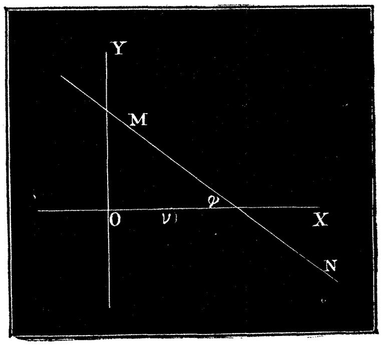
XI. *On a new Form of Tangential Equation.* *By* John Casey, *LL.D., F.R.S., M.R.I.A., Professor of Mathematics in the Catholic University of Ireland.*

Received January 24,—Read February 22, 1877.

INTRODUCTION.

Art. 1. The tangential equation of a curve is, as is well known, a relation among the coefficients in the equation of a variable line, which being fulfilled, the line must be a tangent to the curve.

Fig. 1.



Let O be the origin, OX, OY the axes; and let a variable line MN in any of its positions make an intercept ν on OX and an angle ϕ with it; then the equation of the line is

$$x+y\cot\phi-\nu=0,$$

and ν and ϕ, the quantities which determine the position of the line, may be called its coordinates. From this it follows that any relation between ν and ϕ, such as

$$\nu=f(\phi), \quad \ldots \quad (1)$$

will be the tangential equation of a curve which is the envelope of the line.

This form of equation will be the special subject of this paper. Occasionally our investigations will embrace collateral subjects, when their importance will be such as to justify the digression.

It will be seen that our form of equation admits of easy transformation into all the known forms of equation; that it adapts itself with great facility to the various problems of the Integral Calculus relating to curves, such as Rectification, Curvature, Involutes, &c., and gives its results in very simple forms.

In most of the methods of Modern Geometry, such as Pedals, Parallel Curves, Reciprocation, &c., it solves in a very simple manner problems that are very difficult by any other method. I have illustrated it throughout by numerous examples, most of which are of historical interest. Some of the problems discussed are, I believe, now solved for the first time, among which I may mention the rectification of Bicircular Quartics by Elliptic Functions. To this outline of the subject of this paper I may add that the form of equation is suggested by *Kinematics*. For if we differentiate the equation $\nu=f(\phi)$ with respect to a variable t (denoting the time) we get

$$\frac{d\nu}{dt}=f'(\phi)\frac{d\phi}{dt}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Now if we suppose a rigid body to move so that a fixed point in it, say the centre of gravity, describes a right line, then $\frac{d\nu}{dt}$ will be the linear velocity of the centre of gravity, and $\frac{d\phi}{dt}$ will be the angular velocity with which the body revolves round the same point. Then the equation (2) will be the most general equation of the motion of such a body. It gives linear velocity divided by angular velocity as a function of the angle through which the body has rotated. From this it will be seen that some of our results will have a physical as well as a purely mathematical interest. With these remarks we proceed to the subject of the paper.

CHAPTER I.

Section I.—*Transformation of Cartesian into Tangential Equations.*

2. *Definition.*—We shall find it convenient to call the line OX, on which the variable line makes the intercept ν, and with which it makes the angle ϕ, the *director* line.

3. If the Cartesian equation of a curve be $U=0$, we can by the usual process find the condition that the line $x+y\cot\phi-\nu=0$ touches it; this condition will be our tangential equation. For this purpose the equation of the line may be written in the form

$$y=(\nu-x)t, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where t denotes $\tan\phi$; and eliminating y between this and the equation $U=0$, we shall have an equation in x of the form

$$(A_0, A_1, A_2, \ldots A_n)(x-1)^n=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The discriminant of this will be the tangential equation required. It can be transformed into the usual form of tangential equation by changing ν into $-\frac{\nu}{\lambda}$ and t into $\frac{\lambda}{\mu}$. This is evident by comparing the equations

$$x+y\cot\phi-\nu=0, \quad \lambda x+\mu y+\nu=0.$$

Cor. The usual form of tangential equation can be transformed into our form as follows:—Let

$$\psi(\lambda, \mu, \nu)=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

be the tangential equation, say of the nth degree; divide by λ^n, and change $\frac{\nu}{\lambda}$ into $-\nu$, and $\frac{\mu}{\lambda}$ into $\cot\phi$.

4. The coefficients in equation (4) are deserving of notice. Equated to zero, they are the tangential equations of curves which possess interesting geometrical relations to the curve U. For the purpose of examining their properties, let the curve U be given by the equation

$$(a_0, a_1, a_2 \ldots a_n \between x, y)^n + n(b_1 b_2 \ldots b_n \between x, y)^{n-1} + \frac{n.n-1}{\underline{|2}}(c_2, c_3 \ldots c_n \between x, y)^{n-3} + \&c. = 0; \qquad (6)$$

then substituting in this the value of y from equation (3), and equating the result with equation (4), we get the following system of identities:—

$$\left.\begin{aligned}
A_0 &\equiv (a_0, a_1, a_2 \ldots a_n \between 1, -t)^n = 0,\\
A_1 &\equiv \nu t(a_1, a_2 \ldots a_n \between 1, -t)^{n-1}\\
&\quad + (b_1, b_2, \ldots b_n \between 1, -t)^{n-1} = 0,\\
A_2 &\equiv \nu^2 t^2 (a_2, a_3, \ldots a_n \between 1, -t)^{n-2}\\
&\quad + 2\nu t(b_2, b_3 \ldots b_n \between 1, -t)^{n-2}\\
&\quad + (c_2, c_3, \ldots c_n \between 1, -t)^{n-2} = 0,\\
A_3 &\equiv \nu^3 t^3 (a_3, a_4 \ldots a_n \between 1, -t)^{n-3}\\
&\quad + 3\nu^2 t^2 (b_3, b_4 \ldots b_n \between 1, -t)^{n-3}\\
&\quad + 3\nu t(c_3, c_4 \ldots c_n \between 1, -t)^{n-3}\\
&\quad\quad (d_3, d_4 \ldots d_n \between 1, -t)^{n-3} = 0,\\
&\&c. \qquad \&c. \qquad \&c.
\end{aligned}\right\} \qquad (7)$$

5. The system of identities (7) are remarkable for their symmetry, the equation $A_0=0$ being independent of all but the coefficients of the highest powers of x and y, A_1 of all the homogeneous terms lower than the $(n-1)$th in x and y, &c. Transformed into the usual form of tangential coordinates, they become

$$\left.\begin{aligned}
A_0 &\equiv (a_0, a_1, a_2 \ldots a_n \between \mu, -\lambda)^n = 0,\\
A_1 &\equiv \nu(a_1, a_2, a_3 \ldots a_n \between \mu, -\lambda)^{n-1}\\
&\quad - \mu(b_1, b_2, b_3 \ldots b_n \between \mu, -\lambda)^{n-1} = 0,\\
A_2 &\equiv \nu^2 (a_2, a_3 \ldots a_n \between \mu, -\lambda)^{n-2}\\
&\quad - 2\mu\nu(b_2, b_3 \ldots b_n \between \mu, -\lambda)^{n-2}\\
&\quad + \mu^2 (c_2, c_3 \ldots b_n \between \mu, -\lambda)^{n-2} = 0,\\
A_3 &\equiv \nu^3 (a_3, a_4 \ldots a_n \between \mu, -\lambda)^{n-3}\\
&\quad - 3\mu\nu^2 (b_3, b_4 \ldots b_n \between \mu, -\lambda)^{n-3}\\
&\quad + 3\mu^2\nu(c_3, c_4 \ldots c_n \between \mu, -\lambda)^{n-3}\\
&\quad - \mu^3 (d_3, d_4 \ldots d_n \between \mu, -\lambda)^{n-3} = 0.
\end{aligned}\right\} \qquad (8)$$

6. We shall now examine the geometrical interpretation of the equations (8), first, for the sake of illustration, in special cases, and then we shall give the general results.

We may remark in passing that all the contravariants of curves can be expressed in terms of these tangential curves; for instance, if U be a cubic, the envelope of the line which cuts it in three points, whose distances are in arithmetical progression, is the curve

$$A_0^2 A_3 + 2A_1^3 - 3A_0 A_1 A_2 = 0; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

and if U be a quartic, the envelope of the line which it cuts harmonically is the determinant

$$\begin{vmatrix} A_0, & A_1, & A_2 \\ A_1, & A_2, & A_3 \\ A_2, & A_3, & A_4 \end{vmatrix} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

7. Let the curve U=0 be a conic, then the equation (4) becomes

$$(A_0, A_1, A_2 \between x, 1)^2 = 0.$$

Now if $A_1=0$, it is evident the line $y=(\nu-x)t$ will cut the curve in two points, which are equally distant from the axis of y; but when $n=2$, A_1 becomes

$$\nu(a_1\mu - a_2\lambda) - \mu(b_1\mu - b_2\lambda) = 0; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

that is, a conic section. Hence we have the following theorems, the second of which is the projection of the first, and follows from the equation in λ, μ, ν, as the first does from the corresponding one in ν and t:—

1st. *If a variable line intersect a conic section, and if the locus of its middle point be a right line, its envelope is a conic section.*

2nd. *If a variable line be cut harmonically by a conic section and a pair of lines, its envelope is a conic section touching the pair of lines.*

8. Let U be the cubic

$$(a_0, a_1, a_2, a_3 \between x, y)^3 + 3(b_1, b_2, b_3 \between x, y)^2 + 3(c_2, c_3 \between x, y) + d_3 = 0, \quad . \quad . \quad (12)$$

and the curve A_1 will be

$$\nu t(a_1, a_2, a_3 \between 1-t)^2 + (b_1, b_2, b_3 \between 1-t)^2 = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

This equation is the condition that the locus of the mean centre of the points where the line $x+y\cot\phi-\nu$ meets the curve is the axis of y; and since the axis of y may be any line, we have the following theorem:—*If a variable line intersect a cubic in such a manner that the locus of the mean centre of the points where it meets the cubic is a right line, its envelope is a curve of the third class.*

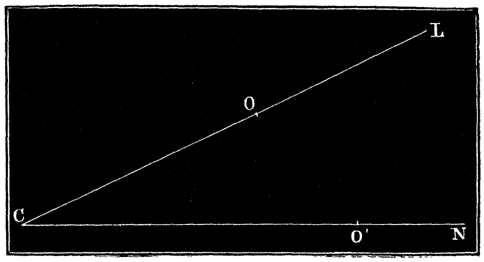
9. The equation (13), expressed in the usual notation of tangential coordinates, is

$$\nu(a_1, a_2, a_3 \between \mu-\lambda)^2 - \mu(b_1, b_2, b_3 \between \mu-\lambda)^2 = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

This is the analytical statement of the theorem we get by projecting that of the last article; and since in projection the line at infinity becomes a finite line, it may be expressed as follows:—*Being given a curve of the third degree*, $U=0$, *and two fixed lines* L *and* N; *then if* O, O′ *be two variable points on* L *and* N *respectively, such that the polar line of* O *with respect to* U *passes through* O′, *the envelope of the line* OO′ *is a curve of the third class.*

10. Since the highest power of ν contained in equation (14) is the first, the tangential cubic A_1 which it represents has one double tangent, namely the line joining the points λ and μ, which we may call the line $(\lambda\mu)$. Similarly the line $(\mu\nu)$ is a single tangent. The same thing can be shown geometrically, as follows:—Let the lines L and N intersect in C, then C is the point whose equation is $\mu=0$. Now since the polar line of O passes through O′, then the polar conic of O′ passes through O; but this conic intersects the line L in two points, and the line joining O′ to each of them is a tangent to A_1. Hence from any point of the line CN can in general two tangents be drawn to A_1; and we shall see immediately that CN itself is a tangent. This agrees with the fact of the curve being of the third class. Let the polar conic of C intersect CL in the points Ω, Ω', then the lines $C\Omega$, $C\Omega'$ are tangents to A_1; in other words, CL is a double tangent, and it is plain that Ω, Ω' are its points of contact. Again, let the polar line of C intersect CN in H, then H is a point of contact, so that CN is a tangent.

Fig. 2.



11. Since the point O′ moves on CN, its polar conic will pass through four fixed points, namely, the four poles of CN with respect to U. Hence any line will be cut in involution by the polar conics of the points O′; and we have the following theorem:—*If from any three points in* CN *three pairs of tangents be drawn to* A_1, *these will meet its double tangent in six points in involution, and the two points of contact of the double tangent belong to the involution.*

12. We find the limiting points of the involution as follows:—Let the pole conic of the line CL with respect to U intersect CN in the points Σ, Σ'; then since the pole-conic is the locus of points whose polar conics touch CL, the polar conics of the points Σ, Σ' will touch CL. Let the points where they touch it be denoted by Δ, Δ', then Δ, Δ' will be the double points of the involution. *Or thus, the double points will be the points of contact of the two conics, which can be drawn through the four poles of* CN *to touch* CL.

13. From the last article, it is plain that each of the lines $\Sigma\Delta$, $\Sigma'\Delta'$ is a pair of coincident tangents to the curve A_1; and since CN is itself a tangent we see that from each of the points Σ, Σ' can be drawn only two tangents to A_1; but the curve is of the third class, therefore it must pass through Σ and Σ'. Hence we have the following theorem:—

The curve A_1 *intersects the line* CN *in the points where the pole-conic of* CL *meets it, and it touches* CN *in the point whose polar conic passes through* C.

14. The polar line of the point C with respect to U will cut CN at its point of contact with A_1. The same polar line will be a tangent to the pole-conic of CL, and will be the polar of the point C with respect to the polar conic of C. *Hence it will with* C *divide harmonically the segment of* CL *included between the points of contact with* A_1.

15. We can get the equation of the line of which A_1 is the envelope as follows:—Since $y=(\nu-x)t$ we have $\nu t=(y+xt)$; and substituting in equation (13) we get

$$a_3xt^3-(2a_2x-a_3y-b_3)t^2+(a_1x-2a_2y-2b_2)t+a_1y+b_1 \quad . \quad . \quad . \quad . \quad (15)$$

which is the required line, and the discriminant with respect to t will be Cartesian equation of A_1. This discriminant is

$$\left.\begin{array}{l} 27a_3^2x^2(a_1y+b_1)^2+4a_3x(a_1x-2a_2y-2b_2)^3 \\ \quad -4(a_1y+b_1)(2a_2x-a_3y-b_3)^3 \\ \quad -(2a_2x-a_3y-b_3)^2(a_1x-2a_2y-2b_2)^2 \\ \quad +18a_3x(2a_2x-a_3y-b_3)(a_1x-2a_2y-2b_2)(a_1y+b_1)=0. \end{array}\right\} \quad . \quad . \quad . \quad (16)$$

This equation is of the fourth degree, as it ought, since the curve has a double tangent.

16. If we denote the equation (15) by T, and since a cusp is a point at which three consecutive tangents intersect, the conditions that there shall be a cusp are that

$$T=0, \quad \frac{dT}{dt}=0, \quad \frac{d^2T}{dt^2}=0\,;$$

and eliminating x and y from these equations, we get the following determinant:—

$$\left|\begin{array}{lll} a_3t-2a_2\,, & a_3 \,, & b_3 \,, \\ a_1-2a_2t, & a_3t-2a_2\,, & b_3t-2b^2\,, \\ a_1t \,, & a_1-2a_2t, & b_1-2b_2t, \end{array}\right| =0 \quad . \quad . \quad . \quad . \quad . \quad (17)$$

This determinant is a cubic in t, showing that there are three cusps. The values of t, got from this equation, if substituted in equation (15), will give us the three cuspidal tangents.

17. If we denote the singularities by the following notation—

Class ν,	Degree μ,
Double tangents τ,	Double points δ,
Cusps $\varkappa$,	Points of inflection ι,

we have the singularities of the curve A_1 as follows:—

$$\left.\begin{array}{lll} \nu=3, & \mu=4, & \tau=1, \\ \varkappa=3, & \iota=0, & \delta=0, \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

Observation.—The curve A_2 for a cubic has properties similar to A_1. They differ only in that the lines CL and CN are interchanged, for CL is a single and CN a double tangent to A_2.

18. If U be the general curve of the nth degree, $A_1=0$ gives the following theorem:—

Given a curve of the n*th degree, and two lines* CL *and* CN, *then if* O, O′ *be two movable points on these lines, such that the polar line of* O *with respect to* U *may pass through* O′, *the envelope of* OO′ *will be a curve of the* n*th class, to which the line* CL *will be a multiple tangent of the order* $(n-1)$.

19. If in the equation for A_1 given in art. 4 we substitute for ν its value $y+xt$, as in art. (15), we shall find the equation of OO′ in the form

$$at^n+nbt^{n-1}+\frac{n.n-1}{\underline{|2}}ct^{n-2}+\&c. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

Hence (see SALMON's 'Higher Curves,' second edition, p. 66) we have

$$\left.\begin{array}{lll}\nu=n, & \mu=2(n-1), & \varkappa=3(n-2), \\ \delta=(n-2)(n-3), & \tau=\frac{1}{2}(n-1)(n-2), & \iota=0.\end{array}\right\} \quad . \quad . \quad . \quad . \quad (20)$$

All this will also follow from the propositions of the following articles, of which this and the preceding are special cases.

20. We will now examine the general case $A_m=0$.

The equation $A_m=0$ gives us the following theorem:—If $U=0$ *be a curve of the* n*th degree, and* CL, CN *two given lines, then if* O, O′ *be two points taken on these lines, such that the* m*th polar of* O *with respect to* U *passes through* O′, *then the envelope of* OO′ *is the curve of the* n*th class* $A_m=0$.

21. The curve A_m touches the line CL in $(n-m)$ points and CN in m points.

Demonstration.—Since the mth polar of O passes through O′, the $(n-m)$th polar of O′ passes through O. Hence we have two ways of generating the curve. Now let the point O′ move along CN until it becomes consecutive to C, and it is evident that the $(n-m)$ points in which its $(n-m)$th polar intersects CL will be points of contact of CL with A_m. In like manner the m points in which the mth polar of a point consecutive to C on the line CL intersects the line CN will be points of contact. Hence the proposition is proved.

Cor. The number of double tangents which A_m has $=\frac{n^2-n-2mn+2m^2}{2}$.

For the line CL is equivalent to

$$\frac{(n-m)(n-m-1)}{2} \text{ double tangents,}$$

and the line CN to

$$\frac{m(m-1)}{2} \text{ double tangents;}$$

∴ we have

$$2\tau=n^2-n-2mn+2m^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

22. The curve A_m is of the degree $2m(n-m)$.

Demonstration.—If O′ be any point on the line CN, then since the $(n-m)$th polar of O′ cuts CL in $(n-m)$ points, the lines drawn from O′ to these points will make up $(n-m)$ tangents, and the line CL itself counts for m tangents. Hence the n tangents which can be drawn from O′ are accounted for. Now if the point O′ itself be on the curve A_m, only $(n-1)$ tangents can be drawn from it, and two of the points in which the line CL is intersected by the polar curve of the $(n-m)$th degree must coincide, that is the polar curve must touch CL. Hence we have to find the points on CN whose polar curves of the $(n-m)$th degree, with respect to U, will touch CL. In order to find the number of solutions of this problem, we will use trilinear coordinates. Let (a, b, c) (a', b', c') be the coordinates of two fixed points on CN, then the coordinates of any variable point on it are $a+ka'$, $b+kb'$, $c+kc'$, and the polar curve of this point of the $(n-m)$th degree, with respect to U, is

$$\left\{(a+ka')\frac{d}{dx}+(b+kb')\frac{d}{dy}+(c+kc')\frac{d}{dz}\right\}^m U=0. \quad . \quad . \quad . \quad . \quad . \qquad (22)$$

Now this equation contains the variables in the degree $n-m$, and its coefficients contain k in the mth degree. Hence the condition that it will touch any given line will contain k in the degree $2m(n-m-1)$; and this is the number of points in which the curve A_m intersects the line CN, but it touches CN in m points; ∴ the total number of points in which the curve meets CN is $2m(n-m)$.

Hence the proposition is proved.

23. The following are the singularities for the curve A_m:—

$$\nu=n, \qquad \mu=2m(n-m), \qquad \iota=0.$$

$$\left.\begin{aligned} 2\tau&=n^2-n-2mn+2m^2,\\ \delta&=2m^2(n-m)^2-10m(n-m)+4n,\\ \varkappa&=6mn-6m^2-3n. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (23)$$

Cor. $\mu+2\tau=n^2-n$, and is therefore the same for the curves A_1, A_2, &c.; that is, it is independent of m.

Cor. 2. The curves A_m, A_{n-m} have the same singularities.

Examples.

(1) Find the tangential equation of the cuspidal cubic $ay^2=x^3$.

Eliminating y between this and the equation $y=(\nu-x)t$, we get

$$x^3-at^2x^2+2a\nu t^2x-a\nu^2t^2=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (24)$$

The discriminant of this is

$$\nu=\frac{4a}{27}t^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (25)$$

which is the required tangential equation.

In the usual notation this is

$$4a\lambda^3+27\mu^2\nu=0. \quad \ldots \quad (26)$$

The equation (24) shows that the sum of the x's of the points where any line cuts $ay^2=x^3$ is proportional to the square of the tangent of the angle which the line makes with the axis of x, and the sum of their reciprocals is proportional to the reciprocal of the intercept which the same line makes on the same axis.

Cor. It is evident that similar theorems hold for the curve $ay^{n-1}=x^n$.

(2) Let the curve be $x^3+y^3-3axy=0$.

The tangential equations are

$$\nu^4-(6a^2\cot\phi)\nu^2-4a^3(1+\cot^3\phi)\nu+3a^4\cot^2\phi=0, \quad \ldots \quad (27)$$

$$\nu^4-6a^2\lambda\mu\nu^2+4a^3(\lambda^3+\mu^3)\nu+3a^4\lambda^2\mu^2=0. \quad \ldots \quad (28)$$

(3) Find the tangential equations of the cissoid.

They are

$$(2a-\nu)^3=27a^2\nu\cot^2\phi, \quad \ldots \quad (29)$$

$$(2a\lambda+\nu)^3+27a^2\mu^2\nu=0. \quad \ldots \quad (30)$$

(4) Find the tangential equation A_1 for a cubic in its canonical form—that is, referred to its three chords of inflection as axes. This question is solved by supposing the coefficients in the equation (12) to vanish, except a_0, a_3, b_2, d_3; then equation (14) becomes the conic

$$a_3\nu\lambda+2b_2\mu^2=0, \quad \ldots \quad (31)$$

and the curve A_1 for the Hessian of the cubic is

$$3(a_0b_3d_3)^{\frac{1}{3}}b_2^2\nu\lambda=(a_0b_3d_3+2b_2^3)\mu^2, \quad \ldots \quad (32)$$

a curve which has double contact with the former.

(5) Find the equations of the curves A_1, A_2, A_3 for the trinodal quartic

$$(a, b, c, f, g, h\between x^{-1}, y^{-1}, z^{-1})^2.$$

$$A_1\equiv\mu^2\lambda(g\lambda-f\mu+c\nu)=0, \quad \ldots \quad (33)$$

$$A_3\equiv\mu^2\nu(a\lambda-h\mu+g\nu)=0, \quad \ldots \quad (34)$$

$$A_2\equiv(a, b, c, -f, 2g, -h\between\lambda, \mu, \nu)^2=0. \quad \ldots \quad (35)$$

(6) The points where the curve A_m intersects the line CN may be found as follows:—If a variable point moves along the line CL the envelope of its polar curve of the mth degree with respect to U will be a curve of the degree $2m(n-m-1)$ which will cut CN in the required points. Similarly the points where it cuts CL may be found.

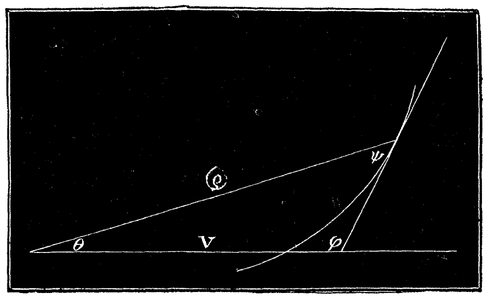
SECTION II.—*Transformation of Polar into Tangential Equations.*

24. The polar equation of a curve being given, to find its tangential equation.

Let the polar equation be $\varrho = F(\theta)$, then

$$\tan \psi = \frac{F(\theta)}{F'(\theta)}. \quad . \quad . \quad . \quad . \quad (1)$$

Fig. 3.



Also we have $\nu \sin \phi = \varrho \sin \psi$, that is, we have

$$\nu \sin \phi = F(\theta) \sin \psi \quad . \quad . \quad . \quad . \quad (2)$$

and

$$\theta + \phi + \psi = \pi. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Then eliminating θ and ψ between equations (1), (2), (3). The result will be the tangential equation.

Ex. Let the polar equation be

$$\varrho^m = a^m \sin m\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

We find, by taking logarithmic differentials,

$$\tan \psi = \tan m\theta;$$

$$\therefore \psi = m\theta,$$

$$\text{and } \nu^m \sin^m \phi = \varrho^m \sin^m \psi = a^m \sin^{m+1} \psi.$$

Hence the tangential equation is

$$\nu \sin \phi = a \left\{ \sin \frac{m(\pi - \phi)}{m+1} \right\}^{\frac{m+1}{m}},$$

or, putting ϕ in place of $\pi - \phi$,

$$\nu \sin \phi = a \left\{ \sin \frac{m\phi}{m+1} \right\}^{\frac{m+1}{m}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

25. The family of curves represented by equation (36) includes several important species. The following Table contains the principal, with their corresponding tangential equations.

Value of m.	Name of curve.	Tangential equation of curve.	
2	Lemniscate....	$\nu \sin \phi = a \left(\sin \frac{2\phi}{3} \right)^{\frac{3}{2}}$	(38)
-2	Equilateral hyp.	$\nu \sin \phi = a (\sin 2\phi)^{\frac{1}{2}}$	(39)
		or $\nu = a \sqrt{2} \cot \phi$	(40)
$-\frac{1}{2}$	Parabola	$\nu = -a \operatorname{cosec}^2 \phi$	(41)
		The parabola has another form of tangential equation, namely, $\nu = a \tan \phi$............	(42)
		The *director* line in this form of equation is the tangent at the vertex. In the other forms it is the axis.	
$\frac{1}{2}$	Cardioide	$\nu = a \sin^3 \left(\frac{\phi}{3} \right)$	(43)
1	Circle	$\nu \sin \phi = a \sin^2 \frac{1}{2} \phi$	(44)
		or $\nu = \frac{a}{2} \tan \frac{1}{2}\phi$	(45)

CHAPTER II.

Section I.—*Transformation of Tangential into Cartesian Equations.*

26. The tangential equations $\nu=f(\phi)$ of a curve being given, to find its Cartesian equations.

Consider two consecutive positions of the line $x+y\cot\phi-\nu=0$, such as LP, L′P intersecting in P; then P is the point of contact of LP with its envelope, and the diameter of the circle described about the infinitesimal triangle LL′P is evidently equal $\frac{\text{LL}'}{d\phi}=\frac{d\nu}{d\phi}=f'(\phi)$; $\therefore$ LQ$=f'(\phi)$. Hence PR$=f'(\phi)\sin^2\phi$ and

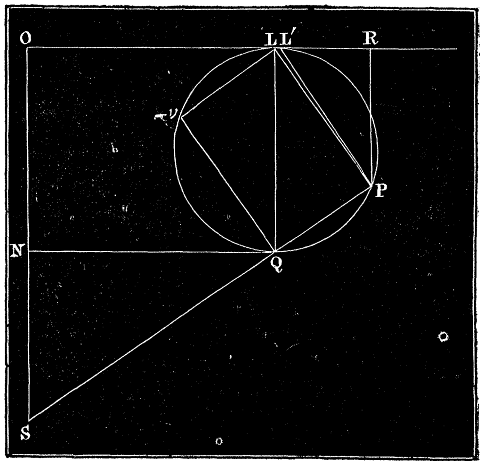
$$\text{OR}=\text{OL}+\text{LR}=f\phi+f'(\phi)\sin\phi\cos\phi\,;$$

$\therefore$ the Cartesian coordinates of the point P are

$$x=f(\phi)+f'(\phi)\sin\phi\cos\phi \quad . \quad . \quad (46)$$

$$y=\quad -f'(\phi)\sin^2\phi \quad . \quad . \quad . \quad . \quad (47)$$

Fig. 4.



These values can be got also by the analytical method of finding envelopes. For differentiating the equation $x+y\cot\phi-f\phi=0$ with respect to ϕ, we get $y=-f'\phi\sin^2\phi$; and substituting this in the equation $x+y\cot\phi-f\phi$, we get $x=f(\phi)+f'(\phi)\sin\phi\cos\phi$.

27. From the results of the last article we get

$$\text{the subtangent}=f'(\phi)\sin\phi\cos\phi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

$$\text{subnormal}=f'(\phi)\sin^2\phi\tan\phi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

28. If the movable line $x+y\cot\phi-\nu=0$ be a double tangent, it is evident that for the same values of ν and ϕ we must have two different values for LP, one value corresponding to each point of contact. Hence, since LP$=f'(\phi)\sin\phi$, we must have two different values for $f'(\phi)$. This will happen when $f'(\phi)$ is given in the form of a fraction whose numerator and denominator each vanish. Thus, suppose the equation of the curve to be given in the form

$$\text{F}(\nu,\phi)=0\,;$$

then we have

$$\frac{d\nu}{d\phi}=-\frac{d\text{F}}{d\phi}\div\frac{d\text{F}}{d\nu}\,;$$

and therefore the conditions for a double tangent are

$$\frac{d\text{F}}{d\phi}=0,\ \frac{d\text{F}}{d\nu}=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (50)$$

which correspond with the conditions in Cartesian coordinates for a double point.

29. Professor CAYLEY considers a "curve as described (see SALMON's 'Higher Curves,' second edition, p. 33) by a point which moves along a line at the same time that the line revolves round the point. There is, then, this peculiarity at a point of inflection, the line first becomes stationary and then reverses the sense of its motion." From this it follows that the line $x+y\cot\phi-\nu$ will cut off a maximum or minimum intercept on the *director* line when it passes through a point of inflection, and also it will make in the same case with the same line a maximum or minimum angle. Hence when

$$x+y\cot\phi-\nu$$

is an inflectional tangent,

$$f(\phi)=\text{maximum or minimum}$$

and

$$\phi=\text{maximum or minimum.}$$

Examples.

(1) If a line of constant length slide along two rectangular lines, to find its envelope. In this case we have evidently

$$\nu=a\cos\phi\,;\ \therefore\ f(\phi)=a\cos\phi.$$

Hence from equations (46), (47) we get

$$\left.\begin{aligned} x=a\cos^3\phi, \qquad y=a\sin^3\phi\,; \\ \therefore\ x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}} \end{aligned}\right\} \quad . \ . \ . \ . \ . \ . \ . \ . \quad (51)$$

(2) If from any point in an ellipse perpendiculars be let fall on the axes, find the envelope of the line joining their feet. In this case $f(\phi)=\dfrac{a^2}{\sqrt{a^2+b^2\tan^2\phi}}$, and the required equation is

$$\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}=1. \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (52)$$

(3) Let $\nu=k\tan^m\phi$; then if we put $\dfrac{(m+1)^{m+1}}{m^m}k=a$, we get the Cartesian equation

$$x^{m+1}=ay^m \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (53)$$

(4) If $\nu=c\left\{1+(\cot\phi)^{\frac{2}{m}}\right\}^{\frac{m}{2}}$, the Cartesian equation is

$$x^{\frac{2}{2-m}}+y^{\frac{2}{2-m}}=c^{\frac{2}{2-m}} \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (54)$$

Compare equation (51).

Cor. If in this example we substitute $-2n$ for m, we get

$$x^{\frac{1}{n+1}}+y^{\frac{1}{n+1}}=c^{\frac{1}{n+1}} \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (55)$$

as the Cartesian equation of the curve

$$\nu=c\left\{1+(\tan\phi)^{\frac{1}{n}}\right\}^{-n}.$$

(5) Let $\nu=\frac{c^2}{(a^2+b^2\tan^2\phi)^{\frac{1}{2}}}$, where $c^2=a^2-b^2$. This curve is the evolute of the ellipse.

The Cartesian equation is

$$(ax)^{\frac{2}{3}}+(by)^{\frac{2}{3}}=c^{\frac{4}{3}} \quad \ldots \quad (56)$$

(6) To find a curve in which the subnormal is constant, let the constant be $2a$; then from equation (49) we have

$$\left.\begin{aligned} f'(\phi)\sin^2\phi\tan\phi&=2a;\\ \therefore f(\phi)&=-a\operatorname{cosec}^2\phi,\end{aligned}\right\} \quad \ldots \quad (57)$$

which is the common parabola (see art. 25).

Cor. In like manner the curve in which the subtangent is constant is

$$\nu=a\log\tan\phi \quad \ldots \quad (58)$$

or

$$\frac{\mu}{\lambda}=e^{\frac{a\lambda}{\nu}} \quad \ldots \quad (59)$$

(7) If in fig. 4 (art. 26) PL be produced to meet OS in T, required to find the curve in which PL : LT in a given ratio, say n : 1. Here we have evidently

$$\frac{f'(\phi)\sin\phi\cos\phi}{f(\phi)}=n,$$

$$\therefore \log f(\phi)=\mathrm{C}+n\log\tan\phi,$$

$$\therefore f(\phi)=k\tan^n\phi, \qquad \text{if } k=e^{\mathrm{C}};$$

∴ the required curve is

$$\nu=k\tan^n\phi \quad \ldots \quad (60)$$

or

$$\mu^n\nu+k\lambda^{n+1}=0. \quad \ldots \quad (61)$$

Section II.—*Transformation of the Tangential into the Intrinsic Equation.*

30. If we differentiate the value of x given in art. 26, we get

$$\frac{dx}{d\phi}=2f'(\phi)\cos^2\phi+f''(\phi)\sin\phi\cos\phi;$$

but

$$\frac{dx}{d\phi}=\frac{dx}{ds}\times\frac{ds}{d\phi}=\cos\phi\frac{ds}{d\phi},$$

$$\therefore \frac{ds}{d\phi}=2f'(\phi)\cos\phi+f''(\phi)\sin\phi, \quad \ldots \quad (62)$$

$$\therefore s=f'(\phi)\sin\phi+\int f'(\phi)\cos\phi\, d\phi.$$

Hence if $\nu=f(\phi)$ be the tangential equation of a curve, its intrinsic equation is

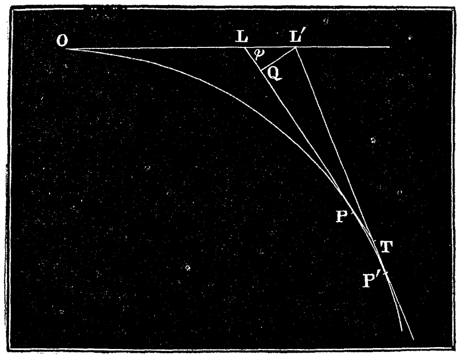
$$s=f'(\phi)\sin\phi+\int f'(\phi)\cos\phi\,d\phi. \qquad (63)$$

The result in equation (62) may be written in a form which in practice we shall find more useful. Thus

$$\frac{ds}{d\phi}=\frac{\frac{d}{d\phi}(f'(\phi)\sin^2\phi)}{\sin\phi}. \qquad (64)$$

31. Equation (63) may be established geometrically as follows:—Let LP, L′P′ be two consecutive positions of the movable line, P, P′ their points of contact with the envelope, and T their point of intersection. Let L′Q be a perpendicular on LP. Now PP′ is an element of the curve, and denoting it by ds, we have

Fig. 5.



$$ds=P'T+TP=P'L'-QP=P'L'-PL+LQ$$
$$=d(LP)+LL'\cos\phi=d(PL)+d\nu\cos\phi;$$
$$\therefore\ s=PL+\int d\nu\cos\phi$$
$$=f'(\phi)\sin\phi+\int f'(\phi)\cos\phi\,d\phi.$$

Which is the same result as before.

Cor. $$s=PL+\int PQ\,d\phi \text{ (see fig. art. 26)}; \qquad (65)$$

$$\therefore\ \frac{ds}{d\phi}=PQ+\frac{d}{d\phi}(PL). \qquad (66)$$

From this it will be seen that the triangle LPQ is an important one in this theory.

Observation.—The geometrical method of proof shows that this theorem holds even when the director line OL (see def. art. 2) is any plane curve; and we shall further on have to make use of this generalization.

32. Before giving examples of the process of this section we will give the following integral reduction.

To reduce $\int\frac{zdz}{\sqrt{1+z^3}}$ to the normal form of elliptic integrals.

Let $z=\sqrt{3}\cot^2\frac{1}{2}\theta-1$, and $\sqrt{1-\frac{2+\sqrt{3}}{4}\sin^2\theta}=\Delta(\theta)$. Then after some easy reductions, we get

$$\int\frac{zdz}{\sqrt{1+z^3}}=\frac{2}{(\sqrt{3}-1)\sqrt[4]{3}}\int\frac{d\theta}{\Delta\theta}-2\sqrt[4]{3}\int\frac{(1+\cos\theta)d\theta}{\sin^2\theta\,\Delta(\theta)}.$$

Now

$$\int\frac{\cos\theta\,d\theta}{\sin^2\theta\,.\,\Delta(\theta)}=-\frac{\Delta\theta}{\sin\theta},$$

and

$$\int\frac{d\phi}{\sin^2\theta\,.\,\Delta\theta}=F(k,\theta)-E(k,\theta)-\cot\theta\,.\,\Delta\theta,$$

where

$$k=\frac{\sqrt{3}+1}{2\sqrt{2}}.$$ (See DURÈGE, 'Theorie der elliptischen Functionen.')

Hence

$$\int\frac{zdz}{\sqrt{1+z^3}}=\frac{1-3^{\frac{1}{2}}}{3^{\frac{1}{4}}}\,\mathrm{F}(k,\theta)+2\,.\,3^{\frac{1}{4}}\mathrm{E}(k,\theta)+2\,.\,3^{\frac{1}{4}}\cot\tfrac{1}{2}\theta\,.\,\Delta(\theta). \qquad (67)$$

Examples.

(1) Let the tangential equation be $\nu=k\tan^n\theta$, to find the intrinsic equation.

Here we have $f(\phi)=k\tan^n\phi$;

$$\therefore\ f'(\phi)\sin^2\phi=nk\tan^{n+1}\phi.$$

Hence from equation (64) $\frac{ds}{d\phi}=n(n+1)k\tan^{n-1}\phi\sec^3\phi$;

$$\therefore\ s=n(n+1)k\int\tan^{n-1}\phi\sec^3\phi\,d\phi. \qquad (68)$$

We can get a formula of reduction for this integral as follows:—Put $\mathrm{P}=\tan^{n-2}\phi\sec^3\phi$, then by differentiation and reduction,

$$\frac{d\mathrm{P}}{d\phi}=(n+1)\tan^{n-1}(\phi)\sec^3\phi+(n-2)\tan^{n-3}\phi\sec^2\phi;$$

$$\therefore\ \int\tan^{n-1}(\phi)\sec^3\phi\,d\phi=\frac{\tan^{n-2}\phi\,.\,\sec^3\phi}{n+1}-\frac{n-2}{n+1}\int\tan^{n-3}\phi\sec^3\phi\,d\phi, \qquad (69)$$

which is the required formula; and the integral will ultimately depend on known forms.

(2) Let the tangential equation be that of the evolute of an ellipse,

$$\nu=\frac{c^2}{\sqrt{a^2+b^2\tan^2\phi}}.$$

We have

$$f'(\phi)\sin^2\phi=\frac{-b^2c^2\tan^3\phi}{(a^2+b^2\tan^2\phi)^{\frac{3}{2}}}.$$

Hence, from equation (64),

$$\frac{ds}{d\phi}=-\frac{3b^2}{a}\cdot\frac{\sin\phi\cos\phi}{\Delta^5(\phi)},$$

where

$$\Delta(\phi)=\sqrt{1-e^2\sin^2\phi};$$

$$\therefore\ s=\frac{b^2}{a}\cdot\frac{1}{\Delta^3(\phi)}; \qquad (70)$$

and this is the intrinsic equation of the evolute of an ellipse.

(3) To find a curve in which the radius of curvature bears a constant ratio to the normal, the given condition is expressed by the equation

$$\frac{f'(\phi)\sin^2\phi}{\cos\phi}=\frac{\frac{1}{a}\frac{d}{d\phi}\left(f'(\phi)\sin^2\phi\right)}{\sin\phi};$$

$$\therefore \frac{\frac{d}{d\phi}(f'(\phi)\sin^2\phi)}{f'(\phi)\sin^2\phi}=\frac{a\sin\phi}{\cos\phi},$$

$$\therefore f'(\phi)\sin^2\phi=\frac{e^c}{(\cos\phi)^a},$$

$$\therefore f(\phi)=e^c\int\frac{d\phi}{\sin^2\phi\,(\cos\phi)^a},$$

$$\therefore \nu=e^c\int\frac{d\phi}{\sin^2\phi\,.\,(\cos\phi)^a}. \qquad (71)$$

If a be any even integer the integration on the right-hand side can be performed. See WILLIAMSON'S 'Integral Calculus.'

(4) To find a curve whose tangential equation is the same as its intrinsic equation.

Here we have $f'(\phi)\sin\phi+\int f'(\phi)\cos\phi\,d\phi=f(\phi)$,

$$\text{or } \frac{d}{d\phi}(f'(\phi)\sin^2\phi)=f'(\phi)\sin\phi;$$

$$\therefore f'(\phi)\sin^2\phi=\text{C}\tan\tfrac{1}{2}\phi,$$

$$\therefore f(\phi)=a\{\tan^2\tfrac{1}{2}\phi+\log\tan^2\tfrac{1}{2}\phi)\}, \qquad (72)$$

where a stands for $\frac{\text{C}}{4}$.

(5) If the tangential equation of a curve be $\nu=f(\phi)$, and the intrinsic equation $s=f'(\phi)$, find the curve.

We have $f'(\phi)\sin\phi+\int f'(\phi)\cos\phi\,d\phi=f'(\phi)$;

$$\therefore \frac{f''(\phi)}{f'(\phi)}=\frac{2\cos\phi}{1-\sin\phi}.$$

Hence $f'(\phi)=\frac{\text{C}_1}{(1-\sin\phi)^2}$;

$$\therefore f(\phi)=\text{C}_2+\text{C}_1\left\{\frac{\cot\left(\frac{\pi}{4}-\frac{\phi}{2}\right)}{2}+\frac{\cot\left(\frac{\pi}{4}-\frac{\phi}{2}\right)^3}{6}\right\} \qquad (73)$$

(6) To find the intrinsic equation of the curve

$$\nu^{\frac{1}{3}}=1+(\cot\phi)^{\frac{1}{3}}.$$

This is the curve whose ordinary tangential equation is

$$\lambda^{\frac{1}{3}}+\mu^{\frac{1}{3}}+\nu^{\frac{1}{3}}=0,$$

or the curve whose trilinear equation is

$$\alpha^{-\frac{1}{2}}+\beta^{-\frac{1}{2}}+\gamma^{-\frac{1}{2}}=0.$$

We have $f'(\phi)\sin^2\phi=-(\tan^{\frac{1}{3}}\phi+1)^2$;

$$\therefore \frac{d}{d\phi}(f'(\phi)\sin^2\phi)=-\tfrac{2}{3}(\cot^{\frac{1}{3}}\phi+\cot^{\frac{2}{3}}\phi)\sec^2\phi;$$

∴ from equation (64) we have

$$s=-\tfrac{2}{3}\int\cot^{\frac{1}{3}}\phi\sec^2\phi\,\mathrm{cosec}\,\phi\,d\phi-\tfrac{2}{3}\int\cot^{\frac{2}{3}}\phi\sec^2\phi\,\mathrm{cosec}\,\phi\,d\phi. \quad . \quad . \quad . \quad (74)$$

We reduce the first of these integrals to the normal form of elliptic integrals as follows:—

Let $z=\cot^{\frac{2}{3}}\phi$, and we find

$$\int\cot^{\frac{1}{3}}\phi\sec^2\phi\,\mathrm{cosec}\,\phi\,d\phi=-\tfrac{3}{2}\int\frac{dz}{z^2\sqrt{1+z^3}}-\tfrac{3}{2}\int\frac{zdz}{\sqrt{1+z^3}}$$

$$=\frac{\tfrac{3}{2}\sqrt{1+z^3}}{z}-\tfrac{9}{4}\int\frac{zdz}{\sqrt{1+z^3}}$$

$$=\tfrac{3}{2}\cdot\frac{1}{\sin^{\frac{1}{3}}\phi\cos^{\frac{2}{3}}\phi}-\tfrac{9}{4}\int\frac{zdz}{\sqrt{1+z^3}}$$

$$=\tfrac{3}{2}\cdot\frac{1}{\sin^{\frac{1}{3}}\phi\cos^{\frac{2}{3}}\phi}-\frac{3^{\frac{5}{4}}}{2}\cot\tfrac{1}{2}\theta\,.\,\Delta\theta+\frac{3^{\frac{3}{4}}(3^{\frac{1}{2}}-1)}{2}\,\mathrm{F}(k,\theta)-\frac{3^{\frac{5}{4}}}{2}\,\mathrm{E}(k,\theta), \quad . \quad . \quad . \quad (75)$$

where θ and ϕ are connected by the equation

$$\cos\theta=\frac{\cos^{\frac{2}{3}}\phi-(\sqrt{3}-1)\sin^{\frac{2}{3}}\phi}{\cos^{\frac{2}{3}}\phi+(\sqrt{3}+1)\sin^{\frac{2}{3}}\phi}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (76)$$

The second integral in equation (74) may be derived from the first by changing the sign and putting $\left(\frac{\pi}{2}-\phi\right)$ for ϕ. Hence we have at once

$$\int\cot^{\frac{2}{3}}\phi\sec^2\phi\,\mathrm{cosec}\,\phi\,d\phi$$

$$=-\tfrac{3}{2}\,\frac{1}{\sin^{\frac{2}{3}}\phi\cos^{\frac{1}{3}}\phi}+\frac{3^{\frac{5}{4}}}{2}\cot\tfrac{1}{2}\theta'\,.\,\Delta\theta-\frac{3^{\frac{3}{4}}(3^{\frac{1}{2}}-1)}{2}\,\mathrm{F}(k,\theta')+\frac{3^{\frac{5}{4}}}{2}\,\mathrm{E}(k,\theta'), \quad . \quad . \quad (77)$$

where θ' is given by the equation

$$\cos\theta'=\frac{\sin^{\frac{2}{3}}\phi-(\sqrt{3}-1)\cos^{\frac{2}{3}}\phi}{\sin^{\frac{2}{3}}\phi+(\sqrt{3}+1)\cos^{\frac{2}{3}}\phi}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (78)$$

and substituting from equations (75) (77) in (74), we get the required intrinsic equation

$$\left.\begin{aligned} s=&\frac{\cos^{\frac{1}{3}}\phi-\sin^{\frac{1}{3}}\phi}{\sin^{\frac{2}{3}}\phi\cos^{\frac{2}{3}}\phi}+3^{\frac{5}{4}}\{\cot\tfrac{1}{2}\theta\,.\,\Delta(\theta)-\cot\tfrac{1}{2}\theta'\,.\,\Delta\theta'\}\\ &-\frac{3^{\frac{3}{4}}(3^{\frac{1}{2}}-1)}{2}\{\mathrm{F}(k,\theta)-\mathrm{F}(k,\theta')\}\\ &+3^{\frac{5}{4}}\{\mathrm{E}(k,\theta)-\mathrm{E}(k,\theta')\},\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (79)$$

where $k=\dfrac{\sqrt{3}+1}{2\sqrt{2}}$, and θ, θ' are given by the equations (76), (78).

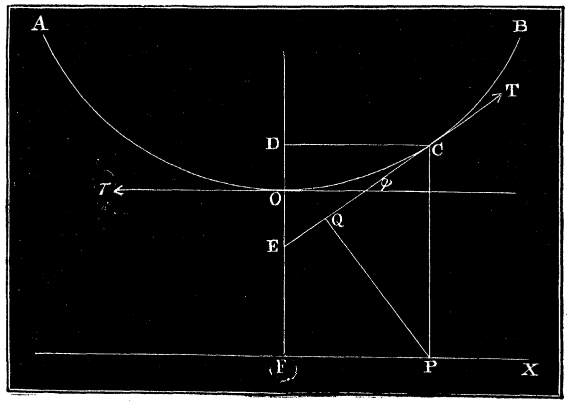
Section III.—*Transformation of the Intrinsic into the Tangential Equation.*

33. We shall have much use to make of the intrinsic equation of the catenary in this

and in subsequent sections; for this reason, and also on account of its extremely elementary character, we give here an investigation of the leading properties of that curve.

Let O be the lowest point of a uniform string AOB, suspended at the points A and B, and let the tension at O be denoted by τ, and at any other point C by T. Then if we consider the equilibrium of the portion O C we find that the forces acting on it are τ, T, and its own weight W; and these are parallel respectively to the sides of the triangle C D E. Hence, by the property of the triangle of forces,

Fig. 6.



$$\frac{\mathrm{W}}{\tau} = \tan \mathrm{E\,C\,D} = \tan \phi,$$

where ϕ is the angle which the tangent at C makes with the tangent at O. Now if s be the length of O C and c the length of a portion whose weight is equal to τ, we have, since the string is uniform,

$$\frac{s}{c} = \frac{\mathrm{W}}{\tau};$$

$$\therefore s = c \tan \phi. \quad (80)$$

34. The equation $s = c \tan \phi$, which we have just obtained, is the intrinsic equation of the catenary; we get the Cartesian equation from it as follows:—Make O F $= c$, and draw FX parallel to C D. Then we shall take these lines as axes. Now let the coordinates of the point C be denoted by x and y, and we have

$$\frac{dy}{ds} = \sin \phi; \text{ but } ds = c \sec^2 \phi \, d\phi, \text{ equation (80)},$$

$$\therefore y = c \sec \phi.$$

Again, we have

$$\frac{dx}{ds} = \cos \phi,$$

$$\therefore dx = c \sec \phi \, d\phi,$$

and

$$x = c \log (\sec \phi + \tan \phi),$$

$$\therefore e^{\frac{x}{c}} = e^{\frac{x}{c}} = 2 \sec \phi.$$

Hence

$$y = \frac{c}{2}\left(e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right); \quad (81)$$

and this is the Cartesian equation of the catenary.

35. From the value

$$x = c \log (\sec \phi + \tan \phi)$$

we get

$$e^{\frac{x}{c}} - e^{-\frac{x}{c}} = 2 \tan \phi.$$

Hence, from equation,

$$s = \frac{c}{2}\left(e^{\frac{x}{c}} - e^{-\frac{x}{c}}\right) \quad \ldots \ldots \quad (82)$$

36. If from the point P we let fall the perpendicular PQ on the tangent at C, we have evidently

$$PQ = y \cos \phi \text{ ; but } y = c \sec \phi,$$

$$\therefore PQ = c. \quad \ldots \ldots \quad (83)$$

Again we have

$$\frac{CQ}{PQ} = \tan \phi,$$

$$\therefore CQ = s. \quad \ldots \ldots \quad (84)$$

Hence the locus of the point Q is the involute of the catenary.

37. From the diagram we have

$$\frac{T}{W} = \operatorname{cosec} \phi$$

and

$$\frac{CP}{CQ} = \operatorname{cosec} \phi = \frac{y}{s};$$

and since s is the length of a portion of the string whose weight is W, y is the length of a portion whose weight is T.

38. The Intrinsic Equation of a curve being given, to find its Tangential Equation.

This problem is the converse of the one solved in art. 30, Section II.

Let $s = F(\phi)$ be the given intrinsic equation,

$$\therefore \frac{ds}{d\phi} = F'(\phi).$$

Hence from equation (64) we have

$$\frac{d}{d\phi}\left(f'(\phi) \sin^2 \phi\right) = F'(\phi) \sin \phi ;$$

$$\therefore f(\phi) = \int \operatorname{cosec}^2 \phi \left\{ \int F'(\phi) \sin \phi d\phi \right\} d\phi.$$

Hence the required tangential equation is

$$\nu = \int \operatorname{cosec}^2 \phi \left\{ \int F'(\phi) \sin \phi d\phi \right\} d\phi. \quad \ldots \ldots \quad (85)$$

Examples.

(1) Find the tangential equation of the catenary.

Here $\quad F(\phi) = c \tan \phi.$ See equation (80).

Hence
$$\int F'(\phi)\sin\phi d\phi = c\sec\phi,$$

$$\therefore \nu = c\left\{\log\tan\left(\frac{\pi}{4}+\frac{\phi}{2}\right)-\operatorname{cosec}\phi\right\}+c \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (86)$$

(2) Find the tangential equation of the involute of the catenary; that is, the tangential equation of the locus of the point Q (see fig. art. 33). The intrinsic equation of the involute of the catenary is

$$s = c\int\tan\phi\, d\phi.$$

Hence we have
$$F(\phi) = c\int\tan\phi d\phi,$$

$$\therefore \int F'(\phi)\sin\phi d\phi = c\{\log(\sec\phi+\tan\phi)-\sin\phi\}.$$

Hence
$$\nu = c\int\frac{\log(\sec\phi+\tan\phi)}{\sin^2\phi} - c\int\frac{d\phi}{\sin\phi};$$

and integrating the first integral by parts, we find it equal

$$-\cot\phi\,.\,\log\{\sec\phi+\tan\phi\}+\int\frac{d\phi}{\sin\phi}.$$

Hence
$$\nu = C - c\cot\phi\,.\,\log(\sec\phi+\tan\phi),$$

where C is the constant of integration, which is evidently equal to c; therefore the required tangential equation is

$$\nu = c\left\{1-\frac{\log(\sec\phi+\tan\phi)}{\tan\phi}\right\}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (87)$$

(3) Let the intrinsic equation be $s = a\sin n\phi$. Then we find

$$\nu = \frac{2na}{n^2-1}\sin^2\frac{n\phi}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (88)$$

This formula fails when $n=1$; but in that case we have $F(\phi) = a\sin\phi$, and we find

$$\nu = \frac{a\phi}{2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (89)$$

(4) Find the equation of a curve, being given

$$\nu = ns.$$

Here we have, if
$$\nu = f(\phi), \quad s = nf(\phi);$$

$$\therefore \frac{d}{d\phi}(f'(\phi)\sin^2\phi) = nf'(\phi)\sin\phi. \quad \text{See equation (64).}$$

$$\therefore f'(\phi)\sin^2\phi = C(\tan\tfrac{1}{2}\phi)^n,$$

$$\therefore f(\phi) = \frac{C}{2}\left\{\frac{(\tan\frac{1}{2}\phi)^{n-1}}{n-1}+\frac{(\tan\frac{1}{2}\phi)^{n+1}}{n+1}\right\},$$

or, since C is an arbitrary constant,

$$f\phi = C\left\{\frac{(\tan\frac{1}{2}\phi)^{n-1}}{n-1} + \frac{(\tan\frac{1}{2}\phi)^{n+1}}{n+1}\right\}. \quad \ldots \ldots \quad (90)$$

(5) It is required to find the equation of two curves A and B, which are so related that the Tangential Equation of A is the same as the Intrinsic of B, and the Tangential of B the same as the Intrinsic of A.

Let $\nu = F(\phi)$ be the curve A,

$\nu = f(\phi)$ „ „ B.

Then by the first condition we have

$$F'(\phi)\sin\phi + \int F'(\phi)\cos\phi d\phi$$

$$= f'(\phi)\sin 2\phi + f''(\phi)\sin^2\phi + 2\int f'(\phi)\cos^2\phi d\phi + \int f''(\phi)\sin\phi\cos\phi d\phi,$$

and by the second condition

$$f(\phi) = F'(\phi)\sin\phi + \int F'(\phi)\cos\phi d\phi\,;$$

$$\therefore f(\phi) = f'(\phi)\sin 2\phi + f''(\phi)\sin^2\phi + 2\int f'(\phi)\cos^2\phi + \int f''(\phi)\sin\phi\cos\phi d\phi.$$

And by differentiating and some easy reduction we get

$$5f''(\phi)\sin 2\phi + 6f'(\phi)\cos 2\phi - f'''(\phi)\cos 2\phi + f'''(\phi) = 0,$$

or

$$\frac{d}{d\phi}\{3f'(\phi)\sin 2\phi - f''(\phi)\cos 2\phi + f''(\phi)\} = 0.$$

Hence

$$3f'(\phi)\sin 2\phi + f''(\phi)\{1 - \cos 2\phi\} = 4C_1,$$

the multiple 4 being put to the arbitrary constant in order to avoid fractions;

$$\therefore 3f'(\phi)\cos\phi + f''(\phi)\sin\phi = \frac{2C_1}{\sin\phi}. \quad \ldots \ldots \quad (\alpha)$$

This may be written

$$\left.\begin{aligned} &\frac{d}{d\phi}(f'(\phi)\sin^3\phi) = 2C_1\sin\phi, \\ &\therefore f'(\phi)\sin^3\phi = -2C_1\cos\phi + 2C_2, \end{aligned}\right\} \quad \ldots \ldots \quad (\beta)$$

$2C_2$ being an arbitrary constant;

$$\therefore f(\phi) = \frac{C_1 - C_2\cos\phi}{\sin^2\phi} + C_2\log\tan\tfrac{1}{2}\phi + C_3.$$

This is the tangential equation of the curve B.

To find the equation of A we have, from equation (β),

$$f'(\phi)\cos\phi = -\frac{2C_1\cos^2\phi}{\sin^3\phi} + \frac{2C_2\cos\phi}{\sin^3\phi},$$

and subtracting this from equation (α) we get

$$2f'(\phi)\cos\phi+f''(\phi)\sin\phi=\frac{2C_1}{\sin^3\phi}-\frac{2C_2\cos\phi}{\sin^3\phi};$$

$$\therefore \frac{dS}{d\phi}=\frac{2C_1}{\sin^3\phi}-\frac{2C_2\cos\phi}{\sin^3\phi},$$

$$\therefore S=\frac{C_1\cos\phi-C_2}{\sin^2\phi}+C_1\log\tan\tfrac{1}{2}\phi+C_4.$$

This is the intrinsic equation of B, and therefore the tangential equations of A and B respectively are

$$\nu=\frac{C_1\cos\phi-C_2}{\sin^2\phi}+C_1\log\tan\tfrac{1}{2}\phi+C_4 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (91)$$

$$\nu=\frac{C_1-C_2\cos\phi}{\sin^2\phi}+C_2\log\tan\tfrac{1}{2}\phi+C_3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (92)$$

CHAPTER III.

SECTION I.—*Evolutes.*

39. If the tangential equation of a curve be

$$\nu=f(\phi),$$

we have proved, in art. 30,

$$\frac{ds}{d\phi}=\frac{\frac{d}{d\phi}(f'(\phi)\sin^2\phi)}{\sin\phi}.$$

Hence if ϱ denote the radius of curvature, we have

$$\varrho\sin\phi=\frac{d}{d\phi}(f'(\phi)\sin^2\phi);$$

$\therefore$ if $\nu=f(\phi)$ be the tangential equation of a curve, the intrinsic equation of its evolute is

$$s\sin\phi=\frac{d}{d\phi}(f'(\phi)\sin^2\phi) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (93)$$

40. If our movable line had been given by the equation

$$y=x\tan\phi+f(\phi),$$

we get in the usual manner

$$x=-f'(\phi)\cos^2\phi, \quad y=f(\phi)-f'(\phi)\sin\phi\cos\phi.$$

Hence

$$\frac{dx}{d\phi}=\cos\phi(2f'(\phi)\sin\phi-f''(\phi)\cos\phi),$$

$$\frac{dy}{d\phi}=\sin\phi(2f'(\phi)\sin\phi-f''(\phi)\cos\phi);$$

$$\therefore \frac{ds}{d\phi} = 2f'(\phi)\sin\phi - f''(\phi)\cos\phi\,;$$

and, as in the last art., we find

$$s\cos\phi = -\frac{d}{d\phi}(f'(\phi)\cos^2\phi. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (94)$$

41. The Tangential Equation of a Curve being given, to find the Tangential Equation of its Evolute. This problem is solved by articles 38 and 39.

For if $\nu = f(\phi)$ be the tangential equation, the intrinsic equation of the evolute is

$$s = 2f'(\phi)\cos\phi + f''(\phi)\sin\phi. \quad \text{(Art. 39.)}$$

Let this be denoted by $F(\phi)$, and, by art. 38, the tangential equation is

$$\nu = \int \operatorname{cosec}^2\phi\{\int F'(\phi)\sin\phi d\phi\}d\phi.$$

Now we have, from the value of $F(\phi)$,

$$F'(\phi)\sin\phi = 3f''(\phi)\sin\phi\cos\phi + f'''(\phi)\sin^2\phi - 2f'(\phi)\sin^2\phi\,;$$

and integrating by parts we easily get

$$\int F'(\phi)\sin\phi d\phi = f''(\phi)\sin^2\phi + f'(\phi)\sin\phi\cos\phi - f(\phi).$$

Multiplying by $\operatorname{cosec}^2\phi$, and integrating again, we get

$$f'(\phi) + f(\phi)\cot\phi.$$

Hence the tangential equation required,

$$\nu = f'(\phi) + f(\phi)\cot\phi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (95)$$

42. The foregoing result may be obtained very simply from geometrical considerations as follows. In fig. 4 (art. 26) the line PS is a tangent to the evolute, and the angle $OSP = \phi$; then we have

$$\begin{aligned} OS &= ON + NS \\ &= LQ + OL\cot\phi \\ &= f'(\phi) + f(\phi)\cot\phi. \end{aligned}$$

Hence if OS be taken as the directing-line, the tangential equation of the envelope of SP is $\nu = f'(\phi) + f(\phi)\cot\phi$; but the envelope of SP is the evolute, and therefore we have the same result as before.

43. The right-hand side of equation (95) may be written

$$\frac{\frac{d}{d\phi}(f(\phi)\sin\phi)}{\sin\phi}.$$

Hence if ν_1, ν_2, ν_3, &c. represents for the successive evolutes what we have denoted by ν for the curve itself, we have

$$\nu_1 \sin\phi = \frac{d}{d\phi}(\nu \sin\phi);$$

similarly

$$\nu_2 \sin\phi = \frac{d}{d\phi}(\nu_1 \sin\phi),$$

$$\therefore\ \nu_2 \sin\phi = \frac{d^2}{d\phi^2}(\nu \sin\phi).$$

Hence in general

$$\nu_n \sin\phi = \frac{d^n}{d\phi^n}(\nu \sin\phi). \quad (96)$$

44. Since

$$\nu = f(\phi), \quad \nu \sin\phi = f(\phi)\sin\phi;$$

and denoting this by $\pi(\phi)$ and the corresponding functions for the evolutes by $\pi_1(\phi)$, $\pi_2(\phi)$, &c., we have, from equation (96),

$$\pi_n(\phi) = \frac{d^n}{d\phi^n}(\pi(\phi)). \quad (97)$$

45. In art. 26 we have found the coordinates of a point on the curve $\nu = f(\phi)$:—

$$x = f(\phi) + f'(\phi)\sin\phi\cos\phi; \quad y = -f'(\phi)\sin^2\phi.$$

These assume, if we substitute from art. 44 for $f(\phi)$ the value $\frac{\pi(\phi)}{\sin\phi}$, the symmetrical form

$$x = \pi(\phi)\sin\phi + \pi'(\phi)\cos\phi, \quad (98)$$

$$y = \pi(\phi)\cos\phi - \pi'(\phi)\sin\phi; \quad (99)$$

and hence, from art. 44, if we denote by x_n, y_n the coordinates of a point on the nth evolute,

$$x_n = \left\{\sin\phi\left(\frac{d}{d\phi}\right)^n + \cos\phi\left(\frac{d}{d\phi}\right)^{n+1}\right\}\pi(\phi), \quad (100)$$

$$y_n = \left\{\cos\phi\left(\frac{d}{d\phi}\right)^n - \sin\phi\left(\frac{d}{d\phi}\right)^{n+1}\right\}\pi(\phi). \quad (101)$$

46. By using LEIBNITZ's theorem, we find, from equation (98),

$$\frac{d^n x}{d\phi^n} = \sin\phi\frac{d^n\pi}{d\phi} + n\cos\phi\frac{d^{n-1}\pi}{d\phi^{n-1}} - \frac{n\,.\,n-1}{\underline{|2}}\sin\phi\frac{d^{n-2}\pi}{d\phi^{n-2}} - \frac{n\,.\,n-1\,.\,n-2}{\underline{|3}}\cos\phi\frac{d^{n-3}\pi}{d\phi^{n-3}} + \&c.$$

$$+\cos\phi\frac{d^{n+1}\pi}{d\phi^{n+1}} - n\sin\phi\frac{d^n\pi}{d\phi^n} - \frac{n\,.\,n-1}{\underline{|2}}\cos\phi\frac{d^{n-1}\pi}{d\phi^{n-1}} + \frac{n\,.\,n-1\,.\,n-2}{\underline{|3}}\sin\phi\frac{d^{n-1}\pi}{d\phi^{n-2}} - \&c.$$

Hence, by equations (100), (101), we get

$$\frac{d^n x}{d\phi^n} = x_n + ny_{n-1} - \frac{n\,.\,n-1}{\underline{|2}}x_{n-2} - \frac{n\,.\,n-1\,.\,n-2}{\underline{|3}}y_{n-3} + \&c. \quad (102)$$

Similarly, from equation (99), we get

$$\frac{d^n y}{d\phi^n}=y_n-ux_{n-1}-\frac{n\,.\,n-1}{\lfloor 2}\,y_{n-2}+\frac{n\,.\,n-1\,.\,n-2}{\lfloor 3}\,x_{n-3}+\&c. \quad . \quad . \quad . \quad (103)$$

47. The Intrinsic Equation of a Curve being given, we can find the Tangential Equation of the Evolute thus:—

Let $s=f(\phi)$ be the given intrinsic equation, then the intrinsic equation of the evolute is

$$s=f'(\phi).$$

(see Whewell, "On the Intrinsic Equation of Curves," Phil. Trans. vol. viii. p. 659); and therefore, by art. 38, equation (85),

$$\nu=\int \operatorname{cosec}^2\phi\{\int f''(\phi)\sin\phi\, d\phi\}d\phi. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (104)$$

Cor. The tangential equation of the second evolute is

$$\nu=\int \operatorname{cosec}^2\phi\{\int f'''(\phi)\sin\phi\, d\phi\}d\phi, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (105)$$

and, in general, of the nth evolute

$$\nu=\int \operatorname{cosec}^2\phi\{\int f^{(n+1)}(\phi)\sin\phi\, d\phi\}d\phi. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (106)$$

Examples.

(1) Find the tangential equation of the evolute of the catenary.

Here we have $f(\phi)=c\tan\phi$;

$$\therefore \int f''(\phi)\sin\phi\, d\phi=c\{\sec\phi\tan\phi-\log(\sec\phi+\tan\phi)\},$$

$$\therefore \int \operatorname{cosec}^2\phi\{\int f''(\phi)\sin\phi\, d\phi\}d\phi$$

$$=c\{\sec\phi+\cot\phi\,.\,\log(\sec\phi+\tan\phi)\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (107)$$

which is the required equation.

The following three examples are illustrations of art. 39.

(2) To find the intrinsic equation of the evolute of the curve $\nu=(1+\cot^{\frac{1}{3}}\phi)^3$:—

$$f(\phi)=(1+\cot^{\frac{1}{3}}\phi)^3;$$

$$\therefore\ s\,.\,\sin\phi=-\tfrac{2}{3}\{\cot^{\frac{1}{3}}\phi+\cot^{\frac{2}{3}}\phi\}\sec^2\phi$$

(see example 6, Section II., Chapter II.),

$$\therefore\ s=-\tfrac{2}{3}\left\{\frac{1}{\sin^{\frac{4}{3}}\phi\cos^{\frac{5}{3}}\phi}+\frac{1}{\sin^{\frac{5}{3}}\phi\cos^{\frac{4}{3}}\phi}\right\}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (108)$$

(3) If the curve be the lemniscate,

$$f\phi=a\left(\sin\frac{2\phi}{3}\right)\operatorname{cosec}\phi \text{ (see art. 25),}$$

and

$$\therefore s = \frac{a}{3\sqrt{\sin\frac{2\phi}{3}}} \text{ (see art. 39)}. \quad (109)$$

is the intrinsic equation of the evolute.

(4) Let the given curve be the equilateral hyperbola, we have

$$f(\phi) = a\sqrt{2\cot\phi};$$

$$\therefore \text{the evolute is } s = a(\operatorname{cosec} 2\phi)^{\frac{3}{2}}. \quad (110)$$

The next five examples are illustrations of art. 43.

(5) Let the curve be $\nu = e^{\sin\phi}$, its evolute will be

$$\nu = e^{\sin\phi}(\cot\phi + \cos\phi). \quad (111)$$

(6) The tangential equations of the successive evolutes of the curve $\nu = a\cos\phi$ are

$$\nu_1 = \frac{a\cos 2\phi}{\sin\phi},$$

$$\nu_2 = -4a\cos\phi = -4\nu;$$

and in general

$$\nu_{2m} = \pm 4^m \nu, \quad (112)$$

$$\nu_{2m+1} = \pm 4^m \nu_1, \quad (113)$$

where the sign + or − is to be used according as m is even or odd.

(7) Find the evolute of the logarithmic curve.

The Cartesian equation of the curve is $y = e^{cx}$,

and the tangential is $\nu = a\log\tan\phi$;

and therefore the tangential equation of its evolute is

$$\nu = a\cot\phi\{\log\tan\phi + \sec^2\phi\}. \quad (114)$$

(8) Let the curve be the polar one, $\rho^m = a^m \sin m\phi$.

The tangential equation is

$$\nu = a\left\{\sin\frac{m\phi}{m+1}\right\}^{\frac{m+1}{m}} \operatorname{cosec}\phi,$$

and the evolute is

$$\nu_1 = \nu\cot\left(\frac{m\phi}{m+1}\right). \quad (115)$$

This result could be easily obtained geometrically.

(9) The tangential equation of the evolute of the curve

$$\nu = k\tan^n\phi$$

is

$$\nu_1 = \nu\{(n+1)\cot\phi + n\tan\phi\}. \quad (116)$$

Hence the tangential equation of the evolute of the common parabola is

$$\nu_1 = \nu(2\cot\phi + \tan\phi). \quad (117)$$

Section II.—*Involutes.*

48. From the equations in art. 43 for the successive evolutes of a curve, we can conversely infer the equations of the successive involutes: thus, let the tangential equation of a curve be

$$\nu=f(\phi),$$

the tangential equations of the successive involutes are

$$\nu_{-1}\sin\phi=\int\nu\sin\phi\,d\phi,$$

or, as it may be written,

$$\nu_{-1}\sin\phi=\int_{d\phi}\nu\sin\phi,$$

$$\nu_{-2}\sin\phi=\iint_{d\phi}\nu\sin\phi,$$

$$\nu_{-3}\sin\phi=\iiint_{d\phi}\nu\sin\phi\,;$$

and in general, for the nth involute,

$$\nu_{-n}\sin\phi=\iiint_{d\phi}^{\ldots(n)}\nu\sin\phi. \qquad (118)$$

Mathematicians have recognized it as legitimate to interpret the symbol of differentiation with a negative index, as denoting integration; therefore we may write the equation (118) as follows:—

$$\nu_{-n}\sin\phi=\left(\frac{d}{d\phi}\right)^{-n}(f(\phi)\sin\phi). \qquad (119)$$

Hence the equation (96) includes the formulæ both for evolutes and involutes, according as n is regarded as positive or negative.

By an extension of the notation of art. 44, the last equation may be written

$$\pi_{-n}(\phi)=\left(\frac{d}{d\phi}\right)^{-n}(\pi(\phi)). \qquad (120)$$

49. If x_{-1}, y_{-1} denote the coordinates of a point on the first involute, x_{-2}, y_{-2} those of a point on the second involute, &c., we have

$$x_{-1}=\cos\phi(\pi(\phi))+\sin\phi\int_{d\phi}\pi(\phi), \qquad (121)$$

$$y_{-1}=-\sin\phi(\pi(\phi))+\cos\phi\int_{d\phi}\pi(\phi)\,; \qquad (122)$$

and, in general,

$$x_{-n}=\left\{\sin\phi\left(\frac{d}{d\phi}\right)^{-n}+\cos\phi\left(\frac{d}{d\phi}\right)^{-(n-1)}\right\}\pi(\phi), \qquad (123)$$

$$y_{-n}=\left\{\cos\phi\left(\frac{d}{d\phi}\right)^{-n}-\sin\phi\left(\frac{d}{d\phi}\right)^{-(n-1)}\right\}\pi(\phi). \qquad (124)$$

50. The Tangential Equation of a Curve being given, to find the Intrinsic Equation of its Involute.

This problem is solved by articles 30 and 48. Thus, if $\nu=\mathrm{F}(\phi)$ be the tangential equation of the involute,

$$\frac{ds}{d\phi}=\frac{\frac{d}{d\phi}(\mathrm{F}'(\phi)\sin^2\phi)}{\sin\phi}\quad\text{(see equation (64));}$$

but by article 48 we have

$$\mathrm{F}(\phi)=\frac{\int f(\phi)\sin\phi d\phi}{\sin\phi};$$

$$\therefore \frac{ds}{d\phi}=f'(\phi)\sin\phi+f(\phi)\cos\phi+\int f(\phi)\sin\phi\, d\phi:$$

that is,

$$\frac{ds}{d\phi}=\frac{d}{d\phi}(f(\phi)\sin\phi)+\int f(\phi)\sin\phi\, d\phi; \quad (125)$$

$$\therefore s=f(\phi)\sin\phi+\iint(f(\phi)\sin\phi\, d\phi)\, d\phi, \quad (126)$$

or, as it may be written,

$$s=\left\{1+\left(\frac{d}{d\phi}\right)^{-2}\right\}(f(\phi)\sin\phi). \quad (127)$$

Hence we have the following theorem:—

If $\nu=f(\phi)$ be the tagential equation of a curve, the intrinsic equation of its involute is

$$s=\left\{1+\left(\frac{d}{d\phi}\right)^{-2}\right\}(f(\phi)\sin\phi).$$

Cor. 1. Since s is the length of the involute, $\frac{ds}{d\phi}$ is the length of the given curve.

Hence from equation (125) we have the following theorem:—If $\nu=f(\phi)$ be the tangential equation of a curve, the length of the curve is given by the equation

$$s=\left\{\frac{d}{d\phi}+\left(\frac{d}{d\phi}\right)^{-1}\right\}f(\phi)\sin\phi. \quad (128)$$

Cor. 2. The equation (127) is equivalent to the following:—

$$s=\int f'(\phi)\sin\phi\, d\phi+\iint(f'(\phi)\cos\phi\, d\phi)\, d\phi; \quad (129)$$

for we have proved, art. 30, that if $\nu=f(\phi)$ be the tangential equation, the intrinsic equation is

$$s=f(\phi)\sin\phi+\int f'(\phi)\cos\phi\, d\phi,$$

and we get the intrinsic equation of the involute from this by integration.

51. From the intrinsic equation to find the tangential of the involute.

Let $s=f(\phi)$ be the given equation, then the intrinsic equation of the involute is

$$s=\int f(\phi)d\phi.$$

Hence from equation (85), art. 38, the tangential equation of the involute is

$$\nu=\int \mathrm{cosec}^2\phi\{f(\phi)\sin\phi\, d\phi\}d\phi. \quad (130)$$

Observation.—Under each of the heads Evolute and Involute it will be observed we have solved three problems, which may be stated briefly as follows:—

Given	To find
Tangential equation of a curve,	Tangential equation of its evolute, involute.
Tangential " "	Intrinsic " " "
Intrinsic " "	Tangential " " "

We have omitted the problems given the intrinsic equation of a curve to find the intrinsic equation of its evolute and involute, because these had been previously solved by WHEWELL (see 'Cambridge Philosophical Transactions,' already cited).

Examples.

Examples 1–3 are illustrations of art. 48, 4 and 5 of art. 50, and 6–8 of art. 51.

(1) Let $\nu = k\tan^n\phi$ be the equation, it is required to find the involute.

From equation (118), art. 48, we have

$$\nu_{-1}\sin\phi = k\int\tan^n\phi\sin\phi\, d\phi.$$

We can get a formula of reduction for this integral as follows:—

$$\text{Put } \mathrm{P} = \tan^{n-1}(\phi)\sin\phi;$$

$$\therefore \frac{d\mathrm{P}}{d\phi} = n\tan^{n-2}\phi\sin\phi + (n-1)\tan^n\phi\sin\phi,$$

$$\therefore \int\tan^n\phi\sin\phi\, d\phi = \frac{\tan^{n-1}(\phi)\sin\phi}{n-1} - \frac{n}{n-1}\int\tan^{n-2}\phi\sin\phi\, d\phi, \quad . \quad . \quad (131)$$

which is the required formula.

Cor. If $\nu = n\tan^{n-2}\phi + (n-1)\tan^n\phi$ be the equation of a curve, the equation of its involute is

$$\nu_{-1} = \tan^{n-1}\phi. \quad . \quad . \quad . \quad . \quad . \quad . \quad (132)$$

Compare equation (116).

(2) Find the involute of the curve

$$\nu = a\log\tan\phi$$

(that is, of the logarithmic curve), we have

$$\nu_{-1}\sin\phi = a\int\sin\phi\,(\log\tan\phi)\,d\phi$$

$$= \mathrm{C} - a\left\{\cos\phi\,.\log\tan\phi + \log\tan\frac{\phi}{2}\right\}. \quad . \quad . \quad . \quad (133)$$

Cor. $$\nu_{-1}\sin\phi + \nu\cos\phi = \mathrm{C} - a\log\tan\frac{\phi}{2}. \quad . \quad . \quad . \quad . \quad (134)$$

(3) Let the curve be

$$\nu = (1 + \cot^{\frac{1}{3}}\phi)^3,$$

then the tangential equation of its involute is

$$\nu_{-1}\sin\phi=\int\sin\phi\,d\phi+3\int\cos^{\frac{1}{3}}\phi\sin^{\frac{2}{3}}\phi\,d\phi$$
$$+3\int\cos^{\frac{2}{3}}\phi\sin^{\frac{1}{3}}\phi\,d\phi+\int\cos\phi\sin\phi\,d\phi. \quad \ldots \quad (135)$$

Of these four integrals, the first and fourth are elementary, and the third is derived from the second by putting $\left(\frac{\pi}{2}-\phi\right)$ in place of ϕ and changing signs. Hence the question will be solved if we integrate

$$\int\cos^{\frac{1}{3}}\phi\sin^{\frac{2}{3}}d\phi.$$

To reduce this to elliptic integrals, let $z^3=\cot^2\phi$, and we easily find

$$\int\cos^{\frac{1}{3}}\phi\sin^{\frac{2}{3}}\phi=\frac{\sin^{\frac{5}{3}}\phi}{\cos^{\frac{2}{3}}\phi}+\int\frac{dz}{z^2\sqrt{1+z^3}}.$$

Now

$$\frac{d}{dz}\left\{\frac{\sqrt{1+z^3}}{z}\right\}=\frac{1}{2}\cdot\frac{z}{\sqrt{1+z^3}}-\frac{1}{z^2\sqrt{1+z^3}}.$$

Hence

$$\int\cos^{\frac{1}{3}}\phi\sin^{\frac{2}{3}}\phi\,d\phi=-\frac{\cos^{\frac{4}{3}}\phi}{\sin^{\frac{1}{3}}\phi}+\frac{1}{2}\int\frac{2dz}{\sqrt{1+z^3}}, \quad \ldots \quad (136)$$

and the question is completely solved. (See art. 32.)

(4) Let $\nu=a\cos\phi$ be the tangential equation of a curve, then (see art. 50) the intrinsic equation of its involute is

$$s=\left\{1+\left(\frac{d}{d\phi}\right)^{-1}\right\}f(\phi)\sin\phi$$
$$=\frac{3a\sin 2\phi}{8}+\frac{a\phi}{4}. \quad \ldots \quad (137)$$

(5) Let $\nu=a\cos\phi$; then we find for the involute

$$s=\frac{2a\sin 3\phi}{9}+\frac{a\phi}{3}. \quad \ldots \quad (138)$$

(6) Let the intrinsic equation of a curve be

$$s=a\cos^3\phi;$$

then (see art. 51) the tangential equation of its involute is

$$\nu=-\frac{a}{4}\int\frac{\cos^4\phi\,d\phi}{\sin^2\phi}$$
$$=\frac{a}{16}(4\cot\phi+2\phi+\sin 2\phi). \quad \ldots \quad (139)$$

Similarly, if $s=a\cos^5\phi$,

$$\nu=\frac{a}{96}\{16\cot\phi+30\phi+9\sin 2\phi-4\sin^3\phi\cos\phi\}. \quad \ldots \quad (140)$$

is its involute.

(7) Let $s=c\tan^2\phi$, then $f\phi=c\tan^2\phi$;

$$\therefore\ \int f(\phi)\sin\phi\,d\phi=c(\cos\phi+\sec\phi),$$

$$\therefore\ \nu=c\int(\cos\phi+\sec\phi)\operatorname{cosec}^2\phi\,d\phi$$

$$=c\{\log(\sec\phi+\tan\phi)-2\operatorname{cosec}\phi\}. \qquad (141)$$

(8) Find the tangential equation of the involute of a circle.

The intrinsic equation of the circle is

$$s=a\phi;$$

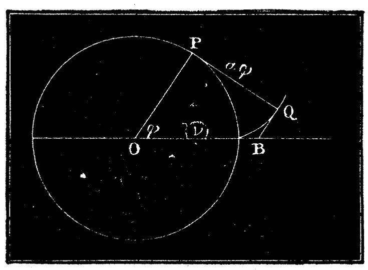
$$\therefore\ \int f(\phi)\sin\phi\,d\phi=a(\sin\phi-\phi\cos\phi),$$

$$\therefore\ \int\operatorname{cosec}^2\phi\{f(\phi)\sin\phi\,d\phi\}d\phi=\frac{a\phi}{\sin\phi};$$

$\therefore$ the tangential equation of the involute of the circle is

$$\nu=\frac{a\phi}{\sin\phi}. \qquad (142)$$

Fig. 7.



We can verify the foregoing result geometrically as follows:—For in the annexed diagram, which represents a circle and its involute, we have $PQ=a\phi$, and $OB=\nu$; and since QB is parallel to PO, we have at once

$$OB\sin\phi=PQ;\ \text{that is, } \nu\sin\phi=a\phi,$$

which proves the proposition.

CHAPTER IV.

Section I.—*Positive Pedals.*

52. If we make the perpendicular to our *director* line the initial line, it is evident that the polar equation of the first positive pedal of the curve

$$\nu=f(\phi)$$

$$\text{is } \varrho=f(\phi)\sin\phi. \qquad (143)$$

Hence the tangential equation of any curve is at once transformed into the polar equation of its first positive pedal by changing ν into ϱ, and multiplying the function on the right-hand side by $\sin\phi$.

Thus the tangential equation of the parabola is

$$\nu=a\tan\phi \text{ (see art. 25)};$$

hence its first positive pedal is

$$\rho=\frac{a\sin^2\phi}{\cos\phi}, \quad \ldots\ldots\ldots \quad (144)$$

which equation represents, as is well known, the cissoid.

Again, the pedal of the logarithmic curve is

$$\rho=a\sin\phi\log\tan\phi, \quad \ldots\ldots\ldots \quad (145)$$

and of the ellipse

$$\rho=\sqrt{a^2\sin^2\phi+b^2\cos^2\phi}. \quad \ldots\ldots\ldots \quad (146)$$

This curve is a bicircular quartic.

53. The tangential equation of the evolute of the curve $\nu=f(\phi)$ is

$$\nu=f(\phi)\cot\phi+f'(\phi).$$

Hence the polar equation of the first positive pedal of the evolute is

$$\rho=f(\phi)\cos\phi+f'(\phi)\sin\phi. \quad \ldots\ldots\ldots \quad (147)$$

54. The foregoing result can be shown geometrically as follows (see fig. art. 26):—

The perpendicular OT on PQ is the radius vector of the pedal of the evolute; but

$$\begin{aligned}\mathrm{OT}&=\mathrm{OL}\cos\phi+\mathrm{LP}\\&=f(\phi)\cos\phi+f'(\phi)\sin\phi.\end{aligned}$$

Cor. 1. The equation (147) may be written

$$\rho=\frac{d}{d\phi}(f(\phi)\sin\phi). \quad \ldots\ldots\ldots \quad (148)$$

This also appears from art. 43; and from the same article we see that the first positive pedal of the nth evolute is

$$\rho_n=\left(\frac{d}{d\phi}\right)^n(f(\phi)\sin\phi). \quad \ldots\ldots\ldots \quad (149)$$

Cor. 2. If in the last equation n be taken as negative, we have the first positive pedal of the nth involute.

Cor. 3. If ρ_1 and ρ_{-1} denote the radii vectores of the first positive pedals of the evolute and involute of $\nu=f(\phi)$, we have

$$\begin{aligned}\rho_1&=f'(\phi)\sin\phi+f(\phi)\cos\phi,\\ \rho_{-1}&=-f(\phi)\cos\phi+\int f''(\phi)\cos\phi\, d\phi;\\ \therefore\ \rho_1+\rho_{-1}&=f'(\phi)\sin\phi+\int f'(\phi)\cos\phi\, d\phi,\\ \therefore\ \rho_1+\rho_{-1}&=s. \quad \ldots\ldots\ldots \quad (150)\end{aligned}$$

Hence we have the following theorem:—

The length of a curve is equal to the sum of the radii vectores of the first positive pedals of its evolute and involute.

Cor. 4. If ϱ denote the radius vector of the first positive pedal of a curve,

$$s=\frac{d\varrho}{d\phi}+\int\varrho\, d\phi. \quad \text{(151)}$$

SECTION II.—*Negative Pedals.*

55. We have seen in art. 52 that the polar equation of the first positive pedal of a curve is obtained from its tangential equation by changing ν into ϱ, and multiplying the function on the right-hand side by $\sin\phi$. Hence, conversely, we have the following theorem:—If $\varrho=F(\phi)$ be the polar equation of a curve, the tangential equation of its first negative pedal is

$$\nu=\frac{F(\phi)}{\sin\phi}. \quad \text{(152)}$$

Thus the polar equation of a parabola is

$$\varrho=4a\tan\phi\sec\phi;$$

∴ the tangential equation of its first negative pedal is

$$\nu=4a\sec^2\phi, \quad \text{(153)}$$

or, in Cartesian coordinates,

$$(x-4a)^3=27a\,y^2, \quad \text{(154)}$$

showing that it is in the semicubical parabola.

56. The equation of the line whose envelope is the negative pedal is

$$x\sin\phi+y\cos\phi-F(\phi)=0.$$

Hence the points where this line meets its envelope are given by the equations

$$x=F(\phi)\sin\phi+F'(\phi)\cos\phi, \quad \text{(155)}$$

$$y=F(\phi)\cos\phi-F'(\phi)\sin\phi; \quad \text{(156)}$$

and by eliminating ϕ between these equations, we get the equation of the pedal.

Cor.

$$\left.\begin{aligned} x^2+y^2&=(F(\phi))^2+(F'(\phi))^2 \\ \text{or}\quad x^2+y^2&=\varrho^2+\left(\frac{d\varrho}{d\phi}\right)^2 \end{aligned}\right\} \quad \text{(157)}$$

Hence the distance from the extremity of ϱ to where the perpendicular to it meets its envelope is $\left(\frac{d\varrho}{d\phi}\right)$.

Examples.

(1) Find the first negative pedal of the cardioide.

The polar equation of this curve is, taking the perpendicular to the cuspidal tangent as the initial line,

$$\varrho=a(1+\sin\phi);$$

$$\therefore F(\phi)=a(1+\sin\phi);$$

$$\therefore x=2a+a\sin\phi,$$

$$y=a\cos\phi,$$

$$\therefore (x-2a)^2+y^2=a^2. \qquad (158)$$

Therefore the pedal is a circle.

(2) Find the negative pedal of $\varrho=k\tan^2\phi\sec\phi$.

$$\therefore \frac{x}{k}=2\tan\phi+4\tan^3\phi,$$

$$\frac{y}{k}=-\tan^2\phi-3\tan^4\phi.$$

Then if we put

$$N=\{\sqrt{k-12y}-k^2\},$$

the result of eliminating ϕ will be

$$\frac{k^4x^2}{2}=\frac{N}{27}\{N+3k^2\}^2. \qquad (159)$$

By differentiating the values of x and y given in equations (155), (156), then squaring and adding &c., we get the length of the first negative pedal,

$$s=F'(\phi)+\int F(\phi)d\phi, \qquad (160)$$

an equation which agrees with equation (151), but expressed in a different notation.

57. If in art. 39 we substitute $\left(\frac{F\phi}{\sin\phi}\right)$ for $f(\phi)$, we get, from equation (93),

$$s=\left\{1+\left(\frac{d}{d\phi}\right)^2\right\}F(\phi).$$

Hence we have the following theorem :—

If $\varrho=F(\phi)$ *be the polar equation of a curve, the intrinsic equation of the evolute of its first negative pedal is*

$$s=\left\{1+\left(\frac{d}{d\phi}\right)^2\right\}F(\phi). \qquad (161)$$

In like manner, from art. 50, *the intrinsic equation of the involute of the first negative pedal is*

$$s=\left\{1+\left(\frac{d}{d\phi}\right)^{-2}\right\}F(\phi). \qquad (162)$$

Cor. *The intrinsic equation of the* n*th evolute is*

$$s=\left\{\left(\frac{d}{d\phi}\right)^{n-1}+\left(\frac{d}{d\phi}\right)^{n+1}\right\}F(\phi), \qquad (163)$$

and of the n*th involute is*

$$s=\left\{\left(\frac{d}{d\phi}\right)^{-(n-1)}+\left(\frac{d}{d\phi}\right)^{-(n+1)}\right\}F(\phi). \qquad (164)$$

Examples.

Find the intrinsic equation of the first negative pedal of an ellipse.

The polar equation of the ellipse,

$$\varrho=\frac{b}{\Delta(\phi)}, \text{ where } \Delta(\phi)=\sqrt{1-e^2\sin^2\phi}.$$

Hence by art. 56, equation (160), the equation of its first negative pedal is

$$s=\frac{be^2\sin\phi\cos\phi}{\Delta^3\phi}+b\int\frac{d\phi}{\Delta\phi},$$

or

$$s=b\left\{\frac{e\sin\phi\cos\phi}{\Delta^3\phi}+\mathrm{F}(e,\phi)\right\}, \quad (165)$$

and the intrinsic equation of the evolute of the pedal is

$$s=\frac{b}{\Delta(\phi)}\left\{1+\frac{e^2\cos2\phi}{\Delta^2\phi}+\frac{b}{4}\left(\frac{e^2\sin^2\phi}{\Delta^2\phi}\right)^2\right\}. \quad (166)$$

58. The converse of the problem solved in art. 56 is, being given the intrinsic equation of a curve to find the polar equation of its first positive pedal.

Let $s=f(\phi)$ be the given intrinsic equation, then we have, from equation (151),

$$\frac{d\varrho}{d\phi}+\int\varrho\, d\phi=f(\phi);$$

$$\therefore\ \frac{d^2\varrho}{d^2\phi}+\varrho=f'(\phi),$$

$$\therefore\ \varrho=\left\{1+\left(\frac{d}{d\phi}\right)^2\right\}^{-1}f'(\phi). \quad (167)$$

Cor. 1. The polar equation of the positive pedal of the evolute is

$$\varrho=\left\{1+\left(\frac{d}{d\phi}\right)^{-2}\right\}^{-1}f(\phi), \quad (168)$$

and of the involute

$$\varrho=\left\{1+\left(\frac{d}{d\phi}\right)^2\right\}^{-1}f(\phi). \quad (169)$$

Cor. 2. The equation (167) may be written

$$\varrho=\sin\phi\int\cos\phi f'(\phi)d\phi-\cos\phi\int\sin\phi f'(\phi)d\phi$$
$$+\mathrm{C}_1\cos\phi+\mathrm{C}_2\sin\phi. \quad (170)$$

See Boole's 'Differential Equations,' where the reader will find illustrations of the cases in which the symbol $\left(1+\left(\frac{d}{d\phi}\right)^2\right)^{-1}$ on the right-hand side of equation (167) may be usefully expanded in ascending powers of $\left(\frac{d}{d\phi}\right)^2$, and thus the integration on the right-hand side rendered unnecessary, for the process then will be performed by differentiation.

Cor. 3. The equation (169) may be written

$$\varrho=\sin\phi\int\cos\phi\, f(\phi)d\phi-\cos\phi\int\sin\phi\, f(\phi)d\phi$$
$$+C_1\cos\phi+C_2\sin\phi. \qquad (171)$$

59. Since $(1+x^{-1})^{-1}=x(1+x)^{-1}$, if we put for x the symbol $\left(\frac{d}{d\phi}\right)^2$, we get

$$\left\{1+\left(\frac{d}{d\phi}\right)^{-2}\right\}^{-1}=\left(\frac{d}{d\phi}\right)^2\left\{1+\left(\frac{d}{d\phi}\right)^2\right\}^{-1}. \qquad (172)$$

$\therefore$ if the right-hand side of equation (169) be differentiated twice with respect to ϕ, we get the right-hand side of equation (168). Hence equation (168) may be written

$$\varrho=\cos\phi\int\sin\phi f(\phi)d\phi-\sin\phi\int\cos\phi f(\phi)d\phi$$
$$+f(\phi)+C_1\cos\phi+C_2\sin\phi. \qquad (173)$$

60. From equation (160) art. 56 we have at once the following theorems.

If we have three polar curves given by the equations

$$\varrho=F(\phi),\quad \varrho=F_1(\phi),\quad \varrho=mF(\phi)+nF_1(\phi);$$

then, 1°, if the corresponding lengths of their negative pedals be denoted by

$$s,\quad s_1,\quad S,$$

we shall have

$$S=ms+ns_1. \qquad (174)$$

2°. If the corresponding lengths of the nth evolutes of their first negative be

$$\sigma,\quad \sigma_1,\quad \text{and } \Sigma,$$

then

$$\Sigma=m\sigma+n\sigma_1. \qquad (175)$$

61. To find the curve whose length bears a constant ratio to the radius vector of its first positive pedal. The given condition is expressed by the equation

$$kf(\phi)\sin\phi=f'(\phi)\sin\phi+\int f'(\phi)\cos\phi\, d\phi;$$

$$\therefore\ k(f'(\phi)\sin\phi+f(\phi)\cos\phi)=2f'(\phi)\cos\phi+f''(\phi)\sin\phi.$$

Hence

$$f(\phi)=\frac{e^{m\phi}}{\sin\phi}, \qquad (176)$$

where $m=k+\frac{1}{k}$;

$$\therefore\ \nu=\frac{e^{m\phi}}{\sin\phi}.$$

This curve is the equiangular spiral; and we infer from the form of its equation that its reciprocal with respect to a circle whose radius is k is another equiangular spiral whose equation in polar coordinates is

$$\rho=k^2e^{-m\phi}. \qquad (177)$$

Cor. The positive and negative pedals of equiangular spirals are also equiangular spirals, and so is the inverse. So that every geometrical transformation of this curve is another curve of the same species.

Section III.—*Reciprocal Curves.*

62. We have seen that the polar equation of the first positive pedal of the curve

$$\nu=f(\phi)$$

is

$$\rho=f(\phi)\sin\phi\,;$$

and the reciprocal of a curve being the inverse of its first positive pedal, then the polar equation of the reciprocal of $\nu=f(\phi)$ is

$$\frac{k^2}{\rho}=f(\phi)\sin\phi. \qquad (178)$$

Thus the reciprocal of the parabola is

$$\frac{k^2}{\rho}=a\tan\phi\sin\phi, \qquad (179)$$

or, in Cartesian coordinates,

$$y^2=\frac{k^2}{a}x, \qquad (180)$$

which is another parabola, as it ought, since the centre of reciprocation is a point on the curve.

63. Since the value of ρ derived from art. 178 is

$$\rho=\frac{k^2}{f(\phi)\sin\phi},$$

we infer, from art. 55, that the equation of the first negative pedal of the curve is

$$\nu=\frac{k^2}{f(\phi)\sin^2\phi}.$$

Hence we have the following theorem:—*If $\nu=f(\phi)$ be the tangential equation of a curve the reciprocal of its first positive pedal or the first negative pedal of its reciprocal is*

$$\nu=\frac{k^2}{f(\phi)\sin^2\phi} \qquad (181)$$

64. If the intrinsic equation be given, say $s=\mathrm{F}(\phi)$, then we have, from equation (167), the polar equation of its reciprocal,

$$\frac{k^2}{\rho}=\left\{1+\left(\frac{d}{d\phi}\right)^2\right\}^{-1}\mathrm{F}'(\phi) \qquad (182)$$

Cor. The polar equation of the reciprocal of the nth evolute is

$$\frac{k^2}{\rho}=\left\{1+\left(\frac{d}{d\phi}\right)^2\right\}^{-1}\mathrm{F}^{(n+1)}(\phi) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (183)$$

65. If in equation (182) we put $\varrho=\psi(\phi)$, we find

$$\mathrm{F}'(\phi)=\left\{1+\left(\frac{d}{d\phi}\right)^2\right\}\frac{k^2}{\psi(\phi)};$$

$$\therefore \mathrm{F}(\phi)=k^2\int\frac{d\phi}{\psi(\phi)}-k^2\frac{\psi'\phi}{(\psi\phi)^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (184)$$

an equation which gives the intrinsic equation of a curve in terms of the polar equation of its reciprocal.

66. If in equation (178) we put $\varrho=\psi(\phi)$, we get

$$f(\phi)=\frac{k^2}{\psi(\phi)\sin\phi}.$$

Hence if $\varrho=\psi(\phi)$ be the polar equation of a curve,

$$\nu=\frac{k^2}{\psi(\phi)\sin\phi} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (185)$$

is the tangential equation of its reciprocal.

Obs.—The problems we have solved in this section may be briefly stated thus:—

Given	To find
Tangential equation of a curve,	Polar equation of its reciprocal.
Polar ,, ,,	Tangential equation of its reciprocal.
Intrinsic ,, ,,	Polar ,, ,,
Polar ,, ,,	Intrinsic ,, ,,

Examples.

(1) Let it be required to find the reciprocal of the catenary.

The intrinsic equation is

$$s=c\tan\phi;$$

$\therefore$ $\mathrm{F}'(\phi)=c\sec^2\phi$, and, substituting in equation (182),

$$\frac{k^2}{\varrho}=c\left\{1+\left(\frac{d}{d\phi}\right)^2\right\}^{-1}\sec^2\phi.$$

Hence, from equation (170), we have

$$\frac{k^2}{\varrho}=c\sin\phi\int\sec\phi d\phi=c\cos\phi\int\sec\phi\tan\phi d\phi+\mathrm{C}_1\cos\phi+\mathrm{C}_2\sin\phi;$$

then performing the integrations, and determining the constants by the condition that

ϱ must be infinite when $\phi=0$, we have $C_1=c$ and $C_2=0$, and the required equation is

$$\frac{k^2}{\varrho}=c\sin\phi\log(\sec\phi+\tan\phi)-2c\sin^2\tfrac{1}{2}\phi\,. \quad (186)$$

(2) Find the reciprocal of the curve

$$\varrho^m=a^m\sin m\phi.$$

$$\therefore\ \psi(\phi)=a(\sin m\phi)^{\frac{1}{m}};$$

$\therefore$ the tangential equation of its reciprocal is

$$\nu=\frac{k^2}{a\sin\phi\,(\sin m\phi)^{\frac{1}{m}}} \quad (187)$$

(3) Find the reciprocal of the cycloid.

The intrinsic equation is

$$s=4a\cos\phi.$$

Hence, from equations (182) and (170),

$$\frac{k^2}{\varrho}=2a\phi\sin\phi-a\cos\phi+C_1\cos\phi+C_2\sin\phi.$$

Now it is evident that ϱ must be infinite when ϕ vanishes, and that $\frac{k^2}{\varrho}$ must be equal to $a\pi$ when $\phi=\frac{\pi}{2}$. Hence $C_1=a$, $C_2=0$, and therefore the required reciprocal curve is

$$\frac{k^2}{\varrho}=2a\phi\sin\phi \quad (188)$$

(4) The reciprocal of the logarithmic curve

$$\nu=a\log\tan\phi$$

is

$$\frac{k^2}{\varrho}=a\sin\phi\log\tan\phi,$$

or, in Cartesian coordinates,

$$x=ye^{\frac{ay}{k^2}} \quad (189)$$

(5) Find the reciprocal of the curve

$$\nu=(1+\cot^{\frac{1}{3}}\phi)^3.$$

Here we have, from equation (178),

$$\frac{k^2}{\varrho}=(1+\cot^{\frac{1}{3}}\phi)^3\sin\phi,$$

$$k^2=(x^{\frac{1}{3}}+y^{\frac{1}{3}})^3;$$

$$\therefore\ x^{\frac{1}{3}}+y^{\frac{1}{3}}=k^{\frac{2}{3}} \quad (190)$$

CHAPTER V.

Section I.—*The Cycloid.*

67. There is one curve which, though we have very seldom mentioned hitherto in our memoir, was the one which led to the discovery of its methods. This curve is the cycloid; and the reason it has not been more frequently used in our illustrations is that we consider its importance demands a chapter to itself. The novelty of the methods and of most of the results is our apology for devoting so much space to its investigation.

68. In the figure, art. 26, it is evident that the point Q is the centre of instantaneous rotation for the line LP, because the motion of the points L and P are respectively at right angles to the lines LQ and PQ respectively, and since the coordinates of the point Q are OL and LQ. Hence the locus of the centres of instantaneous rotation of the line LP, whose position is given at any time by the quantities ν and ϕ, where $\nu=f(\phi)$, is the curve obtained by eliminating ϕ between the equations

$$\left\{\begin{array}{l} x=f(\phi), \\ y=f'(\phi). \end{array}\right\} \quad \ldots \ldots \ldots \quad (191)$$

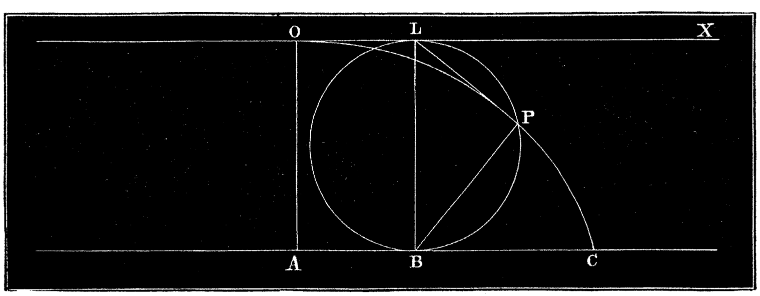
69. In the same fig., if LV, VQ be at right angles to LP, PQ, and since LQ$=f'(\phi)$, the values of LV, VQ will be $f'(\phi)\cos\phi$, $f'(\phi)\sin\phi$; and therefore the motion of the line LP will be given by supposing a curve whose equation is the system

$$\left\{\begin{array}{l} x=f'(\phi)\cos\phi, \\ y=f'(\phi)\sin\phi \end{array}\right\} \quad \ldots \ldots \ldots \quad (192)$$

to roll on the curve whose equation is the system (191), and the line LP will be the axis of y with respect to this rolling curve.

70. Let $f(\phi)=2a\phi$, then $\nu=2a\phi$; let O be the origin, OL$=\nu$, the angle XLP$=\phi$; then if P be the point of contact of LP with its envelope we have, by art. 26, the diameter

Fig. 8.



of the circle touching OX at L and passing through the point P$=f'(\phi)=2a$. Hence if we erect LB at right angles to OX, and PB to LP, the diameter LB of the circle LPB will be constant and equal to $2a$, and the arc LP of the same circle will be equal $2a\phi$; $\therefore$ the arc LP$=$the line OL$=$the line AB. Hence if we make AC$=\pi a$, the arc PB will be equal to the line BC; and ∴ the point P may be considered as fixed in the circle LPB, and the locus of P will be the curve described by a fixed point in the circle LPB rolling on the line AC. In other words the locus of P is a cycloid.

71. Since $f(\phi)=2a\phi$, the equations (191) denote a right line, and the equations (192) the circle $x^2+y^2=4a^2$. Hence the cycloid $\nu=2a\phi$ is the envelope of a fixed diameter of the circle $x^2+y^2=4a^2$, which rolls along the line $y=-2a$. Therefore we have two methods of generating the same cycloid, either as a locus or an envelope.

72. The coordinates of the point P are, from equations (46), (47), the system

$$\text{I.}\quad \begin{cases} x=a(2\phi+\sin 2\phi), & (193) \\ y=-2a\sin^2\phi. & (194) \end{cases}$$

From equation (62) we have the intrinsic equation

$$\text{II.}\quad s=4a\sin\phi, \qquad (195)$$

and from (61)

$$\text{III.}\quad \rho=4a\cos\phi. \qquad (196)$$

If we differentiate the equation $\nu=2a\phi$ we have the differential equation of the cycloid

$$\text{IV.}\quad \frac{d\nu}{d\phi}=2a=\text{constant}. \qquad (197)$$

73. From equation (93), art. 39, the intrinsic equation of the evolute is

$$s=4a\cos\phi=4a\sin\left(\frac{\pi}{2}-\phi\right). \qquad (198)$$

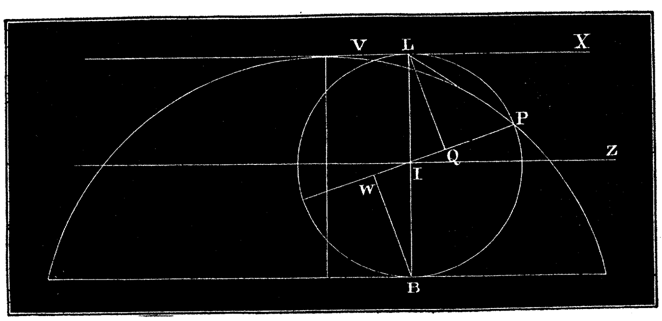
Hence the evolute is another cycloid.

We can show the same thing geometrically; for we have seen that the arc PB=the line BC. Hence denoting BC by ν, and the angle PBC by θ, we have $\nu=2a\theta$, and therefore the envelope of PB is a cycloid.

Cor. If the line PB be produced to R, making BR=BP, then R is the centre of curvature.

74. From L let fall the perpendicular LQ on the diameter VP of the revolving circle,

Fig. 9.



then it is evident that the angle XLQ=2XLP=2ϕ; and denoting this angle by ψ, we have

$$\nu=a\psi,$$

or

$$\frac{d\nu}{a\psi}=a.$$

Hence the envelope of LQ is a cycloid, and it is evident that Q is the point of contact. This is the cycloid that would be described by a fixed point in the circumference of the circle, whose diameter is the line IL, rolling on the line Z.

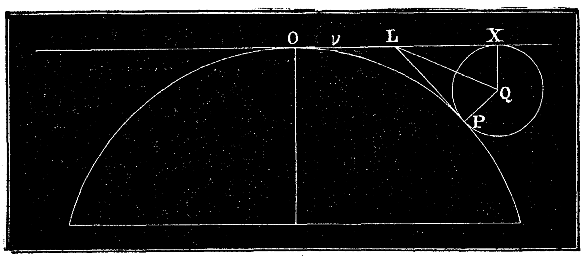
Cor. From B let fall the perpendicular BW on the diameter VP of the rolling circle, then we have IW=IQ; and therefore the locus of the point W is the envelope of VP, and it is the evolute of the cycloid described by Q.

75. The circle whose centre is P, and which touches the line LX, also touches the locus of Q. Hence we have the following theorem:—

If a variable circle has its centre on a given cycloid, and if it touches the tangent at its vertex, its envelope is another cycloid.

76. *If a variable circle touch a given cycloid, and also touch the tangent at the vertex, the locus of its centre is a cycloid.*

Fig. 10.



Or we may give a direct proof of this last theorem: let the angle $XLQ=\frac{1}{2}XLP=\frac{1}{2}\phi$ be denoted by θ; now we have

$$\nu=2a\phi=4a\theta;$$

hence the envelope of LQ is a cycloid. Again, $LP=LQ\cos\theta$, but $LP=2a\sin 2\theta$, $\therefore$ $LQ=4a\sin\theta$; and therefore Q is the point of contact of LQ with its envelope, and the proposition is proved.

77. If LP, L′P′, L″P″ be three fixed tangents to a variable cycloid, we have

$$\nu=2a\phi,\quad \nu'=2a\phi',\quad \nu''=2a\phi''.$$

Hence

$$\frac{\nu'-\nu}{\nu''-\nu'}=\frac{\phi'-\phi}{\phi''-\phi'}=\text{constant}.$$

Hence the tangent at the vertex of the cycloid is divided in a given anharmonic ratio by the three given tangents and the line at infinity. Hence we have the following theorem:—

Being given three fixed tangents to a variable cycloid, the envelope of the tangent at the vertex is a parabola.

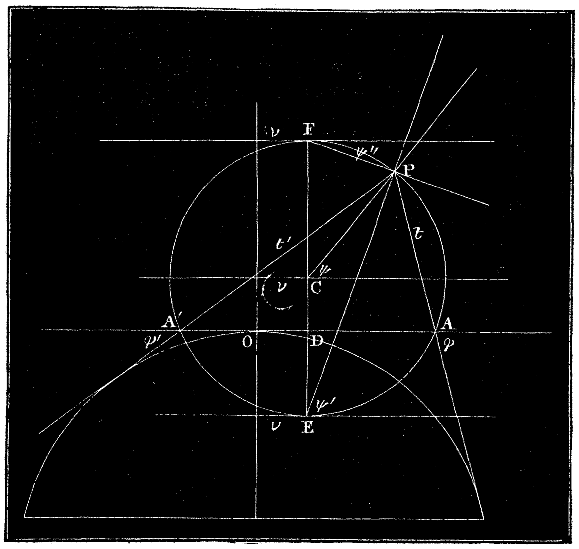
78. If four fixed tangents to a cycloid be given, the tangent at the vertex is a common tangent to two parabolas. Now being given two parabolas they have, in addition to the

common tangent at infinity, three finite common tangents. Hence we have the following theorem:—

Four lines being given, three cycloids can be described to touch them.

79. If two variable tangents $(t\,t')$ to a cycloid intersect at a constant angle, and a circle be described about the triangle formed by $t\,t'$ and the tangent at the vertex of the cycloid, then (1°) *the envelope of the diameter of this circle passing through the points* $t\,t'$ *is a cycloid*; (2°) *the envelopes of the chords passing through the same point, and through the highest and lowest points of the circle, are cycloids.*

Fig. 11.



Let the tangents $t\,t'$ intersect in P, and let C be the centre of the circle APA′; then since the angle APA′ is constant, $\phi+\phi'$ is constant, $\therefore$ $AA'=2a(\phi+\phi')$ is constant. Hence the base and the vertical angle of the triangle APA′ is constant; $\therefore$ the diameter of the circumscribing circle is constant, and it is evident that the loci of the points E, C, F are right lines parallel to AA′.

Again, since D is the middle point of AA′,

$$OD=\tfrac{1}{2}(OA-OA')=a(\phi-\phi')=a(\widehat{FCP}).$$

Hence

$$\nu=a\left(\frac{\pi}{2}-\phi\right),$$

$$\therefore\ \frac{d\nu}{d\phi}=-a=\text{constant};$$

therefore (see art 72, equation (197)) the envelope of CP is a cycloid.

(2) Since $OD=a(\widehat{FCP})=2a(\widehat{FEP})$, we have, for finding the envelope of EP,

$$\nu=2a\left(\frac{\pi}{2}-\psi'\right);$$

therefore it is a cycloid.

(3) The angle $FEP=\psi''$, $\therefore$ $\nu=2a\psi''$, and the envelope of FP is a cycloid.

80. *If* C, C′ *be the centres of the circles* APA′ *and* TPT′, *then* CC′ *is perpendicular to the tangent at the vertex of the cycloid, and equal to the radius of its generating circle.*

Demonstration.—Since $AT=2a\sin\phi$, and $A'T'=2a\sin\phi'$, we have

$$\frac{AT}{A'T'}=\frac{\sin\phi}{\sin\phi'}=\frac{PA'}{PA};$$

$\therefore$ PA . AT=PA′ . A′T′. Hence the radical axis of the two circles is parallel to AA′, and therefore CC′ is perpendicular to AA′.

Again, the radical axis of the two circles passes through P; hence, by a known property of coaxal circles, the rectangle

$$PA \,.\, AT = 2CC' \,.\, PD;$$

that is,

$$PA \,.\, AT = 2CC' \,.\, PA \,.\, \sin\phi,$$

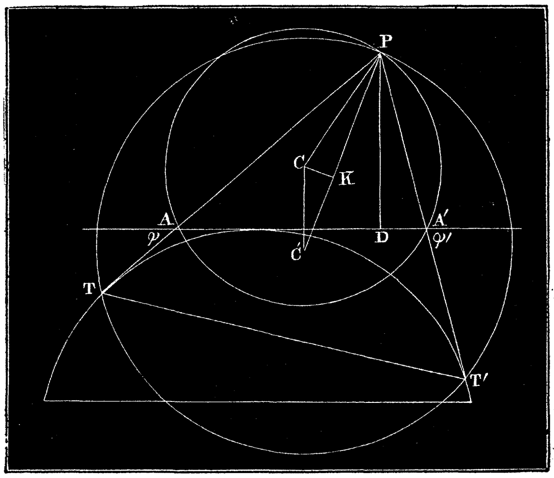
$$\therefore AT = 2CC' \sin\phi;$$

but

$$AT = 2a \sin\phi,$$

$$\therefore CC' = a. \qquad \text{Q. E. D.}$$

Fig. 12.



81. If the angle TPT′ be constant, *the locus of* C′, *the centre of the circle described about the triangle formed by the two tangents and the chord of contact, is a right line.* This is evident, since CC′=a and perpendicular to AA′, and the locus of C is a right line. (See art. 79.)

Section II.—*Intern and Extern Cycloids.*

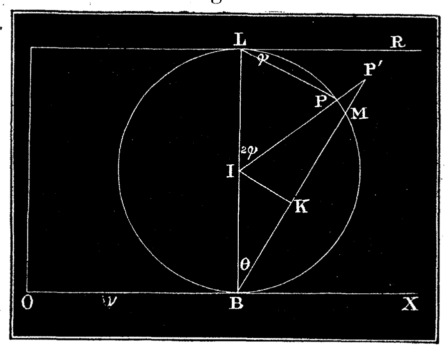
82. *Definition.*—When the extremity of the revolving radius of the generating circle describes a cycloid, a fixed point in the radius describes a curve, which, according as the point is *inside* or *outside* the circle, I shall call the *intern* or *extern* cycloid.

These curves are usually called the *prolate* and the *curtate* cycloid; but the names I have adopted are more suggestive.

83. To find the intrinsic equation of an extern cycloid.

Let BPL be the generating circle of the cycloid, P the point which describes it, and P′ the point which describes the extern cycloid; then denoting IP, IP′ by a, b, and the angles as in the diagram, we have, since $\nu=2a\phi$, the coordinates of the point P′ given by the equations

Fig. 13.



$$\left.\begin{aligned} x &= 2a\phi + b\sin 2\phi, \\ y &= a \;\; + b\cos 2\phi. \end{aligned}\right\} \quad . \;\; . \;\; . \quad (199)$$

If we differentiate these equations with respect to ϕ, then square and add, we get

$$\left(\frac{ds}{d\phi}\right)^2 = 4(a+b)^2 - 16\,ab\sin^2\phi;$$

$$\therefore s = 2(a+b)\,E(c', \phi), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (200)$$

where

$$c'^2 = \frac{4ab}{(a+b)^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (201)$$

Hence the arc of an extern cycloid is equal in length to an elliptic arc.

If we make $c' = \frac{2\sqrt{c}}{1+c}$, we find from equation (201) $c = \frac{a}{b}$; but from the triangle IBP′ we have

$$a : b :: \sin(2\phi - \theta) : \sin\theta.$$

Hence

$$\sin(2\phi - \theta) = c \sin\theta.$$

Hence we can apply LANDEN's formula of transformation to the function on the right of equation (200), and we get (see CAYLEY's 'Elliptic Functions,' p. 329)

$$2(a+b)\,E(c', \phi) = 2bE(c, \theta) + \frac{a^2 - b^2}{b}\,F(c, \theta) + 2a \sin\theta;$$

$$\therefore s = 2bE(c, \theta) + \frac{a^2 - b^2}{b}\,F(c, \theta) + 2a \sin\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad (202)$$

Now since B is the centre of instantaneous rotation, the locus of the point P′ will be at right angles to P′B; that is, the tangent at P′ will be perpendicular to P′B, and the tangent at the highest point will be perpendicular to IB; hence the angle between these tangents will be equal to θ, and therefore the transcendental equation (202) is the intrinsic equation of the extern cycloid.

84. If the point P′ be inside the circle, that is, if the curve be an intern cycloid, the formula (202) will still hold, and be the intrinsic equation of the curve; but the modulus c of the functions E and F will be greater than unity. A simple transformation of that formula will give one in which c is changed into its reciprocal. If we interchange the quantities a and b in equation (200), the value of s remains unaltered; hence when the point P′ is inside, and b less than a, if we wish that the modulus of the functions E and F should be less than unity, instead of formula (202) we shall have the following:—

$$s = 2aE(c, \theta) + \frac{b^2 - a^2}{a}\,F(c, \theta) + 2b \sin\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad (203)$$

In this formula $c = \frac{b}{a}$, and the angle θ is the angle IP′B.

85. If we differentiate the equation (202) we get, after some slight reduction,

$$\frac{ds}{d\theta} = \frac{b(c\cos\theta + \Delta(\theta))^2}{\Delta(\theta)}.$$

Hence if ρ denote the radius of curvature of an extern cycloid, we have

$$\rho = \frac{b(c\cos\theta + \Delta(\theta))^2}{\Delta(\theta)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (204)$$

In the figure, art. 82, we have evidently

$$P'K=b(\Delta(\theta)),\ P'B=b(c\cos\theta+\Delta(\theta));$$

$$\therefore \varrho=\frac{P'B^2}{P'K} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (205)$$

Hence we have the following elegant construction for the centre of curvature of an intern or extern cycloid:—

From the centre I *of the generating circle let fall the perpendicular* IK *on the normal to the curve, and then the third proportional* P′Q *to* P′K *and* P′B *will be the radius of curvature.*

86. Since

$$P'Q=\frac{b(c\cos\theta+\Delta(\theta))^2}{\Delta\theta}$$

and

$$P'B=b(c\cos\theta+\Delta(\theta)),$$

we have

$$BQ=\frac{a\cos\theta\,(c\cos\theta+\Delta(\theta))}{\Delta(\theta)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (206)$$

Again, if the line BP′, that is, the normal to the curve, meet the polar of the point P′, with respect to the generating circle, in the point N, then the line BN is divided harmonically, and we have

$$\frac{1}{BP'}+\frac{1}{BN}=\frac{2}{BM}$$

or

$$\frac{1}{b(c\cos\theta+\Delta(\theta))}+\frac{1}{BN}=\frac{1}{bc\cos\theta};$$

$$\therefore BN=\frac{a\cos\theta\,(c\cos\theta+\Delta(\theta))}{\Delta(\theta)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (207)$$

$$\therefore BQ=BN.$$

Hence we have the following theorem:—*The portion of the normal to an intern or extern cycloid at any point* P′ *of the curve included between the polar of the point* P′ *with respect to the centre of the generating circle and the corresponding centre of curvature is bisected by the centre of instantaneous rotation.*

87. By art. 30, equation (64), if $\nu=f(\theta)$ be the tangential equation of a curve,

$$\frac{ds}{d\theta}=\frac{\frac{d}{d\theta}(f'(\theta)\sin^2\theta)}{\sin\theta};$$

but by art. 85,

$$\frac{ds}{d\theta}=\frac{b(c\cos\theta+\Delta(\theta))^2}{\Delta\theta}.$$

Hence

$$(\theta)\sin^2\theta=a\sin^2\theta-b\cos\theta\,.\,\Delta(\theta)+b;$$

$$\therefore f(\theta)=a\sin^{-1}\{\sin\theta(c\cos\theta+\Delta\theta)\}+\frac{b(\Delta(\theta)-\cos\theta)}{\sin\theta}.$$

Hence the tangential equation of an intern or extern cycloid is

$$\nu=a\sin^{-1}\{\sin\theta(c\cos\theta+\Delta(\theta))\}+b\left(\frac{\Delta\theta-\cos\theta}{\sin\theta}\right). \quad (208)$$

88. In the triangle IBP′, art. 83, we have

$$\sin 2\phi : \sin\theta :: \mathrm{BP}' : b,$$

$$\therefore \sin 2\phi=\sin\theta(c\cos\theta+\Delta(\theta));$$

and from the equation of the cycloid described by the point P we have

$$\nu=2a\phi;$$

∴ eliminating ϕ, we have for the envelope of the line BP′

$$\nu=a\sin^{-1}\{\sin\theta(c\cos\theta+\Delta(\theta))\}; \quad (209)$$

and this is the tangential equation of the evolute of an intern or extern cycloid.

89. The tangential equation (208) can be expressed very simply as follows. For if we take the conic

$$\frac{x^2}{b^2-a^2}+\frac{y^2}{b^2}=1, \quad (210)$$

we easily find its tangential equation to be

$$\nu=\frac{b(\Delta(\theta)-\cos\theta)}{\sin\theta}. \quad (211)$$

Hence we have the following theorem:—

If $\nu=\mathrm{F}(\theta)$ be the tangential equation of the evolute (see equation (209)), and $\nu=\mathrm{G}(\theta)$ the tangential equation of the ellipse (210), then the tangential equation of the intern or extern cycloid is

$$\nu=\mathrm{F}(\theta)+\mathrm{G}(\theta). \quad (212)$$

Cor. 1. The intrinsic equation of the evolute of an intern or extern cycloid is

$$s=\frac{b(c\cos\theta+\Delta(\theta))^2}{\Delta(\theta)}. \quad (213)$$

Cor. 2. If σ, σ', σ'' denote the lengths of an extern or intern cycloid, its evolute, and its auxiliary conic (see equation (210)), taken on the three curves from points whose tangents are parallel to other three points whose tangents also are parallel, then

$$\sigma=\sigma'+\sigma''. \quad (214)$$

Cor. 3. If $\varrho, \varrho', \varrho''$ be the radii of curvature of the same three curves at points whose tangents are parallel,

$$\varrho=\varrho'+\varrho''. \quad (215)$$

Cor. 4. ϱ will be infinite when either ϱ' or ϱ'' is infinite; but ϱ'' will be infinite when the auxiliary conic is a hyperbola and the point of contact at infinity. Now if b is less than a, we have the following theorem:—

An intern cycloid has two points of inflection, the tangents at which are parallel to the asymptotes of the auxiliary conic.

Section III.

90. *If two tangents to a cycloid intersect at a constant angle, the locus of their point of intersection is an extern cycloid.*

Demonstration.—Since the angle APA′ (see fig. art. 79) between the tangents tt' is constant, $\phi+\phi'$ is constant, $\therefore$ AA′$=2a(\phi+\phi')$ is constant; but the diameter of the circle about the triangle APA′

$$=\frac{AA'}{\sin P}=\frac{2a(\phi+\phi')}{\sin(\phi+\phi')};$$

$$\therefore CP=\frac{a(\phi+\phi')}{\sin(\phi+\phi')}.$$

Hence CP is constant. Again we have (see art. 79)

$$\frac{dv}{d\psi}=-a=\text{constant}.$$

Hence if CP were equal to a, the locus of P would be a cycloid; but since $\phi+\phi'$ is always greater than $\sin(\phi+\phi')$, CP is greater than a, and therefore the locus of P is an extern cycloid.

Lemma. *If two tangents,* PT, PT′, *to any given curve be inclined at a constant angle, the circle described about the triangle formed by the two tangents and the chord of contact touches the locus of* P.

Demonstration.—Let P′ be a consecutive point on the locus, then the tangents from P′ touch the curve in the points T, T′. Hence, since the angle TPT′=TP′T′, the quadrilateral is inscribed in a circle, and the line joining the consecutive points is a tangent to the circle. Hence the proposition is proved.

91. *If two tangents to a given cycloid make a given angle, the centre of the circle described about the triangle formed by the two tangents and the chord of contact is the centre of instantaneous rotation for the extern cycloid, which is the locus of the intersection of the tangents.*

Demonstration.—Since the angle TPT′ is given, the locus of P is an extern cycloid, and therefore, by the preceding lemma (see fig. art. 80), C′P is normal to the locus of P.

Again, since P is a point in the revolving radius of a circle whose centre is C and radius a, and we have proved CC′$=a$, the circle rolls on the locus of C′. Hence the proposition is proved.

92. If the angle TPT′ be constant, the radius of the circle TPT′ is a mean propor-

tional between the radius of the circle APA′ and half the chord of curvature at P, passing through the centre of APA′.

Demonstration.—Let the radii of the circles be r, r', and the angle CPC′ be ω; then, by art. 85, we have PK : PC′ : : PC′ : ϱ. But PK$=r\cos\omega$, $\therefore r\cos\omega : r' :: r' : \varrho$;

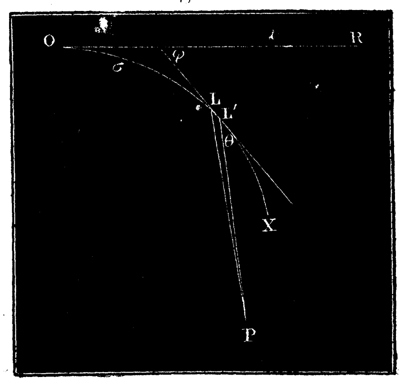
$$\therefore r'^2 = r\times\varrho\cos\omega. \qquad \text{Q. E. D.}$$

CHAPTER VI.

Section I.—*Epicycloids.*

93. The form of tangential equation employed in the previous portion of this memoir may be usefully generalized as follows:—Thus, instead of taking a directing line OX (see art. 1) and a variable line LP, making an angle ϕ with OX at the distance ν from the origin, let us take a directing curve OX, and a variable line LP, making an angle θ with the curve at L, and denoting the arc OL by σ; then any relation between σ and θ, such as $\sigma=f(\theta)$, may be called the tangential equation of the curve which the line LP envelopes.

Fig. 14.



Let us take a consecutive position, L′P of LP, then P is the point where LP touches its envelope, and LL′$=d\sigma$. Let the intrinsic equation of the directing curve be $\sigma=f_1(\phi)$, then the angle LPL′ is easily seen to be $d\phi+d\theta$; and if δ denote the diameter of the circle described about the infinitesimal triangle LPL′, we have

$$\frac{1}{\delta}=\frac{d\phi+d\theta}{d\sigma}.$$

Hence if ϱ denotes the radius of curvature of the directing curve at L, we have

$$\frac{1}{\delta}=\frac{1}{\varrho}+\frac{1}{f'(\theta)};$$

$$\therefore \delta=\frac{\varrho f'(\theta)}{\varrho+f'(\theta)}.$$

Hence

$$\text{LP}=\frac{\varrho f'(\theta)\sin\theta}{\varrho+f'(\theta)}.$$

If s denotes the length of the curve which is the envelope of LP from some fixed point in it up to P, then (see art. 31)

$$ds=\text{LL}'\cos\theta+d(\text{LP})$$

$$=d\sigma\cos\theta+d(\text{LP}).$$

Hence

$$s=\text{LP}+\int\cos\theta d\sigma;$$

$$\therefore\ s=\frac{\varrho f'(\theta)\sin\theta}{\varrho+f'(\theta)}+\int\cos\theta d\sigma\,;$$

that is,

$$s=\frac{f'(\theta)f'_1(\varphi)\sin\theta}{f'(\theta)+f'_1(\varphi)}+\int\cos\theta f'(\theta)d\theta \quad . \quad . \quad . \quad . \quad . \quad (216)$$

If we denote the angle which LP makes with OR by ψ, we have evidently

$$d\psi=d\varphi+d\theta,\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (217)$$

and we have also

$$f(\theta)=f_1(\varphi).\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (218)$$

Hence eliminating θ and φ between the three last equations, there will be a resulting equation between s and ψ, say

$$s=F(\psi),\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (219)$$

and this will be the intrinsic equation of the envelope of LP.

94. Let the directing curve be the catenary, and let the functional symbols f, f_1 be the same; then, since $\sigma=f(\theta)=f_1(\varphi)$, we have $\theta=\varphi$; $\therefore\ \psi=2\theta$.

Now we have, from the intrinsic equation of the catenary,

$$f'_1(\varphi)=c\sec^2\varphi\,;\ \therefore f'(\theta)=c\sec^2\theta.$$

Hence, making these substitutions in equation (216), and putting $\theta=\varphi=\frac{\psi}{2}$, we have the required intrinsic equation

$$s=c\left\{\tfrac{1}{2}\sec\frac{\psi}{2}\tan\frac{\psi}{2}+\log\left(\sec\frac{\psi}{2}+\tan\frac{\psi}{2}\right)\right\}.\quad . \quad . \quad . \quad . \quad . \quad (220)$$

95. Let the directing curve be the cycloid, and let, as before, f, f_1 be the same, then we get the intrinsic equation

$$s=2a(1-\cos\psi),\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (221)$$

a curve which we shall find to be a parallel to the cycloid.

96. The most interesting application of our general equation is where the directing curve is a circle, and the relation between σ and θ is linear; that is,

$$\sigma=\delta_{-1}(\theta),$$

where δ_{-1} is a constant. It will be seen that in this simple case the envelope belongs to the class of curves known as epicycloids and hypocycloids, and it will belong to one or the other according as δ_{-1} is positive or negative, or, what comes to the same thing, according as θ is positive or negative.

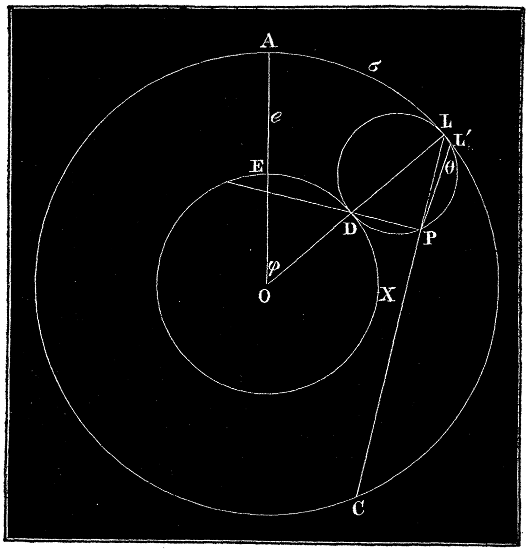
97. Let $\sigma=\delta_{-1}\theta$, and let the radius of the directing circle ALC be ϱ, then, from art. 93,

$$\frac{1}{\delta}=\frac{1}{\varrho}+\frac{1}{\delta_{-1}}\,;\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (222)$$

$$\therefore\ \delta_{-1}(\varrho-\delta)=\delta\varrho.$$

but
$$\varrho\varphi=\delta_{-1}\theta,$$
$$\therefore\ (\varrho-\delta)\varphi=\delta\theta;$$
that is, since LD$=\delta$, the arc ED$=$arc LP; and making the arc EX$=$semicircle LPD, we

Fig. 15.



have the arc XD$=$arc DP. Hence the locus of P is an epicycloid described by the rolling of the circle DPL on the circle EDX.

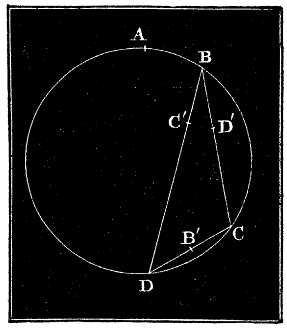
98. *Def.* We shall call the circle DPL the *generating* circle of the epicycloid, and the circle EDX, on which it rolls, the *base*.

It is evident that the motion of the circle DPL with respect to the director circle ALC is that of pure sliding, and its motion with respect to EDX is that of pure rolling.

99. Since L is the centre of similitude of the circles DPL and ALC, we have
$$\frac{\mathrm{LP}}{\mathrm{PC}}=\frac{\delta}{2\varrho-\delta};$$
but, from equation, (222)
$$\frac{\delta}{2\varrho-\delta}=\frac{\delta_{-1}}{2\varrho+\delta_{-1}},$$
$$\therefore\ \frac{\mathrm{LP}}{\mathrm{PC}}=\frac{\delta_{-1}}{2\varrho+\delta_{-1}}.$$
Again, since
$$\mathrm{AL}=\delta_{-1}\theta,$$
and arc
$$\mathrm{LC}=2\varrho\theta,$$
we have
$$\frac{\text{arc AL}}{\text{arc AC}}=\frac{\delta_{-1}}{2\varrho+\delta_{-1}}.$$
Hence
$$\mathrm{LP} : \mathrm{PC} :: \text{arc AL} : \text{arc AC} \quad \ldots\ldots \quad (223)$$
Hence, if a variable arc, AC, *has one extremity,* A, *fixed, and be divided in a given*

ratio in the point L, *the envelope of the chord* LC *is an epicycloid, which it touches in a point* P, *determined by the proportion* LP : PC :: *arc* AL : *arc* AC.

Fig. 16.



100. If A be a fixed point, and B, C, D variable points, and if the ratios be given arc AB : arc BC : arc CD, then, from the last article, the envelope of each side of the triangle BCD is an epicycloid touching the circle in the point A. Hence we have the following theorem:—*If a variable polygon be inscribed in a circle, and if the envelopes of all the sides but one be epicycloids which have a common point of contact with the circle, then the envelope of the remaining side is another epicycloid, having the same point of contact with the circle.*

Cor. If the points of contact of the sides of the triangle BCD, with their respective envelopes, be B′, C′, D′, the three lines BB′, CC′, DD′, are concurrent. This is evident from art. 99.

101. In the figure (art. 97), since the arc XD=arc PD, then denoting XD by σ' and the angle which PD makes with the circle XDE by θ', we have

$$\sigma'=\delta\theta'; \quad \ldots \ldots \quad (224)$$

therefore the envelope of PD is an epicycloid, whose directing circle is the circle XDE.

Hence, the evolute of an epicycloid is another epicycloid, and the director circle of one is the base of the other.

102. If δ_1 denote the diameter of the generating circle of the evolute, we have, as in art. 97,

$$\frac{1}{\delta_1}=\frac{1}{\rho-\delta}+\frac{1}{\delta}. \quad \ldots \ldots \quad (225)$$

But δ_1 and δ denote respectively the diameters of the generating circles of an epicycloid and its involute. *Hence, the difference between the reciprocals of the diameters of the generating circles of an epicycloid and its involute equals reciprocal of radius of directing circle of the epicycloid.*

Cor. *In the equation* $\sigma=\delta_{-1}\theta$, *the constant* δ_{-1} *is the diameter of the generating circle of the involute.*

This follows from the present article combined with equation (222). It was on this account that the negative suffix was put to δ.

103. In the figure (art. 97), if PD meet its envelope in P′, then P′ is the centre, and PP′ the radius of curvature at P; but PD$=\delta\cos\theta$, and P′D$=\delta_1\cos\theta$,

$$\therefore \text{PD} : \text{P'D} :: \delta : \delta_1 \quad \ldots \ldots \quad (226)$$

That is, *the base of an epicycloid divides its radii of curvature in the constant ratio of the diameters of the generating circles of the epicycloid and its evolute.*

104. Let P″ (see fig., art. 97) be the point where LC meets the epicycloid, which is

the involute of the locus of P, then from the last article we have

$$\frac{\text{P}''\text{L}}{\text{PL}}=\frac{\delta_{-1}}{\delta}=\frac{\varrho}{\varrho-\delta} \text{ from art. 97;}$$

$$\because \frac{\text{P}''\text{L}-\text{LP}}{\text{P}''\text{P}}=\frac{\delta}{2\varrho-\delta}=\frac{\text{LP}}{\text{PC}} \text{ from art. 99.}$$

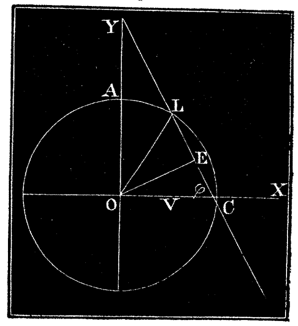
Hence the points P″, P are harmonic conjugates to the points L and C.

Cor. 1. *Every radius of curvature of an epicycloid is divided harmonically by the base of the epicycloid.*

Cor. 2. *If in the figure* (art. 100) B″, C″, D″ *be the points on the involute of the three epicycloids which touch the sides of the triangle* BCD, *corresponding to the points of contact of these epicycloids, then the points* B″, C″, D″ *are collinear.*

105. From art. 99 we see that the arc AL : arc AC :: $\delta_{-1} : 2\varrho+\delta_{-1}$. Hence, letting fall the perpendicular OE, and denoting the angle AOE by ϕ, the angle LOE will be $m\phi$; if $m=\frac{\varrho}{\varrho+\delta_{-1}}$, and if we denote the radius of the circle by a, we have OE$=a\cos m\phi$. Hence the equation of the tangent to the epicycloid is

Fig. 17.



$$x\sin\phi+y\cos\phi=a\cos m\phi \quad . \quad . \quad . \quad . \quad (227)$$

For examples of the case in which the envelope of this line is an algebraic curve, see SALMON's 'Higher Curves,' p. 270.

106. The equation (227) may be written in the form $x+y\cot\phi=a\cos m\phi\operatorname{cosec}\phi$. Hence if the line OX be taken as the director line, the tangential equation of the epicycloid is

$$\nu=a\cos m\phi\,.\operatorname{cosec}\phi. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (228)$$

107. In order to find the intrinsic equation we have $f(\phi)=a\cos m\phi\,.\operatorname{cosec}\phi$.

Hence from equation (64), art. 30, we find

$$\frac{ds}{d\phi}=a(1-m^2)\cos m\phi\,;$$

$$\therefore\ s=\frac{a(1-m^2)}{m}\sin m\phi, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (229)$$

which is the required intrinsic equation.

The same result may be obtained from art. 93, equation (216).

Cor. If we substitute for m its value we find, from art. 97, equation (222) combined with (229), that

$$s=(\delta+\delta_{-1})\sin\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (230)$$

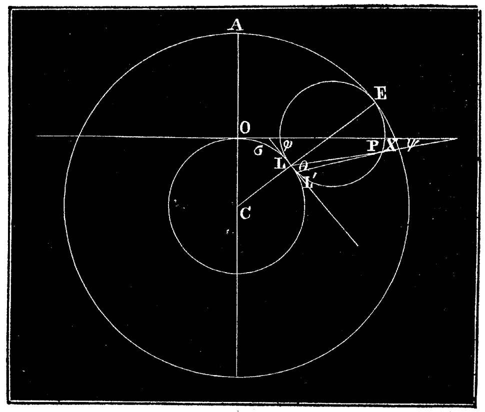
Hence putting $\theta=\frac{\pi}{2}$ and doubling, we have the whole length of the epicycloid from cusp to cusp=twice the sum of the diameters of the generating circles of the curve and its involute.

Section II.—*The Hypocycloid.*

108. Having discussed at considerable length the properties of the epicycloid, we shall treat very briefly those of the hypocycloid. In fact, analytically, the latter curve differs from the former only in the sign of a parameter; hence the properties of one curve are with slight modifications true of the other. The most interesting are those which are found by considering the curves in combination.

109. In the equation $\sigma=\delta_{-1}\theta$ of art. 96 let θ denote the angle which LP makes externally with the tangent to the director circle (which comes to the same thing as to consider θ negative. Now if θ change its sign, since we must regard σ as positive, δ_{-1} must change sign; in other words δ_{-1} has changed direction).

Fig. 18.



Also let ψ denote the angle which LP makes with the tangent to the director circle at the origin; then we have $\psi=\theta-\phi$;

$$\therefore \frac{d\psi}{d\sigma}=\frac{d\theta}{d\sigma}-\frac{d\phi}{d\sigma}.$$

Hence, if $'\delta$ denote the diameter of the circle LPE described about the infinitesimal triangle LL′P, we have

$$\frac{1}{'\delta}=\frac{1}{\delta_{-1}}-\frac{1}{\rho}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (231)$$

and we find, as in art. 97,

$$(\rho+{'\delta})\phi={'\delta}\theta;$$

that is, the arc AE=arc LP; and making the arc AX=semicircle LPE, X will be a fixed point, and we shall have the arc EP=arc EX. Hence the locus of P is the hypocycloid generated by the rolling of the circle EPL on the circle AEX.

110. Since $\frac{1}{'\delta}=\frac{1}{\delta_{-1}}-\frac{1}{\rho}$, see equation (231),

and $\frac{1}{\delta}=\frac{1}{\delta_{-1}}+\frac{1}{\rho}$, see equation (222),

$$\therefore \frac{1}{\delta}+\frac{1}{'\delta}=\frac{1}{\delta_{-1}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (232)$$

Hence if an epicycloid and hypocycloid have the same director circle, and if the generating circles of their involutes be equal to one another, the diameter of the generating circles of their involutes is a harmonic mean between the diameters of the generating circles of the curves themselves.

111. From equation (231) we get

$$\rho+{'\delta}=\frac{\rho^2}{\rho-\delta_{-1}},$$

and from equation (222)

$$\rho-\delta=\frac{\rho^2}{\rho+\delta_{-1}}.$$

Hence $\varrho+\delta$, ϱ, $\varrho-\delta$ are in harmonical progression. *Hence, with the same hypothesis as in the last article, the radius of the common director circle is a harmonic mean between the radii of their bases.*

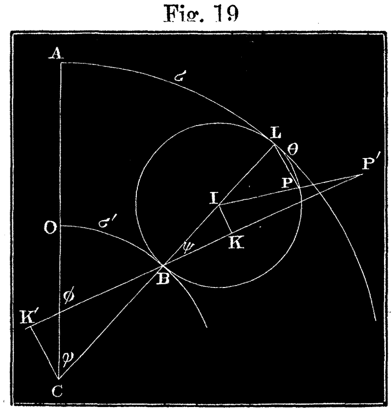
112. Several propositions proved for the cycloid may with scarcely any modification of the demonstration be extended to epi- and hypocycloids. Thus:—1°. *If from the point where the generating circle of an epi- or hypocycloid touches the base a perpendicular be let fall on the revolving radius, the envelope of the perpendicular is an epi- or hypocycloid.* 2°. *If the perpendicular be let fall from the point where the generating circle touches the director circle, the envelope is an epi- or hypocycloid.* 3°. *The envelope of the revolving radius is an epi- or hypocycloid.* 4°. *If two tangents,* PT, PT′, *to an epi- or hypocycloid meeting the director circle in the points* A, A′ *make a constant angle, the locus of the centre of the circle about the triangle* APA′ *is a circle.* 5°. *The envelope of the diameter of this circle which passes through* P *is an epi- or hypocycloid.* 6°. *The envelopes of the chords passing through* P *and through the highest or lowest points are epi- or hypocycloids.*

SECTION III.—*Extern Epicycloids.*

113. In the same manner as we have called the curve described by a fixed point in the revolving radius of the generating circle of a cycloid an in- or extern cycloid, we shall call the curve described by a fixed point in the plane of the generating circle of an epicycloid an in- or extern epicycloid according as the point is inside or outside the circumference of the circle. Similarly we shall have an in- or extern hypocycloid; so that the curve embraces four distinct species; but as they differ only in the magnitude or sign of a parameter, their properties are virtually the same; hence we shall discuss only the extern hypocycloid.

114. Let P′ be the point in the radius IP; then, since B is the centre of instantaneous rotation, BP′ will be a normal to the curve, and P′Z perpendicular to BP′ will be a tangent. The curve will have points of inflection. This follows at once from a beautiful theorem of Professor BALL'S, Astronomer Royal of Ireland:—"That if a plane figure is moving in a plane according to any law, there is always a circle of points rigidly connected with it, such that three consecutive positions of each point are in a right line"*. (See 'Proceedings of the Royal Irish Academy,' December 11, 1871.) Another proof will be given in the course of our investigations.

Fig. 19



Let the equation of the curve described by the point P be

$$\sigma=n\theta;$$

* This circle is called the "circle of inflections." The theorem in the text was originally given by SAVARY in his 'Leçons des Machines.'—November 1877.

then if the diameter of the generating circle be $2a$, we have, from art. 97,

$$n=\frac{2a\varrho}{\varrho-2a}.$$

Now let the angle IBP′ be denoted by ψ and IP′ by b, and we have

$$\sin 2\theta : \sin\psi :: \mathrm{BP}' : b.$$

But

$$\mathrm{BP}'=\mathrm{BK}+\mathrm{KP}'=b(c\cos\psi+\Delta\psi), \text{ if } c=\frac{a}{b};$$

hence

$$2\theta=\sin^{-1}\{\sin\psi(c\cos\psi+\Delta\psi)\}.$$

Again, we have

$$\sigma:\sigma'::\varrho:\varrho-2a;$$

$$\therefore\ \sigma'=2a\theta,$$

$$\therefore\ \sigma'=a\sin^{-1}\{\sin\psi(c\cos\psi+\Delta\psi)\}. \qquad (233)$$

And this is the equation of the evolute of the extern epicycloid.

115. From the diagram we have $\Phi=\phi+\psi$;

$$\therefore\ \frac{d\Phi}{d\sigma'}=\frac{d\phi}{d\sigma'}+\frac{d\psi}{d\sigma'}.$$

Now let a consecutive position of the line P′B intersect P′B in the point Q, and the arc OB in the point B′; then if D denote the diameter of the circle described about the infinitesimal triangle BB′Q, we have

$$\frac{d\Phi}{d\sigma'}=\frac{1}{\mathrm{D}},$$

and from equation (233) we get

$$\frac{d\psi}{d\sigma'}=\frac{\Delta(\psi)}{a(c\cos\psi+\Delta(\psi))};$$

$$\therefore\ \frac{1}{\mathrm{D}}=\frac{1}{\varrho-2a}+\frac{\Delta(\psi)}{a(c\cos\psi+\Delta(\psi))}.$$

Hence

$$\frac{1}{\mathrm{D}\cos\psi}=\frac{1}{(\varrho-2a)\cos\psi}+\frac{\Delta(\psi)}{a\cos\psi(c\cos\psi+\Delta\psi)}. \qquad (234)$$

If the polar of the point P′ with respect to the generating circle meet BP′ in N, we have, see art. 86, equation (207),

$$\mathrm{BN}=\frac{a\cos\psi(c\cos\psi+\Delta\psi)}{\Delta(\psi)}.$$

Hence, from equation (234), if CK′ be perpendicular to BP′, we have

$$\frac{1}{\mathrm{BQ}}=\frac{1}{\mathrm{BK}'}+\frac{1}{\mathrm{BN}}. \qquad (235)$$

Therefore BQ is half the harmonic mean between BK′ and BN; and this gives a geometric construction for the centre of curvature at the point P′.

116. From equation (235) we get

$$BQ=\frac{BK'.BN}{K'N}.$$

Now if we make BQ′=BN, the point Q′ would be the centre of curvature of an extern cycloid, see art. 86. Hence by subtraction

$$QQ'=\frac{BN^2}{K'N}. \qquad (236)$$

That is, *the distance between the centres of curvature of an extern cycloid and epicycloid is a third proportional to the lines* K′N *and* BN. This vanishes, as it ought, when K′ is at infinity.

117. From equation (234) we get the value of D cos ψ, that is, of BQ; thus

$$BQ=\frac{a(\varrho-2a)\cos\psi\,(\varrho\cos\psi+\Delta(\psi))}{ac\cos\psi-(\varrho-a)\Delta(\psi)}$$

and

$$BP'=b(\varrho\cos\psi+\Delta\psi).$$

Hence, remembering that $a=bc$, we get P′Q; that is, the radius of curvature at P′

$$=\frac{b(\varrho-a)\{c\cos\psi+\Delta\psi\}^2}{ac\cos\psi+(\varrho-a)\Delta\psi}. \qquad (237)$$

118. The following geometrical expression for the radius is remarkable for its simplicity and symmetry.

From equation (235),

$$BQ=\frac{BK'.BN}{K'N},$$

and from art. 86,

$$\frac{1}{BP'}=\frac{1}{BK}-\frac{1}{BN};$$

$$\therefore BP'=\frac{BK.BN}{KN},$$

$$\therefore P'Q=BN\left\{\frac{BK}{KN}+\frac{BK'}{K'N}\right\}. \qquad (238)$$

Cor. 1. The anharmonic ratio of the four points,

$$N, K, B, K'=BP':BQ. \qquad (239)$$

Cor. 2.

$$\frac{1}{BP'}+\frac{1}{BQ}=\frac{1}{BK}+\frac{1}{BK'}. \qquad (240)$$

119. If P′ be a point of inflection, the radius of curvature at P′ will be infinite, and

therefore the denominator of the fraction on the right of equation (237) will vanish; and therefore

$$ac\cos\psi=(\varrho-a)(-\Delta\psi). \quad \quad (241)$$

Hence $\Delta(\psi)$ must be negative. But $\Delta(\psi)=\mathrm{P'K}\div b$, $\therefore$ P′K is negative; or the point P′ must lie between B and K, and therefore IP′ is less than IB; that is, the curve must be an intern epicycloid; and, from what we have proved, we see that the angle IP′B must be obtuse.

Again, we have $c\cos\psi=\frac{\mathrm{BK}}{b}$ and $\Delta\psi=\frac{\mathrm{P'K}}{b}$; and therefore the following conditions for determining a point of inflection:—

1st. The angle IP′B must be obtuse.

2nd. BK : KP′ :: $\varrho-a:a$ from equation (241); that is, BK : KP′ :: CI : IB. Hence the triangle IP′B can be constructed; and it follows that there are two points of inflection in each revolution of the generating circle.

120. Since

$$\sigma'=2a\theta \text{ (see art. 114)},$$

and also

$$\sigma'=(\varrho-2a)\phi,$$

we have

$$\phi=\frac{a}{\varrho-2a}\sin^{-1}\{\sin\psi(c\cos\psi+\Delta\psi)\};$$

but

$$\varphi=\psi+\phi,$$

$$\therefore \frac{d\varphi}{d\psi}=\frac{ac\cos\psi+(\varrho-a)\Delta\psi}{(\varrho-2a)\Delta(\psi)}. \quad \quad (242)$$

Again

$$\frac{ds}{d\varphi}=\text{radius of curvature};$$

$$\therefore \frac{ds}{d\varphi}=\frac{b(\varrho-a)\{c\cos\psi+\Delta(\psi)\}^2}{ac\cos\psi+(\varrho-a)\Delta\psi};$$

$$\therefore \frac{ds}{d\psi}=\frac{b(\varrho-a)\{c\cos\psi+\Delta(\psi)\}^2}{(\varrho-2a)\Delta(\psi)}. \quad \quad (243)$$

Now, if s' denote the corresponding arc of the extern cycloid, we have, from art. 85,

$$\frac{ds'}{d\psi}=\frac{b(\cos\psi+\Delta(\psi))^2}{\Delta(\psi)};$$

$$\therefore ds=\left(\frac{\varrho-a}{\varrho-2a}\right)ds' \quad \quad (244)$$

This formula, which connects the extern epicycloid with the extern cycloid, may be also obtained as follows, for we have evidently

$$x=(\varrho-a)\sin\phi+b\sin\left(\frac{\varrho-a}{a}\right)\phi,$$

$$y=(\varrho-a)\cos\phi+b\cos\left(\frac{\varrho-a}{a}\right)\phi;$$

$$\therefore \frac{ds}{d\phi}=\frac{\varrho-a}{a}(a+b)\Delta(c'\theta),\ c'=\frac{2\sqrt{ab}}{a+b}.$$

But

$$\frac{d\phi}{d\theta}=\frac{2a}{\varrho-2a},$$

$$\therefore \frac{ds}{d\theta}=\frac{\varrho-a}{\varrho-2a}2(a+b)\Delta(c'\theta),$$

$$\therefore s=\frac{\varrho-a}{\varrho-2a}+2(a+b)\mathrm{E}(c'\theta); \qquad (245)$$

but

$$s'=2(a+b)\mathrm{E}(c',\theta) \text{ (see art. 83)},$$

$$\therefore s=\left(\frac{\varrho-a}{\varrho-2a}\right)s'. \qquad (246)$$

The same result as before.

CHAPTER VII.

Section I.—*Parallel Curves.*

121. The intercept which a parallel at the distance k from the movable line $x+y\cot\phi-\nu=0$ at either side makes on the directing line is $\nu\pm k\operatorname{cosec}\phi$, the choice of sign depending on the position of the parallel with respect to the origin. Hence we have the following theorem:—

If $\nu=f(\phi)$ be the tangential equation of a curve, the tangential equation of a parallel curve at the distance k is

$$\nu=f(\phi)\pm k\operatorname{cosec}\phi. \qquad (247)$$

Thus the parallel to the parabola is

$$\nu=a\tan\phi\pm k\operatorname{cosec}\phi, \qquad (248)$$

and the parallel to the cissoid

$$(2a-\nu)^3=27a^2\nu\cot^2\phi$$

is

$$\{(2a-\nu)\sin\phi\pm k\}^3=27a^2(\nu\sin\phi\pm k)\cos^2\phi. \qquad (249)$$

122. By the method of art. 26 we get the coordinates of a point on the parallel curve to be

$$x=f(\phi)+f'(\phi)\sin\phi\cos\phi\pm k\sin\phi, \qquad (250)$$

$$y=-f'(\phi)\sin^2\phi\pm k\cos\phi; \qquad (251)$$

3 o 2

and eliminating ϕ between these equations, we have the Cartesian equation of the parallel curve.

123. Since the ordinary tangential equation of a curve is the envelope of the line $\lambda x+\mu y+\nu=0$, the tangential equation of the parallel curve is the envelope of $\lambda x+\mu y+\nu\pm k\sqrt{\lambda^2+\mu^2}=0$. Hence we have the following theorem :—

If the tangential equation of a curve is

$$F(\lambda, \mu, \nu)=0,$$

the tangential equation of the parallel curve is

$$F(\lambda, \mu, \nu\pm k\sqrt{\lambda^2+\mu^2})=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (252)$$

124. If the result of the last article be expanded by TAYLOR'S theorem, it may be written in the form

$$P\pm Qk\sqrt{\lambda^2-\mu^2}=0;$$

or, cleared of radicals,

$$P^2-Q^2k^2(\lambda^2+\mu^2). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (253)$$

Hence the class of the parallel curve is twice the class of the original, and is independent of the sign of k. This shows the figures got by taking k plus and minus are both included in the equation of the parallel curve.

125. As in art. 30 we get for the intrinsic equation of the parallel curve*

$$\frac{ds}{d\phi}=2f'(\phi)\cos\phi+f''(\phi)\sin\phi+k,$$

$$\therefore s=f'(\phi)\sin\phi+\int f'(\phi)\cos\phi\, d\phi+k. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (254)$$

Cor. From the value of $\frac{ds}{d\phi}$ we see that the radius of curvature of the parallel curve differs from the radius of curvature of the original curve by the quantity k, as is otherwise evident.

Examples.

(1) Find the intrinsic equation of the parallel to the curve

$$\nu=\int\frac{d\theta}{\Delta(\theta)},$$

the function on the right being the elliptic integral of the first species.

* The following is an elegant focal property of parallel curves :—*Every single focus of the original curve is a double focus of the parallel curve.*

Demonstration. Let a tangent from a point I meet the original curve in two consecutive points P, P′; then if P, P′ be the centres of two circles, each of which passes through I, the line IP will be a normal to each, and therefore a normal to any curve of which these circles are generators. Now let the point I be one of the circular points at infinity; and since the parallel to any curve is the envelope of a circle of constant radius whose centre moves along the given curve, the line IP will be a normal to the parallel curve at I, and therefore a tangent at I; hence if two tangents be drawn to the original curve from the circular points at infinity, these tangents will touch the parallel curve at the circular points. Hence the theorem is proved.—*November* 1877.

Here we have

$$f'(\theta)=\frac{1}{\Delta(\theta)};$$

$$\therefore s=\frac{\sin\theta}{\Delta\theta}+\int\frac{\cos\theta d\theta}{\Delta\theta}+k\theta$$

$$=\frac{\sin\theta}{\Delta\theta}+\frac{1}{c}\sin^{-1}(c\sin\theta)+k\theta. \quad \ldots \quad (255)$$

(2) Find the intrinsic equation of the parallel to the curve

$$\nu=\int\Delta(\theta)d\theta,$$

or

$$\nu=\mathrm{E}(c\theta).$$

In this case $f'(\theta)=\Delta(\theta)$; and we find, as before,

$$s=\frac{3\sin\theta\,.\,\Delta(\theta)}{2}+\frac{1}{2c}\sin^{-1}(c\sin\theta)+k\theta. \quad \ldots \quad (256)$$

(3) The Cartesian equation of the curve parallel to the parabola is, by the equation of art. 122, the result of eliminating ϕ between the equations

$$x=2a\tan\phi+k\sin\phi,$$

$$y=-a\tan^2\phi-k\cos\phi.$$

This problem may be solved more easily by finding the envelope of the line

$$x+y\cot\phi-a\tan\phi-k\operatorname{cosec}\phi,$$

or, what is the same thing, the line

$$x\sin 2\phi+(a+y)\cos 2\phi-2k\cos\phi+(y-a-2k)=0.$$

Writing this in the form

$$\mathrm{A}\sin 2\phi+\mathrm{B}\cos 2\phi+\mathrm{C}\cos\phi+\mathrm{D}=0,$$

we get, by a known method, the required envelope to be

$$\{432(\mathrm{A}^2+\mathrm{B}^2)\mathrm{D}+9(\mathrm{D}-6\mathrm{B}^2)\mathrm{C}-2\mathrm{D}^3\}^2=4\{12(\mathrm{A}^2+\mathrm{B}^2)+3\mathrm{C}^2+\mathrm{D}^2\}^3. \quad . \quad (257)$$

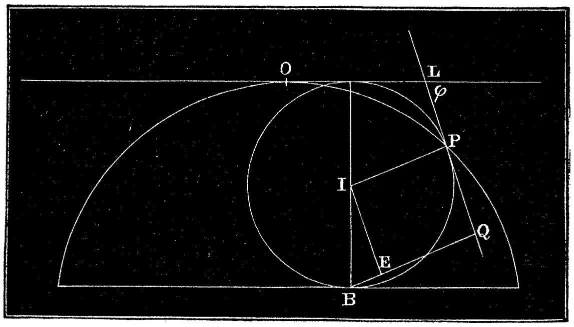
Cor. The characteristics of this curve are

$$\left.\begin{matrix}\mu=4, & \nu=6, & \varkappa=6\\ \delta=4, & \tau=3, & \iota=0\end{matrix}\right\}. \quad \ldots \quad (258)$$

We shall find the reciprocal of this curve in a future article.

(4) *The envelope of a fixed tangent to the generating circle of a cycloid is a parallel to another cycloid whose generating circle has a diameter equal to half the diameter of the former.*

Fig. 20.



Let PQ be the tangent at the point P fixed in the revolving circle; then, since B is the centre of instantaneous rotation, Q, the foot of the perpendicular from B on PQ, will be the point of contact of PQ with its envelope. From I let fall the perpendicular IE on BQ; then it is easy to see that the locus of the point E will be a cycloid whose generating circle will have IB for diameter, and EB will be the normal at E to the cycloid. Now, since EQ=IP =radius of generating circle, we see that the locus of Q is a parallel to the cycloid.

The same result may be shown thus:—It is evident that $OL=a\phi+a\tan\frac{1}{2}\phi$; the equation of the curve which is the envelope of LP is

$$\nu=a\phi+a\tan\tfrac{1}{2}\phi,$$

and the intrinsic equation of this is

$$s=a\phi+2a\sin\phi, \quad \ldots \ldots \ldots \ldots \quad (259)$$

which is a parallel to a cycloid.

Cor. 1. *The envelope of any line in rigid connexion with the generating circle of a cycloid is a parallel to another cycloid.*

Cor. 2. *In like manner the envelope of a fixed tangent* to, *or of any line in rigid connexion* with, *the generating circle of an epicycloid is a parallel to another epicycloid.*

126. From the equation (254) we infer that if $s=F(\phi)$ be the intrinsic equation of a curve, the equation of the parallel to it is

$$s=F(\phi)\pm k(\phi);$$

and hence (see art. 64, equation (182)) the polar equation of the reciprocal of the parallel curve is

$$\frac{a^2}{\varrho}=\left\{1+\left(\frac{d}{d\phi}\right)^2\right\}^{-1}\{F'(\phi)\pm k\}, \quad \ldots \ldots \ldots \quad (260)$$

or

$$\left.\begin{aligned}\frac{a^2}{\varrho}=\sin\phi\int\cos\phi(F'\phi\pm k)\,d\phi-\cos\phi\int\sin\phi(F'(\phi)\pm k)\,d\phi\\ +C_1\cos\phi+C_2\sin\phi.\end{aligned}\right\} \quad \ldots \quad (261)$$

In this equation we have used a^2 as the numerator to ϱ on the left side of the equation instead of k^2 of recent articles in order to avoid confusion of notation.

127. The reciprocal of the parallel curve is found at once from the tangential equation. For it is evident that the polar equation of the reciprocal of the parallel to the curve

$$\nu=f(\phi)$$

is

$$\frac{k^2}{\varrho}=f(\phi)\sin\phi\pm r, \quad \text{(262)}$$

where r is the distance between the curve and its parallel, and k is the radius of the circle of reciprocation.

Examples.

(1) To find the reciprocal of the parallel to a parabola.

We have

$$f(\phi)=a\tan\phi,$$

$$\therefore \frac{k^2}{\varrho}=\frac{a\sin^2\phi}{\cos\phi}\pm r$$

is the required equation. This, expressed in Cartesian coordinates, is

$$(k^2x-ay^2)^2=r^2x^2(x^2+y^2). \quad \text{(263)}$$

This curve has three double points, namely, the origin and the points where the conic k^2x-ay^2 meets the line at infinity.

Again, the curve is evidently the envelope of the conic

$$x^2+y^2+2\mu(k^2x-ay^2)+\mu^2r^2x^2. \quad \text{(264)}$$

The discriminant of this conic is

$$(1-2a\mu)\mu^2k^2.$$

This shows that there are two values of μ, for which the conic breaks up into a pair of lines; hence the curve has four double tangents. Therefore the characteristics of the curve are

$$\mu=4, \qquad \delta=3, \qquad \tau=4,$$
$$\nu=6, \qquad \iota=6, \qquad \varkappa=0.$$

(2) Find the reciprocal of the parallel to the curve

$$\varrho^m=a^m\sin m\theta.$$

From equation (37), art. 24 we have at once the polar equation of the reciprocal

$$\frac{k^2}{\varrho}=a\left\{\sin\frac{m\phi}{m+1}\right\}^{\frac{m+1}{m}}\pm r. \quad \text{(265)}$$

(3) Find the intrinsic equation of the lemniscate.

The tangential equation is (see art. 25)

$$\nu=a\left(\sin\frac{2\phi}{3}\right)^{\frac{3}{2}}\operatorname{cosec}\phi,$$

and, by the method of art. 30, we find the intrinsic equation to be

$$s=\frac{a}{3}\int\frac{d\phi}{\sqrt{\sin\frac{2\phi}{3}}}. \qquad (266)$$

If we put $\sqrt{\sin\frac{2\phi}{3}}=\cos\theta$, this equation becomes

$$s=\frac{a}{\sqrt{2}}\int\frac{d\theta}{\sqrt{1-\frac{1}{2}\sin^2\theta}},$$

or

$$s=\frac{a}{\sqrt{2}}\mathrm{F}\left(\frac{1}{\sqrt{2}}.\theta\right). \qquad (267)$$

(4) Find the reciprocal of the parallel to an ellipse.

Here

$$f(\phi)=\sqrt{a^2+b^2\cot^2\phi},$$

$$\therefore \frac{k^2}{\rho}=\sqrt{a^2\sin^2\phi+b^2\cos^2\phi}\pm r$$

is the required equation.

This curve, in Cartesian coordinates, is

$$4k^4r^2\varrho^2=(a^2x^2+b^2y^2-k^4-r^2\varrho^2)^2. \qquad (268)$$

This curve can by linear transformation be changed into a bicircular quartic. For writing the equation in full by putting x^2+y^2 in place of ϱ^2, and then changing y into $\left(\frac{a^2-r^2}{b^2-r^2}\right)^{\frac{1}{2}}y$, we get

$$(x^2+y^2)^2+\frac{k^4r^2}{a^2-r^2}\left\{\frac{r^2-2(a^2-r^2)(b^2-r^2)}{(a^2-r^2)(b^2-r^2)}\right\}(x^2+y^2)$$
$$+\frac{k^4}{(a^2-r^2)^2}\left\{k^4-\frac{4r^2}{b^2-r^2}(b^2x^2+a^2y^2)\right\}=0. \qquad (269)$$

128. To find the reciprocal of a bicircular quartic, with respect to one of its circles of inversion.

The following method of generating these curves is given in my memoir on "Bicircular Quartics" (see Transactions of the Royal Irish Academy, vol. xxiv. p. 460):—

Let $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ be a conic F, called the focal conic, $(x-f)^2+(y-g)^2-r^2=0$ a circle J; then if from the centre O of the circle we let fall a perpendicular OT on any tangent to F, and take two points, P, P′, in opposite directions from T on OT, such that

$$\mathrm{OT}^2-\mathrm{TP}^2=\mathrm{OT}^2-\mathrm{TP}'^2=r^2,$$

the locus of the points P, P′ is a bicircular quartic. Now, denoting OT by p, and OP by ϱ, we get from this construction

$$2p\varrho=r^2+\varrho^2,$$

or

$$2\{\sqrt{a^2\cos^2\alpha+b^2\sin^2\alpha}-(f\cos\alpha+g\sin\alpha)\}\varrho=r^2+\varrho^2. \qquad (270)$$

Again, since OP . OP$'=r^2$, the points P, P$'$ are inverse points with respect to the circle J, and the perpendicular through P to the line OP will be the polar of P$'$; therefore the envelope of this perpendicular will be the reciprocal of the bicircular quartic. Now, let ϕ be the angle which the perpendicular makes with the axis of x, or the *directing* line, and ν=intercept, then we have $\phi=90-\alpha$, and $\rho=\nu\sin\phi$.

Therefore the tangential equation of the reciprocal of the bicircular quartic is

$$\nu^2\sin^2\phi+r^2=2\{\sqrt{a^2\sin^2\phi+b^2\cos^2\phi}-(f\sin\phi+g\cos\phi)\}\nu\sin\phi. \quad . \quad . \quad (271)$$

Cor. The equation (271) is also the first negative pedal of a bicircular quartic.

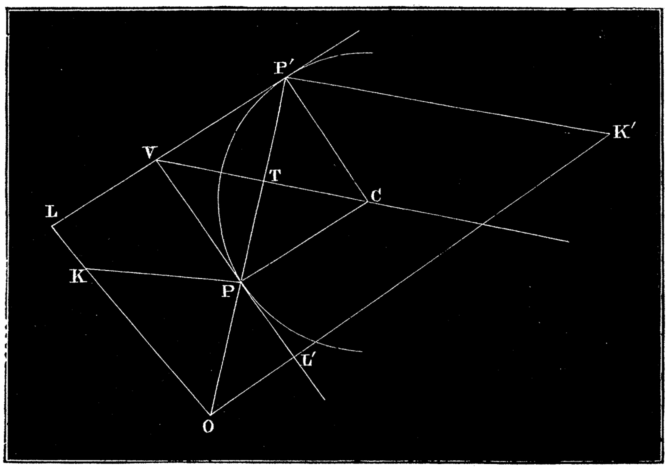
129. If we divide the equation (271) by $\sin^2\phi$, and then change ν into $-\frac{\nu}{\lambda}$ and $\cot\phi$ into $\frac{\mu}{\lambda}$, we get, after a slight reduction,

$$\{r^2(\lambda^2+\mu^2)+\nu^2-2f\nu\lambda-2g\lambda\mu\}^2=4(a^2\lambda^2+b^2\mu^2)\nu^2, \quad . \quad . \quad . \quad . \quad (272)$$

which is in the ordinary form of tangential equations.

130. From the equation (271) it is plain that to each value of ϕ there are two values of ν. This is otherwise evident; for erecting the perpendicular PK and P$'$K$'$ to OP

Fig. 21.



and OP$'$, these perpendiculars will make intercepts on the director line, which will be the required values of ν. Let C be the centre of the generating circle, then C will be a point on the focal conic, and CP, CP$'$ will be normals to the quartic, and PV, P$'$V will be tangents. Now if K$'$ be the point of contact of P$'$K$'$ with its envelope, then the angle P$'$K$'$O=OP$'$V, and therefore OK$'$ is parallel to CP, and OK to CP$'$. Hence, drawing from the point O two parallels to the normals at P, P$'$, they will meet PK, P$'$K$'$ in the points of contact of these lines with their envelopes, and they will intersect the tangents PV, P$'$V perpendicularly in the points L, L$'$.

Cor. 1. The locus of the points L, L$'$ is evidently the first positive pedal of the bicircular quartic.

Cor. 2. The first positive pedal of a bicircular quartic is the inverse of its first negative pedal; for evidently

$$\text{OL}' . \text{OK}'=\text{OP} . \text{OP}'=r^2.$$

Cor. 3. $OK'-OK$=diameter of generating circle of the bicircular quartic; for, denoting the angle $K'OP'=KOP$ by ψ, we have

$$OK'\cos\psi-OK\cos\psi=OP'-OP=2CP\cos\psi;$$

$\therefore OK'-OK=2CP$=diameter of generating circle.

SECTION II.—*Rectification of Bicircular Quartics.*

131. If through the point O (see diagram, art. 130) we draw a consecutive line OQQ′, then the perpendiculars to this line at the points Q, Q′ will pass through the points K, K′, and we have evidently

$$\frac{P'Q'-PQ}{d\phi}=OK'-OK.$$

Hence, denoting the elements P′Q′, PQ of the quartic by ds' and ds, we have

$$\frac{ds'-ds}{d\phi}=2\rho \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (273)$$

if ρ denotes the radius CP of the generating circle (see *Cor.* 3, art. 130).

132. Mr. W. ROBERTS showed, in LIOUVILLE'S Journal, vol. xv. p. 194, "Sur les arcs des Lignes Aplanétiques," that the difference between two arcs of a Cartesian oval is expressed by an arc of an ellipse; and Professor GENOCCHI showed some time afterwards, in TORTOLINI'S 'Annali,' that the arc of a Cartesian oval is the sum of three elliptic arcs. We propose in this section to extend these theorems to bicirculars in general. We will show that GENOCCHI'S theorem is an immediate inference from ROBERTS'S, and that each is only a particular case of a more general theorem which holds for all bicirculars, and which can be expressed in terms of the radii of the generating circles of these curves.

In order that we may not have to be referring to other writings, we shall investigate briefly the leading properties of these curves, referring for a fuller discussion to the author's memoir on Bicirculars.

133. In art. 128, equation (270), we have the polar equation of a bicircular quartic. This, expressed in Cartesian coordinates, is

$$4(a^2x^2+b^2y^2)=(x^2+y^2+2fx+2gy+r^2)^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (274)$$

This equation is the envelope of the conic

$$S+\mu C+\mu^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (275)$$

where S represents the expression $a^2x^2+b^2y^2$, and C the circle $x^2+y^2+2fx+2gy+r^2=0$. Now the discriminant of the equation (275) is

$$\frac{\mu^2f^2}{a^2+\mu}+\frac{\mu^2g^2}{a^2+\mu}=\mu r^2\mu^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (276)$$

a biquadratic equation showing that there are four pairs of double tangents.

If the four values of μ be denoted by μ_1, μ_2, μ_3, μ_4, we have the equations of the four pairs of double tangents to the bicircular (see my memoir "On Bicirculars," art. 47). These pairs of lines are

$$S+\mu_1C+\mu^2,\ S+\mu_2C+\mu_2^2,\ \&c.;$$

and, from the same article, the double points of these pairs of lines are the four centres of inversion of the quartic. Since one value of μ is obviously $=0$ in the foregoing equation, we see that the pair of double tangents drawn from the centre of the circle of inversion J $((x-f)^2+(y-g)^2=r^2)$ will, when that centre is taken as origin, be $a^2x^2+b^2y^2=0$. Hence, if the other centres be taken respectively as origin, the equations of the other pairs of double tangents will be

$$(a^2+\mu_2)x^2+(b^2+\mu_2)y^2=0, \quad \ldots \quad (277)$$

$$(a^2+\mu_3)x^2+(b^2+\mu_3)y^2=0, \quad \ldots \quad (278)$$

$$(a^2+\mu_4)x^2+(b^2+\mu_4)y^2=0. \quad \ldots \quad (279)$$

Now since the pair of lines $a^2x^2+b^2y^2=0$ are the asymptotes of the reciprocal of the conic $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$, we infer that the pairs of lines (277), (278), (279) are the asymptotes of the reciprocals of the other focal conics. Hence we have the following system as the equations of these conics:—

$$\frac{x^2}{a^2+\mu_2}+\frac{y^2}{b^2+\mu_2}=1, \quad \ldots \quad (280)$$

$$\frac{x^2}{a^2+\mu_3}+\frac{y^2}{b^2+\mu_3}=1, \quad \ldots \quad (281)$$

$$\frac{x^2}{a^2+\mu_4}+\frac{y^2}{b^2+\mu_4}=1. \quad \ldots \quad (282)$$

Hence the four focal conics of a bicircular quartic are confocal.

134. Since the equation (276) may be written in the form

$$\frac{f^2}{a^2+\mu}+\frac{g^2}{b^2+\mu}=1+\frac{x^2}{\mu},$$

and this is the discriminant of $\mu F+J$ (where J and F have the values in art. 128; see SALMON'S 'Conics,' p. 324), we infer that the same values of μ which will make $\mu F+J$ a pair of lines will make $S+\mu C+\mu^2$ a pair of lines; the two pairs of lines will have the same double point, their equations referred to that point as origin being

$$\frac{x^2(a^2+\mu)}{a^2}+\frac{y^2(b^2+\mu)}{b^2}=0, \quad \ldots \quad (283)$$

$$x^2(a^2+\mu)+y^2(b^2+\mu)=0. \quad \ldots \quad (284)$$

Hence we have the following theorem:—If F and J be a corresponding focal conic and circle of inversion of a bicircular quartic, and if μ_1, μ_2, μ_3 be the roots of the cubic which is the discriminant of $\mu F+J$, then if F be given in its canonical form $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$, the equations of the other three focal conics are got from this by changing a^2, b^2 respectively into $a^2+\mu_1$, $b^2+\mu_1$; $a^2+\mu_2$, $b^2-\mu_2$; and $a^2+\mu_3$, $b^2+\mu_3$.

135. When $S+\mu C+\mu^2$ represents a pair of lines, the coordinates of the double point are, by the usual process,

$$\frac{-\mu f}{a^2+\mu},\ \frac{-\mu g}{b^2+\mu}$$

if referred to the centre of J as origin, and

$$\frac{a^2 f}{a^2+\mu},\ \frac{b^2 g}{b^2+\mu}$$

if referred to the centre of F as origin. Hence we have the following theorem:—

If $F\equiv\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$ and $J=(x-f)^2+(y-g)^2-r^2=0$ be a corresponding focal conic and circle of inversion of a bicircular quartic, and if μ_1, μ_2, μ_3 be the three roots of the equation which is the discriminant of $\mu F+J$, then the coordinates of the centres of the three other circles of inversion are:—

$$\frac{a^2 f}{a^2+\mu_1},\ \frac{b^2 g}{b^2+\mu_1}, \quad \dots \quad (285)$$

$$\frac{a^2 f}{a^2+\mu_2},\ \frac{b^2 g}{b^2+\mu_2}, \quad \dots \quad (286)$$

$$\frac{a^2 f}{a^2+\mu_3},\ \frac{b^2 g}{b^2+\mu_3}. \quad \dots \quad (287)$$

136. Being given $F\equiv\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$, and $J\equiv(x-f)^2+(y-g)^2-r^2=0$, the equation of the quartic is

$$4(a^2x^2+b^2y^2)-(x^2+y^2+2fx+2gy+r^2)^2=0. \quad \dots \quad (288)$$

Again, being given

$$F'\equiv\frac{x^2}{a^2+\mu_1}+\frac{y^2}{b^2+\mu_1}-1=0,$$

$$J'\equiv\left(x-\frac{a^2 f}{a^2+\mu_1}\right)^2+\left(y-\frac{b^2 g}{b^2+\mu_1}\right)^2-r'^2=0,$$

the equation of the same quartic is

$$4\{(a^2+\mu_1)x^2+(b^2+\mu_1)y^2\}-\left\{x^2+y^2+\frac{2a^2 f}{a^2+\mu_1}x+\frac{2b^2 g}{b^2+\mu_1}y+r'^2\right\}^2. \quad \dots \quad (289)$$

In order to compare the equations (288) and (289), which represent the same curve, they must be referred to the same origin; we will therefore transform (288) to the same origin as (289), and we get

$$4\left\{a^2\left(x+\frac{\mu_1 f}{a^2+\mu_1}\right)^2+b^2\left(y+\frac{\mu_1 g}{b^2+\mu_1}\right)^2\right\}-\left\{\left(x+\frac{\mu_1 f}{a^2+\mu_1}\right)^2+\left(y+\frac{\mu_1 g}{b^2+\mu_1}\right)^2\right.$$
$$\left.+2f\left(x+\frac{\mu_1 f}{a^2+\mu_1}\right)+2g\left(y+\frac{\mu_1 g}{b^2+\mu_1}\right)+r'^2\right\}^2=0. \quad \dots \quad (290)$$

Since the equations (289) and (290) represent the same curve, their absolute terms must be the same. Now if the absolute term in the latter equation be reduced by means of the relation $\frac{f^2}{a^2+\mu_1}+\frac{g^2}{b^2+\mu_1}=1+\frac{r^2}{\mu_1}$, it becomes

$$\left\{\frac{\mu_1{}^2f^2}{(a^2+\mu_1)^2}+\frac{\mu_1{}^2g^2}{(b^2+\mu_1)^2}-r^2\right\}^2,$$

and the absolute term in equation (289) is r'^4; hence we get

$$r^2+r'^2=\left(\frac{\mu_1 f}{a^2+\mu_1}\right)^2+\left(\frac{\mu_1 g}{b^2+\mu_1}\right)^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (291)$$

That is, the sum of the squares of the radii of J and J′ equals square of the distance between their centres. Hence J and J′ cut each other orthogonally.

137. The propositions established in articles 133–136 are those which we shall require for the present investigation. They are proved in the memoir already cited, but by another method. It is useful to recapitulate them here:—

(1) A bicircular quartic is the envelope of a variable circle whose centre moves on a given conic F, called the focal conic, and which cuts a given circle J orthogonally.

(2) The circle J is a circle of inversion of the quartic.

(3) There are four circles of inversion and four focal conics.

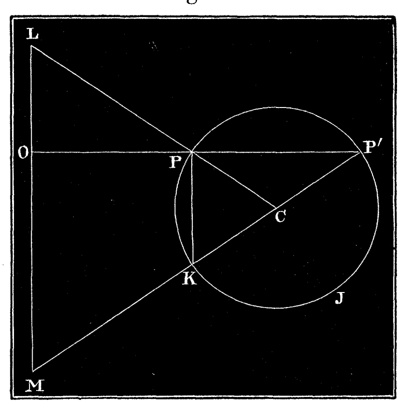
(4) The four focal conics are confocal.

(5) The four circles of inversion are mutually orthogonal.

(6) The centres of the circles of inversion are such that any three will form a self-conjugate triangle with respect to the circle which has the fourth for centre; in other words, the four centres form the angular points and the point of intersection of perpendiculars of a plane triangle.

138. The proposition proved in art. 131 is our fundamental one for the rectification of bicirculars; it will be seen that it is a generalization of Mr. Roberts's theorem already referred to. On account of its importance we will here give an elementary proof of it, but under a slightly different enunciation.

Fig. 22.



If OPP′ be any line cutting a circle J in the points P, P′, then if two circles passing through O touch J in the points P, P′ respectively, the difference between their diameters is equal to the diameter of J.

Demonstration.—Let C be the centre of J. Join CP, CP′, and produce them to meet the line LOM drawn perpendicular to OP. Join PK. Now, evidently, PL=KM;

$\therefore$ P′M−PL=P′K=diameter of J.

Hence the proposition is proved.

Cor. If the point O be within the circle J, we shall have the sum of the diameters equal the diameter of J.

139. If we denote the diameter of J by 2ϱ, and if a line OQQ′ (see last diagram) infinitely near OP make an angle $d\theta$ with OP, then $\frac{P'Q'}{d\theta}=P'M$ and $\frac{PQ}{d\theta}=PL$.

Hence by art. 138 we have

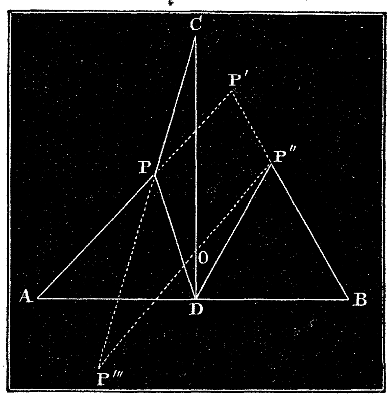
$$P'Q'-PQ=2\varrho d\theta. \qquad (292)$$

Cor. If the point O be inside J we have

$$P'Q'+PQ=2\varrho d\theta. \qquad (293)$$

140. If four circles be mutually orthogonal, and if any figure be inverted with respect to each of the four circles in succession, the fourth inversion will coincide with the original figure *.

Fig. 23.



Demonstration.—It will plainly be sufficient to prove the proposition for a single point, for the general proposition will then follow.

Since the four circles are mutually orthogonal, their four centres will form the angular points and the intersection of the perpendiculars of a plane triangle. Let them be the points A, B, C, O; CO produced will intersect AB perpendicularly in D, and the squares of the radii of the four circles will be equal to the four rectangles

$$AB\,.\,AD, \quad BA\,.\,BD, \quad -CO\,.\,OD, \quad CD\,.\,OD,$$

one of the circles being imaginary, namely the one the square of whose radius is $-CO\,.\,OD$. Now let P be the point we operate on, and let P′ be its inverse with respect to the circle A, and P″ the inverse of P′ with respect to the circle B. Join P″O and CP meeting in P‴. Now since P′ is the inverse of P with respect to the circle A, the square of whose radius is AB . AD, we have the rectangle AB . AD=AP . AP′. The triangle ADP is similar to the triangle AP′B, therefore the angle ADP=angle AP′B; in like manner, the angle BDP″=angle AP′B, therefore the triangles ADP and BDP″ are equiangular, and the rectangle AD . DB=rectangle PD . DP″. Again, because O is the intersection of the perpendiculars of the triangle ABC, the rectangle AD . DB= CD . OD; hence CD . OD=PD . DP″, and the angles CDP and P″DO being the com-

* An important extension of this theorem can be got by combining it with the following proposition, which is proved in art. 95 of my memoir on "Cyclides and Sphero-Quartics":—If a sphero-quartic be projected on one of the planes of circular section of any quadric passing through it by lines parallel to the greatest or least axis of the quadric, the projection will be a bicircular quartic whose centres of inversion will be the projection of the centres of inversion of the sphero-quartic. The extension is as follows. *There exists in sphero-quartics a series of inscribed quadrilaterals* ABCD, *whose sides* AB, BC, CD, DA, *taken in order, pass through the vertices of the four cones of the sphero-quartic.*

plements of equal angles are equal: therefore the triangles P″DO and CDP are equiangular, and the angle OP″D=PCD; hence the four points C, P″, D, P‴ are concyclic, and therefore the point P‴ is the inverse of P″ with respect to the imaginary circle the square of whose radius is −CO . OD, and whose centre is the point O.

Again, the angle ODP=P″DO=OP‴C; therefore the points O, D, P‴, P are concyclic, and P is the inverse of P‴ with respect to the circle whose centre is C, and the square of whose radius is the rectangle CD . OD. Hence the proposition is proved.

Cor. 1. If the point C be at infinity, the point O will coincide with D, and the point P will be the reflection of P‴ with respect to the line AB.

Cor. 2. If the points A, B, C, O be the centres of inversion of a bicircular quartic, and if the point P be on the curve, the points P′, P″, P‴ will also be on the curve.

141. Let the radii of the generating circles of the bicircular quartic which touch it at the four pairs of points (PP′), (P′P″), (P″P‴), (P‴P) be denoted by ϱ, ϱ', ϱ'', ϱ''' respectively. Let the angle which the line APP′ makes with any fixed line in the plane, say the axis of x, be denoted by θ, and the angles which the lines BP′P″, OP″P‴, CPP‴ make with the same line by θ', θ'', θ'''.

Now if the points P, P′, P″, P‴ describe infinitesimal arcs, we have (see art. 139), denoting these arcs by ds, ds', &c.,

$$ds' - ds = 2\varrho\, d\theta\ ,$$

$$ds' - ds'' = 2\varrho'\, d\theta'\ ,$$

$$ds'' + ds''' = 2\varrho''\, d\theta''\ ,$$

$$ds''' - ds = 2\varrho'''\, d\theta'''.$$

Hence

$$ds' = \varrho d\theta + \varrho' d\theta' + \varrho'' d\theta'' + \varrho''' d\theta''' ; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (294)$$

$$\therefore\ s' = \int \varrho d\theta + \int \varrho' d\theta' + \int \varrho'' d\theta'' + \int \varrho''' d''' \ . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (295)$$

Hence the arc of a bicircular quartic is the sum of four similar integrals. We shall find that each of them is expressed in terms of elliptic integrals. This theorem is our generalization of ROBERTS'S and GENOCCHI'S theorems*.

* The following proof of the theorem art. 138 will apply equally to sphero-quartics, and will lead to an important extension of the theorem of this article:—

Let CV, C′V′ be two consecutive tangents to the focal conic F of the bicircular quartic (or, in the case of a sphero-quartic, to the focal sphero-conics), and OPP′, OQQ′ two perpendiculars to CV, C′V′ (see fig. art. 130). If CV, C′V′ intersect the generating circle in the points R, R′, it is evident, from geometrical considerations, that

$$RR' = \tfrac{1}{2}(P'Q' - PQ).$$

But $RR' = \varrho d\theta$ for bicircular quartics and $= \sin \varrho d\theta$ for sphero-quartics; hence, remembering the theorem in the footnote to art. 140, we have, for sphero-quartics,

$$s' = \int \sin \varrho d\theta + \int \sin \varrho' d\theta' + \int \sin \varrho'' d\theta'' + \int \sin \varrho''' d\theta''',$$

and the rectification of sphero-quartics is reduced to elliptic integrals.—*November* 1877.

142. If the bicircular becomes a Cartesian oval, the point C will be at infinity, and we shall have (see cor. 1, art. 140),

$$ds''' - ds = 0.$$

Hence

$$\rho'''d\theta''' = 0,$$

and

$$ds' = \rho d\theta + \rho' d\theta' + \rho'' d\theta'; \quad \ldots \ldots \quad (296)$$

$$\therefore s' = \int \rho d\theta + \int \rho' d\theta' + \int \rho'' d\theta''. \quad \ldots \ldots \quad (297)$$

This is GENOCCHI's theorem.

Cor. By integrating the equation

$$ds' - ds = 2\rho d\theta$$

we get

$$s' - s = 2\int \rho d\theta, \quad \ldots \ldots \quad (298)$$

which is ROBERTS's theorem.

143. To reduce the integral $\int \rho d\theta$ to the normal form of elliptic integrals.

Let the focal conic of the bicircular be $F \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$, and the corresponding circle of inversion $J \equiv (x-f)^2 + (y-g)^2 - k^2 = 0$. The equation of a tangent to the conic is

$$x \cos\theta + y \sin\theta = \sqrt{a^2\cos^2\theta + b^2\sin^2\theta},$$

or say

$$x \cos\theta + y \sin\theta = p;$$

therefore if x' y' be the point of contact, we have

$$\frac{x'}{a^2} = \frac{\cos\theta}{p},$$

$$\frac{y'}{b^2} = \frac{\sin\theta}{p}.$$

Now if ϕ be the eccentric angle, $x' = a\cos\phi$, $y' = b\sin\phi$;

$$\therefore \frac{\cos\phi}{a} = \frac{\cos\theta}{p},$$

$$\frac{\sin\phi}{b} = \frac{\sin\theta}{p};$$

$$\therefore \tan\theta = \frac{b}{a}\tan\phi.$$

Hence

$$d\theta = \frac{abd\phi}{a^2\sin^2\phi + b^2\cos^2\phi} \quad \ldots \ldots \quad (299)$$

Again, since ρ is the radius of the generating circle whose centre is the point $(x'\ y')$ on the focal conic F, and which cuts the circle J orthogonally, we have

$$\varrho=\sqrt{(a\cos\phi-f)^2+(b\sin\phi-g)^2-k^2}\,; \quad . \quad . \quad . \quad . \quad . \quad . \qquad (300)$$

$$\therefore\ \varrho d\theta=\frac{\varrho^2 d\theta}{\varrho}=\frac{ab\{(a\cos\phi-f)^2+(b\sin\phi-g)^2-k^2\}d\phi}{\varrho\{a^2\sin^2\phi+b^2\cos^2\phi\}}$$

$$=-\frac{abd\phi}{\varrho}+\frac{ab\{a^2+b^2+f^2+g^2-k^2-2af\cos\phi-2bg\sin\phi\}d\phi}{\varrho\{a^2\sin^2\phi+b^2\cos^2\phi\}}.$$

Put $\sin\phi=\frac{2z}{1+z^2}$; then $\cos\phi=\frac{1-z^2}{1+z^2}$, and $d\phi=\frac{2dz}{1+z^2}$. Making these substitutions, we get

$$\varrho d\theta=-\frac{2abdz}{\sqrt{Z}}+\frac{2ab\{(a^2+b^2+f^2+g^2-k^2)(1+z^2)^2-2af(1-z^4)-4bgz(1+z^2)\}dz}{\{4a^2z^2+b^2(1-z^2)\}\sqrt{Z}}, \quad . \qquad (301)$$

where Z stands for the quartic

$$\{a(1-z^2)-f(1+z^2)\}^2+\{2bz-g(1+z^2)\}^2-k^2(1+z^2)^2\ . \quad . \quad . \quad . \qquad (302)$$

144. In order to reduce still further the expression (301), we must decompose

$$2ab\,\frac{\{(a^2+b^2+f^2+g^2-k^2)(1+z^2)^2-2af(1-z^4)-4bgz(1+z^2)\}}{4a^2z^2+b^2(1-z^2)^2}$$

into simpler fractions, or say the fraction Φ into simpler fractions.

Let us, for shortness, put t^2 for the expression $a^2+b^2+f^2+g^2-k^2$. Making this substitution, and denoting the eccentricity of the conic $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1$ by e, we find, by dividing &c.,

$$\Phi=\frac{2a}{b}\,(t^2+2af)-\frac{2ab\{4b^3gz^3+4a^2\{e^2t^2+af(1+e^2)\}z^2+4b^3gz+b^2t^2+4ab^2f\}}{\{b^2z^2+a^2(1+e)^2\}\{b^2z^2+a^2(1-e)^2\}}.$$

Then decomposing the fraction still remaining, and substituting, we get, after some reduction,

$$\int\varrho d\theta=\frac{2a}{b}\{(a+f)^2+g^2-k^2\}\int\frac{dz}{\sqrt{Z}}$$

$$-\frac{4b^2g}{a}\int\frac{zdz}{\{(1+e)z^2+(1-e)\}\sqrt{Z}}-\frac{4b^2g}{a}\int\frac{zdz}{\{(1-e)z^2+(1+e)\}\sqrt{Z}}$$

$$-\frac{b\{8afe+(1-e)(1+3e)t^2\}}{2ae(1+e)}\int\frac{dz}{\{(1+e)z^2+(1-e)\}\sqrt{Z}}$$

$$-\frac{b\{8afe-(1+e)(1-3e)t^2\}}{2ae(1-e)}\int\frac{dz}{\{(1-e)z^2+(1-e)\}\sqrt{Z}} \quad . \quad . \quad . \quad . \qquad (303)$$

Since Z is a quartic function of the variable, each of these integrals belongs to the domain of the elliptic integrals. (See CAYLEY'S 'Elliptic Functions.')

145. If the bicircular quartic be a Cartesian oval, the focal conic F is a circle; and taking the line joining the centres of J and F as the axis of x, we may write their equations in the forms

$$F\equiv x^2+y^2-a^2=0, \quad J\equiv(x-f)^2+y^2-k^2=0,$$

and we get

$$\int\rho d\theta=\int\sqrt{a^2+f^2-k^2-2af\cos\theta}\,d\theta; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (304)$$

and this represents an arc of an ellipse. Hence ROBERTS'S and GENOCCHI'S theorems are proved.