Riemann Hypothesis and Robin Inequality

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Abstruct

In the Riemann hypothesis, the Robin inequality is one of the most important and famous criterion, but it is still unsolved completely yet. In this paper we prove that the Robin inequality holds unconditionally. The main idea for it is the relation between the sum of divisors function and the Hardy-Ramanujan number.

Keywords; Riemann hypothesis; Robin inequality; Sum of divisors function; Hardy-Ramanujan number.

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Contents

1. Introduction

The function $\zeta(s)$ defined by an absolute convergent Dirichlet's series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{1}$$

in complex half-plane $\operatorname{Re} s > 1$ is called the Riemann's zeta function ([3]). The Riemann's zeta function has a simple pole with the residue 1 at $s = 1$ and except the point $s = 1$ the function $\zeta(s)$ is analytically continued to whole complex plane. And $\zeta(s)$ is expressed for $\operatorname{Re} s > 1$ as

$$\zeta(s) = \prod_p \left(1 - p^{-s}\right)^{-1}, \tag{2}$$

where infinite product runs over all the prime numbers. Also for $\operatorname{Re} s > 1$ the function $\zeta(s)$ satisfies the functional equation

$$\zeta(s) = 2 \cdot (2\pi)^{s-1} \cdot \Gamma(1-s) \cdot \sin\left(\frac{\pi s}{2}\right) \cdot \zeta(1-s), \tag{3}$$

where $\Gamma(s)$ is the gamma function ([3])

$$\Gamma(s) = \int_0^{+\infty} e^{-s} x^{s-1} dx . \tag{4}$$

From the infinite product of $\zeta(s)$ the Riemann's zeta function has no zeros in $\operatorname{Re} s > 1$ and from the functional equation of $\zeta(s)$ it has trivial zeros $-2, -4, -6, \cdots$ in $\operatorname{Re} s < 0$. The zeros of $\zeta(s)$ in $0 \le \operatorname{Re} s \le 1$ are called the nontrivial zeros of $\zeta(s)$ ([3]). In 1859 G. Riemann conjectured that all the nontrivial zeros of $\zeta(s)$ would lie on the line $\operatorname{Re} s = 1/2$. This is just the Riemenn's hypothesis. There have been published many research results on the RH, but the RH is unsolved until now ([1~4]).

To study the RH we will here consider the sum of divisors function. The sum of divisors function is one of the important arithmetical functions, but its properties are not well known in the RH. In the past, the study of the sum of divisors function had been mostly limited to the relation with the Euler's function and to the relation with the perfect numbers, but for the RH it has been studied after Robin inequality in 1984 ([6]). The Robin inequality is one of the most famous theorems for the RH. Recently, The Robin inequality has been studied in many papers, but it is still unsolved completely yet ([1,2]). In this paper we prove that the Robin inequality holds unconditionally. To do this, first, we obtain a new condition equivalent to the Robin inequality. This condition is a generalization of the Robin inequality. And it is easy to consider rather than Robin one. Next, we show that the new condition holds unconditionally. We present a new idea for the proof of the new condition. The idea is to introduce a notion, which would be called a sigma-index of the natural number. The sigma-index is a certain function expressed by the sum of divisors function. According to the sigma-index, the Robin inequality and the new condition are in the essential difference. For example, the sigma-index of the Robin inequality must be smaller than 1, but one of the new condition needs a bounded above. On the basis of the idea, we work with a standpoint that any natural number has the three-dimensional structure. By the standpoint, we pass three steps for the completion of the proof. The every step is accompanied with the process reducing the dimension of the natural number in the sigma-index. **The first step** is the relation of the sum of divisors function and the Hardy-Ramanujan number ([3,7]). This relation is also one of the important properties for the sum of divisors function. In other words, we plan to reduce the sigma-index of any natural number to the Hardy-Ramanujan number of the general type. **The second step** is an optimization problem of a certain exponential function with the sun of divisors function. This problem is related with the existence of the optimum points of the given exponential function under the constraint of the inequalities. In other words, we would like to estimate the maximum value of the sigma-index of the general Hardy-Ramanujan number by the Hardy-Ramanujan number of the special type. This step needs to do it by a new method and a lot of work. From the result of the second step, we get an estimate on the difference between consecutive primes. **The third step** is related with an inequality on the sum of divisors function. This inequality is also generalized than the Robin inequality, but it is a one equivalent to the Robin inequality. In other words, we intend to determine the sigma-index of the special Hardy-Ramanujan number by the prime number $p = 2$. The above three steps are accompanied by the courses such as the restriction of the prime factor pattern, the limitation of the exponent pattern, the decrease of the exponent length of the natural number. Consequently, the new condition is proved and so the Robin inequality would be held unconditionally.

Let N be the set of the natural numbers. The function $\sigma(n) = \sum_{d|n} d$ is called the sum of divisors function of n ([3,5]). Then the function $\sigma(n)$ is multiplicative on the coprime numbers.

It is well known that the RH is true if and only if it holds that, for any $n \geq 5041$,

$$\sigma(n) \leq e^{\gamma} \cdot n \cdot \log\log n, \tag{5}$$

where $\gamma = 0.577\cdots$ is Euler's constant. This (5) is called the Robin inequality or the Robin criterion. In the paper [2], they showed that the Robin inequality holds for any odd $n \geq 5041$. But the Robin inequality is determinately related with the even numbers. In particular, it is essentially related with the Hardy-Ramanujan number ([2]). In deed, the Robin inequality in the case of all odd numbers is only a corollary of the result on the case of all Hardy-Ramanujan numbers.

To consider the Robin inequality we need to take three steps. That reason is explained as follows. As said above, we are able to say that any natural number has three-dimentional structure. In fact, suppose that $n = q_1^{\lambda_1} \cdot q_2^{\lambda_2} \cdots q_m^{\lambda_m}$ is a prime factorization of $n \in N$, where $q_1, q_2, \cdots q_m$ are distinct primes and $\lambda_1, \lambda_2, \cdots, \lambda_m$ are non-negative integers. We assume that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \geq 1$. And we put $\omega(n) = m$, $\bar{\lambda}(n) = (\lambda_1, \lambda_2, \cdots, \lambda_m)$ and $\bar{q}(n) = (q_1, q_2, \cdots, q_m)$, which would be called an exponent length, an exponent pattern and a prime factor pattern of $n \in N$, respectively. Here $\omega(n) = \sum_{p|n} 1$ ([4]) is the number of the prime factors of a given n. Then we could write any natural number n and the set N as

$$n = n\left(\bar{q}(n),\ \bar{\lambda}(n),\ \omega(n)\right) \tag{6}$$

and

$$N = \bigcup_{\omega(n)} \bigcup_{\bar{\lambda}(n)} \bigcup_{\bar{q}(n)} n\left(\bar{q}(n), \bar{\lambda}(n), \omega(n)\right) \tag{7}$$

respectively. Hence we can say that any natural number n has the three-parameters. Of course, both $\bar{\lambda}(n)$ and $\bar{q}(n)$ are dependent on $\omega(n) = m$. But, if we take such the standpoint at the consideration of the Robin inequality, then we would be able to prove it more easily.

To prove the Robin inequality we put

$$H(n) = \frac{\exp\left(\exp\left(e^{\gamma} \cdot \sigma(n)/n\right)\right)}{n}. \tag{8}$$

Here for $n = q_1^{\lambda_1} \cdot q_2^{\lambda_2} \cdots q_m^{\lambda_m}$ it holds that

$$\frac{\sigma(n)}{n} = \prod_{i=1}^{m} \frac{1 - q_i^{-\lambda_i - 1}}{1 - q_i^{-1}}. \tag{9}$$

This $H(n)$ we would like to call a sigma-index of natural number $n \in N$.

The aim of this paper is just to reduce the dimension of $n \in N$ in the sigma-index $H(n)$, and to obtain the upper estimate of the sigma-index $H(n)$. First of all, the Hardy-Ramanujan number is used.

2. Hardy-Ramanujan number

Let $p_1 = 2,\ p_2 = 3,\ p_3 = 5, \cdots, p_n, \cdots$ be the first primes ([4]). Here then p_n is n-th prime number. If $\overline{\lambda}(n)$ and $\omega(n)$ are fixed in a given number $n = q_1^{\lambda_1} \cdot q_2^{\lambda_2} \cdots q_m^{\lambda_m}$, then we put $r_0(n) = p_1^{\lambda_1} \cdot p_2^{\lambda_2} \cdots p_m^{\lambda_m}$ to n. If $n = r_0(n)$ then the number n is called a Hardy-Ramanujan number ([2]). In other words, the Hardy-Ramanujan number is just a natural number of such forms as $p_1^{\lambda_1} \cdot p_2^{\lambda_2} \cdots p_m^{\lambda_m}$ with $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_m \ge 1$. The Hardy-Ramanujan number has special properties. In particular, the Hardy-Ramanujan number has the close relations with the sum of divisors function. We put

$$S(\overline{\lambda}, m) = \{\ n \in N \mid\ \overline{\lambda} = \overline{\lambda}(n) = (\lambda_1, \lambda_2, \cdots, \lambda_m),\ \omega(n) = m\}, \qquad (10)$$

and

$$HR(m) = \{\ n \in N \mid\ n = r_0(n), \omega(n) = m\}. \qquad (11)$$

Then $S(\overline{\lambda}, m)$ consists of the natural numbers with $\overline{\lambda}(n) = (\lambda_1, \lambda_2, \cdots, \lambda_m)$ and $\omega(n) = m$. And the set $HR(m)$ consists of the Hardy-Ramanujan number with $\omega(n) = m$.

In this section we will show a relation with the sigma-index and the Hardy-Ramanujan number. This relation says that one can reduce the dimension of the natural number at the proof of the Robin inequality. In other words, the sigma-index of any natural number is estimated upper by the Hardy-Ramanujan number.

We have

Theorem 1. For any $n \in S(\overline{\lambda}, m)$ we have $H(n) \le H(r_0(n))$.

Proof. There are three steps for the proof of the theorem 1.

① For any $n \in S(\overline{\lambda}, m)$ we have $r_0(n) \le n$, that is,

$$r_0(n) = \min_{(q_1, q_2, \cdots q_m)} S(\overline{\lambda}, m). \tag{12}$$

This shows a property of the Hardy-Ramanujan number. That is, the Hardy-Ramanujan number is a unique minimum element in the set $S(\overline{\lambda}, m)$.

There are also two steps for the proof of ①.

- For any $n = q_1^{\lambda_1} \cdot q_2^{\lambda_2} \cdots q_m^{\lambda_m} \in S(\overline{\lambda}, m)$ we have

$$q_1^{\lambda_1} \cdot q_2^{\lambda_2} \cdots q_m^{\lambda_m} \ge (q_1')^{\lambda_1} \cdot (q_2')^{\lambda_2} \cdots (q_m')^{\lambda_m}, \tag{13}$$

where $\{q_1', q_2', \cdots, q_m'\}$ is to be rearranged in the order of their magnitude size from $\{q_1, q_2, \cdots, q_m\}$, namely, $q_1' \le q_2' \le \cdots \le q_m'$.

We will prove (13) by the mathematical induction with respect to m.

It is clear that (13) holds as $m = 1$.

Let $m = 2$. Then for any $n \in S(\overline{\lambda}, 2)$ there exist prime numbers q_1, q_2 and integers λ_1, λ_2 such that $\lambda_1 \ge \lambda_2 \ge 1$ and $n = q_1^{\lambda_1} \cdot q_2^{\lambda_2}$. If $q_1 < q_2$ then we put $q_1' = q_1, q_2' = q_2$. If $q_1 > q_2$ then we have $q_1^{\lambda_1 - \lambda_2} \ge q_2^{\lambda_1 - \lambda_2}$ since $\lambda_1 \ge \lambda_2 \ge 1$. Here $=$ is only $\lambda_1 = \lambda_2$. Therefore we have $q_1^{\lambda_1} \cdot q_2^{\lambda_2} \ge q_2^{\lambda_1} \cdot q_1^{\lambda_2}$. Now we put $q_1' = q_2, q_2' = q_1$.

Assume that (13) holds also as $m-1$ and let's see it in the case of m.

Let $n \in S(\overline{\lambda}, m)$ and $n = q_1^{\lambda_1} \cdot q_2^{\lambda_2} \cdots q_m^{\lambda_m}$. And assume that q_i is the smallest prime in $\{q_1, q_2, \cdots, q_m\}$ and we put $q_1' = q_i$. We will repeat continually such the process as $m = 2$. Then we have the following inequality;

$$\begin{aligned} & q_1^{\lambda_1} \cdots q_{i-1}^{\lambda_{i-1}} \cdot q_i^{\lambda_i} \cdot q_{i+1}^{\lambda_{i+1}} \cdots q_m^{\lambda_m} = \\ & \quad = q_1^{\lambda_1} \cdots q_{i-1}^{\lambda_{i-1}} \cdot \left(q_i^{\lambda_i}\right) \cdot \left(q_{i+1}^{\lambda_{i+1}} \cdots q_m^{\lambda_m}\right) \geq \\ & \quad \geq q_1^{\lambda_1} \cdots q_{i-2}^{\lambda_{i-2}} \cdot \left(q_i^{\lambda_{i-1}} \cdot q_{i-1}^{\lambda_i}\right) \cdot \left(q_{i+1}^{\lambda_{i+1}} \cdots q_m^{\lambda_m}\right) \geq \\ & \quad \geq q_1^{\lambda_1} \cdots \left(q_i^{\lambda_{i-2}} \cdot q_{i-2}^{\lambda_{i-1}} \cdot q_{i-1}^{\lambda_i}\right) \cdot \left(q_{i+1}^{\lambda_{i+1}} \cdots q_m^{\lambda_m}\right) \geq \\ & \quad \geq \cdots \quad \cdots \geq \\ & \quad \geq q_i^{\lambda_1} \cdot \left(q_1^{\lambda_2} \cdots q_{i-3}^{\lambda_{i-2}} \cdot q_{i-2}^{\lambda_{i-1}} \cdot q_{i-1}^{\lambda_i}\right) \cdot \left(q_{i+1}^{\lambda_{i+1}} \cdots q_m^{\lambda_m}\right) = \\ & \quad = \left(q_1'\right)^{\lambda_1} \cdot \left(q_1^{\lambda_2} \cdots q_{i-3}^{\lambda_{i-2}} \cdot q_{i-2}^{\lambda_{i-1}} \cdot q_{i-1}^{\lambda_i} \cdot q_{i+1}^{\lambda_{i+1}} \cdots q_m^{\lambda_m}\right). \end{aligned} \tag{14}$$

We now put $n_1 = \left(q_1^{\lambda_2} \cdots q_{i-3}^{\lambda_{i-2}} \cdot q_{i-2}^{\lambda_{i-1}} \cdot q_{i-1}^{\lambda_i} \cdot q_{i+1}^{\lambda_{i+1}} \cdots q_m^{\lambda_m}\right)$. And then it holds that $\lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_m \geq 1$ and $\omega(n_1) = m-1$. Therefore by the assumption of the induction we have

$$\left(q_1^{\lambda_2} \cdots q_{i-3}^{\lambda_{i-2}} \cdot q_{i-2}^{\lambda_{i-1}} \cdot q_{i-1}^{\lambda_i} \cdot q_{i+1}^{\lambda_{i+1}} \cdots q_m^{\lambda_m}\right) \geq \left(q_2'\right)^{\lambda_2} \cdot \left(q_3'\right)^{\lambda_3} \cdots \left(q_m'\right)^{\lambda_m}, \tag{15}$$

where q_2' is the smallest prime in $\{q_1, \cdots, q_{i-1}, q_{i+1}, \cdots, q_m\}$ and so on.

From (14) and (15), we get

$$q_1^{\lambda_1} \cdot q_2^{\lambda_2} \cdots q_m^{\lambda_m} \geq \left(q_1'\right)^{\lambda_1} \cdot \left(q_2'\right)^{\lambda_2} \cdots \left(q_m'\right)^{\lambda_m}.$$

This is (15).

● Now we will see $r_0(n) = \min\limits_{(q_1, q_2, \cdots q_m)} S(\overline{\lambda}, m)$.

Since $\{q_1', q_2', \cdots, q_m'\}$ in the inequality (15) are distinct primes and, generally, $q_1' \geq p_1, q_2' \geq p_2, \cdots, q_m' \geq p_m$, we have

$$\begin{aligned} n &= q_1^{\lambda_1} \cdot q_2^{\lambda_2} \cdots q_m^{\lambda_m} \geq \\ & \geq \left(q_1'\right)^{\lambda_1} \cdot \left(q_2'\right)^{\lambda_2} \cdots \left(q_m'\right)^{\lambda_m} \geq \\ & \geq p_1^{\lambda_1} \cdot p_2^{\lambda_2} \cdots p_m^{\lambda_m} = r_0(n). \end{aligned} \tag{16}$$

This shows that $r_0(n) = \min_{(q_1, q_2, \cdots q_m)} S(\bar{\lambda}, m)$.

② For any $n \in S(\bar{\lambda}, m)$ we have $\frac{\sigma(n)}{n} \le \frac{\sigma(r_0(n))}{r_0(n)}$, i.e.

$$\frac{\sigma(r_0(n))}{r_0(n)} = \max_{(q_1, q_2, \cdots q_m)} \left\{ \frac{\sigma(n)}{n} \right\}. \tag{17}$$

This shows a relation between the sun of divisors function and the Hardy-Ramanujan number. This relation is one of many interesting properties of the sum of divisors function.

There are also two steps for the proof of ②.

● For any $n = q_1^{\lambda_1} \cdot q_2^{\lambda_2} \cdots q_m^{\lambda_m} \in S(\bar{\lambda}, m)$ it holds that

$$\frac{\sigma\left(q_1^{\lambda_1} \cdot q_2^{\lambda_2} \cdots q_m^{\lambda_m}\right)}{q_1^{\lambda_1} \cdot q_2^{\lambda_2} \cdots q_m^{\lambda_m}} \le \frac{\sigma\left((q_1')^{\lambda_1} \cdot (q_2')^{\lambda_2} \cdots (q_m')^{\lambda_m}\right)}{(q_1')^{\lambda_1} \cdot (q_2')^{\lambda_2} \cdots (q_m')^{\lambda_m}}, \tag{18}$$

where $\{q_1', q_2', \cdots, q_m'\}$ is also to be rearranged as $q_1' \le q_2' \le \cdots \le q_m'$ from $\{q_1, q_2, \cdots, q_m\}$.

We will prove (18) by the mathematical induction with respect to m, too. This is the same as in the front of (13). But we will note it again. It is clear that (18) holds as $m = 1$. Let $m = 2$. Then for any $n \in S(\bar{\lambda}, 2)$ there exist prime numbers q_1, q_2 and integers λ_1, λ_2 such that $\lambda_1 \ge \lambda_2 \ge 1$ and $n = q_1^{\lambda_1} \cdot q_2^{\lambda_2}$. If $q_1 < q_2$ then we put $q_1' = q_1, q_2' = q_2$. If $q_1 > q_2$ then we will see

$$\frac{\sigma\left(q_1^{\lambda_1} \cdot q_2^{\lambda_2}\right)}{q_1^{\lambda_1} \cdot q_2^{\lambda_2}} \le \frac{\sigma\left(q_2^{\lambda_1} \cdot q_1^{\lambda_2}\right)}{q_2^{\lambda_1} \cdot q_1^{\lambda_2}} = \frac{\sigma\left((q_1')^{\lambda_1} \cdot (q_2')^{\lambda_2}\right)}{(q_1')^{\lambda_1} \cdot (q_2')^{\lambda_2}}, \tag{19}$$

where $q_1' = q_2$, $q_2' = q_1$ and $=$ is only $\lambda_1 = \lambda_2$, too. So assume that $\lambda_1 > \lambda_2 \ge 1$. Then, since the function $\sigma(n)$ is multiplicative on any coprime numbers ([4]), we have

$$\frac{\sigma\left(q_1^{\lambda_1}\cdot q_2^{\lambda_2}\right)}{q_1^{\lambda_1}\cdot q_2^{\lambda_2}}=\frac{\sigma\left(q_1^{\lambda_1}\right)}{q_1^{\lambda_1}}\cdot\frac{\sigma\left(q_2^{\lambda_2}\right)}{q_2^{\lambda_2}}\le\frac{\sigma\left(q_2^{\lambda_1}\cdot q_1^{\lambda_2}\right)}{q_2^{\lambda_1}\cdot q_1^{\lambda_2}}=\frac{\sigma\left(q_2^{\lambda_1}\right)}{q_2^{\lambda_1}}\cdot\frac{\sigma\left(q_1^{\lambda_2}\right)}{q_1^{\lambda_2}} \Leftrightarrow$$

$$\Leftrightarrow \left(\frac{1-q_1^{-\lambda_1-1}}{1-q_1^{-1}}\right)\cdot\left(\frac{1-q_2^{-\lambda_2-1}}{1-q_2^{-1}}\right)\le\left(\frac{1-q_2^{-\lambda_1-1}}{1-q_2^{-1}}\right)\cdot\left(\frac{1-q_1^{-\lambda_2-1}}{1-q_1^{-1}}\right) \Leftrightarrow \tag{20}$$

$$\Leftrightarrow \left(\frac{1-q_1^{-\lambda_1-1}}{1-q_1^{-\lambda_2-1}}\right)\le\left(\frac{1-q_2^{-\lambda_1-1}}{1-q_2^{-\lambda_2-1}}\right).$$

On the other hand, for any real numbers with $1\le b<a$, the function $f(t)=\frac{1-t^{-a-1}}{1-t^{-b-1}}$ is decreasing in the interval $(1,+\infty)$ ([2]).

In fact, since we can write the function $f(t)$ as $f(t)=\frac{1-t^{-a-1}}{1-t^{-b-1}}=\frac{t-t^{-a}}{t-t^{-b}}$, we have

$$f'(t)=\frac{(a+1)t^{b+1}-(b+1)t^{a+1}-(a-b)}{t^{a+b+1}\left(t-t^{-b}\right)^2}. \tag{21}$$

We put $g(t)=(a+1)t^{b+1}-(b+1)t^{a+1}-(a-b)$, then since $t>1$ and $1\le b<a$ we have $g'(t)=(a+1)(b+1)\left(t^b-t^a\right)<0$. This shows that the function $g(t)$ is decreasing in the interval $(1,+\infty)$. And it is clear that $g(1)=0$ and $g(t)<0$ $\left(t\in(1,+\infty)\right)$. From this we have $f'(t)<0$ $\left(t\in(1,+\infty)\right)$. Therefore the function $f(t)$ is decreasing in the interval $(1,+\infty)$ and we have

$$f(q_1)=\frac{1-q_1^{-\lambda_1-1}}{1-q_1^{-\lambda_2-1}}\le\frac{1-q_2^{-\lambda_1-1}}{1-q_2^{-\lambda_2-1}}=f(q_2). \tag{22}$$

Now we put $q_1'=q_2,\ q_2'=q_1$.

Assume that (18) holds also as $m-1$ and let's see it in the case of m.

Let $n \in S\left(\overline{\lambda}, m\right)$ and $n = q_1^{\lambda_1} \cdot q_2^{\lambda_2} \cdots q_m^{\lambda_m}$. Let q_i be the smallest prime in $\{q_1, q_2, \cdots, q_m\}$ and we put $q_1' = q_i$. We will repeat continually such the process as $m = 2$. Then we have the following inequality;

$$
\begin{aligned}
&\frac{\sigma\left(q_1^{\lambda_2} \cdots q_{i-1}^{\lambda_{i-1}} \cdot q_i^{\lambda_i} \cdot q_{i+1}^{\lambda_{i+1}} \cdots q_m^{\lambda_m}\right)}{\left(q_1^{\lambda_2} \cdots q_{i-1}^{\lambda_{i-1}} \cdot q_i^{\lambda_i} \cdot q_{i+1}^{\lambda_{i+1}} \cdots q_m^{\lambda_m}\right)} = \\
&= \frac{\sigma\left(q_1^{\lambda_2} \cdots q_{i-2}^{\lambda_{i-2}}\right)}{q_1^{\lambda_2} \cdots q_{i-2}^{\lambda_{i-2}}} \cdot \frac{\sigma\left(q_{i-1}^{\lambda_{i-1}} \cdot q_i^{\lambda_i}\right)}{q_{i-1}^{\lambda_{i-1}} \cdot q_i^{\lambda_i}} \cdot \frac{\sigma\left(q_{i+1}^{\lambda_{i+1}} \cdots q_m^{\lambda_m}\right)}{\left(q_{i+1}^{\lambda_{i+1}} \cdots q_m^{\lambda_m}\right)} \le \\
&\le \frac{\sigma\left(q_1^{\lambda_2} \cdots q_{i-2}^{\lambda_{i-2}}\right)}{q_1^{\lambda_2} \cdots q_{i-2}^{\lambda_{i-2}}} \cdot \frac{\sigma\left(q_i^{\lambda_{i-1}} \cdot q_{i-1}^{\lambda_i}\right)}{q_i^{\lambda_{i-1}} \cdot q_{i-1}^{\lambda_i}} \cdot \frac{\sigma\left(q_{i+1}^{\lambda_{i+1}} \cdots q_m^{\lambda_m}\right)}{\left(q_{i+1}^{\lambda_{i+1}} \cdots q_m^{\lambda_m}\right)} = \\
&= \frac{\sigma\left(q_1^{\lambda_2} \cdots q_{i-2}^{\lambda_{i-2}} \cdot q_i^{\lambda_{i-1}} \cdot q_{i-1}^{\lambda_i} \cdot q_{i+1}^{\lambda_{i+1}} \cdots q_m^{\lambda_m}\right)}{q_1^{\lambda_2} \cdots q_{i-2}^{\lambda_{i-2}} \cdot q_i^{\lambda_{i-1}} \cdot q_{i-1}^{\lambda_i} \cdot q_{i+1}^{\lambda_{i+1}} \cdots q_m^{\lambda_m}} \le \\
&\le \frac{\sigma\left(q_1^{\lambda_2} \cdots q_i^{\lambda_{i-2}} \cdot q_{i-2}^{\lambda_{i-1}} \cdot q_{i-1}^{\lambda_i} \cdot q_{i+1}^{\lambda_{i+1}} \cdots q_m^{\lambda_m}\right)}{q_1^{\lambda_2} \cdots q_i^{\lambda_{i-2}} \cdot q_{i-2}^{\lambda_{i-1}} \cdot q_{i-1}^{\lambda_i} \cdot q_{i+1}^{\lambda_{i+1}} \cdots q_m^{\lambda_m}} \le \\
&\le \cdots \quad \cdots \le \\
&\le \frac{\sigma\left(q_i^{\lambda_1} \cdot \left(q_1^{\lambda_2} \cdots q_{i-3}^{\lambda_{i-2}} \cdot q_{i-2}^{\lambda_{i-1}} \cdot q_{i-1}^{\lambda_i} \cdot q_{i+1}^{\lambda_{i+1}} \cdots q_m^{\lambda_m}\right)\right)}{q_i^{\lambda_1} \cdot \left(q_1^{\lambda_2} \cdots q_{i-3}^{\lambda_{i-2}} \cdot q_{i-2}^{\lambda_{i-1}} \cdot q_{i-1}^{\lambda_i} \cdot q_{i+1}^{\lambda_{i+1}} \cdots q_m^{\lambda_m}\right)} = \\
&= \frac{\sigma\left(\left(q_1'\right)^{\lambda_1} \cdot \left(q_1^{\lambda_2} \cdots q_{i-3}^{\lambda_{i-2}} \cdot q_{i-2}^{\lambda_{i-1}} \cdot q_{i-1}^{\lambda_i} \cdot q_{i+1}^{\lambda_{i+1}} \cdots q_m^{\lambda_m}\right)\right)}{\left(q_1'\right)^{\lambda_1} \cdot \left(q_1^{\lambda_2} \cdots q_{i-3}^{\lambda_{i-2}} \cdot q_{i-2}^{\lambda_{i-1}} \cdot q_{i-1}^{\lambda_i} \cdot q_{i+1}^{\lambda_{i+1}} \cdots q_m^{\lambda_m}\right)} = \\
&= \frac{\sigma\left(\left(q_1'\right)^{\lambda_1}\right)}{\left(q_1'\right)^{\lambda_1}} \cdot \frac{\sigma\left(q_1^{\lambda_2} \cdots q_{i-3}^{\lambda_{i-2}} \cdot q_{i-2}^{\lambda_{i-1}} \cdot q_{i-1}^{\lambda_i} \cdot q_{i+1}^{\lambda_{i+1}} \cdots q_m^{\lambda_m}\right)}{\left(q_1^{\lambda_2} \cdots q_{i-3}^{\lambda_{i-2}} \cdot q_{i-2}^{\lambda_{i-1}} \cdot q_{i-1}^{\lambda_i} \cdot q_{i+1}^{\lambda_{i+1}} \cdots q_m^{\lambda_m}\right)}
\end{aligned}
\tag{23}
$$

We put $n_2 = \left(q_1^{\lambda_2} \cdots q_{i-3}^{\lambda_{i-2}} \cdot q_{i-2}^{\lambda_{i-1}} \cdot q_{i-1}^{\lambda_i} \cdot q_{i+1}^{\lambda_{i+1}} \cdots q_m^{\lambda_m}\right)$ and then it holds that $\lambda_2 \ge \lambda_3 \ge \cdots \ge \lambda_m \ge 1$ and $\omega\left(n_2\right) = m-1$. Therefore by the assumption of the induction we have

$$\frac{\sigma\left(q_1^{\lambda_2}\cdots q_{i-3}^{\lambda_{i-2}}\cdot q_{i-2}^{\lambda_{i-1}}\cdot q_{i-1}^{\lambda_i}\cdot q_{i+1}^{\lambda_{i+1}}\cdots q_m^{\lambda_m}\right)}{\left(q_1^{\lambda_2}\cdots q_{i-3}^{\lambda_{i-2}}\cdot q_{i-2}^{\lambda_{i-1}}\cdot q_{i-1}^{\lambda_i}\cdot q_{i+1}^{\lambda_{i+1}}\cdots q_m^{\lambda_m}\right)}\leq \leq\frac{\sigma\left(\left(q_2'\right)^{\lambda_2}\cdot\left(q_3'\right)^{\lambda_3}\cdots\left(q_m'\right)^{\lambda_m}\right)}{\left(q_2'\right)^{\lambda_2}\cdot\left(q_3'\right)^{\lambda_3}\cdots\left(q_m'\right)^{\lambda_m}} \tag{24}$$

where q_2' is the smallest prime in $\{q_1,\cdots,q_{i-1},\ q_{i+1},\cdots,q_m\}$ and so on.

From (23) and (24), we get

$$\begin{aligned}&\frac{\sigma\left(q_1^{\lambda_1}\cdots q_{i-1}^{\lambda_{i-1}}\cdot q_i^{\lambda_i}\cdot q_{i+1}^{\lambda_{i+1}}\cdots q_m^{\lambda_m}\right)}{q_1^{\lambda_1}\cdots q_{i-1}^{\lambda_{i-1}}\cdot q_i^{\lambda_i}\cdot q_{i+1}^{\lambda_{i+1}}\cdots q_m^{\lambda_m}}\leq\\ &\leq\frac{\sigma\left(\left(q_1'\right)^{\lambda_1}\right)}{\left(q_1'\right)^{\lambda_1}}\cdot\frac{\sigma\left(q_1^{\lambda_2}\cdots q_{i-3}^{\lambda_{i-2}}\cdot q_{i-2}^{\lambda_{i-1}}\cdot q_{i-1}^{\lambda_i}\cdot q_{i+1}^{\lambda_{i+1}}\cdots q_m^{\lambda_m}\right)}{\left(q_1^{\lambda_2}\cdots q_{i-3}^{\lambda_{i-2}}\cdot q_{i-2}^{\lambda_{i-1}}\cdot q_{i-1}^{\lambda_i}\cdot q_{i+1}^{\lambda_{i+1}}\cdots q_m^{\lambda_m}\right)}\leq\\ &\leq\frac{\sigma\left(\left(q_1'\right)^{\lambda_1}\right)}{\left(q_1'\right)^{\lambda_1}}\cdot\frac{\sigma\left(\left(q_2'\right)^{\lambda_2}\cdot\left(q_3'\right)^{\lambda_3}\cdots\left(q_m'\right)^{\lambda_m}\right)}{\left(q_2'\right)^{\lambda_2}\cdot\left(q_3'\right)^{\lambda_3}\cdots\left(q_m'\right)^{\lambda_m}}=\\ &=\frac{\sigma\left(\left(q_1'\right)^{\lambda_1}\cdot\left(q_2'\right)^{\lambda_2}\cdots\left(q_m'\right)^{\lambda_m}\right)}{\left(q_1'\right)^{\lambda_1}\cdot\left(q_2'\right)^{\lambda_2}\cdots\left(q_m'\right)^{\lambda_m}}.\end{aligned} \tag{25}$$

This is (18).

● Next we will see $\dfrac{\sigma(r_0(n))}{r_0(n)}=\max\limits_{(q_1,\,q_2,\cdots q_m)}\left\{\dfrac{\sigma(n)}{n}\right\}$.

Since $\{q_1',\ q_2',\ \cdots,q_m'\}$ in the inequality (18) are distinct primes and, in general, $q_1'\geq p_1,\ q_2'\geq p_2,\cdots,q_m'\geq p_m$, we have for any i $(1\leq i\leq m)$

$$\begin{aligned}&\frac{\sigma\left(\left(q_i'\right)^{\lambda_i}\right)}{\left(q_i'\right)^{\lambda_i}}=\left(1+\frac{1}{\left(q_i'\right)^1}+\frac{1}{\left(q_i'\right)^2}+\cdots+\frac{1}{\left(q_i'\right)^{\lambda_i}}\right)\leq\\ &\leq\left(1+\frac{1}{p_i^1}+\frac{1}{p_i^2}+\cdots+\frac{1}{p_i^{\lambda_i}}\right)=\frac{\sigma\left(p_i^{\lambda_i}\right)}{p_i^{\lambda_i}}.\end{aligned} \tag{26}$$

Therefore we have

$$
\begin{aligned}
\frac{\sigma(n)}{n} &= \frac{\sigma\left(q_1^{\lambda_1}\cdot q_2^{\lambda_2}\cdots q_m^{\lambda_m}\right)}{q_1^{\lambda_1}\cdot q_2^{\lambda_2}\cdots q_m^{\lambda_m}} \le \\
&\le \frac{\sigma\left((q_1')^{\lambda_1}\cdot (q_2')^{\lambda_2}\cdots (q_m')^{\lambda_m}\right)}{(q_1')^{\lambda_1}\cdot (q_2')^{\lambda_2}\cdots (q_m')^{\lambda_m}} = \\
&= \frac{\sigma\left((q_1')^{\lambda_1}\right)}{(q_1')^{\lambda_1}}\cdot\frac{\sigma\left((q_2')^{\lambda_2}\right)}{(q_2')^{\lambda_2}}\cdots\frac{\sigma\left((q_m')^{\lambda_m}\right)}{(q_m')^{\lambda_m}} \le \\
&\le \frac{\sigma\left(p_1^{\lambda_1}\right)}{p_1^{\lambda_1}}\cdot\frac{\sigma\left(p_2^{\lambda_2}\right)}{p_2^{\lambda_2}}\cdots\frac{\sigma\left(p_m^{\lambda_m}\right)}{p_m^{\lambda_m}} = \\
&= \frac{\sigma\left(p_1^{\lambda_1}\cdot p_2^{\lambda_2}\cdots p_m^{\lambda_m}\right)}{p_1^{\lambda_1}\cdot p_2^{\lambda_2}\cdots p_m^{\lambda_m}} = \frac{\sigma(r_0(n))}{r_0(n)}.
\end{aligned}
\tag{27}
$$

This shows that $\frac{\sigma(r_0(n))}{r_0(n)} = \max_{n\in S(\bar{\lambda},\, m)}\left\{\frac{\sigma(n)}{n}\right\}$.

③ For any $n \in S(\bar{\lambda},\, m)$ we have $H(n) \le H(r_0(n))$.

In fact, from ① and ②, for any $n \in S(\bar{\lambda},\, m)$ we have

$$
\begin{aligned}
H(n) &= \frac{\exp\left(\exp\left(e^{\gamma}\cdot\sigma(n)/n\right)\right)}{n} \le \\
&\le \frac{\exp\left(\exp\left(e^{\gamma}\cdot\sigma(r_0(n))/r_0(n)\right)\right)}{n} \le \\
&\le \frac{\exp\left(\exp\left(e^{\gamma}\cdot\sigma(r_0(n))/r_0(n)\right)\right)}{r_0(n)} = H(r_0(n))
\end{aligned}
\tag{28}
$$

This is the proof of the theorem 1. □

Note. By above the theorem 1, $H(r_0(n))$ is related only with the exponent pattern $\bar{\lambda}(n)$ and the exponent length $\omega(n)$. Thus $H(r_0(n))$ has two-parameters. Hence the consideration of the sigma-index $H(n)$ on any $n \ge 2$

is reduced to one on the set $\bigcup_m HR(m)$ of the Hardy-Ramanujan number.

3. Optimization problem

In this section we will consider the opimization problem of the sigma-index with the real variable. By this consideration, we have obtained an estimate for the difference between the consecutive primes. This estimate is a new result at the distribution of the prime numbers. This section is the second step for the proof of the Robin inequality.

We here assume that $\overline{\lambda} = (\lambda_1, \lambda_2, \cdots, \lambda_m)$ are real numbers and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \geq 1$. Of course, let $p_1 = 2, p_2 = 3, \cdots, p_m, \cdots$ be consecutive primes. We will choose $p_m \geq 5$ arbitrarily and fix it. We define functions $F(\overline{\lambda})$ and $H(\overline{\lambda})$ respectively by

$$F(\overline{\lambda}) = F(\lambda_1, \lambda_2, \cdots, \lambda_m) = \prod_{i=1}^{m} \frac{1 - p_i^{-\lambda_i - 1}}{1 - p_i^{-1}}, \tag{29}$$

$$H(\overline{\lambda}) = H(\lambda_1, \lambda_2, \cdots, \lambda_m) = \frac{\exp\left(\exp\left(e^{-\gamma} \cdot F(\overline{\lambda})\right)\right)}{p_1^{\lambda_1} \cdot p_2^{\lambda_2} \cdots p_m^{\lambda_m}}, \tag{30}$$

where $\gamma = 0.577\cdots$ is Euler's constant ([3,4]).

The aim of this section is to show an existence of the optimum points of the function $H(\overline{\lambda})$ in the m-dimensional real space R^m and to estimate the optimum points.

3.1. An existence of the optimum points of the function $H\left(\overline{\lambda}\right)$

Here we will show that the function $H\left(\overline{\lambda}\right)$ has an optimum points in the space R^m. The maximum value theorem of the continuous function is used here. We get

Theorem 2. There exist $\overline{\lambda}_0=\left(\lambda_1^0,\,\lambda_2^0,\cdots,\lambda_m^0\right)\in R^m$ such that for any $\left(\lambda_1,\,\lambda_2,\cdots,\lambda_m\right)\in R^m$ we have $H\left(\overline{\lambda}\right)\le H\left(\overline{\lambda}_0\right)$, that is,

$$H\left(\lambda_1^0,\,\lambda_2^0,\cdots,\lambda_m^0\right)=\max_{(\lambda_1,\,\lambda_2,\cdots,\lambda_m)\in R^m}\frac{\exp\left(\exp\left(e^{-\gamma}\cdot\prod_{i=1}^{m}\frac{1-p_i^{-\lambda_i-1}}{1-p_i^{-1}}\right)\right)}{p_1^{\lambda_1}\cdot p_2^{\lambda_2}\cdots p_m^{\lambda_m}}. \tag{31}$$

Proof. There is the proof of the theorem 2 in the paper [9]. □

3.2. The estimate of the optimum points of the function $H\left(\overline{\lambda}\right)$

Here we will estimate the optimum points $\overline{\lambda}_0=\left(\lambda_1^0,\,\lambda_2^0,\cdots,\lambda_m^0\right)$ of the function $H\left(\overline{\lambda}\right)$ obtained from the theorem 2. The optimization problem of the function $H\left(\overline{\lambda}\right)$ with the constraints of the certain inequalities is discussed here. We obtain

Theorem 3. Assume that $p_m\ge 5$. Then for the optimum points $\overline{\lambda}_0=\left(\lambda_1^0,\,\lambda_2^0,\cdots,\lambda_m^0\right)\in R^m$ of the function $H\left(\overline{\lambda}\right)$ in the space R^m we have;

① There exist some λ_i^0 in $\left\{\lambda_1^0,\,\lambda_2^0,\cdots\lambda_m^0\right\}$ such that $\lambda_i^0=1$. In particular, we have $\lambda_m^0=1$.

② There exist some λ_i^0 in $\left\{\lambda_1^0,\,\lambda_2^0,\cdots\lambda_m^0\right\}$ such that $\lambda_i^0>1$. In particular, we have $\lambda_1^0>1$.

③ There exists a number k such that

$$\lambda_1^0 > \lambda_2^0 > \cdots > \lambda_k^0 > \lambda_{k+1}^0 = \cdots = \lambda_m^0 = 1 . \tag{32}$$

In particular, for any $i\left(1 \le i \le k\right)$ we have

$$\lambda_i^0 = \left(\frac{\log p_m}{\log p_i} + \frac{\log\log p_m}{\log p_i} - 1 \right) + \mathrm{O}\left(\frac{1}{\log p_i \cdot \log p_m} \right). \tag{33}$$

Proof. See [9]. □

The last bigger number k than 1 in the optimum points $\{\lambda_1^0, \lambda_2^0, \cdots, \lambda_m^0\}$ of the function $H\left(\bar{\lambda}\right)$ is especially important. We will here discuss λ_k, p_k and k in detail. In the furture, we assume that $p_m \ge 5$. We have

Theorem 4. For the number k such that $\lambda_1^0 > \lambda_k^0 > \lambda_{k+1}^0 = 1$ we have;

$$\text{①}\quad \lambda_k^0 = 1 + \mathrm{O}\left(\frac{1}{\log p_m} \right), \tag{34}$$

$$\text{②}\quad p_k = \sqrt{p_m \cdot \log p_m} \cdot \left(1 + \mathrm{O}\left(\frac{1}{\log p_m} \right) \right), \tag{35}$$

$$\text{③}\quad k = 2\sqrt{m} \cdot \left(1 + \mathrm{O}\left(\frac{\log\log p_m}{\log p_m} \right) \right). \tag{36}$$

Proof. One can find the proof of the theorem 4 in the paper [9]. □

Note. In the proof of the theorem 2, we have taken a certain suitable constant $a > 1$ determining the region $\prod \subset R_+^m$ such that there exist the optimum points $\bar{\lambda}_0 = \left(\lambda_1^0, \lambda_2^0, \cdots \lambda_m^0\right) \in R^m$ of the function $H\left(\bar{\lambda}\right)$.

Let's estimate the size of the constant $a > 1$.

In general, since $\lambda_1^0 \ge \lambda_2^0 \ge \cdots \ge \lambda_m^0 \ge 1$, it is sufficient to take a constant $a > 1$ such that $1 < \lambda_1^0 \le a$. On the other hand, since

$$\lambda_i^0 = \left(\frac{\log p_m}{\log p_i} + \frac{\log\log p_m}{\log p_i} - 1 \right) + \mathrm{O}\left(\frac{1}{\log p_i \cdot \log p_m} \right),$$

we get

$$\lambda_1^0 \le \frac{\log\left(p_m \cdot \log p_m\right)}{\log p_1} + 1 . \tag{37}$$

Hence we can take the constant $a > 1$ as

$$a = \frac{\log p_m + \log\log p_m}{\log p_1} + 1 . \tag{38}$$

4. Estimate of some quantities

The aim of this section is to apply the result obtained from the optimization problem to the estimate of some quantities.

By the theorem 2, the theorem 3 and the theorem 4, for the optimum points $\overline{\lambda}_0 = \left(\lambda_1^0, \lambda_2^0, \cdots, \lambda_m^0\right) \in R^m$ in m-dimensional real space R^m of the function $H\left(\overline{\lambda}\right)$, the function value $H\left(\lambda_1^0, \lambda_2^0, \cdots, \lambda_m^0\right)$ is dependent only on p_m. So we can put

$$C_m = H\left(\lambda_1^0, \lambda_2^0, \cdots, \lambda_m^0\right) = \frac{\exp\left(\exp\left(e^{-\gamma} \cdot F\left(\overline{\lambda}_0\right)\right)\right)}{p_1^{\lambda_1^0} \cdot p_2^{\lambda_2^0} \cdots p_m^{\lambda_m^0}} . \tag{39}$$

In this connection, we will put

$$\begin{cases} n_0 = p_1^{\lambda_1^0} p_2^{\lambda_2^0} \cdots p_k^{\lambda_k^0} \cdot p_{k+1}^1 \cdots p_m^1 , \quad n_0' = n_0 \cdot p_m^{-1} , \\ \overline{\lambda}_0' = \left(\lambda_1^0, \lambda_2^0, \cdots, \lambda_{m-1}^0\right) \in R^{m-1}, \\ C_{m-1}' = H\left(\overline{\lambda}_0'\right) = H\left(\lambda_1^0, \lambda_2^0, \cdots, \lambda_{m-1}^0\right) \end{cases} \tag{40}$$

and

$$C_{m-1} = \max_{(\lambda_1, \lambda_2, \cdots, \lambda_{m-1}) \in R^m} H\left(\lambda_1, \lambda_2, \cdots, \lambda_{m-1}\right). \tag{41}$$

Then it is clear that $C'_{m-1} \le C_{m-1}$.

Let $\bar{\lambda}' = \left(\lambda'_1, \lambda'_2, \cdots, \lambda'_{m-1}\right)$ be the optimum points of the function $H\left(\lambda_1, \lambda_2, \cdots, \lambda_{m-1}\right)$ with $(m-1)$-variable in the space R^{m-1}. In general, then we have

$$\lambda'_1 > \lambda'_2 > \cdots > \lambda'_{k-1} > \lambda'_k = \cdots = \lambda'_{m-1} = 1. \tag{42}$$

Rarely, the last bigger number than 1 in $\left\{\lambda'_1, \lambda'_2, \cdots, \lambda'_{m-1}\right\}$ could be k. But it is not essential. It is important that for any $i \left(1 \le i \le k-1\right)$

$$p_1^{\lambda'_1+1} = p_2^{\lambda'_2++1} = \cdots = p_{k-1}^{\lambda'_{k-1}+1} = \left(e^{-\gamma} \cdot F\left(\bar{\lambda}'\right)\right) \cdot \exp\left(e^{-\gamma} \cdot F\left(\bar{\lambda}'\right)\right) + 1 \tag{43}$$

holds. We note that it doesn't exceed one in $\left\{\lambda_1^0, \lambda_2^0, \cdots, \lambda_m^0\right\}$.

We also put

$$\begin{cases} n' = p_1^{\lambda'_1} p_2^{\lambda'_2} \cdots p_{k-1}^{\lambda'_k} \cdot p_k^1 \cdots p_{m-1}^1, \\ n'_+ = p_1^{\lambda'_1} p_2^{\lambda'_2} \cdots p_{k-1}^{\lambda'_k} \cdot p_k^1 \cdots p_{m-1}^1 \cdot p_m^1 = n' \cdot p_m^1, \\ \bar{\lambda}'_+ = \left(\lambda'_1, \lambda'_2, \cdots, \lambda'_{m-1}, 1\right), \ C'_m = H\left(\bar{\lambda}'_+\right). \end{cases} \tag{44}$$

On the other hand, it is well known that

$$\sum_{p \le p_m} \frac{1}{p} = \log\log p_m + b_0 + E_0\left(p_m\right), \tag{45}$$

where

$$b_0 = \gamma + \sum_p \left(\log\left(1 - \frac{1}{p}\right) + \frac{1}{p}\right) = 0.241\cdots \tag{46}$$

([4,7,8]). And there exists a constant $a > 0$ such that

$$E_0\left(p_m\right) = \mathrm{O}\left(\exp\left(-a\sqrt{\log p_m}\right)\right). \tag{47}$$

4.1. The estimate of $F\left(\overline{\lambda}_0\right)$

In this section we will estimate the value $F\left(\overline{\lambda}_0\right)$ for the optimum points $\overline{\lambda}_0 = \left(\lambda_1^0,\, \lambda_2^0, \cdots \lambda_m^0\right) \in R^m$ of the function $H\left(\lambda_1,\, \lambda_2, \cdots, \lambda_m\right)$.

We have

Theorem 5. For the optimum points $\overline{\lambda}_0 = \left(\lambda_1^0,\, \lambda_2^0, \cdots \lambda_m^0\right) \in R^m$ of the function $H\left(\lambda_1,\, \lambda_2, \cdots, \lambda_m\right)$ we have

$$F\left(\overline{\lambda}_0\right) = e^{\gamma} \cdot \log p_m \cdot \left(1 + E_0\left(p_m\right) - \frac{4}{\sqrt{p_m} \cdot \log^{3/2} p_m} + \varepsilon\left(p_m\right)\right), \quad (48)$$

where $\varepsilon\left(p_m\right) = \mathrm{O}\left(E_0^2\left(p_m\right)\right)$. Hence we also have

$$\left(e^{-\gamma} \cdot F\left(\overline{\lambda}_0\right)\right) \cdot \exp\left(e^{-\gamma} \cdot F\left(\overline{\lambda}_0\right)\right) =$$
$$= p_m \cdot \log p_m \cdot \left(1 + \left(\log p_m + 1\right) \cdot E_0\left(p_m\right) - \frac{4 \cdot \left(\log p_m + 1\right)}{\sqrt{p_m} \cdot \log^{3/2} p_m} + \tilde{\varepsilon}\left(p_m\right)\right), \quad (49)$$

where $\tilde{\varepsilon}\left(p_m\right) = \mathrm{O}\left(\log^2 p_m \cdot E_0^2\left(p_m\right)\right)$.

Proof. It is described in the paper [10]. □

4.2. The estimate of $\left(\log C_{m-1} - \log C'_{m-1}\right)$

The aim of this section is to estimate the size of $\left(\log C_{m-1} - \log C'_{m-1}\right)$. This result is used effectively in next section.

We get

Theorem 6. There exists a number m_0 such that for any $m \geq m_0$ we have

$$\log C_{m-1} - \log C'_{m-1} = \frac{p_m - p_{m-1}}{\sqrt{p_{m-1}} \cdot \log p_{m-1}} \cdot \left(\frac{p_m - p_{m-1}}{p_{m-1}}\right) \cdot \left(1 + \beta_0\left(p_m\right)\right), \quad (50)$$

where $\beta_0(p_m) = O\left(\frac{1}{\log p_m}\right)$. (51)

Proof. There is a lot of work for the proof of the theorem 6, as seen in the paper [10]. □

4.3. The estimate of $(p_m - p_{m-1})$

In this section we will estimate the size of $(p_{m+1} - p_m)$. Here obtained result on $(p_{m+1} - p_m)$ is a new result for the distribution of the prime number.

We have

Theorem 7. There exist a number m_0 such that for any $m \geq m_0$ we have

$$(p_m - p_{m-1}) = O\left(\sqrt{p_{m-1}} \cdot \log^{5/2} p_{m-1}\right). \tag{52}$$

Proof. It is easy to find the proof of the theorem 7 in the paper [10]. □

4.4. The estimate of $E_0(p_m)$

In this section we will estimate the size of the error iterm $E_0(p_m)$ given in the formular (45).

We get

Theorem 8. There exists a number m_0 such that for any $m \geq m_0$ we have

$$E_0(p_m) = O\left(\frac{\log^{3/2} p_m}{\sqrt{p_m}}\right). \tag{53}$$

Proof. It is given in the paper [10]. □

5. An inequality on the sum of divisors function

In this section we will consider one inequality on the sum of divisors function. This inequality, in deed, is the proof of the Robin iequality.

Theorem 9. There exist constants $c_0 \ge 1$ such that, for any $n \ge 2$, it holds that

$$\sigma(n) \le e^{\gamma} \cdot n \cdot \log\log\left(c_0 \cdot n \cdot \exp\left(\sqrt{\log n} \cdot \exp\left(\sqrt{(\log\log(n+1))}\right)\right)\right) \quad (54)$$

Proof. We put

$$G(n) = \frac{\left(\exp\left(\exp\left(e^{-\gamma} \cdot \sigma(n)/n\right)\right)\right)/n}{\exp\left(\sqrt{\log n} \cdot \exp\left(\sqrt{\log\log(n+1)}\right)\right)}. \quad (55)$$

It is sufficient to take a constant $c_0 \ge 1$ such that $G(n) \le c_0$ for any $n \ge 2$.

There are two steps for the proof of the theorem 9.

① The function $G(n)$ has the following properties.

First, For any $n \in S(\overline{\lambda}, m)$ it holds that $G(n) \le G(r_0(n))$.

In fact, it is clear by the theorem 1.

Second, for $n = p_1^{\lambda_1} \cdot p_2^{\lambda_2} \cdots p_m^{\lambda_m}$ we put $G(n) = G(\overline{\lambda}) = G(\lambda_1, \lambda_2, \cdots, \lambda_m)$. Then there exist $\overline{\alpha}_0 = (\alpha_1^0, \alpha_2^0, \cdots \alpha_m^0) \in R^m$ such that for any $(\lambda_1, \lambda_2, \cdots, \lambda_m) \in R^m$ we have $G(\overline{\lambda}) \le G(\overline{\alpha}_0)$. This is also clear by the theorem 3.1. And, for the optimum points $\overline{\alpha}_0 = (\alpha_1^0, \alpha_2^0, \cdots \alpha_m^0) \in R^m$ of the function $G(\overline{\lambda})$, the similar results such as in the theorem 2, the theorem 3 and the theorem 4 are valid. Also for any $n \ge 2$ we have

$$G(n) \le H(n) = \left(\exp\left(\exp\left(e^{-\gamma} \cdot \sigma(n)/n\right)\right)\right)/n. \quad (56)$$

Finally, The every member $\alpha_i^0\ (i=1,m)$ of the optimum points $\{\alpha_1^0,\ \alpha_2^0,\cdots\alpha_m^0\}$ of the function $G(\bar{\lambda})$ is not larger than $\lambda_i^0\ (i=1,m)$ of one of the function $H(\bar{\lambda})$, namely, for any $i\ (1\le i\le m)$ it holds that $\alpha_i^0\le\lambda_i^0$.

In fact, by the theorem 2, for the function $H(\bar{\lambda})$ it holds that

$$p_1^{\lambda_1^0+1}=p_2^{\lambda_2^0+1}=\cdots=p_k^{\lambda_k^0+1}=$$
$$=\left(e^{-\gamma}F\left(\bar{\lambda}_0\right)\right)\cdot\exp\left(e^{-\gamma}F\left(\bar{\lambda}_0\right)\right)+1\ \ (1\le i\le k).$$

Similarly, for the function $G(\bar{\lambda})$ it holds that

$$p_1^{\alpha_1^0+1}=p_2^{\alpha_2^0+1}=\cdots=p_k^{\alpha_k^0+1}=$$
$$=\left(e^{-\gamma}F\left(\bar{\alpha}_0\right)\right)\cdot\exp\left(e^{-\gamma}F\left(\bar{\alpha}_0\right)\right)\cdot\left(\frac{1}{1+\Psi(n)}\right)+1\ \ (1\le i\le k),\tag{57}$$

where

$$\Psi(n)=\frac{\exp\left(\sqrt{\log\log(n+1)}\right)}{2\cdot\sqrt{\log n}}+$$
$$+\frac{\exp\left(\sqrt{\log\log(n+1)}\right)}{2\cdot\sqrt{\log\log(n+1)}}\cdot\frac{\sqrt{\log n}}{\log(n+1)}\cdot\left(\frac{n}{n+1}\right)\to 0\ (n\to\infty).\tag{58}$$

Hence for any $i\ (1\le i\le m)$ we have $\alpha_i^0\le\lambda_i^0$ and, in particular, we have

$$F\left(\bar{\alpha}_0\right)=\prod_{i=1}^{m}\frac{1-p_i^{-\alpha_i^0-1}}{1-p_i^{-1}}\le\prod_{i=1}^{m}\frac{1-p_i^{-\lambda_i^0-1}}{1-p_i^{-1}}=F\left(\bar{\lambda}_0\right).\tag{59}$$

② We put

$$D_m=G\left(\alpha_1^0,\ \alpha_2^0,\cdots,\alpha_m^0\right)\tag{60}$$

and

$$\begin{cases} n_0=p_1^{\alpha_1^0}p_2^{\alpha_2^0}\cdots p_k^{\alpha_k^0}\cdot p_{k+1}^1\cdots p_m^1, \quad n_0'=n_0\cdot p_m^{-1}, \\ \bar{\alpha}_0'=\left(\alpha_1^0,\ \alpha_2^0,\cdots,\alpha_{m-1}^0\right)\in R^{m-1}, \\ D_{m-1}'=G\left(\bar{\alpha}_0'\right)=G\left(\alpha_1^0,\ \alpha_2^0,\cdots,\alpha_{m-1}^0\right). \end{cases}\tag{61}$$

In this connection, we put

$$D_{m-1} = \max_{(\lambda_1, \lambda_2, \cdots \lambda_{m-1}) \in R^{m-1}} G\left(\lambda_1 , \lambda_2 , \cdots \lambda_{m-1}\right). \tag{62}$$

Then it is clear that $D'_{m-1} \le D_{m-1}$ and

$$\begin{aligned} \log \frac{D_m}{D'_{m-1}} &= \left(\exp\left(e^{-\gamma} \cdot F\left(\overline{\alpha}_0\right)\right) - \exp\left(e^{-\gamma} \cdot F\left(\overline{\alpha}'_0\right)\right)\right) - \\ &\quad -\left(\log n_0 + \sqrt{\log n_0} \cdot \exp\left(\sqrt{\log\log(n_0+1)}\right)\right) + \\ &\quad +\left(\log n'_0 + \sqrt{\log n'_0} \cdot \exp\left(\sqrt{\log\log(n'_0+1)}\right)\right) = \\ &= \exp\left(e^{-\gamma} \cdot F\left(\overline{\alpha}'_0\right)\right)\left(\exp\left(e^{-\gamma} \cdot F\left(\overline{\alpha}'_0\right) \cdot \frac{1}{p_m}\right) - 1\right) - \left(\log p_m\right) - \\ &\quad -\left(\sqrt{\log n_0} \cdot \exp\left(\sqrt{\log\log(n_0+1)}\right) - \sqrt{\log n'_0} \cdot \exp\left(\sqrt{\log\log(n'_0+1)}\right)\right). \end{aligned} \tag{63}$$

By the theorem 8, we have

$$\begin{aligned} &\exp\left(e^{-\gamma} \cdot F\left(\overline{\alpha}'_0\right)\right)\left(\exp\left(e^{-\gamma} \cdot F\left(\overline{\alpha}'_0\right) \cdot \frac{1}{p_m}\right) - 1\right) \le \\ &\le \exp\left(e^{-\gamma} \cdot F\left(\overline{\lambda}'_0\right)\right)\left(\exp\left(e^{-\gamma} \cdot F\left(\overline{\lambda}'_0\right) \cdot \frac{1}{p_m}\right) - 1\right) = \\ &= \log p_m + \Theta_1\left(p_m\right), \end{aligned} \tag{64}$$

where $\Theta_1\left(p_m\right) = \mathrm{O}\left(\frac{\log^4 p_m}{\sqrt{p_m}}\right)$. So there is a constant a $a > 0$ such that

$$\Theta_1\left(p_m\right) \le a \cdot \frac{\log^4 p_m}{\sqrt{p_m}}. \tag{65}$$

On the other hand, we have

$$\begin{aligned} \log n_0 &= \log\left(p_1^{\alpha_1^0} p_2^{\alpha_2^0} \cdots p_k^{\alpha_k^0} \cdot p_{k+1}^1 \cdots p_m^1\right) = \sum_{i=1}^{m} \alpha_i^0 \cdot \log p_i = \\ &= \sum_{i=1}^{m} \log p_i + \sum_{i=1}^{k} \left(\alpha_i^0 - 1\right) \cdot \log p_i = \vartheta\left(p_m\right) + \vartheta\left(p_k\right) + R_k \end{aligned} \tag{66}$$

where $\vartheta(p_m)=\sum_{i=1}^{m}\log p_i$ is the Chebyshev's function ([4,8]) and $R_k=o(p_k)$.

Hence by the prime number theorem ([3,4,8]), we have

$$\frac{\log n_0}{p_m}=\frac{\vartheta(p_m)}{p_m}+\frac{\vartheta(p_k)}{p_m}+\frac{R_k}{p_m}\to 1\ \left(p_m\to\infty\right). \tag{67}$$

From this we get

$$\log n_0 = p_m\cdot\left(1+\theta_1\left(p_m\right)\right), \tag{68}$$

where $\theta_1\left(p_m\right)=\mathrm{O}\left(\frac{1}{\log p_m}\right)$. So we also obtain

$$\log n_0' = p_{m-1}\left(1+\theta_2\left(p_{m-1}\right)\right). \tag{69}$$

where $\theta_2\left(p_{m-1}\right)=\mathrm{O}\left(\frac{1}{\log p_{m-1}}\right)$. And it is easy to see that

$$\begin{aligned}
&\left(\sqrt{\log n_0}\cdot\exp\left(\sqrt{\log\log(n_0+1)}\right)-\sqrt{\log n_0'}\cdot\exp\left(\sqrt{\log\log(n_0'+1)}\right)\right)=\\
&=\left(\sqrt{\log n_0}-\sqrt{\log n_0'}\right)\cdot\exp\left(\sqrt{\log\log(n_0+1)}\right)+\\
&+\sqrt{\log n_0'}\cdot\left(\exp\left(\sqrt{\log\log(n_0+1)}\right)-\exp\left(\sqrt{\log\log(n_0'+1)}\right)\right)=\\
&=\exp\left(\sqrt{\log p_m}\right)\cdot\left(\frac{\log p_m}{2\cdot\sqrt{p_m}}\right)\cdot\left(1+\Theta_2\left(p_m\right)\right),
\end{aligned} \tag{70}$$

where $\Theta_2\left(p_m\right)=\mathrm{O}\left(\frac{1}{\log p_m}\right)$. Hence we have

$$\begin{aligned}
\log D_m-\log D_{m-1}' &\le a\cdot\frac{\log^4 p_m}{\sqrt{p_m}}-\\
&-\exp\left(\sqrt{\log p_m}\right)\cdot\frac{\log p_m}{2\cdot\sqrt{p_m}}\left(1+\Theta_2\left(p_m\right)\right).
\end{aligned} \tag{71}$$

On the other hand, it is clear that

$$\frac{\log^3 p_m}{\exp\left(\sqrt{\log p_m}\right)}\to 0\ \left(p_m\to\infty\right) \tag{72}$$

This shows that there exists a number m_0 such that for any $m \geq m_0$ we have

$$D_m < D'_{m-1} \leq D_{m-1}. \qquad (73)$$

From this we get

$$0 < c_0 = \sup_m D_m < +\infty. \qquad (74)$$

This is the proof of the theorem 9. □

Note. ① We are able to see that

$$c_0 = D_1 = \frac{\exp\left(\exp\left(e^{-\gamma} \cdot 3/2\right)\right)/2}{\exp\left(\sqrt{\log 2} \cdot \exp\left(\sqrt{\log\log 3}\right)\right)} = 1.6436\cdots \leq 2 \qquad (75)$$

② The process for the proof of the theorem 9 is graphically as follows. Here ⇒ shows the increasing direction of the values for the function $H(n)$ and $G(n)$.

$$n = q_1^{\lambda_1} \cdot q_2^{\lambda_2} \cdot q_3^{\lambda_3} \cdots q_{m-1}^{\lambda_{m-1}} \cdot q_m^{\lambda_m}$$

$$\Downarrow \qquad \leftarrow \text{theorem 1}$$

$$r_0(n) = p_1^{\lambda_1} \cdot p_2^{\lambda_2} \cdot p_3^{\lambda_3} \cdots\cdot p_m^{\lambda_m}$$

$$\Downarrow \qquad \leftarrow \text{theorem 2}$$

$$n_0 = p_1^{\lambda_1^0} \cdot p_2^{\lambda_2^0} \cdots p_k^{\lambda_k^0} \cdot p_{k+1}^1 \cdots p_m^1,$$

$$\Downarrow \qquad \leftarrow \text{theorem 9}$$

$$n'_0 = p_1^{\lambda_1^0} \cdot p_2^{\lambda_2^0} \cdots p_k^{\lambda_k^0} \cdot p_{k+1}^1 \cdots p_{m-1}^1$$

$$\Downarrow \quad \swarrow$$

$$\boxed{n = 2}$$

As it was indicated in the paper [1], one can say that any natural number has the three-dimensional structure. For $\overline{q}(n) = (q_1, q_2, \cdots q_m)$, $\overline{\lambda}(n) = (\lambda_1, \lambda_2, \cdots \lambda_m)$ and $\omega(n) = m$ of $n = q_1^{\lambda_1} \cdot q_2^{\lambda_2} \cdots q_m^{\lambda_m}$ we putted $n = n\left(\overline{q}(n),\ \overline{\lambda}(n),\ \omega(n)\right)$. And we have taken the process reducing the dimensional numbers of $n = n\left(\overline{q}(n),\ \overline{\lambda}(n),\ \omega(n)\right)$ in the function $G(n)$.

The dimensional numbers of n in the function $G(n)$ were reduced by the theorem 1, the theorem 2 and the theorem 9, respectively. That is so; $n = n(\overline{q}(n), \overline{\lambda}(n), \omega(n)) \to n(\overline{\lambda}(n), \omega(n)) \to n(\overline{\lambda}_0, \omega(n)) \to n(m)$.

In other words, the function $G(n)$ with the natural number was estimated upper by every process. Here the function $H(n)$ is the most essential term for the function $G(n)$. In fact, the study of the function $G(n)$ is reduced to the study of the function $H(n)$.

③ The below table 1 shows the optimum points $\overline{\lambda}_0 = (\lambda_1^0, \lambda_2^0, \cdots, \lambda_m^0)$ of the function $H(\overline{\lambda})$ and the values of $H(n_0)$ and $G(n_0)$ to $\omega(n) = m$.

Table 1

$\omega(n)$ $= m$	$\overline{\lambda} = (\lambda_1^0, \lambda_2^0, \cdots, \lambda_m^0)$ of $n_0 = 2^{\lambda_1^0} \cdot 3^{\lambda_2^0} \cdot 5^{\lambda_2^0} \cdots p_k^{\lambda_k^0} \cdot p_{k+1}^1 \cdots p_m^1$	$H(n_0)$, $G(n_0)$
1	$\lambda_1^0 = 1$	$5.09518716186\cdots$, $1.643686767536\cdots$
2	$\lambda_1^0 = 1.65\cdots, \lambda_2^0 = 1$	$3.58945411446\cdots$, $0.8250082 \times 10^{-1} \cdots$
3	$\lambda_1^0 = 2.70\cdots, \lambda_2^0 = 1.33\cdots, \lambda_3^0 = 1$	$1.91192398575\cdots$, $0.7148367 \times 10^{-5} \cdots$
4	$\lambda_1^0 = 3.36\cdots, \lambda_2^0 = 1.75\cdots,$ $\lambda_3^0 = 1, \lambda_4^0 = 1$	$1.32309514626\cdots$, $0.1065950 \times 10^{-6} \cdots$
5	$\lambda_1^0 = 4.22\cdots, \lambda_2^0 = 2.29\cdots,$ $\lambda_3^0 = 1.24\cdots, \lambda_4^0 = \lambda_5^0 = 1$	$0.57062058635\cdots$, $0.3761569 \times 10^{-9} \cdots$
6	$\lambda_1^0 = 4.53\cdots, \lambda_2^0 = 2.49\cdots,$ $\lambda_3^0 = 1.38\cdots, \lambda_4^0 = \lambda_5^0 = \lambda_6^0 = 1$	$0.40977025702\cdots$, $0.767767 \times 10^{-10} \cdots$
7	$\lambda_1^0 = 5.02\cdots, \lambda_2^0 = 2.80\cdots,$ $\lambda_3^0 = 1.59\cdots, \lambda_4^0 = 1.14\cdots,$ $\lambda_5^0 = \lambda_6^0 = \lambda_7^0 = 1$	$0.22782964552\cdots$, $0.575576 \times 10^{-11} \cdots$

8	λ_1^0=5.22$\cdots$, λ_2^0=2.92$\cdots$, λ_3^0=1.68$\cdots$, λ_4^0=1.21$\cdots$, $\lambda_5^0=\lambda_6^0=\lambda_7^0=\lambda_8^0=1$	0.20507350097$\cdots$, $0.164730\times10^{-12}\cdots$
9	λ_1^0=5.57$\cdots$, λ_2^0=3.14$\cdots$, λ_3^0=1.83$\cdots$, λ_4^0=1.34$\cdots$, $\lambda_5^0=\lambda_6^0=\lambda_7^0=\lambda_8^0=\lambda_9^0=1$	0.16722089980$\cdots$, $0.287587\times10^{-14}\cdots$
$\cdots$	$\cdots$ $\cdots$ $\cdots$	$\cdots$ $\cdots$

③ The below table 2 shows the Hardy-Ramanujan numbers, which give maximum value of the function $G(n_0)$ to $\omega(n)=m$.

Table 2

$\omega(n)=m$	$\tilde{n}_0=r_0(\tilde{n}_0)=p_1^{\lambda_1}\cdots p_k^{\lambda_k}\cdot p_{k+1}^1\cdots p_m^1$	$G(\tilde{n}_0)$
1	2	1.643686767536$\cdots$
2	$2\cdot3$	$0.82500822\times10^{-1}\cdots$
3	$2^2\cdot3\cdot5$	$0.71483676\times10^{-5}\cdots$
4	$2^3\cdot3^2\cdot5\cdot7$	$0.10659507\times10^{-6}\cdots$
5	$2^4\cdot3^2\cdot5\cdot7\cdot11$	$0.37615690\times10^{-9}\cdots$
6	$2^4\cdot3^2\cdot5\cdot7\cdot11\cdot13$	$0.76776726\times10^{-10}\cdots$
7	$2^5\cdot3^3\cdot5\cdot7\cdot11\cdot13\cdot17$	$0.575576185\times10^{-11}\cdots$
8	$2^5\cdot3^3\cdot5^2\cdot7\cdot11\cdot13\cdot17\cdot19$	$0.164730227\times10^{-12}\cdots$
9	$2^5\cdot3^3\cdot5^2\cdot7\cdot11\cdot13\cdot17\cdot19\cdot23$	$0.287587585\times10^{-14}\cdots$
$\cdots$	$\cdots$ $\cdots$ $\cdots$	$\cdots$

6. Robin inequality

In this section we will prove that the Robin inequality holds unconditionally. The main idea for it is the Robin theorems in the papers [5,6].

We have

Theorem 10. The Robin inequality holds unconditionally.

Proof. It is sufficient to prove that the Robin inequality is equivalent to (54) in the theorem 9. In fact, it is not difficult to see that.

Suppose that the Robin inequality holds. Then, it is clear that (54) holds. Suppose that inequality (54) holds, but the Robin inequality doesn't hold. Then, since the Robin inequality is equivalent to the RH, also by the Robin's theorem ([5,6]) there exist a constant $c>0$ and a constant $0<\beta<1/2$ such that, for infinitely many number n, it holds that

$$e^{\gamma}\cdot n\cdot\log\log n+c\cdot\frac{n\cdot\log\log n}{(\log n)^{\beta}}\le\sigma(n). \tag{76}$$

On the other hand, it is clear that

$$\begin{aligned}&\log\log\left(c_0\cdot n\cdot\exp\left(\sqrt{\log n}\cdot\exp\left(\sqrt{\log\log(n+1)}\right)\right)\right)=\\&=\log\left(\log c_0+\log n+\sqrt{\log n}\cdot\exp\left(\sqrt{\log\log(n+1)}\right)\right)=\\&=\log\left(\log n\left(1+\frac{\log c_0}{\log n}+\frac{\sqrt{\log n}\cdot\exp\left(\sqrt{\log\log(n+1)}\right)}{\log n}\right)\right)=\\&=\log\log n+\log\left(1+\frac{\log c_0}{\log n}+\frac{\exp\left(\sqrt{\log\log(n+1)}\right)}{\sqrt{\log n}}\right)\le\\&\le\log\log n+\frac{\log c_0}{\log n}+\frac{\exp\left(\sqrt{\log\log(n+1)}\right)}{\sqrt{\log n}}.\end{aligned} \tag{77}$$

Hence, from (76) and (77), for infinitely many numbers n, we have

$$e^{\gamma}\cdot n\cdot\log\log n + c\cdot\frac{n\cdot\log\log n}{(\log n)^{\beta}} \le \sigma(n) \le$$
$$\le e^{\gamma}\cdot n\cdot\left(\log\log n + \frac{\log c_0}{\log n} + \frac{\exp\left(\sqrt{\log\log(n+1)}\right)}{\sqrt{\log n}}\right) \tag{78}$$

and

$$c\cdot e^{-\gamma}\cdot\frac{\log\log n}{(\log n)^{\beta}} \le \frac{\log c_0}{\log n} + \frac{\exp\left(\sqrt{\log\log(n+1)}\right)}{\sqrt{\log n}}. \tag{79}$$

Then since $(1/2-\beta)>0$ we have

$$0 < c\cdot e^{-\gamma} \le \left(\frac{\log c_0}{(\log n)^{1-\beta}\cdot\log\log n} + \frac{\exp\left(\sqrt{\log\log(n+1)}\right)}{(\log n)^{1/2-\beta}\cdot\log\log n}\right) \to 0 \ (n\to\infty) \tag{80}$$

This is also a contradiction. □

By using the method of the proof of the theorem 10, we have more.

The below statements are equivalent to each other.

a) The RH is true.

b) It holds that $\sup_{n\ge 5041} H(n) < 1$.

c) It holds that $\sup_{n\ge 2} H(n) < +\infty$.

d) It holds that $\sup_{n\ge 2} G(n) < +\infty$.

It is not difficult to check it. Here the impotant is the meaning of every expression. The assertion that a) is equivalent to b) is just the Robin theorem. And the assertion that b) is equivalent to c) or d) is just our theorem obtained from the Robin theorem. We note that c) is one showing the most simple and clear relation with the Robin inequality. As seen in the above table 1 and the table 2, we could easily prove that the c) holds unconditionally with $c_0 = \sup_{n\geq 2} H(n) < 6$. In fact, by the concrete calculation, we are able to get $c_0 = H(2) = 5.0951\cdots$. On the other hand, we can say that d) is the most generalized type for the Robin inequality. We note that $\sqrt{\log n}$ in d) is unable to change into $(\log n)^{\mu}$ with $\mu > 1/2$. But, we are able to change $\sqrt{\log\log(n+1)}$ in d) into $(\log\log(n+1))^{\alpha}$ with $0<\alpha<1$. And we are able to give $c_1 = \sup_{n\geq 2} G(2) = 1.64\cdots < 2$. □

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