On mean values of Dirichlet series

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Key words: Riemann zeta-function, zeroes of the zeta-function, measure theory.

MSC class: 11M26.

1. Introduction.

In present article some properties of Dirichlet series of a kind

$$f(s)=\sum_{n=1}^{\infty}a_n n^{-s}, s=\sigma+it, (9)$$

are considered, where coefficients a_n are complex numbers. These properties concern mean values and an order of Dirichlet series (see [1, 2]). As a mean value is called the following expression

$$\overline{\lim_{T\to\infty}}\frac{1}{T}\int_0^T|f(\sigma+it)|^2dt.$$

In [1, p. 325] they are studied mean values of the series having a finite order. In the same place function $\mu(\sigma)$ denoting an order of Dirichlet series on straight line $\operatorname{Re}s=\sigma$ is entered. In the theory of Riemann zeta-function exists, thereupon a hypothesis named Lindelöf hypothesis, asserting, that $\mu(\sigma)=0$ on a semi plane of mean values (see [2, p. 137]). The question on communication of mean values with various analytical properties of Dirichlet series has been studied in [1,2,4,6]. In works [2,3,4,6,7,8,9,10] mean values of Dirichlet series have been enclosed to a question on an arrangement of zeros of Dirichlet series and arithmetic appendices are given also. Generally consideration of mean values of Dirichlet series is connected with the big difficulties. Only in a few special cases exact results are known. Rather easily, but the proof of that is not trivial at $\sigma>1/2$

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T |\zeta(\sigma+it)|^2\,dt = \zeta(2\sigma)$$

(see [2]). Using functional equation, Hardy and Littlewood have proved, under the same conditions, the following result (see [2, p. 56]):

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T |\zeta(\sigma+it)|^4\,dt = \frac{\zeta^4(2\sigma)}{\zeta(4\sigma)}.$$

For mean values of higher degrees of the zeta-function it is known very little (see [2, p.60], also [12]). In connection with the Lindelöf Hypothesis it is known, that it is equivalent to the statement

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T |\zeta(\sigma+it)|^{2k}\,dt = \sum_{n=1}^{\infty}\tau_k^2(n)n^{-2\sigma},\ (2)$$

for any natural k and $\sigma > 1/2$ (see [2, p. 136]). In the work [1] connection of mean values of Dirichlet series with convergence and regularity is studied. The theorem asserting is proved, that the Dirichlet series (1) converges in a semi plane of mean values if it represents regular function on this semi plane. In the present work the converse problem in more general conditions is considered somewhat. Namely, considering a semi plane of mean values, we define it as a semi plane $\sigma > \sigma_m$ where the series below converges:

$$\sum_{n=1}^{\infty}|a_n|^2 n^{-2\sigma}\ .$$

Not assuming that Dirichlet series defines function of a finite order and that there is a finite limit

$$\overline{\lim_{T\to\infty}}\frac{1}{T}\int_0^T |f(\sigma+it)|^2\,dt$$

(see [1])), we prove, that for one class of Dirichlet series from regularity of the sum of the series (1) in above certain semi plane follows, that it has there not only a finite order (see [1, p. 334]), but for it is carried out as well a relation similar (2), true at $\sigma > \sigma_m$ for any natural *k*. We shall define at first a class of Dirichlet series, for which we shall establish our results.

Let r be a natural number, and $N(r)$ - denotes set of all such natural numbers for which the canonical factorization contains only prime numbers not exceeding r. We shall say, that the series (1) belong to class J, if for every r the series

$$\sum_{n\in N(r)} a_n n^{-\sigma}$$

converges absolutely at $\sigma > \sigma_m$. We shall notice, that many well-known Dirichlet series as the zeta - and L - functions which coefficients are multiplicative functions, belong to class J.

Theorem. *Let a function* $f(s)$ *of the class* J *defined, for* $\sigma > \sigma_0$, *by an absolutely converging Dirichlet series (1), be regular on semi plane* $\sigma > \sigma_m$ *where the series*

$$\sum_{n=1}^{\infty} |a_n|^2 n^{-2\sigma} .$$

converges. Then at $\sigma > \sigma_m$*:*

1) for any natural k the relation

$$\overline{\lim_{T\to\infty}} \frac{1}{T} \int_0^T |f(\sigma+it)|^{2k} dt < C(\sigma,k),$$

takes place, where $C(\sigma,k)$ *is some positive constant;*

2) $\mu(\sigma) = 0$;

3) the series (1) converges.

The theorem proof leans on some new results in the theory of distribution of curves in the infinite dimensional unite cube.

2. The basic auxiliary lemmas.

Definition 1. *Let* $\sigma : N \to N$ *be any one to one mapping of a set of natural numbers. If there will be natural number* m *such, that* $\sigma(n) = n$ *for any* $n > m$ *then we say, that* σ *is finite permutation. A subset* $A \subset \Omega$ *we call finite-symmetric if for any element* $\theta = (\theta_n) \in A$ *and any finite permutation* σ *we have* $\sigma\theta = (\theta_{\sigma(n)}) \in A$.

Let Σ denote a set of all finite permutations. This is a group which contains each group of permutations of degree n as a subgroup (we consider each permutation σ of degree n as finite permutation in sense of definition 1, for which $\sigma(m)=m$ when $m>n$). Set Σ is countable set and we can arrange its elements in a sequence.

Let $\omega\in\Omega$, $\Sigma(\omega)=\{\sigma\omega\,|\,\sigma\in\Sigma\}$ and $\Sigma'(\omega)$ means the closed set of all limit points of the sequence $\Sigma(\omega)$. For real t we denote $\{t\Lambda\}=(\{t\lambda_n\})$, where $\Lambda=(\lambda_n)$. Below we denote μ product of linear Lebesgue measures m set on the segment [0,1]: $\mu=m\times m\times\cdots$.

Lemma 1. *Let* $A\subset\Omega$ *be a finite-symmetric subset of zero measure and* $\Lambda=(\lambda_n)$ *be any unbounded monotonously increasing sequence of positive real numbers every finite subfamily of elements of which linearly independent over the field of rational numbers. Let* $B\supset A$ *any open subset with* $\mu(B)<\varepsilon$ *and*

$$E_0=\{0\le t\le 1\,|\,\{t\Lambda\}\in A\wedge\Sigma'\{t\Lambda\}\subset B\}.$$

Then, we have $m(E_0)\le 6c\varepsilon$ *where* c *the absolute constant and* m *denotes line Lebesgue measure.*

Proof. Let ε is any small positive number. As numbers λ_n are linearly independent, for any finite permutation σ we have $(t_1\lambda_n)\ne(t_2\lambda_{\sigma(n)})$, when $t_1\ne t_2$. Really, otherwise we would receive equality $t_1\lambda_s=t_2\lambda_s$, for enough big natural s, i.e. $(t_1-t_2)\lambda_s=k, k\in Z$. Further, writing down the same equality for some other whole $r>m$ we have an equality

$$k_1/\lambda_r-k/\lambda_s=\frac{k_1\lambda_s-k\lambda_r}{\lambda_r\lambda_s}=0,$$

which contradicts linear independence of numbers λ_n. Hence for any pair of various numbers t_1 and t_2 one have $(\{t_1\lambda_n\})\notin\{(\{t_2\lambda_{\sigma(n)}\})\,|\,\sigma\in\Sigma\}$. By the lemma condition there will be a family of open spheres $B_1,B_2,...$ (in Tikhonov's topology) such, that each sphere does not contain any other one from this family (the sphere, containing in other one can be rejected), thus

$$A \subset B \subset \bigcup_{j=1}^{\infty} B_j, \sum \mu(B_j) < 1.5\varepsilon.$$

Now we take some permutation $\sigma \in \Sigma$ defined by equalities $\sigma(1) = n_1, \ ..., \sigma(k) = n_k$ where the natural numbers n_k are picked up as is indicated below. At first we take N such, that

$$\mu(B'_N) < 2\varepsilon_1,$$

where B'_N is a projection of the sphere B_1 in the subspace of first N coordinate axes and $\mu(B_1) = \varepsilon_1$. We can cover B'_N by cubes with ribs δ and a total measure not exceeding $3\varepsilon_1$. We shall put $k = N$ and define the numbers $n_1, ..., n_k$, using following inequalities

$$\lambda_{n_1} > 1, \lambda_{n_2}^{-1} < (1/4)\delta\lambda_{n_1}^{-1}, \lambda_{n_3}^{-1} < (1/4)\delta\lambda_{n_2}^{-1}, ..., \lambda_{n_k}^{-1} < (1/4)\delta\lambda_{n_{k-1}}^{-1}, \delta < 1. (3)$$

Now we take any cube with rib δ and the centre $(\alpha_m)_{1 \le m \le k}$. Then point $(\{t\lambda_{n_m}\})$ will belong to this cube, if

$$|\{t\lambda_{n_m}\} - \alpha_m| \le \frac{\delta}{2}.$$

From definition of a fractional part at $m = 1$ for some whole r we have:

$$\frac{r + \alpha_1 - \delta/2}{\lambda_{n_1}} \le t \le \frac{r + \alpha_1 + \delta/2}{\lambda_{n_1}}. (4)$$

The measure of a set of such t does not exceed the value $\delta\lambda_{n_1}^{-1}$. The number of such intervals corresponding to different values of $r = [t\lambda_{n_1}] \le \lambda_{n_1}$ does not exceed

$$[\lambda_{n_1}] + 2 \le \lambda_{n_1} + 2.$$

The total measure of corresponding intervals does not exceed

$$\le (\lambda_{n_1} + 2)\delta\lambda_{n_1}^{-1} \le (1 + 2\lambda_{n_1}^{-1})\delta.$$

Now we shall consider one of intervals (4); taking $m = 2$, we shall have

$$\frac{s + \alpha_2 - \delta/2}{\lambda_{n_2}} \le t \le \frac{s + \alpha_2 + \delta/2}{\lambda_{n_2}}, (5)$$

with $s=[t\lambda_{n_2}]\le\lambda_{n_2}$. As we consider conditions (4) and (5) simultaneously we should estimate a total measure of those intervals (5) which have nonempty intersection with intervals of a kind (4), using conditions (3). The number of intervals of a kind (5) with lengths $\lambda_{n_2}^{-1}$, having with one interval of a kind (4) nonempty intersection, does not exceed the size

$$[\delta\lambda_{n_1}^{-1}\lambda_{n_2}]+2\le\delta\lambda_{n_1}^{-1}\lambda_{n_2}+2.$$

Then the measure of set of values t for which conditions (4) and (5) are simultaneously satisfied does not exceed

$$(\lambda_{n_1}+2)(2+\delta\lambda_{n_1}^{-1}\lambda_{n_2})\delta\lambda_{n_2}^{-1}.$$

It is possible to continue these reasoning considering all conditions of a kind

$$\frac{l+\alpha-\delta/2}{\lambda_{n_m}}\le t\le\frac{l+\alpha+\delta/2}{\lambda_{n_m}}, m=1,\dots,k.$$

Then we find the following estimation for the measure $m(\delta)$ of a set of those t for which the points $(\{t\lambda_{n_m}\})$ fall in considered cubes with a rib δ:

$$m(\delta)\le(2+\lambda_{n_1})(2+\delta\lambda_{n_1}^{-1}\lambda_{n_2})\dots(2+\delta\lambda_{n_{k-1}}^{-1}\lambda_{n_k}).$$

Spending simple transformations, we find, considering conditions (3):

$$m(\delta)\le(2+\lambda_{n_1})(2+\delta\lambda_{n_1}^{-1}\lambda_{n_2})\cdots(2+\delta\lambda_{n_{k-1}}^{-1}\lambda_{n_k})\delta\lambda_{n_k}^{-1}\le\delta^k\prod_{m=1}^{\infty}(1+2m^{-2}).$$

By summing over all such cubes, for a measure of set of those t for which $(\{t\lambda_{n_m}\})\in B_1$, we receive such an upper bound $\le 3c\varepsilon, c>0$. We shall notice, that the sequence $\Lambda=(\lambda_n)$ defined above depends on δ. We for each sphere B_k shall fix some sequence Λ_k, using conditions (3). Considering all such spheres we denote $\Sigma_0=\{\Lambda_k\mid k=1,2,\dots\}$. As the set A is finite-symmetric the measure of set of values t interesting us can be estimated using any sequence Λ_k since, as it has been shown above, the sets $\Sigma(\{t\Lambda\})$ for different values t have empty intersections. Further, set $\Sigma(\{t\Lambda\})$ has, for any point $t\in E_0$, nonempty intersection only with finite number of spheres B_k. Really, otherwise, some limit point θ (which contains in open set B) $\Sigma(\Lambda)$ belongs to, say, B_s. Let d - minimal distance from point θ to the bound of B_s. Then for infinite number of indexes n_k, since some number k, all spheres will be contained in a sphere of radius $< d/2$, with centre θ. Hence, for enough large k, all such spheres will contained in the sphere B_s that contradicts the assumption. From here, in turn follows, that the set E_0 can be presented in the form of union of subsets $E_k, k=1,2,...$, where

$$E_k = \{t \in E_0 \mid \Sigma(t\Lambda) \bigcap \bigcup_{m>k} B_m = \varnothing\}.$$

Then,

$$E_k = \{t \in E_0 \mid \Sigma(t\Lambda) \subset \bigcup_{k\le m} B_k\}, E_0 = \bigcup_{k=1}^{\infty} E_k; E_k \subset E_{k+1} (k \ge 1).$$

So, we have

$$m(E_0) \le \limsup_{\Lambda\in\Sigma_0} m(E(\Lambda)) \le \sum_k \limsup_{\Lambda\in\Sigma_0} m(E^{(k)}) \le 3c(\varepsilon_1 + \varepsilon_2 + \ldots) = 3c\varepsilon,$$

Where $E(\Lambda) = \{t \in E_0 \mid (\{t\Lambda\}) \in B\}$ and $E^{(k)} = \{t \in E_0 \mid (\{t\Lambda\}) \in B_k\}$. The proof of the lemma 1 is finished.

The following step of our auxiliary tools-construction of some trigonometrical series. Let $\alpha > \sigma_m, \delta = \alpha - \sigma_m, \sigma > \alpha$, and

$$f_r(s) = \sum_{n\in N(2^r)} a_n n^{-s},$$

where *r=1,2, …*. We shall write down canonical factorization of number *n* in the form of $n = \prod_{p\backslash n} p^{\alpha_p}$. We have

$$f_r(s) = \sum a_n n^{-\sigma} e^{-2\pi i t \sum_{p\backslash n} \lambda_p \alpha_p},$$

где $\lambda_p = (\log p)/2\pi$. Now we shall enter on the consideration the function

$$g_r(\theta) = \sum_{n\in N(2^r)} a_n n^{-\sigma} e^{-2\pi i \sum_{p\backslash n} \alpha_p \theta_p}; g_0(\theta) = 1.$$

From uniqueness of canonical factorization follows

$$\int_{\Omega} |g_r(\theta) - g_{r-1}(\theta)|^2 \mu(d\theta) = \sum_{n \in N(2^r)\setminus N(2^{r-1})} |a_n|^2 n^{-2\sigma}. \quad (6)$$

Let

$$g(\theta) = 1 + \sum_{r \geq 1} |g_r(\theta) - g_{r-1}(\theta)|.$$

In the conditions of the theorem we have $g(\theta) \in L_1(\Omega, \Sigma, \mu, R)$. Really, from (6), applying Cauchy inequality, we receive

$$\sum_{r=1}^{\infty} \int_{\Omega} |g_r(\theta) - g_{r-1}(\theta)| \mu(d\theta) \leq \sum_{r=1}^{\infty} 2^{-(r-1)\delta/2} C^{1/2}(\alpha - \delta/2); \quad C(\sigma) = \sum_{n=1}^{\infty} |a_n|^2 n^{-2\sigma},$$

and the demanded result follows from the theorem of Fatou (see [1, p. 387]). If we enter the designation $F(\theta) = \sum_r (g_r(\theta) - g_{r-1}(\theta))$ then we shall have $F(\theta) \in L_1(\Omega, \Sigma, \mu, R)$.

Lemma 2. *Let E be a subset of such real numbers $t, 0 \leq t \leq 1$ where the series $\sum_{r \geq 1} |g_r(\{t\Lambda\}) - g_{r-1}(\{t\Lambda\})|$ is divergent. Then $\mu(E) = 0$.*

Proof. Let A_0 to denote a set of points of divergence of $g(\theta)$. Under Egorov's theorem (see [5, p. 166]) this series converges almost uniformly out of some set $\Omega_1', \mu(\Omega_1') = 0$. We can assume the set $A_0 \cup \Omega_1'$ to be finite-symmetric (otherwise it is possible to take a set of all finite permutations of all its elements). There will be found some countable family of spheres with a total measure not exceeding ε the union of which contains the set $A_0 \cup \Omega_1'$. For every natural *n* we define the set $\Sigma_n'(t\Lambda)$ as a set of all limit points of the sequence

$$\Sigma_n(\omega) = \{\sigma\omega \mid \sigma \in \Sigma \wedge \sigma(1) = 1 \wedge \cdots \wedge \sigma(n) = n\}.$$

Let

$$D^{(n)} = \{t \mid \{t\Lambda\} \in A_0 \wedge \ \Sigma'_n(\{t\Lambda\}) \subset \bigcup_r B_r\}, \lambda_n = (1/2\pi)\log p_n, \ n = 1, 2, \ldots.$$

We have $D^{(n)} \subset D^{(n+1)}$. Hence, if we shall designate $D = \bigcup_n D^{(n)}$ we shall receive $m(D) \leq \sup m(D^{(n)})$. The set $\Sigma'_n(\{t\Lambda\})$ is a closed set. Clearly, that if we shall "restrict" sequences $\{t\Lambda\}$, leaving only components $\{t\Lambda_n\}$ with indexes, bigger than n and we shall designate the truncated sequence as $\{t\Lambda\}'$ the set $\Sigma'(\{t\Lambda\}')$ also will be closed.

Now we consider products $[0,1]^n \times \{t\Lambda\}'$ for every t. We have, as $g_r(\theta)$ converges absolutely,

$$\{t\Lambda\} \in [0,1]^n \times \{t\Lambda\}' \subset A_0.$$

The below-mentioned example shows, that the equality $A_0 = \Omega$ from this does not follow. Let $I = [0,1]; U = [0;1/2]; V = [1/2;1]$ and

$$X_0 = U \times U \times \ldots, X_1 = V \times U \times \ldots,$$

$$X_2 = I \times V \times U \times \ldots, X_{s+1} = I^s \times V \times U \times \ldots, \ldots.$$

Clearly, that $\mu(X_s) = 0$ for all s. Then $\mu(X) = 0$, where

$$X = \bigcup_{s=0}^{\infty} X_s.$$

Apparently from the construction of X, the equality

$$X = [0,1]^s \times X$$

is true for any s.

As the set $[0,1]^n \times \{\{t\Lambda\}'\}$ is closed, there is only finite set R of natural numbers such, that $[0,1]^n \times \{\{t\Lambda\}'\} \subset \bigcup_{r\in R} B_r$. Consider now the set of restricted points θ' of spheres B_r. Let $B_r' = \{\theta' \mid \theta \in B_r\}$. Then the intersection of them, being an open set, contains the point $\{t\Lambda\}'$. So, we have

$$[0,1]^n \times \{\{t\Lambda\}'\} \subset [0,1]^n \times \bigcap_{r\in R} B_r' \subset \bigcup_{r\in R} B_r,$$

for each considered point t. The similar relation is true in the case when the point $\{t\Lambda\}$ shall be replaced by any limit point ω of the sequence $\Sigma(\{t\Lambda\})$ because $\omega \in B_r$. If by B' to denote the union of all open sets of a kind $\bigcap_{r\in R} B_r'$ we shall receive a relation

$$\{t\Lambda\} \in [0,1]^n \times \{\{t\Lambda\}'\} \subset A \subset [0,1]^n \times B' \subset \bigcup_r B_r,$$

for each considered values of t, or

$$\{\omega\} \in [0,1]^n \times \{\omega\}' \subset A \subset [0,1]^n \times B' \subset \bigcup_r B_r,$$

for each limit point ω. From this it follows, that $\mu(B') \le \varepsilon$. The set B' is open set and $\Sigma'(\{t\Lambda\}') \in B'$. Now we can apply the lemma 3 and receive an estimation $m(D^{(n)}) \le 6c\varepsilon$. Thus, we have $m(D) \le 6c\varepsilon$. Hence, taking $n = y_k, k = 1,2,3,...$, we can find such limit point $\omega_k \in \Omega \setminus \bigcup_r B_r$ of the sequence $\Sigma_n(\{t\Lambda\})$, for which the series

$$\sum_{r\ge1} |g_r(\omega_k) - g_{r-1}(\omega_k)|$$

converges for all values $t \in E \setminus D$. As the set $\Omega \setminus \bigcup_r B_r$ is closed, the limit point $\overline{\omega} = (\{t\Lambda\})$ of the sequence (ω_k) will belong to the set $\Omega \setminus \bigcup_r B_r$. Therefore, the series

$$\sum_{r\ge1} |g_r(\overline{\omega}) - g_{r-1}(\overline{\omega})|$$

converges for all $t \in E \setminus D$, i.e. $E \subset D$. Then, for all t, except for values from some set of a measure not exceeding $12c\varepsilon$, last series converges. Owing to randomness of ε, last result shows convergence of the series of the lemma 2 for almost all considered t. The lemma 2 is proved.

Clearly, that condition $0 \le t \le 1$ can be omitted now, i.e. the result of the lemma 2 is true for all real t.

Lemma 3. *Let a series of analytical functions*

$$\sum_{n=1}^{\infty} f_n(s)$$

be given in one-connected domain G of a complex s-plane and converges absolutely almost everywhere in G in the Lebesgue sense and a function

$$\Phi(\sigma,t) = \sum_{n=1}^{\infty} | f_n(s) |$$

is summable function in G. Then, the given series converges in uniformly in any compact subdomain of G; in particular the sum of this series is an analitical function in G.

Proof. It is enough to show, that the theorem is true for any rectangle C in the domain G. Let C be a rectangle in G and C' be another rectangle inside C, moreover, their sides are parallel to the co-ordinate axes. We can assume, that on a contour of these rectangles the series converges almost everywhere, according to the theorem of Fubini (see [5, p. 208]). From the Lebesgue theorem on the bouded convergence (see [14, p. 293]):

$$(2\pi i)^{-1}\int_C \frac{\Phi_0(s)}{s-\xi}ds = \sum_{n=1}^{\infty}(2\pi i)^{-1}\int_C \frac{f_n(s)}{s-\xi}ds,$$

where the integrals are taken in the Lebesgue sense and $\Phi_0(s)=\Phi_0(\sigma,t)$ is the sum of the given series at convergence points. As on the right part of equality the integrals exist in the Riemann sense by applying the Cauchy formula, we receive

$$\Phi_1(\xi)=(2\pi i)^{-1}\int_C \frac{\Phi_0(s)}{s-\xi}ds = \sum_{n=1}^{\infty} f_n(s),$$

where $\Phi_1(\xi)=\Phi_0(\xi)$. almost everywhere and ξ is any point on or in a contour. Further, members of the series in C' are estimated as follows

$$|f_n(\xi)|\le (2\pi)^{-1}\int_C \frac{|f_n(s)|}{|s-\xi|}|ds| \le (2\pi\delta)^{-1}\int_C |f_n(s)||ds|,$$

where δ. designates the minimum distance between the sides C and C'. A series

$$\sum_{n=1}^{\infty}\int_C |f_n(s)||ds|$$

converges in the consent with theorem of Lebesgue on monotonous convergence (see [14, P. 290]). Hence, the series $\sum_{n=1}^{\infty} f_n(\xi)$ converges uniformly in inside of C'. The lemma 3 is proved.

3. Application of a theorem of Croneker.

The following lemma is best known theorem of Croneker.

Lemma 4. *Let* $\alpha_1,\alpha_2,...,\alpha_N$ - *are the he real numbers linearly independent over the field of rational numbers;* γ - *a sub domain of* N - *dimensional unite cube with a volume* Γ *in Jordan sense. Let, further,* $I_\gamma(T)$ - *a measure of set* $t\in(0,T)$, *for which* $(\alpha_1 t,\alpha_2 t,...,\alpha_N t)\in\gamma \bmod 1$.

Then

$$\lim_{T\to\infty}\frac{I_\gamma(T)}{T}=\Gamma.$$

Proof of this lemma is given in [4, p. 345]. At performance of the statement of a lemma 4 for each parallelepiped of the unite cube say, that the curve $(\alpha_1 t, \alpha_2 t, ..., \alpha_N t)$ is uniformly distributed $\bmod 1$ (see [4, p. 348]).

The following theorem is generalization of a lemma 4.

Lemma 5. *Let curve* $(\alpha_1 t, \alpha_2 t, ..., \alpha_N t)$ *be uniformly distributed* $\bmod 1$. *Then for any integrable function* $F(x)$ *in Riemann sense the following relation is true*

$$\lim_{T\to\infty} T^{-1}\int_0^T F(\{t\alpha_1\}, ..., \{t\alpha_N\})dt = \int_0^1 \cdots \int_0^1 F(x_1, ... x_N)dx_1 ... dx_N.$$

Under conditions of the lemma 1 projection of the curve $(\{t\Lambda\})$ in the finite dimensional unite cube is uniformly distributed $\bmod 1$.

Lemma 6. *For any continuous function* $F(x)$ *in the infinite-dimensional unite cube* Ω *the relation*

$$\lim_{T\to\infty} T^{-1}\int_{\{t\Lambda\}\in B} F(\{t\Lambda\})dt = \int_B F(\theta)d\theta,$$

is satisfied, where B *is any sphere.*

Proof. We take a sphere with radius $0 < r$.

$$B(0,r) = \left\{ x\in\Omega \mid d(x,0) = \sum_{n=1}^{\infty} e^{1-n}|x_n| < r \right\}.$$

As $|x_n| \le 1$ then for natural number N we have

$$\sum_{n=N+1}^{\infty} e^{1-n}|x_n| \le e^{-N}\sum_{n=0}^{\infty} e^{-n} < e^{1-N}.$$

Taking any small number $\varepsilon > 0$ we at $N \ge \log e\varepsilon^{-1}$ имеем:

$$\sum_{n=1}^{N} e^{1-n}|x_n| \le d(x,0) \le \sum_{n=1}^{N} e^{1-n}|x_n| + \varepsilon.$$

Therefore

$$B_N(0, r-\varepsilon)\times[0,1]\times\cdots \subset B(0,r) \subset B_N(0,r)\times[0,1]\times\cdots,$$

where $B_N(0,r)$ means a projection of the sphere B to the first N co-ordinate axes.

$$\mu_N(r)-\mu_N(r-\varepsilon)=\int\limits_{r-\varepsilon\le\sum\limits_{n=1}^{N} e^{1-n}|x_n|\le r} dx_1\cdots dx_N=2^N\int\limits_{r-\varepsilon\le u\le r} du\int\limits_{\sum_{n\le N} e^{1-n}u_n=u}\frac{ds}{\|\nabla\|}\le$$

$$\le\varepsilon 2^N\int\limits_M\frac{ds}{\|\nabla\|},$$

where the last integral is a surface integral on surface M which is defined by the equation

$$\sum_{n=1}^{N} e^{1-n}u_n=u,\quad 0\le u_k\le 1,\ (7)$$

with value u, supplying a maximum, and ∇ - a gradient of linear function on the left part of the last equality, i.e.

$$\|\nabla\|=\sqrt{1+e^{-2}+\cdots+e^{2-2N}}\ .$$

Defining u_1 from (7) it is received

$$\int\limits_M\frac{ds}{\|\nabla\|}\le\int\limits_0^1\cdots\int\limits_0^1 du_2\cdots du_N=1.$$

So,

$$\mu_N(r)-\mu_N(r-\varepsilon)\le\varepsilon 2^N.\ (8)$$

Taking the least whole number N with a condition $N\ge\log e\varepsilon^{-1}$, i.e. $N=[\log e\varepsilon^{-1}]+1$ it is received $\varepsilon\ \le e^{2-N}$. Then from (8) follows

$$\mu_N(r)-\mu_N(r-\varepsilon)\le 2^N e^{2-N}\to 0,$$

as $N\to\infty$.

As function $F(\theta)$ is continuous, it is bounded. Taking ε sufficiently small, we deduce justice of a lemma 6 from the previous lemma. Lemma 6 is proved.

4. Proof of the theorem.

Before to pass to a conclusion of the statement of the theorem we shall prove a lemma.

Lemma 7. *Let $L_1(\Omega,\Sigma,\mu,R)$ to mean Lebesgue's class of summable functions. Then for any natural k the relatiosn below takes place:*

1) $g^{2k}(\theta)\in L_1(\Omega,\Sigma,\mu,R);$

2) there will be found a constant $c(\sigma,k)$ *such, that*

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \left|1+F(\{t\Lambda\})\right|^{2k} dt \le c(\sigma,k).$$

Proof. As it has been above shown, set of points of divergence of the series

$$\sum_{r\ge 1}\left|g_r(\overline{\omega})-g_{r-1}(\overline{\omega})\right|$$

has special structure. Namely, if A_0 - set of points of divergence for any natural n it is had:

$$A_0=[0,1]^n\times A_0';\ \mu(A_0')=\mu(A_0').$$

Hence, its each open covering looks like

$$B=[0,1]^n\times B',$$

where $B'\subset\Omega,(\mu(B')\le\varepsilon)$ is an open set (hence, B is finite-symmetrical). Defining sets D and $D^{(n)}$ as in a lemma 2, we deduce the relation $m(D)\le 12c\varepsilon$. Under Egorov's theorem, a considered series converges almost uniformly out of some finite-symmetrical set $\Omega_1,\mu(\Omega_1)=0$. Let $\Omega_1\subset B\subset\Omega.$ Then $g(\theta)$ is bounded out of B. As on the set $\Omega\setminus B$ the series $g(\theta)$ converges uniformly, then for any $\delta>0$ will be found H such that the following inequality is fair

$$\left|g(\theta)-(1+\sum_{r\le H}|\,g_r(\theta)-g_{r-1}(\theta)\,|)\right|\le\delta.\ (9)$$

From absolute convergence $g_r(\theta)$ we conclude, that there exist a constant L for which an inequality below is satisfied:

$$1+\sum\nolimits_{r\le H}|\,g_r(\theta)-g_{r-1}(\theta)\,|\le L$$

for all $\theta\in\Omega$. Hence,

$$\int_{\Omega\setminus B} g(\theta)\mu(d\theta)=\int_{\Omega'}(1+\sum\nolimits_{r\le H}|\,g_r(\theta)-g_{r-1}(\theta)\,|)\mu(d\theta)+O(L\eta),\ (10)$$

with a number $\eta>0$ set beforehand, independently from other parameters, satisfying the inequality $\mu(\Omega')-\mu(\Omega\setminus B)\le\eta$. Then by the lemma 6 one had:

$$\int_{0,\{t\Lambda\}\in\Omega'}^{T}(1+|\,g_r(\{t\Lambda\})-g_{r-1}(\{t\Lambda\})\,|)dt=T\int_{\Omega'}(1+|\,g_r(\theta)-g_{r-1}(\theta)\,|)\mu(d\theta)+o(T).\ (11)$$

Further, there will be found the open set of a measure $\leq 2\eta$ containing the union $A_0 \cup (\Omega' \setminus (\Omega \setminus B))$. Under the remarks made above concerning structure of a set of points of divergence, it is possible suppose last open set to be finite-symmetrical, thus

$$m(\{t \leq T \mid \{t\Lambda\} \in \Omega' \setminus (\Omega \setminus B)\}) \leq 12c\eta T. \quad (12)$$

From relations (9), (10), (11) and (12) considering randomness of δ and η concude:

$$\int_{0 \leq t \leq T, \{t\Lambda\} \in \Omega \setminus B} g(\{t\Lambda\})dt = T \int_{\Omega \setminus B} g(\theta)\mu(d\theta) + o(T). \quad (13)$$

Let now G denote a maximal value of $g(\theta)$ in $\Omega \setminus B$. We shall define the sets $\Omega_k = \{\theta \in \Omega \setminus B \mid G2^{k-1} < g(\theta)\}$. Let Ω'_k be an open set covering Ω_k which has measure $\leq 2\mu(\Omega_k)$. Then, from Parseval's equality one deduces:

$$G^2 2^{2k-2} \mu(\Omega_k) \leq C(\sigma),$$

or

$$\mu(\Omega_k) \leq C(\sigma) G^{-2} 2^{2-2k}.$$

As Ω'_k is open set under the remarks made above, we have:

$$m(\{t \mid \{t\Lambda\} \in \Omega'_k\}) \leq 12c\mu(\Omega_k).$$

According to the lemma 2, we shall have

$$\int_{\{t\Lambda\} \in \Omega \setminus \Omega'} g(\{t\Lambda\})dt \leq 24cC(\sigma)TG^{-2} \sum_{k=1}^{\infty} G2^k 2^{2-2k} \leq 96cC(\sigma)G^{-1}T.$$

Therefore, $g(\{t\Lambda\})$ is a summable function and as G can grow unboundedly (the case $A_0 = \varnothing$ is trivial) considering (13), we have:

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T g(\{t\Lambda\})dt = \int_{\Omega} g(\theta)\mu(d\theta).$$

Thus, the lemma is proved for the case $k = 1$.

Now we shall prove, that $(1 + F(\theta))^{2k} \in L_1(\Omega, \Sigma, \mu, R)$. We have:

$$\left|1 + F(\theta)\right|^{2k} \leq 2k + 2k\left|\sum_r r^{-1} r g_r(\theta)\right|^{2k} \leq 2k + 2k\left(\sum_r r^{-\frac{2k}{2k-1}}\right)^{2k-1} \left(\sum_r r^{2k} \left|g_r^{2k}(\theta)\right|\right).$$

Then

$$\int_{\Omega} |1+F(\theta)|^{2k} \mu(d\theta) \le 2k + c_k \sum_r r^{2k} \int_{\Omega} |g_r^k(\theta)|^2 \mu(d\theta). \ (9)$$

From definition of function g_r we receive

$$g_r^k(\theta) = \sum_{n_1} \cdots \sum_{n_k} a_{n_1} \cdots a_{n_k} (n_1 \cdots n_k)^{-\sigma} e^{-2\pi i \sum_{p \backslash n_1 \cdots n_k} \alpha_p \theta_p} =$$

$$= \sum_{2^{kr-k} \le n < 2^{kr}-1} b_n n^{-\sigma} e^{-2\pi i \sum_{p \backslash n} \alpha_p \theta_p},$$

where $b_n = \sum_{n_1 \cdots n_k = n} a_{n_1} \cdots a_{n_k}$. Hence,

$$|b_n|^2 \le \tau_k(n) \sum_{n_1 \cdots n_k = n} |a_{n_1} \cdots a_{n_k}|^2$$

(Here $\tau_k(n)$ expresses number of factorizations of n into product of k natural factors) and we have:

$$\int_{\Omega} |g_r^k(\theta)|^2 \mu(d\theta) = \sum_{2^{kr-k} \le n < 2^{kr}-1} |b_n|^2 n^{-2\sigma} \le c_0(\varepsilon) 2^{2kr\varepsilon} \left(\sum_{2^{r-1} \le n < 2^r - 1} |a_n|^2 n^{-2\sigma} \right)^k \le$$

$$\le c_0(\varepsilon) C^k (\alpha - (\delta/2)) 2^{2kr\varepsilon - (kr-k)\delta/2},$$

As for any positive ε the following inequality is carried out

$$\tau_k(n) \le c_0(\varepsilon) n^{\varepsilon}$$

with some positive constant $c_0(\varepsilon)$ depending only on ε (see [11, p. 34]). Then from (9) taking into account the subsequent inequality it is received, при $\varepsilon < \delta/8$:

$$\int_{\Omega} |1+F(\theta)|^{2k} \mu(d\theta) \le c_0(\varepsilon) C^k \left(2^{2k\varepsilon} + \sum_{r \ge 2} r^{2k} 2^{-k(r-2)\delta/4} \right), \ C = C(\alpha - (\delta/2)).$$

Now the statement of the lemma 7, at any k, turns out by similar reasoning. The proof of the lemma 7 is finished.

Let's pass now to the theorem proof. We shall consider function of complex argument $s = \sigma + it$:

$$h(s) = 1 + \sum_{r=1}^{\infty} ((h_r(s) - h_{r-1}(s)),$$

where $h_r(\sigma + it) = \sum_{n \in N(2^r)} a_n n^{-\sigma} n^{-it}$. Apparently from expression for g_r, h_r at every $\sigma > \sigma_m$

coincides with $g_r(t\Lambda)$. We take any great real number $T>0$ and we shall consider function $h(s)$ in the rectangle K, with angular points $\sigma \pm iT, a \pm it$, where $a>\sigma_0$ (as usually, σ_0 denotes an absciss of absolute convergence). It is easy to see, that conditions of a lemma 3 are satisfied for the series $1+\sum_{r=1}^{\infty}((h_r(s)-h_{r-1}(s))$. By the lemma 3 this series represents an analytical function which coincides with $f(s)$ on the semi-plane $\mathrm{Re}\, s>\sigma_0$ owing to its regularity in any one-connected domain placed in the semiplane $\mathrm{Re}\, s>\sigma_m$, not containing zeroes of $f(s)$, which have nonempty intersection with the semiplane $\mathrm{Re}\, s>\bar{\sigma}$ of absolute convergence. So, by the principle of analytical continuation, it coinsides with $f(s)$, owing to randomness of T, in the all semiplane $\mathrm{Re}\, s>\sigma_m$, with exception of on no mire than coutale number of segments being parallel to the real axes over which we take a cross cats. Therefore, everywhere in the taken rectangle we have $|f(s)|=|h(s)|$.

The theorem follows now from the lemma 7 of present work, theorems 9.44 and 9.55 of [2].

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