MATH 491 Senior Project I
Asymptotical Methods in Analysis

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Abstract

This project is devoted to the simplest notions of asymptotic analysis and their properties.

Contents

1 Introduction

Asymptotic analysis is a method of classifying limiting behavior mostly by using the language of asymptotic notations. Asymptotic notations (such as big-$\mathcal{O}$ and little-o) have been developed to handle statements related to order of growth. The first who introduced and systematically used them was Edmund Landau in early 1900's. If we have a function, then to approximate it, we mostly care about the large terms in its expression. In other words, we care about the term with the largest order of magnitude and omit the small valued terms (i.e. errors). This kind of analysis is considered to belong to asymptotics. The question is, how close can we get to the exact values of a given function using asymptotic expansions at extremely large values? There are 3 main asymptotic notations that we will use: $\sim$, o and $\mathcal{O}$. It should be mentioned that the symbol $\mathcal{O}(x)$ first appeared in the second volume of Bachmann's treatise on number theory (Bachmann 1894). Landau took this notation from Bachmann's book (Landau 1909, p. 883; Derbyshire 2004, p. 238). However, the symbol $o(x)$ indeed originate with Landau (1909) in place of the earlier notation $\{x\}$ (Narkiewicz 2000, p. XI).

2 The Symbols $\sim$, o and $\mathcal{O}$

Asymptotic notations that we are going to use are order relations: $\mathcal{O}$-notation (big-oh), o-notation (little-oh) and equivalence: '$\sim$' notation :

Definition 2.1. $f(x) = \mathcal{O}\{g(x)\}$ $(x \to \infty) \Leftrightarrow \exists A > 0, \exists x_0, \forall x > x_0,\ |f(x)| \leq A|g(x)|$. *In other words,* $f(x) = \mathcal{O}\{g(x)\}$, *as* $x \to \infty$, *means that* $\left|\frac{f(x)}{g(x)}\right|$ *is bounded for sufficiently large* x. *In this case, we say "f is of order not exceeding g".*

Definition 2.2. $f(x) = o\{g(x)\}$ $(x \to \infty) \Leftrightarrow \forall A > 0, \exists x_0, \forall x > x_0,\ |f(x)| < A|g(x)|$. *In other words,* $f(x) = o\{g(x)\}$, *means that* $\frac{f(x)}{g(x)} \to 0$ *as* $x \to \infty$. *In this case, we say "f is of order (strictly) less than g".*

Definition 2.3. $f(x) \sim g(x)$ $(x \to \infty) \Leftrightarrow \lim\limits_{x\to\infty} \dfrac{f(x)}{g(x)} = 1$. *In this case, we say "f is asymptotically equal to g".*

In all three cases, simply we can say: "*the function f(x) behaves asymptotically similarly to the function g(x)*". Comparing all three definitions, we note that each is a particular case of first one, and *when applicable* each is more informative than the first one. Explicitly, The $\sim$ formula gives much stronger assertion than the o-formula and then the $\mathcal{O}$-formula.

Let us prove the following lemma:

Lemma 2.4. *If* $\lim\limits_{x\to\infty} \dfrac{f(x)}{g(x)} = c$, *then* $f(x) = \mathcal{O}(g(x))$ *as* $x \to \infty$.

Proof. By definition of limit, for all $\varepsilon > 0$, there exists a $\delta > 0$ such that $\left|\frac{f(x)}{g(x)} - c\right| < \varepsilon$ for all large values of x. Thus by triangle inequality we have

$$\left|\frac{f(x)}{g(x)}\right| - |c| \leq \left|\frac{f(x)}{g(x)} - c\right| < \varepsilon \Rightarrow$$

$$\left|\frac{f(x)}{g(x)}\right| \leq |c| + \varepsilon \Rightarrow |f(x)| \leq (|c| + \varepsilon)|g(x)|.$$

□

Claim 2.5. *The converse is not always true. That is, there exist functions when $f(x) = \mathcal{O}(g(x))$ is true by the formal definition of $\mathcal{O}$-notation, but the above relation is not true.*

Proof. For example, consider the functions ($[x]$ denotes the integer part of x):

$$f(x) = \begin{cases} 2x & \text{if } [x] \text{ is even,} \\ x & \text{if } [x] \text{ is odd.} \end{cases}$$

$$g(x) = \begin{cases} x & \text{if } [x] \text{ is even,} \\ 2x & \text{if } [x] \text{ is odd.} \end{cases}$$

So, for $x \neq 0$, the quotient function $\frac{f(x)}{g(x)}$ defined as

$$\frac{f(x)}{g(x)} = \begin{cases} 1 & \text{if } [x] \text{ is even,} \\ 1/2 & \text{if } [x] \text{ is odd.} \end{cases}$$

If $[x]$ is even, then $f(x) = 2x$ and $g(x) = x$. By definition, $f(x) = \mathcal{O}(g(x))$ as $x \to \infty$. If $[x]$ is odd, then $f(x) = x$ and $g(x) = 2x$. By definition, $f(x) = \mathcal{O}(g(x))$ as $x \to \infty$.

Evidently, this discrete piecewise function is not continuous. Thus, $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)}$ does not exists. □

2.1 *Basic Properties*

Some properties, elementary operations, relations and special cases of these symbols are the following:

- The relation '$\sim$' is an equivalence relation, that is, it is reflexive, symmetric and transitive.
- '$\mathcal{O}$' is transitive but not symmetric, i.e. $\mathcal{O}(x) = \mathcal{O}(x^2)$ but $\mathcal{O}(x^2) \neq \mathcal{O}(x)$.
- Product: $\mathcal{O}(f(x)).\mathcal{O}(g(x)) = \mathcal{O}(f(x)g(x))$
- Sum: $\mathcal{O}(f(x)) + \mathcal{O}(g(x)) = \mathcal{O}(|f(x)| + |g(x)|) = \mathcal{O}(\max\{f(x), g(x)\})$
- Multiplication by a constant: $\mathcal{O}(k.f(x)) = k.\mathcal{O}(f(x)) = \mathcal{O}(f(x)) \quad (k \neq 0)$
- Addition of a constant: $\mathcal{O}(k + f(x)) = k + \mathcal{O}(f(x)) = \mathcal{O}(f(x))$
- $\mathcal{O}(f(x)) + \mathcal{O}(f(x)) = \mathcal{O}(f(x))$
- Composition: $\mathcal{O}(\mathcal{O}(f(x))) = \mathcal{O}(f(x))$
- $(f(x) + g(x))^k = \mathcal{O}(f(x) + g(x))^k = \mathcal{O}((f(x))^k) + \mathcal{O}((g(x))^k) \quad (k > 0)$
- The above properties also true for o-symbol.

- If $f(x) = o\{g(x)\}$ then, $f(x) = \mathcal{O}\{g(x)\}$ or simply:
- $o\{g(x)\} = \mathcal{O}\{g(x)\}$ but $\mathcal{O}\{g(x)\} \neq o\{g(x)\}$.
- If $f(x) \sim g(x)$, then $f(x) = g(x)\{1 + o(1)\}$.
- $f(x) = o(1)$ $(x \to \infty)$ means that f tends to 0 at ∞.
- $f(x) = \mathcal{O}(1)$ $(x \to \infty)$ means that $|f|$ is bounded at ∞.

Example 2.6. *The following relations hold:*

$$(i)\ (x+1)^2 \sim x^2,$$

$$(ii)\ \frac{1}{x^2} = o\left(\frac{1}{x}\right),$$

$$(iii)\ \sinh x = \mathcal{O}(e^x)\ \textit{as}\ x \to \infty.$$

Proof.

$$\begin{aligned}
&(i) \quad \lim_{x\to\infty} \frac{(x+1)^2}{x^2} = \lim_{x\to\infty} \frac{x^2+2x+1}{x^2} = \lim_{x\to\infty} 1 + \frac{2}{x} + \frac{1}{x^2} = 1.\\
&(ii) \quad \lim_{x\to\infty} \frac{1/x^2}{1/x} = \lim_{x\to\infty} \frac{1}{x} = 0.\\
&(iii) \quad \frac{e^x - e^{-x}}{2} = \frac{e^x(1-e^{-2x})}{2} \leq \frac{e^x}{2}.
\end{aligned}$$

□

2.2 *Implied Constant*

Sometimes, the symbol $\mathcal{O}$ is associated with a definite interval $[a, \infty)$ instead of a neightbourhood of ∞. Thus,

$$f(x) = \mathcal{O}\{g(x)\}\ \textit{when}\ x \in [a, \infty)$$

simply means that $\left|\frac{f(x)}{g(x)}\right|$ is bounded throughout $a \leq x < \infty$. However, other symbols $\sim$ and o cannot be used in this way.

The statement above assure that there is a number K such that

$$|f(x)| \leq K|g(x)| \quad (x \geq a),$$

without giving an information concerning the actual size of K. Of course, if it holds for a certain value of K, then it also holds for every larger value. The least number is the supremum (*least upper bound*) of $\left|\frac{f(x)}{g(x)}\right|$ in the interval $[a, \infty)$, which is called the **implied constant** of the $\mathcal{O}$ term for this interval.

The notations $o(g)$ and $\mathcal{O}(g)$ can also be used to denote *classes of functions* f with the properties $f(x) = o(g(x))$ and $f(x) = \mathcal{O}(g(x))$ respectively, or even they can be used to denote *unspecified functions* with these properties. Therefore, the statement "$f(x) = \mathcal{O}(g(x))$" is a slightly abuse of notation: we are not really asserting the equality of two functions. For this reason, some authors prefer a *set notation* and write $f \in \mathcal{O}(g)$, thinking of $\mathcal{O}(g)$ as the set of all functions that have the same property.

Example 2.7. [1] *If p has any fixed value, real or complex, prove that*

$$(i)\ x^p = o(e^x),\ (ii)\ e^{-x} = o(x^p).$$

(iii) *Prove also that* $\ln x = o(x^p)$, *provided that* $Re p > 0$.

Proof. Intuitively we know that $\ln x$ grows much more slowly than any power of x and any power of x grows slowly than e^x.
(i) Let $p = a + ib$ be a complex number where $a, b \in \mathbb{R}$.

$$x^p = x^{a+bi} \Rightarrow |x^p| = x^a$$

we know that, for positive x,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... > 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... + \frac{x^c}{c!}$$

where c is an integer greater than a. Therefore,

$$\lim_{x\to\infty} \frac{x^a}{e^x} \leq \lim_{x\to\infty} \frac{x^a}{1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... + \frac{x^c}{c!}} = 0 \Leftrightarrow x^p = o(e^x)\ as\ x \to \infty.$$

(ii) We will use $x^p = o(e^x)$. Take $p = -q$, we get $x^{-q} = o(e^x) \Rightarrow$

$$\lim_{x\to\infty} \frac{x^{-q}}{e^x} = 0 \Rightarrow \lim_{x\to\infty} \frac{e^{-x}}{x^q} = 0 \Leftrightarrow e^{-x} = o(x^q)\ as\ (x \to \infty).$$

(iii) Changing variable $y = \ln x$, and $y = t/a$ we have

$$\lim_{x\to\infty} \frac{\ln x}{x^a} = \lim_{y\to\infty} \frac{y}{e^{ay}} = \frac{1}{a} \lim_{t\to\infty} \frac{t}{e^t} = 0\ by\ (i).$$

□

Example 2.8. *Show that*

$$\cos\{\mathcal{O}(x^{-1})\} = \mathcal{O}(1), \quad \sin\{\mathcal{O}(x^{-1})\} = \mathcal{O}(x^{-1}),$$

and

$$\cos\{x + \alpha + o(1)\} = \cos(x + \alpha) + o(1),$$

where α *is a real constant.*

[1] In the following exercises 2.7 - 2.10, it is assumed that large positive values of the independent variable x are considered.

Proof. The first two relations are trivial, because $|\cos x| \leq 1$ and $|\sin x| \leq 1$ for real x. To prove the third one, note that:

$$\cos(x+\alpha+h(x)) - \cos(x+\alpha) = 2\sin\frac{h(x)}{2}\sin(x+\alpha+\frac{1}{2}h(x)).$$

If $h(x) = o(1)$, then $\left|\sin\frac{h(x)}{2}\right| \leq \frac{|h(x)|}{2} = o(1)$ and we get $\cos(x+\alpha+h(x)) = \cos(x+\alpha) + o(1), x \to \infty$, which is equivalent to desired relation. □

Example 2.9. *Show that*

$$(i)\ \mathcal{O}(x)\mathcal{O}(y) = \mathcal{O}(xy), \quad (ii)\ \mathcal{O}(x)o(y) = o(xy), \quad (iii)\ \mathcal{O}(x) + \mathcal{O}(y) = \mathcal{O}(|x|+|y|).$$

Proof. (i) Let $f(x) = \mathcal{O}(x)$ and $g(y) = \mathcal{O}(y)$ be two functions of x and y respectively. We have:

$$\begin{aligned} f(x) = \mathcal{O}(x) \quad &\Leftrightarrow \quad |f(x)| \leq A|x|, \\ g(y) = \mathcal{O}(y) \quad &\Leftrightarrow \quad |g(y)| \leq B|y|, \end{aligned}$$

for some $A, B > 0$. Multiplying side by side:

$$|f(x)||g(y)| \leq AB|x||y| \Rightarrow |f(x)g(y)| \leq AB|xy| \Leftrightarrow f(x)g(y) = \mathcal{O}(xy).$$

(ii) Let $f(x) = \mathcal{O}(x)$ and $g(y) = o(y)$ then,

$$\begin{aligned} f(x) = \mathcal{O}(x) \quad &\Leftrightarrow \quad |f(x)| \leq A|x|, \\ g(y) = o(y) \quad &\Leftrightarrow \quad \lim_{y\to\infty}\frac{g(y)}{y} = 0 \\ \lim_{x,y\to\infty}\frac{|f(x)g(y)|}{|xy|} \quad &\leq \quad A\lim_{y\to\infty}\frac{|g(y)|}{|y|} = 0 \\ (because\ \frac{|f(x)|}{|x|} \leq A \quad &and \quad \frac{|g(y)|}{|y|} \to 0) \Rightarrow \\ \lim_{x,y\to\infty}\frac{f(x)g(y)}{xy} = 0 \quad &\Leftrightarrow \quad f(x)g(y) = o(xy)\ as\ x \to \infty. \end{aligned}$$

(iii) Let $f(x) = \mathcal{O}(x)$ and $g(y) = \mathcal{O}(y)$ be two functions. We have:

$$\begin{aligned} f(x) = \mathcal{O}(x) \quad &\Leftrightarrow \quad |f(x)| \leq A|x|, \\ g(y) = \mathcal{O}(y) \quad &\Leftrightarrow \quad |g(y)| \leq B|y|. \end{aligned}$$

for some $A, B > 0$. Adding side by side:

$$|f(x)| + |g(y)| \leq A|x| + B|y| \leq C|x| + C|y| = C\{|x|+|y|\}\ for\ C > A, B.$$

Since $|f(x)+g(y)| \leq |f(x)| + |g(y)|$, we get:

$$|f(x)+g(y)| \leq C\{|x|+|y|\} \Leftrightarrow f(x)+g(y) = \mathcal{O}(|x|+|y|).$$

□

Example 2.10. *What are the implied constants in the relations*

$$(i)\ (x+1)^2 = \mathcal{O}(x^2), \quad (ii)\ (x^2 - \frac{1}{2})^{1/2} = \mathcal{O}(x), \quad (iii)\ x^2 = \mathcal{O}(e^x),$$

for the interval $[1, \infty)$*?*

Proof. (i) We have

$$\lim_{x\to\infty} \left| \frac{(x+1)^2}{x^2} \right| = 1 \quad where\ x \in [1, \infty)$$

Since $\frac{(x+1)^2}{x^2}$ is a decreasing function of $x \in [1, \infty)$, it takes its maximum value at $x = 1$. Thus

$$\frac{(x+1)^2}{x^2} \leq \frac{(1+1)^2}{1^2} = 4,$$

and the implied constant is 4.

(ii) We have

$$\lim_{x\to\infty} \left| \frac{\sqrt{x^2 - (1/2)}}{x} \right| = \lim_{x\to\infty} \left| \sqrt{1 - \frac{1}{2x^2}} \right| = 1.$$

As $x \in [1, \infty)$, $\sqrt{1 - \frac{1}{2x^2}} \in [\sqrt{1/2}, 1)$ means that its implied constant is 1.

(iii) We have $\lim\limits_{x\to\infty} \dfrac{x^2}{e^x} = 0$. Let $f(x) = \dfrac{x^2}{e^x}$. In order to find its maximum point, we can find the roots of its derivative for $x \in [1, \infty)$:

$$\left(\frac{x^2}{e^x} \right)' = \frac{2x - x^2}{e^x} = 0$$

for $x = x_0 = 0$ and $x = x_1 = 2$. The function gets its local maximum values at x_0 or x_1. Since $x_0 = 0 \notin [1,\infty)$, but $x_1 = 2 \in [1,\infty)$, $f(2) = 4/e^2 \approx 0.54$ is the implied constant. □

3 Extensions of Asymptotic Relations

3.1 *Extension in Real Variables*

For asymptotic approximations, there is no need for the asymptotic variable x to be continuous; it can pass to infinity through any set of values. For instance:

Example 3.1.

$$\sin\left(\pi n + \frac{1}{n}\right) = \mathcal{O}\left(\frac{1}{n}\right) \quad (n \to \infty)$$

provided that n *is an integer.*

Proof. We have

$$\begin{aligned}
\sin\left(\pi n + \frac{1}{n}\right) &= \sin(\pi n)\cos\left(\frac{1}{n}\right) + \cos(\pi n)\sin\left(\frac{1}{n}\right) = (-1)^n \sin\left(\frac{1}{n}\right) \\
Since \lim_{x\to 0}\frac{\sin x}{x} = 1 &\quad , \quad we\ have \lim_{n\to\infty}\frac{\sin\left(\frac{1}{n}\right)}{1/n} = 1.\ So, \\
\frac{\left|\sin\left(\pi n + \frac{1}{n}\right)\right|}{\left|\frac{1}{n}\right|} &= \frac{\sin\left(\frac{1}{n}\right)}{1/n}\ is\ bounded. \\
Hence, \sin\left(\pi n + \frac{1}{n}\right) &= \mathcal{O}\left(\frac{1}{n}\right) \quad (n \to \infty)
\end{aligned}$$

□

It is not obligatory to consider the behavior of functions only as $x \to \infty$. We can also do this when x tends to any finite point. For example:

Example 3.2. *Show that* $\frac{x^2 - c^2}{x^2} \sim \frac{2(x-c)}{c} = \mathcal{O}(x-c) = o(1)$ *as* $x \to c \quad (c \neq 0)$.

Proof.

$$\lim_{x\to c}\frac{\frac{x^2-c^2}{x^2}}{\frac{2(x-c)}{c}} = \lim_{x\to c}\frac{c(x+c)}{2x^2} = \frac{2c^2}{2c^2} = 1 \Leftrightarrow \frac{x^2-c^2}{x^2} \sim \frac{2(x-c)}{c} \quad (x \to c).$$

$$\left|\frac{2(x-c)}{c}\right| \leq A|x-c|\ for\ \forall A \geq \frac{c}{2} \Leftrightarrow \frac{2(x-c)}{c} = \mathcal{O}(x-c) \quad (x \to c).$$

$$\lim_{x\to c}\frac{2(x-c)}{c} = 0 \Leftrightarrow \frac{2(x-c)}{c} = o(1) \quad (x \to c).$$

□

The limiting point, here c, is called **the distinguished point** of the asymptotic or order relation.

Example 3.3. *Show that*

$$\ln\{1 + \mathcal{O}(z)\} = \mathcal{O}(z) \qquad (z \to 0).$$

Proof. Let $g(z) = \mathcal{O}(z)$, The Maclaurin expansion of $\ln(1+z)$ is:

$$\ln(1+z) = \sum_{s=1}^{\infty}\frac{(-1)^{s+1}}{s}z^s, \quad |z| < 1$$

So, if $|g(z)| < 1$, $g(z) = \mathcal{O}(z)$, we have:

$$\ln\{1 + g(z)\} = \sum_{s=1}^{\infty}\frac{(-1)^{s+1}}{s}g^s(z).$$

Setting $a_s = \frac{(-1)^{s+1}}{s}$ and $n = 1$, we get by Theorem 4.3

$$\ln\{1 + g(z)\} = \mathcal{O}(g(z)),\ z \to 0.$$

Substituting $g(z) = \mathcal{O}(z)$, we get:

$$\ln\{1 + \mathcal{O}(z)\} = \mathcal{O}\{\mathcal{O}(z)\} = \mathcal{O}(z).$$

□

3.2 *Extension in Complex Variables*

We can extend previous definitions to the complex case.

Let $\mathbf{S}$ be an infinite sector $\alpha \leq \arg z \leq \beta$.

Definition 3.4. *If* $\exists K > 0$, *not depending on* $\arg z$, *such that:*

$$|f(z)| \leq K|g(z)| \quad (z \in \boldsymbol{S}(R)),$$

where $\boldsymbol{S}(R)$ *denotes the intersection of* $\boldsymbol{S}$ *with the annulus* $|z| \geq R$, *then we say that*

$$f(z) = \mathcal{O}\{g(z)\} \text{ as } z \to \infty \text{ in } \boldsymbol{S},$$

or

$$f(z) = \mathcal{O}\{g(z)\} \text{ in } \boldsymbol{S}(R).$$

*The set **S**(R) is called an **infinite annular sector** or, simply, **annular sector**.*
The least number K *satisfying the above equation is called the **implied constant** for **S**(R).*

Theorem 3.5. *Let* $\sum_{s=0}^{\infty} a_s z^s$ *converge when* $|z| < r$. *Then for fixed* n,

$$\sum_{s=n}^{\infty} a_s z^s = \mathcal{O}(z^n),\ z \to 0,$$

in any disk $|z| \leq p$ *such that* $p < r$.

Proof. Let $p' \in (p, r)$. Then $a_s p'^s \to 0$ as $s \to \infty$; hence $|a_s p'^s|$ is bounded by a constant $A > 0$:

$$|a_s|p'^s \leq A \quad (s = 0, 1, 2, 3...).$$

Using this inequality, we obtain

$$\left|\sum_{s=n}^{\infty} a_s z^s\right| \leq \sum_{s=n}^{\infty} |a_s z^s| \leq \sum_{s=n}^{\infty} \frac{A}{p'^s}.|z|^s =$$

$$= A.\sum_{s=n}^{\infty} \left(\frac{|z|}{p'}\right)^s = A\left[\left(\frac{|z|}{p'}\right)^n + \left(\frac{|z|}{p'}\right)^{n+1} + \left(\frac{|z|}{p'}\right)^{n+2} + ...\right] =$$

$$= A\frac{\left(\frac{|z|}{p'}\right)^n}{1-\frac{|z|}{p'}} = \frac{Ap'^{(1-n)}}{p'-|z|}|z|^n \le \frac{Ap'^{(1-n)}}{p'-p}|z|^n.$$

Since, $p' > p$, $K = \dfrac{Ap'^{(1-n)}}{p'-p}$ is a positive constant (independent of z). That means:

$$\sum_{s=n}^{\infty} a_s z^s = \mathcal{O}(z^n), \ z \to 0.$$

□

Example 3.6. *Show that* $e^{-\sinh z} = o(1)$ *as* $z \to \infty$ *in the half-strip* $\{z\colon Rez \ge 0, |Imz| \le \frac{1}{2}\pi - \delta < \frac{1}{2}\pi\}$.

Proof. Let $\mathbf{H}$ be the half strip and $z = x + iy, \ x, y \in \mathbb{R}$ with $x \ge 0$, $|y| \le \frac{\pi}{2} - \delta < \frac{\pi}{2}$.
We have

$$\begin{aligned} \lim_{z\to\infty} |e^{-\sinh z}| &= \lim_{z\to\infty} \left| e^{-\frac{e^z - e^{-z}}{2}} \right| = \\ \lim_{z\to\infty} \left| e^{\frac{e^{-x-iy} - e^{x+iy}}{2}} \right| &= \lim_{z\to\infty} e^{Re\left(\frac{e^{-x-iy} - e^{x+iy}}{2}\right)}. \end{aligned}$$

Note that

$$\begin{aligned} Re\left(\frac{e^{-x-iy} - e^{x+iy}}{2}\right) &= Re\left(\frac{e^{-x}e^{-iy} - e^{x}e^{iy}}{2}\right) = \\ \frac{e^{-x}\cos(-y) - e^{x}\cos y}{2} &= \frac{\cos y}{2}(e^{-x} - e^{x}). \end{aligned}$$

Hence, we get:

$$\lim_{z\to\infty} |e^{-\sinh z}| = \lim_{z\to\infty} e^{\frac{\cos y}{2}(e^{-x}-e^{x})}$$

While, $|y| \le \frac{\pi}{2} - \delta$, we have $\cos y \ge \cos(\frac{\pi}{2} - \delta) = \sin\delta$. Therefore

$$\begin{aligned} \lim_{z\to\infty} e^{\frac{\cos y}{2}(e^{-x}-e^{x})} &= \lim_{z\to\infty} e^{-\frac{\cos y}{2}(e^{x}-e^{-x})} \le \\ \lim_{z\to\infty} e^{-\frac{\sin\delta}{2}(e^{x}-e^{-x})} &= 0. \\ \text{Since, } \lim_{z\to\infty} |e^{-\sinh z}| &= 0, \text{ we get } \lim_{z\to\infty} e^{-\sinh z} = 0. \end{aligned}$$

This is equivalent to $e^{-\sinh z} = o(1)$ as $z \to \infty$ in $\mathbf{H}$.

□

4 Integration and Differentiation of Asymptotic and Order Relations

4.1 *Integration*

Asymptotic and order relations may be integrated if the integrals are convergent. For example,

Proposition 4.1. *Let f is an integrable function such that $f(x) \sim x^v$ as $x \to \infty$, for $v \in \mathbb{C}$. If we integrate the function f, we get:*

$$\int_a^x f(t)dt \sim \begin{cases} a \text{ constant}, & \text{if } Rev < -1; \\ \dfrac{x^{v+1}}{v+1}, & \text{if } Rev > -1; \\ \ln x, & \text{if } v = -1. \end{cases}$$

and

$$\int_x^\infty f(t)dt \sim -\frac{x^{v+1}}{v+1} \quad Rev < -1$$

Proof. (i) First, let $Rev > -1$. Since $f(x) \sim x^v$, we can write $f(x) = x^v\{1 + \delta(x)\}$, where $|\delta(x)| < \varepsilon$ when $x > x_0 > a > 0$, for any $\varepsilon > 0$. Then,

$$\begin{aligned} \int_a^x f(t)dt &= \int_a^{x_0} f(t)dt + \int_{x_0}^x f(t)dt \\ &= \int_a^{x_0} f(t)dt + \int_{x_0}^x t^v\{1+\delta(t)\}dt \\ &= \int_a^{x_0} f(t)dt + \int_{x_0}^x t^v dt + \int_{x_0}^x t^v\delta(t)dt \\ &= \int_a^{x_0} f(t)dt + \frac{x^{v+1} - {x_0}^{v+1}}{v+1} + \int_{x_0}^x t^v\delta(t)dt. \end{aligned}$$

and by multiplying both sides by $\dfrac{v+1}{x^{v+1}}$ we get:

$$\begin{aligned} \frac{v+1}{x^{v+1}}\int_a^x f(t)dt &= \frac{v+1}{x^{v+1}}\int_a^{x_0} f(t)dt + \frac{x^{v+1} - {x_0}^{v+1}}{x^{v+1}} + \frac{v+1}{x^{v+1}}\int_{x_0}^x t^v\delta(t)dt \\ &= \frac{v+1}{x^{v+1}}\int_a^{x_0} f(t)dt + 1 - \frac{{x_0}^{v+1}}{x^{v+1}} + \frac{v+1}{x^{v+1}}\int_{x_0}^x t^v\delta(t)dt \\ \frac{v+1}{x^{v+1}}\int_a^x f(t)dt - 1 &= \frac{v+1}{x^{v+1}}\int_a^{x_0} f(t)dt - \frac{{x_0}^{v+1}}{x^{v+1}} + \frac{v+1}{x^{v+1}}\int_{x_0}^x t^v\delta(t)dt \\ \left|\frac{v+1}{x^{v+1}}\int_a^x f(t)dt - 1\right| &\le \left|\frac{v+1}{x^{v+1}}\int_a^{x_0} f(t)dt\right| + \left|\frac{{x_0}^{v+1}}{x^{v+1}}\right| + \left|\frac{v+1}{x^{v+1}}\int_{x_0}^x t^v\delta(t)dt\right|. \end{aligned}$$

Since $\int_a^{x_0} f(t)dt$ is a constant (*not depending on variable* x), the first two terms on the right-hand side of the last equation vanish as $x \to \infty$, then we have:

$$\begin{aligned}\left|\frac{v+1}{x^{v+1}}\int_a^x f(t)dt - 1\right| &\leq \left|\frac{v+1}{x^{v+1}}\int_{x_0}^x t^v\delta(t)dt\right| \\ &\leq \left|\frac{v+1}{x^{v+1}}\right|\int_{x_0}^x |t^v|.\delta(t)dt.\end{aligned}$$

Since $|\delta(t)| < \varepsilon$, $|t^v| = t^{Rev}$ and $|x^{v+1}| = x^{Rev+1}$,

$$\begin{aligned}\left|\frac{v+1}{x^{v+1}}\int_a^x f(t)dt - 1\right| &< \frac{|v+1|}{x^{Rev+1}}\int_{x_0}^x t^{Rev}.\varepsilon dt \\ &< \frac{|v+1|}{x^{Rev+1}}\varepsilon.\left[\frac{t^{Rev+1}}{Rev+1}\right]_{t=x_0}^x \\ &< \frac{|v+1|}{x^{Rev+1}}\varepsilon.\frac{x^{Rev+1} - {x_0}^{Rev+1}}{Rev+1} \\ &< |v+1|\varepsilon.\frac{1 - \frac{{x_0}^{Rev+1}}{x^{Rev+1}}}{Rev+1}.\end{aligned}$$

Since $\dfrac{{x_0}^{Rev+1}}{x^{Rev+1}} > 0$,

$$\left|\frac{v+1}{x^{v+1}}\int_a^x f(t)dt - 1\right| < |v+1|.\frac{\varepsilon}{Rev+1},$$

and that means:

$$\lim_{x\to\infty}\frac{v+1}{x^{v+1}}\int_a^x f(t)dt = 1 \Leftrightarrow \int_a^x f(t)dt \sim \frac{x^{v+1}}{v+1} \quad (x \to \infty).$$

(*ii*) If $v = -1$, then $f(x) \sim \frac{1}{x}$. We write $f(x) = \frac{1}{x}(1 + \delta(x))$ where $|\delta(x)| < \varepsilon$. So,

$$\left|\int_a^x f(t)dt - \ln x\right| = \left|\int_a^x \frac{\delta(t)}{t}dt - \ln a\right| \leq \varepsilon(\ln x + |\ln a|) \leq (\varepsilon + 1)\ln x.$$

Other cases are similar.

□

Example 4.2. *Show that if f(x) is continuous and* $f(x) = o\{g(x)\}$ *as* $x \to \infty$, *where g(x) is a positive nondecreasing function of x, then* $\int_a^x f(t)dt = o\{xg(x)\}$.

Proof. We have

$$f(x) = o\{g(x)\}\ as\ (x \to \infty) \Leftrightarrow \lim_{x\to\infty}\frac{f(x)}{g(x)} = 0 \Rightarrow$$

$$\forall\varepsilon > 0, \exists\delta > 0\ such\ that\ \left|\frac{f(x)}{g(x)}\right| < \varepsilon \Rightarrow |f(x)| < \varepsilon|g(x)|,\ for\ x > x_0 \Rightarrow$$

$$\left|\int_{x_0}^{x} f(t)dt\right| \leq \int_{x_0}^{x} \varepsilon|g(t)|dt \leq \varepsilon g(x)(x-x_0) \leq \varepsilon g(x)x \Rightarrow$$

$$\left|\int_{a}^{x} f(t)dt\right| \leq \left|\int_{a}^{x_0} f(t)dt\right| + \left|\int_{x_0}^{x} f(t)dt\right| \leq$$

$$\left|\int_{x_0}^{x} f(t)dt\right| + \left|\int_{x_0}^{x} f(t)dt\right| \leq 2\varepsilon x g(x) \ for \ a < x_0 < x,$$

$$\left|\int_{x_0}^{x} f(t)dt\right| \leq 2\varepsilon x g(x) \Rightarrow \lim_{x\to\infty} \frac{\int_{x_0}^{x} f(t)dt}{xg(x)} = 0 \Leftrightarrow \int_{x_0}^{x} f(t)dt = o\{xg(x)\}.$$

□

Example 4.3. *If u and x lie in [1,∞), show that*

$$\int_x^\infty \frac{dt}{t(t^2+t+u^2)^{\frac{1}{2}}} = \frac{1}{x} + \mathcal{O}\left(\frac{1}{x^2}\right) + \mathcal{O}\left(\frac{u^2}{x^3}\right).$$

Proof.

$$\begin{aligned}
\int_x^\infty \frac{dt}{t(t^2+t+u^2)^{1/2}} - \frac{1}{x} &= \int_x^\infty \frac{dt}{t(t^2+t+u^2)^{1/2}} - \int_x^\infty \frac{1}{t^2}dt \\
&= \int_x^\infty \left(\frac{1}{t(t^2+t+u^2)^{1/2}} - \frac{1}{t^2}\right) dt \\
&= \int_x^\infty \frac{t-(t^2+t+u^2)^{1/2}}{t^2(t^2+t+u^2)^{1/2}}dt \\
&= \int_x^\infty \frac{t^2-(t^2+t+u^2)}{t^2(t^2+t+u^2)^{1/2}(t+(t^2+t+u^2)^{1/2})}dt \\
&= \int_x^\infty -\frac{t+u^2}{t^2(t^2+t+u^2)^{1/2}(t+(t^2+t+u^2)^{1/2})}dt
\end{aligned}$$

Taking modulus, we obtain

$$\begin{aligned}
\left|\int_x^\infty \frac{dt}{t(t^2+t+u^2)^{1/2}} - \frac{1}{x}\right| &= \int_x^\infty \frac{t+u^2}{t^2(t^2+t+u^2)^{1/2}(t+(t^2+t+u^2)^{1/2})}dt \\
&= \int_x^\infty \frac{t+u^2}{t^3 \underbrace{(1+\frac{1}{t}+\frac{u^2}{t^2})^{1/2}(t+(t^2+t+u^2)^{1/2})}_{\text{greater than 1}}}dt
\end{aligned}$$

Since $(1+\frac{1}{t}+\frac{u^2}{t^2})^{1/2}(t+(t^2+t+u^2)^{1/2})$ is not less than 1 for $u, t \in [1,\infty)$, we have

$$\left|\int_x^\infty \frac{dt}{t(t^2+t+u^2)^{1/2}} - \frac{1}{x}\right| \leq \int_x^\infty \frac{t+u^2}{t^3}dt = \int_x^\infty \frac{dt}{t^2} + \int_x^\infty \frac{u^2}{t^3}dt = \frac{1}{x} + \frac{u^2}{2x^2}$$

Since $\frac{1}{x} = \mathcal{O}\left(\frac{1}{x^2}\right)$ and $\frac{u^2}{2x^2} = \mathcal{O}\left(\frac{u^2}{x^3}\right)$, given equality holds. □

4.2 *Differentiation*

Even if functions are differentiable, differentiation of asymptotic relations is not always permissible. For instance, if $f(x) = x + \sin x$, then $f(x) \sim x$ as $x \to \infty$, but it is not true for the derivatives, i.e. $f'(x) \not\sim 1$, because $\lim_{x\to\infty} f'(x)$ does not exists. In order to assure the differentiation, monotonicity conditions for the derivative must be satisfied:

Theorem 4.4. *Let $f(x)$ be continuously differentiable and $f(x) \sim x^p$ as $x \to \infty$, where $p \geq 1$ is a constant. Then $f'(x) \sim px^{p-1}$, provided that $f'(x)$ is nondecreasing for large x.*

Proof. Let $f(x) = x^v\{1 + \alpha(x)\}$, such that $|\alpha(x)| \leq \varepsilon$ when $x > x_0 > 0$, where $\varepsilon \in (0, 1)$ an arbitrary number. If $h > 0$, then

$$\begin{aligned} hf'(x) &\leq f(x+h) - f(x) = \int_x^{x+h} f'(t)dt = \int_x^{x+h} (t^p\{1 + \alpha(t)\})'dt \\ &= \int_x^{x+h} pt^{p-1}dt + (x+h)^p\alpha(x+h) - x^p\alpha(x) \\ &\leq hp(x+h)^{p-1} + 2\varepsilon(x+h)^p \end{aligned}$$

By setting $h = \varepsilon^{1/2}x$, we get an upper bound for the derivative

$$f'(x) \leq px^{p-1}\{(1 + \varepsilon^{1/2})^{p-1} + 2p^{-1}\varepsilon^{1/2}(1 + \varepsilon^{1/2})^p\} \quad (x > x_0).$$

and similarly we get a lower bound for the derivative as follows

$$f'(x) \geq px^{p-1}\{(1 - \varepsilon^{1/2})^{p-1} - 2p^{-1}\varepsilon^{1/2}\} \quad (x > x_0/(1 - \varepsilon^{1/2})).$$

So,

$$[\{(1-\varepsilon^{1/2})^{p-1} - 2p^{-1}\varepsilon^{1/2}\} - 1]px^{p-1} \leq f'(x) - px^{p-1} \leq [\{(1+\varepsilon^{1/2})^{p-1} + 2p^{-1}\varepsilon^{1/2}(1+\varepsilon^{1/2})^p\} - 1]px^{p-1}$$

Since both expressions in square brackets tend to 0 as $\varepsilon \to 0$, we obtain the desired assertion. □

Theorem 4.5. *Let $f(z)$ be analytic in a region containing a closed annular sector $\boldsymbol{S}$, and*

$$f(z) = \mathcal{O}(x^p) \ (\text{ or } f(z) = o(z^p))$$

as $z \to \infty$ in $\boldsymbol{S}$, where $p \in \mathbb{R}$ fixed. Then

$$f^{(m)}(z) = \mathcal{O}(z^{p-m}) \ (\text{ or } f^{(m)}(z) = o(z^{p-m}))$$

as $z \to \infty$ in any closed annular sector $\boldsymbol{S}$' properly interior to $\boldsymbol{S}$ and having the same vertex.

Proof. To prove the statement, we will use *Cauchy's integral formula* for mth derivative of an analytic function, given by

$$f^{(m)}(z) = \frac{m!}{2\pi i}\int_{\mathcal{C}} \frac{f(t)dt}{(t-z)^{m+1}}, \quad \mathcal{C} : |t - z| = |z| \sin\delta,$$

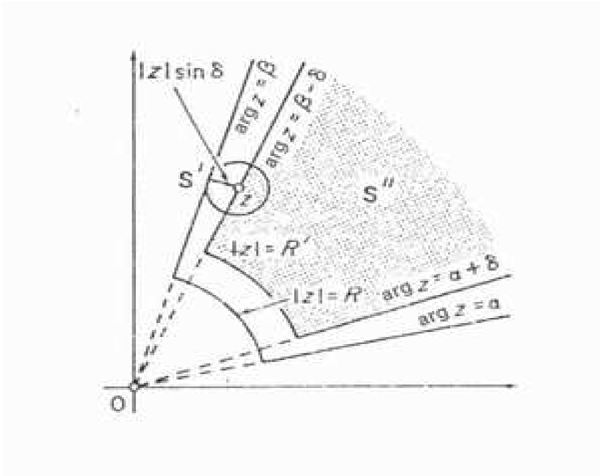

Figure 1: Annular sectors $\mathbf{S}, \mathbf{S}''$.

Let $\mathbf{S}$ be defined by $\alpha \le \arg z \le \beta$, $|z| \ge R$, and let $\mathbf{S}''$ be the closed annular sector defined by

$$\alpha + \delta \le \arg z \le \beta + \delta, \quad |z| \ge R'$$

where $\delta \in (0, \pi/2)$ and $R' = R/(1 - \sin\delta)$; see Figure **??**. By taking δ small enough we can ensure that $\mathbf{S}''$ contains **S'**.

$$0 < |z|(1 - \sin\delta) \le |t| \le |z|(1 + \sin\delta).$$

Hence $t \in \mathbf{S}$ whenever $z \in \mathbf{S}''$. Further, if K is the implied constant of $f(z) = \mathcal{O}(z^p)$ for $\mathbf{S}$, then

$$\begin{aligned} |f^{(m)}(z)| &\le \left|\frac{m!}{2\pi i}\int_{\mathcal{C}} \frac{f(t)dt}{(t-z)^{m+1}}\right| \le \frac{m!}{2\pi}\left|\int_{\mathcal{C}} \frac{f(t)dt}{(t-z)^{m+1}}\right| \\ &= \frac{m!}{2\pi}.\frac{2\pi K|z|^p(1 \pm \sin\delta)^p}{(|z|\sin\delta)^m} = K'|z^{p-m}| \end{aligned}$$

where K' is constant. The proof in the case when the symbol $\mathcal{O}$ replaced by o is similar.

The above inequality shows that the implied constant in $\mathbf{S}''$, does not exceed K'. However, as $\delta \to 0$, $K' \to \infty$. Therefore, the equation is NOT valid in $\mathbf{S}$. ☐

Example 4.6. *Suppose that $f(x) = x^2 + \mathcal{O}(x)$ as $x \to \infty$, and $f'(x)$ is continuous and nondecreasing for all sufficiently large x. Show that $f'(x) = 2x + \mathcal{O}(x^{1/2})$.*

Proof. We have

$$f'(x) = \lim_{h \to 0} \frac{f(x) - f(x-h)}{h} \ge \frac{f(x) - f(x-h)}{h}.$$

Let $\alpha(x) = f(x) - x^2 = \mathcal{O}(x)$ as $x \to \infty$, so $f(x-h) = \alpha(x-h) + (x-h)^2$ and $f(x+h) = \alpha(x+h) + (x+h)^2$. Then

$$f'(x) \geq \frac{\alpha(x) + x^2 - \alpha(x-h) - (x-h)^2}{h} = 2x - h + \frac{\alpha(x) - \alpha(x-h)}{h}$$

Similarly,

$$f'(x) \leq \frac{\alpha(x+h) + (x+h)^2 - \alpha(x) - x^2}{h} = 2x + h + \frac{\alpha(x+h) - \alpha(x)}{h}.$$

Since $\alpha(x) = f(x) - x^2 = \mathcal{O}(x),$ we have $|\alpha(x)| \leq Kx, \quad K > 0 \Rightarrow$

$$\begin{aligned} -Kx &\leq \alpha(x) \leq Kx, \text{ also} \\ -K(x-h) &\leq \alpha(x-h) \leq K(x-h) \text{ and} \\ -K(x+h) &\leq \alpha(x+h) \leq K(x+h). \end{aligned}$$

Therefore,

$$\begin{aligned} 2x + h + \frac{K(x+h) + Kx}{h} \geq & \quad f'(x) \quad \geq 2x - h + \frac{-Kx - K(x-h)}{h} \\ 2x + h + K + \frac{2Kx}{h} \geq & \quad f'(x) \quad \geq 2x - h + K - \frac{2Kx}{h} \\ h + K + \frac{2Kx}{h} \geq & \quad f'(x) - 2x \quad \geq -h + K - \frac{2Kx}{h} \end{aligned}$$

Since $K > 0$,

$$h + K + \frac{2Kx}{h} \geq \quad f'(x) - 2x \quad \geq -h - K - \frac{2Kx}{h} \Leftrightarrow$$

$$|f'(x) - 2x| \leq h + K + \frac{2Kx}{h}.$$

Choosing $h = x^{1/2}$ and assuming $x > 1$, we get:

$$\begin{aligned} |f'(x) - 2x| &\leq x^{1/2} + K + 2Kx^{1/2} = x^{1/2}(2K+1) + K, \\ |f'(x) - 2x| &\leq x^{1/2}(2K+1) + Kx^{1/2}, \\ |f'(x) - 2x| &\leq x^{1/2}(3K+1) \Leftrightarrow \\ f'(x) - 2x &= \mathcal{O}(x^{1/2}) \Rightarrow f'(x) = 2x + \mathcal{O}(x^{1/2}). \end{aligned}$$

□

Example 4.7. *Let f(z) be analytic in a region containing a closed annular sector **S**, and*

$$f(z) \sim z^p \text{ as } z \to \infty \text{ in } \boldsymbol{S},$$

where p is any nonzero real or complex constant. Show that $f^{'}(z) \sim pz^{p-1}$ *as* $z \to \infty$ *in any closed annular sector **C** properly interior to **S** and having the same vertex.*

Proof. We have

$$f(z) = (1 + o(1))z^p, \text{ as } z \to \infty, \ z \text{ in } \mathbf{C},$$

or same as

$$f(z) = z^p + g(z), \text{ where } g(z) = o(z^p).$$

By Theorem 4.4,

$$g(z) = o(z^p) \text{ implies } g'(z) = o(z^{p-1}).$$

Hence,

$$f'(z) = pz^{p-1} + g'(z) = pz^{p-1} + o(z^{p-1})$$

if and only if

$$f'(z) \sim pz^{p-1} \text{ as } z \to \infty \text{ in } \mathbf{C}$$

□

5 Asymptotic Solution of Transcendental Equations:

Transcendental equations are equations involving functions other then simple polynomials, which can not be expressed as a combination of elementary functions. These equations may contain trigonometric, algebraic, exponential, logarithmic and inverse trigonometric etc. terms. It is rarely possible to express the solutions of these functions explicitly. Therefore, we will use asymptotic approach to approximate the values of variables.

5.1 *Real Variables*

Example 5.1. *Consider the equation:*

$$x \tan x = 1$$

Proof. The aim is the determination of the large positive roots of the equation:

$$\tan x = \frac{1}{x}$$

Taking inverse of the tangent function, since the period of the tangent function is π, we get

$$x = \tan^{-1}\left(\frac{1}{x}\right) + n\pi$$

where $n \in \mathbb{Z}$ and the inverse tangent function has its principal value. So,

$$\tan^{-1}\left(\frac{1}{x}\right) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

is bounded. Further,

$$\lim_{n\to\infty} \frac{x}{n\pi} = \lim_{n\to\infty} \frac{\tan^{-1}\left(\frac{1}{x}\right)}{n\pi} + 1 = 1$$

We derive

$$x \sim n\pi \text{ as } n \to \infty.$$

This is the first approximation to the root. Note that,

$$\tan^{-1} x = \int \frac{dx}{1+x^2} = \int \left(1 - x^2 + x^4 - \ldots\right) = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots$$

To derive higher approximations we expand $\tan^{-1}(1/x)$ in a form appropriate for large x, given by

$$\tan^{-1}\left(\frac{1}{x}\right) = \frac{1}{x} - \frac{1}{3x^3} + \frac{1}{5x^5} - \frac{1}{7x^7} + \ldots \quad (x > 1)$$

Hence

$$x = \tan^{-1}\left(\frac{1}{x}\right) + n\pi = n\pi + \mathcal{O}\left(\frac{1}{x}\right) = n\pi + \mathcal{O}\left(\frac{1}{n}\right)$$

gives us a higher approximation.
The next two substitutions produce

$$x = n\pi + \frac{1}{n\pi} + \mathcal{O}\left(\frac{1}{n^3}\right), \quad x = n\pi + \frac{1}{n\pi} - \frac{4}{3(n\pi)^3} + \mathcal{O}\left(\frac{1}{n^5}\right) \ldots$$

and so on. Continuation of this process produces a sequence of approximations with errors of diminishing asymptotic order.

□

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