pected greater things from his pregnant and ſagacious Wit. For he was ſcarce 20 Years of Age when he held theſe Correſpondencies with Mr. *Crabtrie*. And at the Age of 23. he was killed at *Marſton-Moor-Battle*, on *July* 2. 1644. fighting for King *Charles* I. His Father was *Henry Gaſcoigne* Eſq; of *Middleton*, between *Leeds* and *Wakefield*.

IV. *An Attempt towards the Improvement of the Method of approximating, in the Extraction of the Roots of Equations in Numbers. By* Brook Taylor, *Secretary to the Royal Society.*

IN *Phil. Tran.* No. 210. Dr. *Halley*, now Secretary of the Royal Society, has publiſh'd a very compendious and uſeful Method of extracting the Roots of affected Equations of the common Form, in Numbers. This Method proceeds by aſſuming the Root deſired nearly true to one or two Places in Decimals (which is done by a Geometrical Conſtruction, or by ſome other convenient way) and correcting the Aſſumption by comparing the Difference between the true Root and the aſſumed, by means of a new Equation whoſe Root is that Difference, and which he ſhews how to form from the Equation propoſed, by Subſtitution of the Value of the Root ſought, partly in known and partly in unknown Terms.

In doing this he makes uſe of a Table of Products (which he calls *Speculum Analyticum*,) by which he computes the Coefficients in the new Equation for finding the Difference mentioned. This Table, I obſerved, was formed in the ſame Manner from the Equation pro-

propos'd, as the Fluxions are, taking the Root sought for the only flowing Quantity, its Fluxion for Unity, and after every Operation dividing the Product successively by the Numbers 1, 2, 3, 4, *&c.* Hence I soon found that this Method might easily and naturally be drawn from *Cor.* 2. *Prop.* 7. of my *Methodus Incrementorum*, and that it was capable of a further degree of Generality; it being Applicable, not only to Equations of the common Form, (*viz.* such as consist of Terms wherein the Powers of the Root sought are positive and integral, without any Radical Sign) but also to all Expressions in general, wherein any thing is proposed as given which by any known Method might be computed; if *vice versâ*, the Root were consider'd as given: such as are all Radical Expressions of Binomials, Trinomials, or of any other Nomial, which may be computed by the Root given, at least by Logarithms, whatever be the Index of the Power of that Nomial; as likewise Expressions of Logarithms, of Arches by the Sines or Tangents, of Areas of Curves by the *Abscissa's* or any other Fluents, or Roots of Fluxional Equations, *&c.*

For the sake of this great Generality, it may not be improper to shew how this Method is derived from the foresaid *Corollary.* Therefore z and x being two flowing Quantities (whose Relation to one another may be exprest by any Equation whatsoever) by this *Corollary*, while z by flowing uniformly becomes $z + v$, x will

become $x + \frac{\dot{x}}{1 \cdot \dot{z}} v + \frac{\ddot{x}}{1 \cdot 2 \dot{z}^2} v^2 + \frac{\dddot{x}}{1 \cdot 2 \cdot 3 \dot{z}^3} v^3 +$ *&c.*

or $x + \frac{\dot{x} v}{1} + \frac{\ddot{x} v^2}{1 \times 2} + \frac{\dddot{x} v^3}{1 \cdot 2 \cdot 3} +$ *&c.* for $\dot{z}$ putting 1.

Hence if y be the Root of any Expression formed of y and known Quantities, and supposed equal to nothing, and

and z be a part of y, and x be formed of z and the known Quantities, in the ſame manner as the Expreſſion made equal to nothing is formed of y; and let y be equal to $z + v$: the difference v will be found by Extracting the Root of this expreſſion $x + \frac{\dot{x}v}{1} + \frac{\ddot{x}v^2}{1.2} + \frac{\dddot{x}v^3}{1.2.3}$ + &c. = 0. For in this Caſe z being become $z + v = y$, x, which is now become $x + \dot{x}v + \frac{\ddot{x}v^2}{2}$ + &c. muſt become equal to nothing.

The Root v in the Equation $x + \frac{\dot{x}v}{1} + \frac{\ddot{x}v^2}{1.2} + \frac{\dddot{x}v^3}{1.2.3}$ + &c. = 0, is to be found upon the Suppoſition of its being very ſmall with reſpect to z, (as it muſt be, if z be taken tolerably exact) by which means the Terms $\frac{\dddot{x}v^3}{1.2.3} + \frac{\ddddot{x}v^4}{1.2.3.4}$ + &c. may be neglected, upon account of their ſmallneſs with reſpect to the other Terms, ſo as to leave the Equation $x + \frac{\dot{x}v}{1} + \frac{\ddot{x}v^2}{1.2} = 0$, for finding the firſt approximation of v.

By extracting the Root of this Equation, we have $v = \sqrt{\frac{\dot{x}^2}{\ddot{x}^2} - \frac{2x}{\ddot{x}}} - \frac{\dot{x}}{\ddot{x}}$. That is,

Firſt, $\sqrt{\frac{\dot{x}^2}{\ddot{x}^2} - \frac{2x}{\ddot{x}}} - \frac{\dot{x}}{\ddot{x}}$, if $x + \dot{x}v + \frac{\ddot{x}v^2}{2} = 0$.

Sec. $\sqrt{\frac{\dot{x}^2}{\ddot{x}^2} + \frac{2x}{\ddot{x}}} - \frac{\dot{x}}{\ddot{x}}$, if $-x + \dot{x}v + \frac{\ddot{x}v^2}{2} = 0$.

Thirdly

3. $\frac{\dot{x}}{\ddot{x}} - \sqrt{\frac{\dot{x}^2}{\ddot{x}^2} - \frac{2x}{\ddot{x}}}$, if $x - \dot{x}v + \frac{\ddot{x}v^2}{2}$, &c. $= 0$.

4. $\frac{\dot{x}}{\ddot{x}} - \sqrt{\frac{\dot{x}^2}{\ddot{x}^2} + \frac{2x}{\ddot{x}}}$, if $-x - \dot{x}v + \frac{\ddot{x}v^2}{2}$, &c. $= 0$.

This approximation gives v exact to twice as many places as there are true Figures in z, and therefore trebles the number of true Figures in the Expression of y by $z + v$, which may be taken for a new Value of z, for computing a second v, seeking other Values of x, $\dot{x}$, $\ddot{x}$, &c. Tho' when z is tolerably exact (which it may be esteem'd when it contains two or three or more true Figures in the Value of y, according to the Number of Figures the Root is proposed to be computed to,) the Calculation may be restor'd without so much trouble, only by taking $\sqrt{\frac{\dot{x}^2}{\ddot{x}^2} \pm \frac{2x}{\ddot{x}} - \frac{2\dddot{x}}{2.3\ddot{x}}v^3} - \frac{2\ddddot{x}}{1.2.3.4\ddot{x}}v^4$ &c. instead of $\sqrt{\frac{\dot{x}^2}{\ddot{x}^2} \pm \frac{2x}{\ddot{x}}}$ taking every time for v its Value last computed.

From the same Equation $x + \dot{x}v + \frac{\ddot{x}v^2}{2} + \frac{\dddot{x}v^3}{1.2.3}$ + &c. $= 0$, may be gather'd also a rational Form, *viz.* $v = \frac{-x}{\dot{x} - \frac{x\ddot{x}}{2\dot{x}}}$. For neglecting the Terms $\frac{\dddot{x}v^3}{1.2.3}$, &c. we have $v = \frac{-x}{\dot{x} + \frac{\ddot{x}}{2}v}$ which is nearly $= \frac{-x}{\dot{x}}$. Therefore in the Divisor instead of v writing $\frac{-x}{\dot{x}}$ we have more exactly $v = \frac{-x}{\dot{x} - \frac{\ddot{x}x}{2\dot{x}}}$, that is

1. $\frac{-x}{\dot{x} - \frac{\ddot{x}x}{2\dot{x}}}$, when $x + \dot{x}v + \frac{\ddot{x}v^2}{2}$ &c. $= 0$.

2. $\frac{x}{\dot{x} + \frac{\ddot{x}x}{2\dot{x}}}$, when $-x + \dot{x}v + \frac{\ddot{x}v^2}{2}$ &c. $= 0$.

3. $\frac{x}{\dot{x} - \frac{\ddot{x}x}{2\dot{x}}}$, when $x - \dot{x}v + \frac{\ddot{x}v^2}{2}$ &c. $= 0$.

4. $\frac{-x}{\dot{x} + \frac{\ddot{x}x}{2\dot{x}}}$, when $-x - \dot{x}v + \frac{\ddot{x}v^2}{2}$ &c. $= 0$.

This *Formula* will alſo triplicate the number of true Figures in z. And the Calculation may be repeated, after every Operation, taking for a Diviſor $\dot{x} + \frac{\ddot{x}}{2}v + \frac{\dddot{x}v^2}{1.2.3} + \frac{\ddddot{x}v^3}{1.2.3.4}$ + &c. inſtead of $\dot{x} + \frac{\ddot{x}x}{2\dot{x}}$.

Dr. *Halley* has fully explain'd the manner of uſing both theſe *Formula's* in Æquations of the common Form; wherefore I ſhall be the ſhorter in explaining two or three Examples of another ſort.

Ex. I. Let it be propoſed to find the Root of this Equation $\overline{y^2 + 1}|^{\sqrt{2}} + y - 16 = 0$. In this Caſe, for y writing z, and for 0 writing x, we have $\overline{z^2 + 1}|^{\sqrt{2}} + z$

$+ z - 16 = x$. Whence by taking the Fluxions, we have $\dot{x} = 2\sqrt{2} \times z \times \overline{z^2 + 1}|^{\sqrt{2}-1} + 1$, and $\ddot{x} = \overline{2\sqrt{2} \times 8 - 4\sqrt{2}\, z^2} \times \overline{z^2 + 1}|^{\sqrt{2}-2}$. For finding the first Figures of the Root y, for $\sqrt{2}$ take $\frac{3}{2}$, and we have the Equation $\overline{y^2 + 1}|^{\frac{3}{2}} + y - 16 = 0$, which being expanded gives $y^6 + 3y^4 + 2y^2 + 32y - 255 = 0$.

By this Equation I find that for the first supposition we may take $z = 2$. Therefore in order to find v, let us now make $\sqrt{2} = \frac{7}{5}$, (which is nearer than before) and we have $x = \overline{z^2 + 1}|^{\frac{7}{5}} + z - 16 = \overline{2^2 + 1}|^{\frac{7}{5}} - 14 = 5^{\frac{7}{5}} - 14 = -4{,}48$; $\dot{x} = 10{,}66$; $\ddot{x} = 4{,}72$. Whence by the second rational Form $v = \dfrac{4{,}48}{10{,}66 + \dfrac{4{,}72 \times 4{,}48}{2 \times 10{,}66}} = 0{,}38$; which must be too big, because $\frac{7}{5} < \sqrt{2}$, and therefore will require a larger Value of y to exhaust the Equation, than where $\sqrt{2}$ is exact. For the second supposition therefore, let us take $z = 2{,}3$, and make $\sqrt{2} = 1{,}4142136$, and by help of the Logarithms we shall have $\overline{z^2 + 1}|^{\sqrt{2}} = 13{,}47294$, whence $x = -0{,}22706$; $\dot{x} = 14{,}93429$, and $\ddot{x} = 5{,}18419$. Hence by the 2*d.* irrational *Formula* $v = \sqrt{\dfrac{14{,}93429^2}{5{,}18419^2} + \dfrac{0{,}45412}{5{,}18419}} - \dfrac{14{,}93429}{5{,}18419} = 0{,}01516$, which gives $y = z + v = 2{,}31516$, which is true to six Places. If you desire it more exact than to the extent of the Tables of Logarithms, taking $z = 2{,}31516$ for the next supposition, the Calculation must be repeated by computing of $\overline{zz + 1}|^{\sqrt{2}}$ to a sufficient number of Places; which must be done by the Binomial Series, or by making a Loga-

 rithm

rithm on purpoſe, true to as many places as are neceſſary.

Ex. II. For another Example, let it be required to find the Number whoſe Logarithm is 0, 29, ſuppoſing we had no other Table of Logarithms but Mr. *Sharps* of 200 Logarithms to a great many places. This amounts to the reſolving this Equation $ly = 0,29$, or $ly - 0,29 = 0$. Hence therefore we have $x = lx - 0,29$, $\dot{x} = \frac{a}{z}$ (a being the *Modulus* belonging to the Table we uſe, *viz.* 0, 4342944819, *&c.*) $\ddot{x} = \frac{-a}{z^2}$, $\dddot{x} = \frac{2a}{z^3}$, $\ddddot{x} = \frac{-6a}{z^4}$ *&c.* In this Caſe becauſe $\ddot{x}$ has a negative Sign, changing the Signs of all the Coefficients, the Canon for v will be found in the fourth Caſe, which in the irrational Form gives $v = \frac{\dot{x}}{\ddot{x}} - \sqrt{\frac{\dot{x}^2}{\ddot{x}^2} + \frac{2x}{\ddot{x}} - \frac{2\dddot{x}}{2.3\ddot{x}}v^3 - \frac{2\ddddot{x}}{2.3.4\ddot{x}}v^4 \text{ \&c.}} = z - \sqrt{z^2 + \frac{2lz - 0,58}{a} \times z^2 + \frac{2v^3}{3z} - \frac{2v^4}{4z^2} + \frac{2v^5}{5z^3} \text{ \&c.}}$ In this Caſe to avoid often dividing by z, it will be moſt convenient to compute $\frac{v}{z}$, which is got from this Equation $\frac{v}{z} = 1 - \sqrt{1 + \frac{2lz - 0,58}{a} + \frac{2v^3}{3z^3} - \frac{2v^4}{4z^4} + \frac{2v^5}{5z^5}, \text{ \&c.}}$ The neareſt Logarithm, in the Tables propoſed, to the propoſed Logarithm 0, 29 is 0, 2900346114, its Number being 1, 95. Therefore for the firſt ſuppoſition taking $z = 1,95$, we have x ($= lz - 0,29 = 0,2900346114 - 0,29$) $=$

0,0000346114, and $\frac{2lz - 0,58}{a} = \frac{0,0000692228}{0,4342944819} =$ 0,000159391 39, and $1 + \frac{2lz - 0,58}{a} = 1,00015939139$ Whence for the first approximation we have $\frac{v}{z} = 1 - \sqrt{1,00015939139} = -0,00007969247$, and $v = -0,00015540032$, and $y = z + v = 1,94984459968$. Which is true to eleven places, and may easily be corrected by the Terms $\frac{2v^3}{3z}$ *&c.* which I leave to the Readers curiosity.

Being upon the Subject of Approximations, it may not be amiss to set down here two Approximations I have formerly hit upon. The one is a Series of Terms for expressing the Root of any Quadratick Equation: and the other is a particular Method of Approximating in the invention of Logarithms, which has no occasion for any of the Transcendental Methods, and is expeditious enough for making the Tables without much trouble.

A general Series for expressing the Root of any Quadratick Equation.

Any Quadratick Equation being reduc'd to this Form $xx - mqx + my = 0$, the Root x will be exprest by this Series of Terms.

$$x = \frac{y}{q} + A \times \frac{1}{\frac{mq^2}{y} - 2} + B \times \frac{1}{a^2 - 2} + C \times \frac{1}{b^2 - 2}$$

$+ D \times \frac{1}{c^2 - 2}$ *&c.* Which must be thus interpreted.

1. The Capital Letters A, B, C, *&c.* stand for the whole Terms with their Signs, preceding those where-in

in they are found, as $B = A \times \frac{1}{\frac{mq^2}{y} - 2}$.

2. The little Letters *a*, *b*, *c*, &c. in the Divisors, are equal to the whole Divisors of the Fraction in the Terms immediately preceding; thus $b = a^2 - 2$.

For an Example of this, let it be required to find $\sqrt{2}$. Putting $\sqrt{2} = x + 1$, we have $x^2 + 2x - 1 = 0$, which being compared with the general *Formula*, gives $mq = -2$, and $my = -1$: therefore for *m* taking -1, we have $q = 2$, and $y = 1$, which Values substituted in the Series give $x = \frac{1}{2} - \frac{1}{2 \times 6} - \frac{1}{2 \times 6 \times 34} - \frac{1}{2 \times 6 \times 34 \times 1154} - \frac{1}{2 \times 6 \times 34 \times 1154 \times 1331714}$, &c. The Fractions here wrote down giving the Root true to twenty three Places.

A new Method of computing Logarithms.

This Method is founded upon these Considerations.

1. That the Sum of the Logarithms of any two Numbers is the Logarithm of the Product of those two Numbers Multiplied together.

2. That the Logarithm of Unite is nothing; and consequently that the nearer any Number is to Unite, the nearer will its Logarithm be to 0. 3*dly*. That the Product by Multiplication of two Numbers, whereof one is bigger, and the other less than Unite, is nearer to Unite than that of the two Numbers which is on the same side of Unite with its self; for Example the two Numbers being $\frac{2}{3}$ and $\frac{4}{3}$, the Product $\frac{8}{9}$ is less than Unite, but nearer to it than $\frac{2}{3}$, which is also less than Unite. Upon these Considerations, I found the present Approximation;

proximation; which will be best explain'd by an Example. Let it therefore be proposed to find the Relation of the Logarithms of 2 and of 10. In order to this, I take two Fractions $\frac{128}{100}$ and $\frac{8}{10}$, *viz.* $\frac{\overline{2}|^7}{10|^2}$ and $\frac{\overline{2}|^3}{10|^1}$ whose Numerators are Powers of 2, and their Denominators Powers of 10; one of them being bigger, and the other less than 1. Having set these down in Decimal Fractions in the first Column of the Table annext, against them in the second Column I set A and B for their Logarithms, expressing by an Equation the manner how they are Compounded of the Logarithms of 2 and 10, for which I write *l* 2 and *l* 10. Then Multiplying the two Numbers in the first Column together, I have a third Number 1,024, against which I write C for its Logarithm, expressing likewise by an Equation in what manner C is formed of the foregoing Logarithms A and B. And in the same manner the Calculation is continued; only observing this *Compendium*, that before I Multiply the two last Numbers already got in the Table, I consider what Power of one of them must be used to bring the Product the nearest to Unite that can be. This is found, after we have gone a little way in the Table, only by Dividing the Differences of the Numbers from Unite one by the other, and taking the Quotient with the nearest, for the Index of the Power wanted. Thus the two last Numbers in the Table being 0, 8 and 1, 024, their Differences from Unit are 0, 200 and 0, 024; therefore $\frac{0,200}{0,024}$ gives 9 for the Index; wherefore Multiplying the ninth Power of 1, 024 by 0, 8, I have the next Number 0, 990352031429, whose Logarithm is D = 9 C + B. In seeking the Index in this manner by Division of the Differences, the Quotient ought generally to be taken

with

with the leaſt: but in the preſent caſe it happens to be the moſt, becauſe inſtead of the Difference between 0, 8 and 1, we ought ſtrictly to have taken the difference between the reciprocal 1, 25 and 1, which would have given the Index 10; and that would be too big, becauſe the Product by that means would have been bigger than 1, as 1,024 is. Whereas this Approximation requires that the Numbers in the firſt Column be alternately greater and leſs than 1, as may be ſeen in the Table.

When I have in this manner continued the Calculation, till I have got the Numbers ſmall enough, I ſuppoſe the laſt Logarithm to be equal to nothing. Which gives me an Equation, from which having got away the Letters by means of the foregoing Equations, I have the relation of the Logarithms propoſed. In this manner if I ſuppoſe $G = 0$, I have $2136\, l\, 2 - 643\, l\, 10 = 0$. Which gives the Logarithm of 2 true in ſeven Figures, and too big in the Eighth; which happens becauſe the Number correſponding with G is bigger than Unite.

There is another Expedient which renders this Calculation ſtill ſhorter. It is founded upon this Conſideration, that when x is very ſmall $\overline{1+x}|^n$ is very nearly $1+nx$. Hence if $1+x$, and $1-z$ are the two laſt Numbers already got in the firſt Column of the Table, and their Powers $\overline{1+x}|^m$ and $\overline{1-z}|^n$ are ſuch as will make the Product $\overline{1+x}|^m \times \overline{1+z}|^n$ very near to Unite, m and n may be found thus: $\overline{1+x}|^m = 1+mx$, and $\overline{1-z}|^n = 1-nz$, and conſequently $\overline{1+x}|^m \times \overline{1-z}|^n = 1+mx-nz-mnzx$, or (neglecting $mnzx$) $1+mx-nz$. Make this equal to 1, and we have $m:n::z:x::l\,\overline{1-z}:l\,\overline{1+x}$. Whence $x\,l\,\overline{1-z} + z\,l\,\overline{1+x} = 0$. To give an Example of the Application of this, let 1, 024 and 0, 990352 be the laſt Numbers in the Table, their Logarithms being C and D. Then we have

1, 024

$1,024 = 1 + x$, and $0,990352 = 1 - z$, and consequently $x = 0,024$, and $z = 0,009648$. Whence the Ratio $\frac{z}{x}$ in the least Numbers is $\frac{201}{500}$. So that for finding the Logarithms proposed we may have $500\,D + 201\,C = 48510\,l2 - 14603\,l10 = 0$, which gives $l2 = 0,3010307$, which is too big in the last Figure; but it is nearer the truth, than what is got from the Logarithm F supposed equal to nothing. So that by this means we have saved four Multiplications, which were necessary to find the Number 9989595 &c. correspondent to F, and which must have been had if we would make the Logarithm true to the same Number of places without this *Compendium.*

1,28000000000	$A = 7\,l2 - 2\,l10$	$l2 > 0,28$
0,80000000000	$B = 3\,l2 - l10$	$< 0,33$
1,02400000000	$C = B + A = 10\,l2 - 3\,l10$	$> 0,300$
0,99035203429	$D = 9\,C + B = 93\,l2 - 28\,l10$	$< 0,30107$
1,00433627664	$E = 2\,D + C = 196\,l2 - 59\,l10$	$> 0,301020$
0,99895953610	$F = 2\,E + D = 485\,l2 - 146\,l10$	$< 0,3010309$
1,00016289416	$G = 4\,F + E = 2136\,l2 - 643\,l10$	$> 0,30102996$
0,99993628187	$H = 6\,G + F = 13301\,l2 - 4004\,l10$	$< 0,301029997$
1,00003544121	$I = 2\,H + G = 28738\,l2 - 8651\,l10$	$> 0,3010299951$
0,99997172083	$K = I + H = 42039\,l2 - 12655\,l10$	$< 0,3010299959$
1,00000716104	$L = K + I = 70777\,l2 - 21306\,l10$	$> 0,30102999562$
0,99999320351	$M = 3\,L + K = 254370\,l2 - 76573\,l10$	$< 0,30102999567$
1,00000036451	$N = M + L = 325147\,l2 - 97879\,l10$	$> 0,301029995663$
0,99999976468	$O = 18\,N + M = 6107016\,l2 - 1838395\,l10$	$< 0,3010299956640$
Com. Ar. 235313		
$0 = 364511\,O + 235313\,N = 2302585825187\,l2 - 693147400972\,l10$		$> 0,30102999566398$

I have computed this Table so far, that the Reader may see in what manner this Method Approximates; this whole Work, as it appears, costing a little more than three Hours time.

V. *Proprietates quædam simplices* Sectionum Conicarum *ex natura* Focorum *deductæ; cum Theoremate generali de Viribus Centripetis; quorum ope Lex Virium Centripetarum ad Focos Sectionum tendentium, Velocitates Corporum in illis revolventium, & Descriptio Orbium facillime determinantur. Per* Abr. *de* Moivre. R. S. Soc.

SIt DE Axis Transversus Ellipseos, AO Axis alter, & C centrum Sectionis. Sit P punctum quodvis in circumferentia ejus; PQ Tangens curvæ ad P, occurrens Axi Transverso ad Q; puncta S, F Foci; CP, CK semidiametri Conjugatæ; PH Semilatus rectum ad diametrum PC; PG normalis ad Tangentem, cui occurrat HG, perpendicularis ipsi PCH, in puncto G, ut fiat PG radius Curvaturæ Ellipseos in puncto P: sint etiam ST, CR, FV perpendiculares in Tangentem PQ demissæ: Jungatur SO, & demittatur in Axem normalis PL. His positis, Dico quod,

I. *Rectangulum sub distantiis ab utroque Ellipseos Foco, sive* $SP \times PF$ *æquale est quadrato Semidiametri* CK.

Demonstratio.

$PSq = PCq + CSq - 2CS \times CL$ *per* 13. II. Elem.
$PFq = PCq + CSq + 2CS \times CL$ *per* 12. II. Elem.
Unde $PSq + PFq = 2PCq + 2CSq$.
Jam $PS + PF = DE = 2CD$; ac propterea
$PSq + PFq + 2PS \times PF = 4CDq$. Quare