Introduction to Physical Chemistry – Lecture 5 Supplement: Derivation of the Speed of Sound in Air

I. LECTURE OVERVIEW

We can combine the results of Lecture 5 with some basic techniques in fluid mechanics to derive the speed of sound in air. For the purposes of the derivation, we will assume that air is an ideal gas.

II. DERIVATION OF THE SPEED OF SOUND IN AN IDEAL GAS

Consider a sound wave that is produced in an ideal gas, say air. This sound may be produced by a variety of methods (clapping, explosions, speech, etc.). The central point to note is that the sound wave is defined by a local compression and then expansion of the gas as the wave passes by. A sound wave has a well-defined velocity v, whose value as a function of various properties of the gas (P, T, etc.) we wish to determine.

So, consider a sound wave travelling with velocity v, as illustrated in Figure 1. From the perspective of the sound wave, the sound wave is still, and the air ahead of it is travelling with velocity v. We assume that the air has temperature T and P, and that, as it passes through the sound wave, its velocity changes to $v + dv$, and its temperature and pressure change slightly as well, to $T + dT$ and $P + dP$, respectively (see Figure 1).

Now, let us consider a cross-sectional area perpendicular to the air flow, with area A (see Figure 2). In front of the sound wave, the volume of air that flows through the cross-sectional area over a time interval dt is $Avdt$, with a total mass of $\rho Avdt$. Therefore, the mass flow rate of air through the cross-sectional area is ρAv.

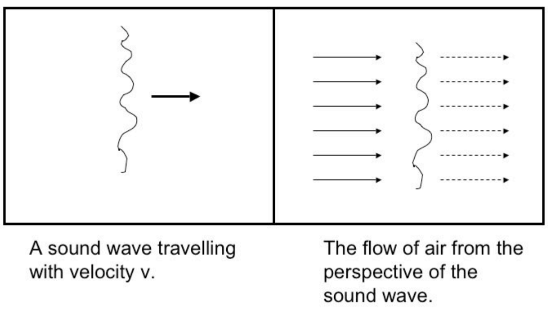

FIG. 1: Diagram of a sound wave travelling through air. One figure is from the perspective of a stationary observer, the other is from the perspective of the sound wave.

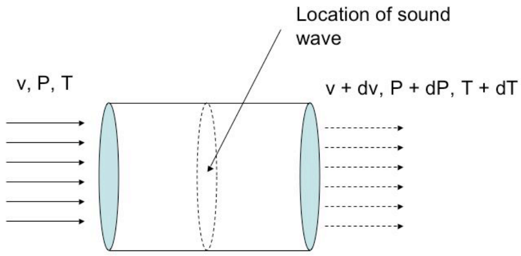

FIG. 2: An imaginary cross-sectional tube running from one side of the sound wave to the other.

Imagine that our cross-sectional area is the opening of a tube that exits behind the sound wave, where the air density is $\rho + d\rho$, and velocity is $v + dv$. Since there is no mass accumulation inside the tube, then applying the principle of conservation of mass we have,

$$\begin{aligned} \rho Av &= (\rho + d\rho)A(v + dv) \Rightarrow \\ \rho v &= \rho v + vd\rho + \rho dv + dvd\rho \Rightarrow \\ 0 &= vd\rho + \rho dv \Rightarrow \\ \rho dv &= -vd\rho \end{aligned} \tag{1}$$

where we set $dvd\rho = 0$, since it is the product of two infinitesimals, hence is much smaller than the other terms and may be neglected (the product of two very small numbers is an even smaller number. For example, $0.001 \times 0.001 = 10^{-6}$, which is $1,000$ times smaller than 0.001).

Let us now turn our attention to Figure 3, where we see a small volume element crossing through the sound wave. The front face experiences a back pressure of $P + dP$, hence a total force of $(P + dP)dydz$. The back face experiences a front pressure of P, hence a total force of $Pdydz$, so that the net force on the cube is $-dPdydz$, in the x-direction. Now, just before the front of the cube hits the sound wave and begins to compress/stretch in the x-direction, it has dimensions dx, dy, dz, and hence the air inside the cube has mass $\rho dxdydz$. Also, before the front face hits sound wave, there is no net force on the cube. Similarly, once the whole cube has crossed through the sound wave, there is no net force. The only time during which a net force is exerted on the cube is while it is crossing through the sound wave.

The total time it takes the cube to cross through the sound wave is given by $dt = dx/v$. Since the net force on the cube is $-dPdydz$, the cube experiences an acceleration of $-dPdydz/(\rho dxdydz) = -dP/(\rho dx)$, and hence

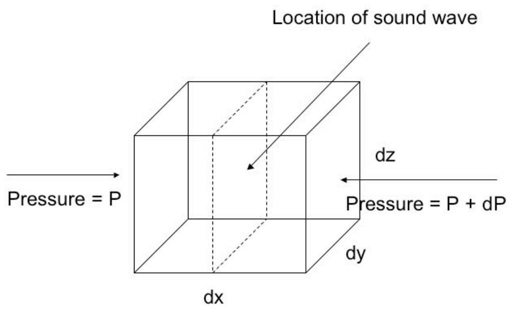

FIG. 3: A force diagram on an infinitesimal cube as it passes through the sound wave.

$dv = -dP/(\rho dx) \times dx/v = -dP/(\rho v)$, so that,

$$\rho v dv = -dP \tag{2}$$

Therefore, since $\rho dv = -v d\rho$, we have, $v^2 d\rho = dP$, and so,

$$v = \sqrt{\frac{dP}{d\rho}} \tag{3}$$

Now, to compute $dP/d\rho$, we note that as air crosses through the sound wave, there is no heat added or removed from the system. Therefore, we can assume that all changes to the thermodynamic variables of the air occur under adiabatic conditions. From Lecture 5, we have, for an adiabatic compression/expansion of an ideal gas,

$$P\bar{V}^{\gamma} = Constant \tag{4}$$

Now, $\rho = \mu_g/\bar{V}$, where μ_g is the molar mass of the gas, so that $\bar{V} = \mu_g/\rho$. We then obtain,

$$P\rho^{-\gamma} = Constant \tag{5}$$

so that,

$$\ln P - \gamma \ln \rho = Constant \tag{6}$$

Differentiating with respect to ρ, we get,

$$\begin{aligned} &\frac{1}{P}\frac{dP}{d\rho} - \frac{\gamma}{\rho} = 0 \Rightarrow \\ &\frac{dP}{d\rho} = \gamma \frac{P}{\rho} \\ &= \gamma \frac{P\bar{V}}{\mu_g} = \frac{\gamma}{\mu_g} RT \end{aligned} \tag{7}$$

and so, the speed of sound is given by,

$$v_{sound} = \sqrt{\frac{\gamma}{\mu_g} RT} \tag{8}$$

Note that the hotter the gas, the faster the speed of sound. Also, note that the heavier the gas, the slower the speed of sound.

For air, $\gamma = 1.4$, and $\mu_g = 28.8 g/mol$. At $25°C = 298K$, we get, $v_{sound} = 347 m/s = 1,250 km/hr$.