L \propto t = V.t$

From the definition of the curvature,

$1/R = d\theta/dL \propto L = cL,$

Where c is the coefficient $1/(R_c L_s)$

At $t = 0, \qquad 1/R = 0$

$\theta = \int d\theta/dL.dL$

$= \int cL.dL$

$= cL^2/2$

$\theta_s = cL_s^2/2$

$= L_s / (2R_c)$

$\Rightarrow \qquad 1/R = 1/R_c = 2\theta_s / L_s$ at $t = t_s$

Or $\qquad L_s = 2R_c.\, \theta_s \qquad (1)$

And $\qquad \theta = \theta_s.(t/ t_s)^2 \qquad (2)$

$t_s = L_s / V \qquad (3)$

$\boldsymbol{v} = V.\boldsymbol{T} \qquad (4)$

$\boldsymbol{T} = (\cos\theta).\boldsymbol{X} + (\sin\theta).\boldsymbol{Y} \qquad (5)$

$\boldsymbol{z} = \int \boldsymbol{v}\, dt$ where $\int$ is integrating from $t = 0$ to $t = t \qquad (6)$

From (2) $\Rightarrow t / t_s = (\theta / \theta_s)^{½}$

$\Rightarrow \qquad dt = t_s\, d\theta / 2(\theta_s.\theta)^{½}$

$dt = L_s / V \,.\, d\theta / 2(\theta_s.\theta)^{½} \qquad (7)$

Eqns (4), (5), (6) & (7) =>

$\boldsymbol{z} = L_s. \int \{ (\cos\theta).\boldsymbol{X} + (\sin\theta).\boldsymbol{Y}) \} \, d\theta / \{2(\theta_s. \theta)^{½} \}$

$x = L_s. \int \{\cos\theta / 2(\theta_s. \theta)^{½}\} \, d\theta$

Now $L_s / 2\theta_s^{½} = 2R_c. \theta_s / 2\theta_s^{½} = R_c. \theta_s^{½}$

$x = R_c. \theta_s^{½} \int \{\cos\theta / \theta^{½}\} \, d\theta$ (8)

Then expand cos θ according to power series expansion (Taylor series)

$\cos\theta = 1 - \theta^2/2! + \theta^4/4! - \theta^6/6! + \ldots$

$x = R_c. \theta_s^{½} \int \{\cos\theta / \theta^{½}\} \, d\theta$

$= R_c. \theta_s^{½} \int \{(1 - \theta^2/2! + \theta^4/4! - \theta^6/6! + \ldots) / \theta^{½}\} \, d\theta$

$= R_c. \theta_s^{½} \int \{(\theta^{-1/2} - \theta^{3/2}/2! + \theta^{7/2}/4! - \theta^{11/2}/6! + \ldots)\} \, d\theta$

$= R_c. \theta_s^{½} \int \text{PowerSeries}(\theta) \, d\theta$

$= R_c. \theta_s^{½} (2/1.\theta^{1/2} - 2/5.\theta^{5/2}/2! + 2/9.\theta^{9/2}/4! - 2/13.\theta^{13/2}/6! + \ldots)$ (9)

Similarly to (8):

$y = R_c. \theta_s^{½} \int \{\sin\theta / \theta^{½}\} \, d\theta$

Then expand sin θ according to power series expansion (Taylor series)

$\sin\theta = \theta - \theta^3/3! + \theta^5/5! - \theta^7/7! + \ldots$

$y = R_c. \theta_s^{½} \int \{\sin\theta / \theta^{½}\} \, d\theta$

$= R_c. \theta_s^{½} \int \{(\theta - \theta^3/3! + \theta^5/5! - \theta^7/7! + \ldots) / \theta^{½}\} \, d\theta$

$= R_c. \theta_s^{½} \int \{(\theta^{1/2} - \theta^{5/2}/3! + \theta^{9/2}/5! - \theta^{13/2}/7! + \ldots)\} \, d\theta$

$= R_c. \theta_s^{½} \int \text{PowerSeries}(\theta) \, d\theta$

$= R_c. \theta_s^{½} (2/3.\theta^{3/2} - 2/7.\theta^{7/2}/3! + 2/11.\theta^{11/2}/5! - 2/15.\theta^{15/2}/7! + \ldots)$ (10)

While $\theta = \theta_s.(L/ L_s)^2$, or

$\theta = L^2 / (2R_c.L_c)$

For computation, a spreadsheet can be performed for computed values of x and y for different and uniform incremental values of θ between 0 and θ_s.

1.3 Alternative and Derivation As Fresnel Integral

Follow the same process as in the last sub-section but simplify in a different manner:

For constant speed,

$L \propto t = V.t$

From the definition of the curvature,

$1/R = d\theta/dL \propto L = cL,$

Where c is the coefficient $1/(R_cL_s)$

At $t = 0$, $1/R = 0$

$\theta = \int d\theta/dL.dL$

$= \int cL.dL$

$= cL^2/2$

$\theta_s = cL_s^2/2$

$= L_s / (2R_c)$

=> $1/R = 1/R_c = 2\theta_s / L_s$ at $t = t_s$

Or $L_s = 2R_c. \theta_s$ (11)

And $\theta = \theta_s.(t/ t_s)^2$ (12)

$L = V.t$

⇨ $t = L / V$

$t_s = L_s / V$

$dt = dL / V$ or $V.dt = dL$ (13)

⇨ $\theta = \theta_s.(L^2 / L_s^2)$ (14)

⇨ $\theta = L^2 / (2R_c.L_s)$ (15)

$\boldsymbol{v} = V.\boldsymbol{T}$ (16)

$\boldsymbol{T} = (\cos\theta).\boldsymbol{X} + (\sin\theta).\boldsymbol{Y}$ (17)

$\boldsymbol{z} = \int \boldsymbol{v}\, dt$ where ~ is integrating from t = 0 to t = t (18)

$\boldsymbol{z} = \int V\{ (\cos\theta).\boldsymbol{X} + (\sin\theta).\boldsymbol{Y} \}\, dt$

$x = \int V \cos\theta\, dt$

$= \int \cos [L^2 / (2R_c.L_s)]\, dL$ (19)

The format is similar to Fresnel integral C(L)

Then expand cos θ according to power series expansion (Taylor series)

$x = \int \text{PowerSeries}(L)\, dL$

The expansion is however less convenient than the previous sub-section, unless $(2R_c.L_s) = 1$

Similarly to (8):

$y = \int V \sin\theta\, dt$

$= \int \sin [L^2 / (2R_c.L_s)]\, dL$ (20)

This format is similar to Fresnel integral S(L)

Then expand sin θ according to power series expansion (Taylor series)

$y = \int \text{PowerSeries}(L)\, dL$

1.4 Fresnel integral

$x = C(L) = \int \cos L^2\, dL$

$y = S(L) = \int \sin L^2\, dL$

$x = \int \cos L^2\, dL$

$\cos\theta = 1 - \theta^2/2! + \theta^4/4! - \theta^6/6! + \ldots$

$x = \int (1 - L^4/2! + L^8/4! - L^{12}/6! + \ldots)\, dL$

$= L - L^5/(5.2!) + L^9/(9.4!) - L^{13}/(13.6!) + \ldots$

$y = \int \sin L^2 \, dL$

$\sin\theta = \theta - \theta^3/3! + \theta^5/5! - \theta^7/7! + \ldots$

$y = \int (L^2 - L^6/3! + L^{10}/5! - L^{14}/7! + \ldots) \, dL$

$= L^3/3 - L^7/(7.3!) + L^{11}/(11.5!) - L^{15}/(15.7!) + \ldots$

1.5 Simplifying By Transformation / Geometric Similarity (Mapping Euler Spiral To Cornu Curve)

The integral in the last section can be simplified if we match the original spiral curve by a smaller spiral curve of a scale-down version of which $2R_c'L_s' = 1$, where R_c' and L_s' are the scaled down radius and spiral length respectively. This can be done by scaling down by factor $\sqrt{(2R_cL_s)}$ so that:

$R_c' = R_c / \sqrt{(2R_cL_s)}$

$= \sqrt{(R_c / (2L_s)}$

$L_s' = L_s / \sqrt{(2R_cL_s)}$

$= \sqrt{(L_s / (2R_c)}$

Then

$2R_c'L_s' = 2. \sqrt{(R_c / (2L_s)} . \sqrt{(L_s / (2R_c)}$

$= 2 / 2$

$= 1$

Example 1

Given $R_c = 300$m,

$L_s = 100$m,

Then

$\theta_s = L_s / (2R_c)$

$= 100 / (2 \text{ x } 300)$

$= 0.1667$ radian, i.e. 9.5493 degrees

$2R_cL_s = 60{,}000$

We scale down the Euler spiral by $\sqrt{60{,}000}$, i.e.$100\sqrt{6}$ to the Cornu spiral that has:

$R_c' = 3/\sqrt{6}$m,

$L_s' = 1/\sqrt{6}$m,

$2R_c'L_s' = 2 \times 3/\sqrt{6} \times 1/\sqrt{6}$

$= 1$

And

$\theta_s = L_s' / (2R_c')$

$= 1/\sqrt{6} / (2 \times 3/\sqrt{6})$

$= 0.1667$ radian, i.e. 9.5493 degrees

The two same angles θ_s confirm the geometric similarity. The locus of the scale-down curve can thus be determined from Fresnel Integral.

Example 2

Given $R_c = 50$m,

$L_s = 100$m,

Then

$\theta_s = L_s / (2R_c)$

$= 100 / (2 \times 50)$

$= 1$ radian, i.e. 57.296 degrees

$2R_cL_s = 10{,}000$

We scale down by $\sqrt{10{,}000}$, i.e.100 to the transition spiral to the Cornu spiral that has:

$R_c' = 0.5$m,

$L_s' = 1$m,

$2R_c'L_s' = 2 \times 0.5 \times 1$

$= 1$

And

$\theta_s = L_s' / (2R_c')$

$= 1/100 / (2 \times 1/200)$

$= 1$ radian, i.e. 57.296 degrees

The two same angles θ_s confirm the geometric similarity. The locus of the scale-down curve can thus be determined from Fresnel Integral.

1.6 Other Properties Of Cornu Spiral

Cornu Spiral is a special case of the transition spiral / Euler spiral which has 2Rc.Ls = 1

$$\theta_s = L_s / 2R_c \quad = L_s^2$$

And

$$\theta = \theta_s.(L^2 / L_s^2)$$

$$= L^2$$