# PHOTOMETRY

by

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# Photometry: The Nature of Luminance

by JR Johnston

useful in photometry

Of all the basic and derived physical units only those from photometry are likely to give much trouble to the physiologist.

There are two main reasons for this. First, many different units of fundamentally different origin have been used in the past so that even recent textbooks and papers can be found to use obscure or unfamiliar units. Second, the nature of the important quantity luminance is intrinsically difficult to understand. In this account the nature of photometric quantities is clarified and the relationships of antique and recent units stated.

## Luminous Flux: The Lumen (lm)

This is the basic unit of light: it is 1/683 watt of green light of frequency 540 x $10^{12}$ Hz (1 watt = 1 joule $sec^{-1}$).

The colour is chosento be at the peak of the eye's sensitivity curve. A lumen of any other colour, say red light, is of a higher wattage.

The energy of a single photon of this frequency is given by Planck's formula

$$E = h\nu$$

where $h = 6.6 \times 10^{-34}$ Js and $\nu$ is the frequency of the photon (Hz).

So $E = 6.6 \times 10^{-34} \times 540 \times 10^{12}$

$= 3.5 \times 10^{-19}$ J

and a flux of 1/683 W corresponds to $1/683 \times 1/3.5 \times 10^{19}$ or $4 \times 10^{15}$ photons per sec.

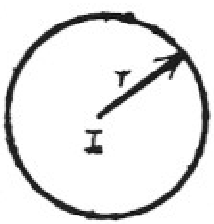
Mixtures of different colours must be summed according to their relative spectral efficiencies. Incandescent (tungsten) lamps emit about 13 lm $W^{-1}$ i.e., 13 lumens of light per watt of electricity dissipated. A fluorescent lamp produces about 30 lm $W^{-1}$. So a lumen correspnds to a steady stream of light and is otherwise unqualified.

<u>Luminous Intensity: The Candela (cd)</u>

A source which emits 1 lm $sr^{-1}$ in a particular direction has an intensity of 1 cd in that direction. A point source which emits P(lm) into space has therefore a mean intensity of $P/4\pi$ cd, although it may vary with direction. If it does so vary then the ratio of the infinitesimal flux dP going through infinitesimal angle $d\Omega (dP/d\Omega)$ must be used. The candela is by convention the fundamental unit of photometry although logically the lumen should be. (In 1979 the Conférence Générale des Poids et Mesures accepted a recommendation of the Comité Consultatif de Photométrie et Radiométrie which had been made through the Comité International des Poids et Mesures that "the candela is the luminous intensity, in a given direction, of a source which is emitting monochromatic radiant energy of frequency $540 \times 10^{12}$ herts and whose radiant intensity in that direction is 1/683 watt per steradian." An ordinary candle is approximately a 1 cd source.

<u>Illumination: The lux (lx)</u>

Consider an isotropic source of intensity I at the centre of a sphere of radius r.




Then it will radiate $4\pi I$ (lm) onto the spherical surface of area $4\pi r^2$.

Therefore it will radiate $\frac{4\pi I}{4\pi r^2}$ lm $m^{-2}$

$$= \frac{I}{r^2} \text{ lm m}^{2} \qquad = \frac{I}{r^2} \text{ lx}$$

This is the <u>illumination</u> <u>E</u> of the surface.

If r is sufficiently large then the same formula, $E = I/r^2$ (lx) will describe the illumination of a plane surface without great error. So a 100 W globe (which radiates 1300 lm and has a mean intensity of $1300/4\pi$ cd) will produce illumination of a surface 3 m distant

$$\frac{1300}{4\pi} \cdot \frac{1}{3^2}$$

$$= 11.5 \text{ lx}$$

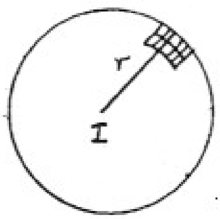
The perceived brightness of the illuminated surface depends on its nature: black and white pieces of paper may both have the same illumination (11.5 lx in this case) but will appear quite different.

The illuminance units phot and foot-candle may be encountered in the literature. They are explained in the section on luminance. The troland, a unit of retinal illumination is also sometimes used (Appendix 4)

Luminance: The Candela per square metre (cd $m^{-2}$)

Luminance is the most difficult concept to understand in photometry. These are two fundamentally different kinds of luminance unit. The first (Class I) are linked directly with illumination, the second (Class II) with fluxes.

Class I. Consider a sphere made of ground glass of 1 m-radius and with a 1 cd source at its centre, then the illumination of the surface is 1 lx.

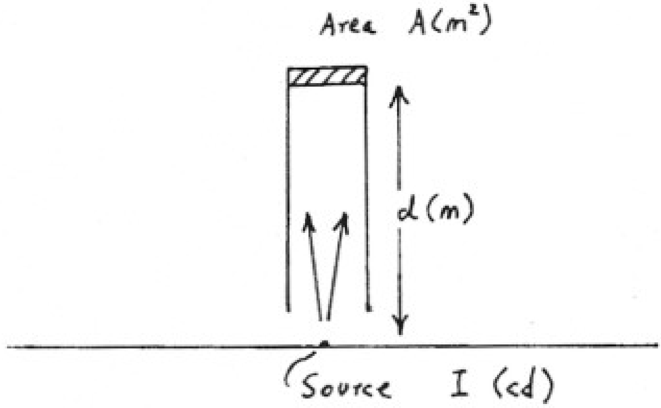



If the glass transmits all the light and scatters it externally and uniformly (where "uniformly" means "lambertian", to be explained later) then the surface is defined to have a luminance of 1 apostilb. For spheres of radii 1 cm and 1 ft, luminance units of the lambert and foot-lambert are similarly defined as corresponding to surfaces of illumination 1 lm $cm^{-2}$ (1 phot) and 1 lm $ft^{-2}$ (1 foot-candle) respectively. These results can be summarized:

| Source Intensity I | Sphere Radius r | Illumination E | Luminance L |
|---|---|---|---|
| 1 cd | 1 cm | 1 phot | 1 lambert |
| 1 cd | 1 m | 1lx | 1 apostilb |
| 1 cd | 1 ft | 1 ft - cd | 1 ft - lambert |

In many ways these are convenient units because they are easy to imagine and reproduce in practice. For example, a candle flame has a radius of about 1 cm and intensity about 1 cd: its luminance is therefore about 1 lambert. However, they have an aethereal existence: what are the dimensions of a lambert? How would a photo-electric lambert meter be built? To answer these questions it is necessary to consider a second way of looking at luminance.

Class II. Since luminance can be intimately associated with illumination (e.g., 1 lx can correspond to 1 apostilb) it might be thought sufficient to measure the light radiating from a given surface. It is indeed true (as will be shown) that a surface which radiates a total of 1 lm $m^{-2}$ has a mean luminance of 1 apostilb. But in general, surfaces vary in brightness according to the direction from which they are viewed so the flux in a particular direction only must be considered. But since there are infinitely (indeed, $\aleph_2$ in Cantor's classification of transfinite numbers) many directions from a given point on a surface, only an infinitesimal number of lumens can go in a specific direction. A more subtle approach is required.




Place a black tube of length d(m) normal to a surface which has light source of I (cd) on it. From the definition of I, the flux at the end of the tube will be

$$\text{Flux (P)} = \text{Luminous Intensity (I)} \times \text{Solid angle } (\Omega)$$
$$= I \times A/d^2 \text{ (lm)}$$

Now replace the source by a large number (n) of smaller sources I/n (cd). Provided d is large compared to A, the flux will remain the same: Each little source on the area A of surface can be regarded as radiating into the same solid angle $A/d^2$ and reaching the end of the tube. The total flux is a measure of the surface's illuminating ability. It might be, for example that 1 $cm^2$ will radiate 1 lm into 0.1 sr.

It is usual to normalize this to unit area and unit solid angle by dividing by the area and the solid angle. (In the same way, an object which passes through 1 cm in 1 ms has a quotable speed of 100 km/s).

so 1 lm into 0.1 sr from 1 $cm^2$ becomes $10^5$ lm $sr^{-1}$ $m^{-2}$ or equivalently $10^5$ cd $m^{-2}$. In general, the formula:

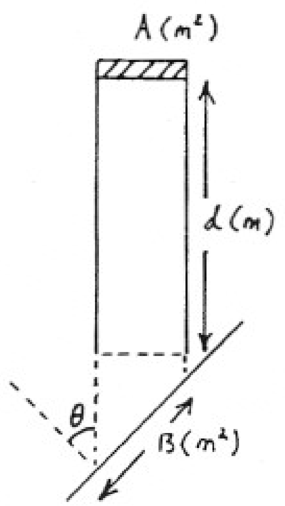
$$\text{Flux} = I \times A/d^2$$

becomes:

$$\text{Flux per area per solid angle} = \frac{I \times A/d^2}{A/d^2 \times A}$$

$$= I/A$$

The quantity I/A is the luminance (L) of the surface. Now consider a surface placed at an angle θ to the black tube and suppose the surface to radiate the same flux per area. Since A ($m^2$) radiated $\frac{IA}{d^2}$ lm, the new area $B = \frac{A}{\cos\theta}$ $m^2$ radiates $\frac{IA}{d^2\cos\theta}$ lm.




The luminance is defined with respect to the apparent area A; the solid angle is still $A/d^2$ so the new luminance is $\frac{IA}{d^2\cos\theta} \cdot \frac{1}{A} \cdot \frac{1}{A/d^2} = \frac{I}{A\cos\theta}$

Luminance is a measure of the light emitting power of a surface and it correlates with perceived brightness in the same way that sound pressure level does with loudness. Notice that if I at each point is independent of θ then luminance should vary with viewing angle .

This is indeed the case for a surface made up of isotropic point sources. A board covered with tungsten light globes will be brighter seen on edge than from in front: while I at each point on the surface is the same regardless of direction, the apparent area of the source will diminish. For many surfaces however L is independent of viewing angle and by implication I is proportional to cosθ ie

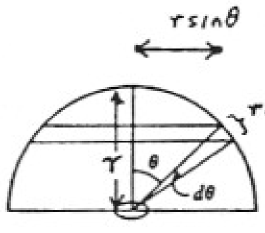
$$L = \frac{I_o \cos\theta}{A \cos\theta} = \frac{I_o}{A} = \text{constant}$$

where $I_o$ is the luminous intensity for the direction $\theta = 0$. The statement $I = I_o \cos\theta$ is an expression of Lambert's Law (see Appendix 2).

<u>Two Classes of Luminance Units and Their Conversion</u>

We now have two kinds of luminance unit. Class I derives from the generation of surfaces of given luminance (lambert, apostilb, foot-lambert) and Class II from analysis of surfaces as illuminants ($cd/m^2$ and its homologues $cd/cm^2$ and $cd/ft^2$). To connect the two systems it is first necessary to calculate the total flux emitted by a surface whose luminance is known in terms of Class II units.

Place an area $A(m^2)$ of Lambertian surface of luminance L ($cd\ m^{-2}$) at the centre of a hemisphere of radius r(m) where r is much larger than the dimension A.



The surface A will radiate according to Lambert's Law into the hemisphere. Consider the flux radiated into the annulus of width $rd\theta$ and radius $r \sin\theta$. This has an area of $2\pi r \sin\theta . r d\theta$ and subtends a solid angle

$$d\Omega = \frac{2\pi r^2 \sin\theta \, d\theta}{r^2} = 2\pi \sin\theta \, d\theta$$

From the definition of luminance

$$L = \frac{dP}{A \cos\theta \, d\Omega}$$

$$dP = L A \cos\theta \, . \, 2\pi \sin\theta \, d\theta$$

and the total flux radiated is

$$P = 2\pi \, LA \int_0^{\pi_2} \sin\theta \cos\theta \, d\theta$$

$$= 2\pi \, LA \left[ -\tfrac{1}{4} \cos 2\theta \right]_0^{\pi_2}$$

$$= 2\pi \, LA \left[ \tfrac{1}{4} + \tfrac{1}{4} \right]$$

$$= \pi L A$$

So a flat source of luminance $1 (cd \, m^{-2})$ and area $1 \, (m^2)$ radiates a total of $\pi$ (lm) into space. It must therefore receive incident light at $\pi (1 \, m \, m^{-2})$ which means it has luminance, in Class I terms, of $\pi$(apostilb).

The table given for Class I units can now be extended:

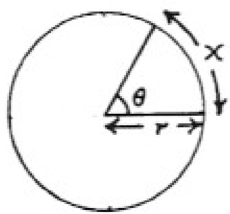
| Illumination | Luminance Class I | Luminance Class II |
|---|---|---|
| 1 phot | 1 lambert | $= 1/\pi$ cd $cm^{-2}$ |
| 1 lux | 1 apostilb | $= 1/\pi$ cd m-2 |
| 1 ft cd | 1 ft-lambert | $= 1/\pi$ cd $ft^{-2}$ |

The only photometric units acceptable as SI units are lumen, candela, lux and candela/metre$^2$. Units from earlier times which may be encountered in the literature are given in Appendix 3.

# APPENDIX 1

## Solid Angles

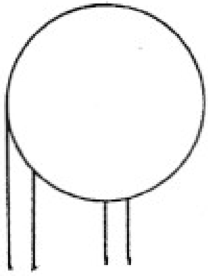
The most convenient measure of plane (ordinary) angles is radians. If in a circle of radius r an arc of length x is marked off, then the angle subtended is defined to be x/r radians.




From the formula ($2\pi r$) for the circumference of a circle it can be seen that the whole circle subtends $2\pi$ rad and therefore 1 rad = $57^o$. Solid angle is defined similarly: The solid angle (in steradians) subtended by an area A on a sphere of radius r is $A/r^2$ (sr). Since the area of a sphere is $4\pi r^2$, the whole sphere subtends $4\pi$ sr.

## Appendix 2: Lambert's Law

In the 18th Century, J.H. Lambert noticed that the sun appears to be a disc of uniform brightness. This is a remarkable observation: an area in the centre of the disc and an area on the limb of equal apparent size correspond to regions on the sun's surface which may be greatly different in area.



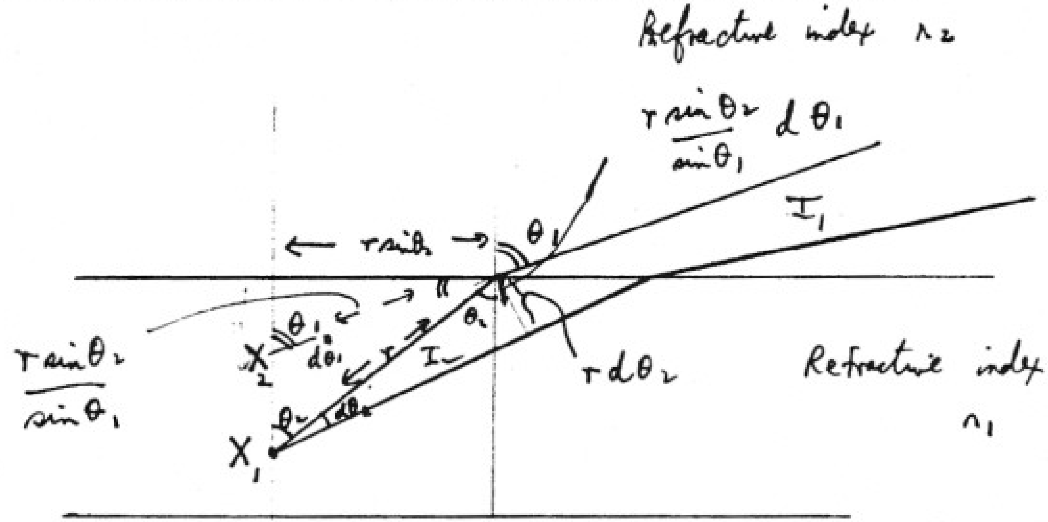
Lambert concluded rightly that more light is emitted per area normal to the surface than at an angle.

From consideration of the definition of luminance it can be concluded that each point on the surface must in fact radiate so that (Lambert's Law):

$$I = I_o \cos\theta$$

Since Lambert's time much effort has been expended on deriving Lambert's Law from the fundamental laws of physics. Whether this has been done or is even possible is by no means clear. Attached are some relevant quotations or extracts from the literature of the past century.

# A plausible derivation of Lamberts' Law




Source at $X_1$ appears to be at $X_2$. Consider the flux through the annular cone formed by $d\theta_2$ & area $dA_1$, solid angle $d\Omega_1$:

$$dA_1 = 2\pi r \sin\theta_2 \frac{r\sin\theta_2}{\sin\theta_1} d\theta_1$$

$$\therefore d\Omega_1 = \frac{2\pi r^2 \sin^2\theta_2 \, d\theta_1}{\sin\theta_1} \cdot \left(\frac{\sin\theta_1}{r\sin\theta_2}\right)^2$$

$$= \frac{2\pi r^2 \sin^2\theta_2 \, d\theta_1}{\sin\theta_1} \frac{\sin^2\theta_1}{r^2\sin^2\theta_2}$$

$$= 2\pi \sin\theta_1 \, d\theta_1$$

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From outside, the flux appears to be of area $dA_2$ & solid angle $d\Omega_2$.

$$dA_2 = 2\pi r \sin\theta_2 \cdot r\,d\theta_2$$

$$\therefore d\Omega_2 = \frac{2\pi r^2 \sin\theta_2 \, d\theta_2}{r^2}$$

$$= 2\pi \sin\theta_2 \, d\theta_2$$

From conservation of energy

$$I_1 \, d\Omega_1 = I_2 \, d\Omega_2$$

$$\therefore I_1 = I_2 \frac{d\Omega_2}{d\Omega_1}$$

$$= I_2 \cdot \frac{2\pi \sin\theta_2 \, d\theta_2}{2\pi \sin\theta_1 \, d\theta_1}$$

$$= I_2 \frac{\sin\theta_2 \, d\theta_2}{\sin\theta_1 \, d\theta_1}$$

$$= I_2 \frac{n_1}{n_2} \sin\theta_1 \cdot \frac{n_1}{n_2} \frac{\cos\theta_1 \, d\theta_1}{\cos\theta_2} \cdot \frac{1}{\sin\theta_1 \, d\theta_1}$$

$$= I_2 \left(\frac{n_1}{n_2}\right)^2 \frac{\cos\theta_1}{\cos\theta_2}$$

$$= I_2 \left(\frac{n_1}{n_2}\right)^2 \frac{\cos\theta_1}{\sqrt{1 - \left(\frac{n_1}{n_2}\right)^2 \sin^2\theta_1}}$$

$$n_1 \sin\theta_1 = n_2 \sin\theta$$

$$n_1 \cos\theta_1 \, d\theta_1 = n_2 \cos \, d\theta_2$$

$$\therefore \sin\theta_2 = \frac{n_1}{n_2} \sin\theta_1$$

$$d\theta_2 = \frac{n_1}{n_2} \frac{\cos\theta_1}{\cos\theta_2}$$

$$\cos\theta_2 = \sqrt{1 - \left(\frac{n_1}{n_2}\right)^2 \sin^2}$$

For $\left(\frac{n_1}{n_2}\right)^2 \ll 1$

$$I_1 \simeq I_2 \left(\frac{n_1}{n_2}\right)^2 \cos\theta_1$$

# APPENDIX 3

## Outdated Photometric Units

### Illumination

| | | |
|---|---|---|
| 1 sea-mile candle | $= (1/6080)^2$ ft cd | |
| 1 nox | $= 10^{-3}$ lux | |
| 1 troland | = 1 photon | = 1 luxon |

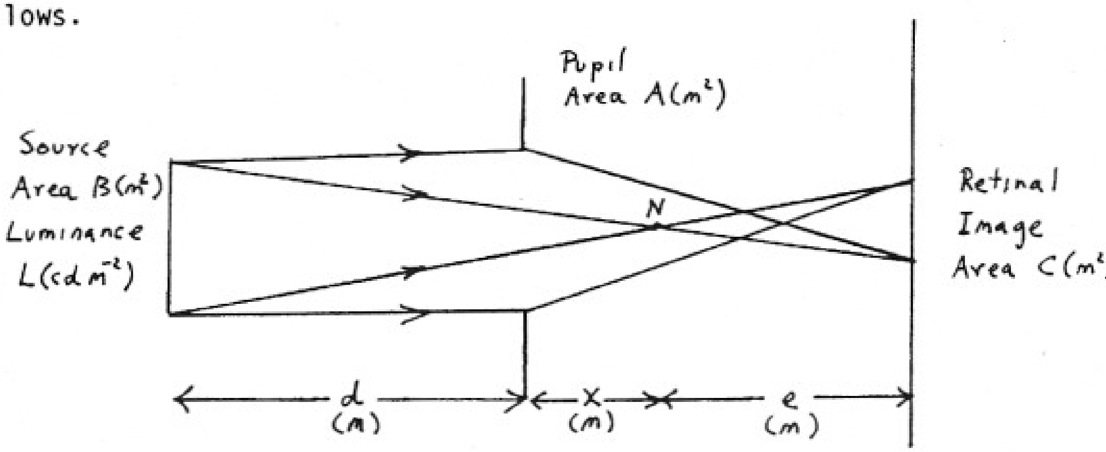
### Luminance

| | |
|---|---|
| 1 skot | $= 10^{-3}$ apostilb |
| 1 blondel | = 1 apostilb |
| 1 equivalent phot | = 1 lambert |
| 1 equivalent lux | = 1 apostilb |
| 1 equivalent ft.cd | = 1 ft.lambert |
| 1 stilb | $= 1$ cd $cm^{-2}$ |
| 1 nit | $= 1$ cd $m^{-2}$ |
| 1 bougie hectomètre carré | $= 10^{-4}$ cd $m^{-2}$ |

APPENDIX 4

## The Troland

The troland is defined to be the retinal illumination in lux produced by a 1 cd $m^{-2}$ surface viewed with a 1 $mm^2$ pupil. It can be calculated as follows.




Since L = P/BΩ, the flux incident on the pupil from the source is

$$P = LB\Omega \text{ (lm)}$$

Suppose the eye to transmit a fraction k of incident light then the flux producing the retinal image is

$$P = k\,L\,B\,\Omega$$

$$= \frac{k\,L\,B\,A}{d^2}$$

From the lens formula

$$\frac{C}{e^2} = \frac{B}{(d + x)^2}$$

where x is the distance from the pupil to the nodal point of the eye and is a few mm (the nodal point of an optical system is that point through which an incident ray passes undeviated, as shown in the diagram. See "Longhurst: Geometrical and Physical Optics").

Approximately, $B = \dfrac{Cd^2}{e^2}$

$$P = \frac{k\ L\ A}{d^2} \cdot \frac{C\ d^2}{e^2}$$

$$= \frac{k\ L\ A\ C}{e^2} \quad (\mathrm{lm})$$

and the illumination of the retinal image is

$$E = \frac{P}{C} = \frac{k\ L\ A}{e^2} \quad (\mathrm{lx})$$

Taking $k = 0.5$, $e = 17$ mm, $A = 1\ \mathrm{mm}^2$, $L = 1\ \mathrm{cd\ m}^{-2}$

$$E = \frac{0.5 \times 1 \times 10^{-6}}{(17 \times 10^{-3})^2}\ \mathrm{lx} \quad = \quad 0.0017\ \mathrm{lx}$$

So 1 troland is approximately a retinal illumination of 0.0017 lx but it will depend on given individual's values for k and e.

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