Map Projections Used by the U.S. Geological Survey

GEOLOGICAL SURVEY BULLETIN 1532

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By JOHN P. SNYDER

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PREFACE TO SECOND EDITION

This study of map projections is intended to be useful to both the reader interested in the philosophy or history of the projections and the reader desiring the mathematics. Under each of the sixteen projections described, the nonmathematical phases are presented first, without interruption by formulas. They are followed by the formulas and tables, which the first type of reader may skip entirely to pass to the nonmathematical discussion of the next projection. Even with the mathematics, there are almost no derivations, very little calculus, and no complex variables or matrices. The emphasis is on describing the characteristics of the projection and how it is used.

This bulletin is also designed so that the user can turn directly to the desired projection, without reading any other section, in order to study the projection under consideration. However, the list of symbols may be needed in any case, and the random-access feature will be enhanced by a general understanding of the concepts of projections and distortion. As a result of this intent, there is some repetition which will be apparent as the book is read sequentially.

Many of the formulas and much of the history and general discussion are adapted from a source manuscript I prepared shortly before joining the U.S. Geological Survey. The relationship of the projections to the Survey has been incorporated as a result of the generous cooperation of several Survey personnel. These include Alden P. Colvocoresses, William J. Jones, Clark H. Cramer, Marlys K. Brownlee, Tau Rho Alpha, Raymond M. Batson, William H. Chapman, Atef A. Elassal, Douglas M. Kinney (ret.), George Y. G. Lee, Jack P. Minta (ret.), and John F. Waananen. Joel L. Morrison of the University of Wisconsin/Madison and Allen J. Pope of the National Ocean Survey also made many helpful comments.

Many of the inverse formulas, and a few others, have been derived in conjunction with this study. Many of the formulas may be found in other sources; however, many, especially inverse formulas, are frequently omitted or are included in more cumbersome form elsewhere. All formulas adapted from other sources have been tested for accuracy.

For the more complicated projections, equations are given in the order of usage. Otherwise, major equations are given first, followed by subordinate equations. When an equation has been given previously, it is repeated with the original equation number, to avoid the need to leaf back and forth. A compromise in this philosophy is the placing of numerical examples in appendix A. It was felt that placing these with the formulas would only add to the difficulty of reading through the mathematical sections.

The need for a working manual of this type has led to an unexpectedly early exhausting of the supply of the first edition of this bulletin. In this new edition there are minor revisions and corrections noted to date. These primarily consist of corrections to equations (15–10) on p. 129 and (20–22) on p. 204 and replacement of the inverse van der Grinten algorithm on p. 215–216 with that developed by Rubincam. The former algorithm is also accurate, but very cumbersome. In addition, historical notes have been corrected on p. 23, 144, and 219.

Further corrections and comments by users are most welcome. It is hoped that this study provides a practical reference for those concerned with map projections.

John P. Snyder

Reston, Va.
May 1983

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SYMBOLS

If a symbol is not listed here, it is used only briefly and identified near the formulas in which it is given.

Az = azimuth, as an angle measured clockwise from the north.
a = equatorial radius or semimajor axis of the ellipsoid of reference.
b = polar radius or semiminor axis of the ellipsoid of reference.
$= a(1-f) = a(1-e^2)^{1/2}$.
c = great circle distance, as an arc of a circle.
e = eccentricity of the ellipsoid.
$= (1-b^2/a^2)^{1/2}$.
f = flattening of the ellipsoid.
h = relative scale factor along a meridian of longitude.
k = relative scale factor along a parallel of latitude.
n = cone constant on conic projections, or the ratio of the angle between meridians to the true angle, called l in some other references.
R = radius of the sphere, either actual or that corresponding to scale of the map.
S = surface area.
x = rectangular coordinate: distance to the right of the vertical line (Y axis) passing through the origin or center of a projection (if negative, it is distance to the left). In practice, a "false" x or "false easting" is frequently added to all values of x to eliminate negative numbers.
y = rectangular coordinate: distance above the horizontal line (X axis) passing through the origin or center of a projection (if negative, it is distance below). In practice, a "false" y or "false northing" is frequently added to all values of y to eliminate negative numbers.
z = angular distance from North Pole of latitude ϕ, or $(90° - \phi)$, or colatitude.
z_1 = angular distance from North Pole of latitude ϕ_1, or $(90° - \phi_1)$.
z_2 = angular distance from North Pole of latitude ϕ_2, or $(90° - \phi_2)$.
ln = natural logarithm, or logarithm to base e, where e = 2.71828.
θ = angle measured counterclockwise from the central meridian, rotating about the center of the latitude circles on a conic or polar azimuthal projection, or beginning due south, rotating about the center of projection of an oblique or equatorial azimuthal projection.
θ' = angle of intersection between meridian and parallel.

Symbols – Continued

λ = longitude east of Greenwich (for longitude west of Greenwich, use a minus sign).

λ_0 = longitude east of Greenwich of the central meridian of the map, or of the origin of the rectangular coordinates (for west longitude, use a minus sign). If ϕ_1 is a pole, λ_0 is the longitude of the meridian extending down on the map from the North Pole or up from the South Pole.

λ' = transformed longitude measured east along transformed equator from the north crossing of the Earth's Equator, when graticule is rotated on the Earth.

ρ = radius of latitude circle on conic or polar azimuthal projection, or radius from center on any azimuthal projection.

ϕ = north geodetic or geographic latitude (if latitude is south, apply a minus sign).

ϕ_0 = middle latitude, or latitude chosen as the origin of rectangular coordinates for a projection.

ϕ' = transformed latitude relative to the new poles and equator when the graticule is rotated on the globe.

ϕ_1, ϕ_2 = standard parallels of latitude for projections with two standard parallels. These are true to scale and free of angular distortion.

ϕ_1 (without ϕ_2) = single standard parallel on cylindrical or conic projections; latitude of central point on azimuthal projections.

ω = maximum angular deformation at a given point on a projection.

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1. All angles are assumed to be in radians, unless the degree symbol (°) is used.
2. Unless there is a note to the contrary, and if the expression for which the arctan is sought has a numerator over a denominator, the formulas in which arctan is required (usually to obtain a longitude) are so arranged that the Fortran ATAN2 function should be used. For hand calculators and computers with the arctan function but not ATAN2, the following conditions must be added to the limitations listed with the formulas:

   For arctan (A/B), the arctan is normally given as an angle between $-90°$ and $+90°$, or between $-\pi/2$ and $+\pi/2$. If B is negative, add $\pm 180°$ or $\pm\pi$ to the initial arctan, where the $\pm$ takes the sign of A, or if A is zero, the $\pm$ arbitrarily takes a + sign. If B is zero, the arctan is $\pm 90°$ or $\pm\pi/2$, taking the sign of A. Conditions not resolved by the ATAN2 function, but requiring adjustment for almost any program, are as follows:

   (1) If A and B are both zero, the arctan is indeterminate, but may normally be given an arbitrary value of 0 or of λ_0, depending on the projection, and

   (2) If A or B is infinite, the arctan is $\pm 90°$ (or $\pm\pi/2$) or 0, respectively, the sign depending on other conditions.

   In any case, the final longitude should be adjusted, if necessary, so that it is an angle between $-180°$ (or $-\pi$) and $+180°$ (or $+\pi$). This is done by adding or subtracting multiples of 360° (or 2π) as required.
3. Where division is involved, most equations are given in the form $A = B/C$ rather than $A = \frac{B}{C}$. This facilitates typesetting, and it also is a convenient form for conversion to Fortran programing.

ACRONYMS

AGS	American Geographical Society
GRS	Geodetic Reference System
HOM	Hotine (form of ellipsoidal) Oblique Mercator
IMC	International Map Committee
IMW	International Map of the World
IUGG	International Union of Geodesy and Geophysics
NASA	National Aeronautics and Space Administration
NGS	National Geographic Society
SOM	Space Oblique Mercator
SPCS	State Plane Coordinate System
UPS	Universal Polar Stereographic
USC&GS	United States Coast and Geodetic Survey
USGS	United States Geological Survey
UTM	Universal Transverse Mercator
WGS	World Geodetic System

Some acronyms are not listed, since the full name is used throughout this bulletin.

MAP PROJECTIONS USED BY THE U.S. GEOLOGICAL SURVEY

By JOHN P. SNYDER

ABSTRACT

After decades of using only one map projection, the Polyconic, for its mapping program, the U.S. Geological Survey (USGS) now uses sixteen of the more common map projections for its published maps. For larger scale maps, including topographic quadrangles and the State Base Map Series, conformal projections such as the Transverse Mercator and the Lambert Conformal Conic are used. On these, the shapes of small areas are shown correctly, but scale is correct only along one or two lines. Equal-area projections, especially the Albers Equal-Area Conic, and equidistant projections which have correct scale along many lines appear in the *National Atlas*. Other projections, such as the Miller Cylindrical and the Van der Grinten, are chosen occasionally for convenience, sometimes making use of existing base maps prepared by others. Some projections treat the Earth only as a sphere, others as either ellipsoid or sphere.

The USGS has also conceived and designed several new projections, including the Space Oblique Mercator, the first map projection designed to permit mapping of the Earth continuously from a satellite with low distortion. The mapping of extraterrestrial bodies has resulted in the use of standard projections in completely new settings.

With increased computerization, it is important to realize that rectangular coordinates for all these projections may be mathematically calculated with formulas which would have seemed too complicated in the past, but which now may be programed routinely, if clearly delineated with numerical examples. A discussion of appearance, usage, and history is given together with both forward and inverse equations for each projection involved.

INTRODUCTION

The subject of map projections, either generally or specifically, has been discussed in thousands of papers and books dating at least from the time of the Greek astronomer Claudius Ptolemy (about A.D.150), and projections are known to have been in use some three centuries earlier. Most of the widely used projections date from the 16th to 19th centuries, but scores of variations have been developed during the 20th century. Within the past 10 years, there have been several new publications of widely varying depth and quality devoted exclusively to the subject (Alpha and Gerin, 1978; Hilliard and others, 1978; Lee, 1976; Maling, 1973; McDonnell, 1979; Pearson, 1977; Rahman, 1974; Richardus and Adler, 1972; Wray, 1974). In 1979, the USGS published *Maps for America,* a book-length description of its maps (Thompson, 1979).

In spite of all this literature, there has been no definitive single publication on map projections used by the USGS, the agency responsible for administering the National Mapping Program. The USGS has relied on map projection treatises published by the former Coast and Geodetic Survey (now the National Ocean Survey). These publications do not include sufficient detail for all the major projections used by the USGS. A widely used and outstanding treatise of the Coast and Geodetic Survey (Deetz and Adams, 1934), last revised in 1945, only touches upon the Transverse Mercator, now a commonly used projection for preparing maps. Other projections such as the Bipolar Oblique Conic Conformal, the Miller Cylindrical, and the Van der Grinten, were just being developed, or, if older, were seldom used in 1945. Deetz and Adams predated the extensive use of the computer and pocket calculator, and, instead, offered extensive tables for plotting projections with specific parameters.

Another classic treatise from the Coast and Geodetic Survey was written by Thomas (1952) and is exclusively devoted to the five major conformal projections. It emphasizes derivations with a summary of formulas and of the history of these projections, and is directed toward the skilled technical user. Omitted are tables, graticules, or numerical examples.

In this bulletin, the author undertakes to describe each projection which has been used by the USGS sufficiently to permit the skilled mathematically oriented cartographer to use the projection in detail. The descriptions are also arranged so as to enable a lay person interested in the subject to learn as much as desired about the principles of these projections without being overwhelmed by mathematical detail. Deetz and Adams' work sets an excellent example in this combined approach.

Since this study is limited to map projections used by the USGS, several map projections frequently seen in atlases and geography texts have been omitted. The general formulas and concepts are useful, however, in studying these other projections. Many tables of rectangular or polar coordinates have been included for conceptual purposes. For values between points, formulas should be used, rather than interpolation. Other tables list definitive parameters for use in formulas.

The USGS, soon after its official inception in 1879, apparently chose the Polyconic projection for its mapping program. This projection is simple to construct and had been promoted by the Survey of the Coast, as it was then called, since Ferdinand Rudolph Hassler's leadership of the early 1800's. The first published USGS topographic "quadrangles," or maps bounded by two meridians and two parallels, did not carry a projection name, but identification as "Polyconic projection" was added to later editions. Tables of coordinates published by the USGS appeared by 1904, and the Polyconic was the only projection mentioned by Beaman (1928, p. 167).

Mappers in the Coast and Geodetic Survey, influenced in turn by military and civilian mappers of Europe, established the State Plane Coordinate System in the 1930's. This system involved the Lambert Conformal Conic projection for States of larger east-west extension and the Transverse Mercator for States which were longer from north to south. In the late 1950's, the USGS began changing quadrangles from the Polyconic to the projection used in the State Plane Coordinate System for the principal State on the map. The USGS also adopted the Lambert for its series of State base maps.

As the variety of maps issued by the USGS increased, a broad range of projections became important: The Polar Stereographic for the map of Antarctica, the Lambert Azimuthal Equal-Area for maps of the Pacific Ocean, and the Albers Equal-Area Conic for *National Atlas* (USGS, 1970) maps of the United States. Several other projections have been used for other maps in the *National Atlas*, for tectonic maps, and for grids in the panhandle of Alaska. The mapping of extraterrestrial bodies, such as the Moon, Mars, and Mercury, involves old projections in a completely new setting. The most recent projection promoted by the USGS and perhaps the first to be originated within the USGS is the Space Oblique Mercator for continuous mapping using artificial satellite imagery (Snyder, 1981).

It is hoped that this study will assist readers to understand better not only the basis for maps issued by the USGS, but also the principles and formulas for computerization, preparation of new maps, and transferring of data between maps prepared on different projections.

MAP PROJECTIONS—GENERAL CONCEPTS

1. CHARACTERISTICS OF MAP PROJECTIONS

The general purpose of map projections and the basic problems encountered have been discussed often and well in various books on cartography and map projections. (Robinson, Sale, and Morrison, 1978; Steers, 1970; and Greenhood, 1964, are among recent editions of earlier standard references.) It is necessary to mention the concepts, but to do so concisely, although there are some interpretations and formulas that appear to be unique.

For almost 500 years, it has been conclusively established that the Earth is essentially a sphere, although there were a number of intellectuals nearly 2,000 years earlier who were convinced of this. Even to the scholars who considered the Earth flat, the skies appeared hemispherical, however. It was established at an early date that attempts to prepare a flat map of a surface curving in all directions leads to distortion of one form or another.

A map projection is a device for producing all or part of a round body on a flat sheet. Since this cannot be done without distortion, the cartographer must choose the characteristic which is to be shown accurately at the expense of others, or a compromise of several characteristics. There is literally an infinite number of ways in which this can be done, and several hundred projections have been published, most of which are rarely used novelties. Most projections may be infinitely varied by choosing different points on the Earth as the center or as a starting point.

It cannot be said that there is one "best" projection for mapping. It is even risky to claim that one has found the "best" projection for a given application, unless the parameters chosen are artificially constricting. Even a carefully constructed globe is not the best map for most applications because its scale is by necessity too small. A straightedge cannot be satisfactorily used for measurement of distance, and it is awkward to use in general.

The characteristics normally considered in choosing a map projection are as follows:

1. *Area.* Many map projections are designed to be *equal-area,* so that a coin, for example, on one part of the map covers exactly the same area of the actual Earth as the same coin on any other part of the map. Shapes, angles, and scale must be distorted on most parts of such a map, but there are usually some parts of an equal-area map which are designed to retain these characteristics correctly, or very nearly so.

Less common terms used for equal-area projections are *equivalent, homolographic*, or *homalographic* (from the Greek *homalos* or *homos* ("same") and *graphos* ("write")); *authalic* (from the Greek *autos* ("same") and *ailos* ("area")), and *equiareal.*

2. *Shape.* Many of the most common and most important projections are *conformal* or *orthomorphic* (from the Greek *orthos* or "straight" and *morphē* or "shape"), in that normally the shape of every *small* feature of the map is shown correctly. (On a conformal map of the entire Earth there are usually one or more "singular" points at which shape is still distorted.) A large landmass must still be shown distorted in shape, even though its small features are shaped correctly. An important result of conformality is that relative angles at each point are correct, and the local scale in every direction around any one point is constant. Consequently, meridians intersect parallels at right (90°) angles on a conformal projection, just as they do on the Earth. Areas are generally enlarged or reduced throughout the map, but they are relatively correct along certain lines, depending on the projection. Nearly all large-scale maps of the Geological Survey and other mapping agencies throughout the world are now prepared on a conformal projection.

3. *Scale.* No map projection shows scale correctly throughout the map, but there are usually one or more lines on the map along which the scale remains true. By choosing the locations of these lines properly, the scale errors elsewhere may be minimized, although some errors may still be large, depending on the size of the area being mapped and the projection. Some projections show true scale between one or two points and every other point on the map, or along every meridian. They are called "equidistant" projections.

4. *Direction.* While conformal maps give the relative local directions correctly at any given point, there is one frequently used group of map projections, called *azimuthal* (or *zenithal*), on which the directions or azimuths of all points on the map are shown correctly with respect to the center. One of these projections is also equal-area, another is conformal, and another is equidistant. There are also projections on which directions from two points are correct, or on which directions from all points to one or two selected points are correct, but these are rarely used.

5. *Special characteristics.* Several map projections provide special characteristics that no other projection provides. On the Mercator projection, all rhumb lines, or lines of constant direction, are shown as straight lines. On the Gnomonic projection, all great circle paths–the shortest routes between points on a sphere–are shown as straight lines. On the Stereographic, all small circles, as well as great circles, are shown as circles on the map. Some newer projections are specially designed for satellite mapping. Less useful but mathematically intrigu-

ing projections have been designed to fit the sphere conformally into a square, an ellipse, a triangle, or some other geometric figure.

6. *Method of construction.* In the days before ready access to computers and plotters, ease of construction was of greater importance. With the advent of computers and even pocket calculators, very complicated formulas can be handled almost as routinely as simple projections in the past.

While the above features should ordinarily be considered in choosing a map projection, they are not so obvious in recognizing a projection. In fact, if the region shown on a map is not much larger than the United States, for example, even a trained eye cannot often distinguish whether the map is equal-area or conformal. It is necessary to make measurements to detect small differences in spacing or location of meridians and parallels, or to make other tests. The type of construction of the map projection is more easily recognized with experience, if the projection falls into one of the common categories.

There are three types of developable[1] surfaces onto which most of the map projections used by the USGS are at least partially geometrically projected. They are the cylinder, the cone, and the plane. Actually all three are variations of the cone. A cylinder is a limiting form of a cone with an increasingly sharp point or apex. As the cone becomes flatter, its limit is a plane.

If a cylinder is wrapped around the globe representing the Earth, so that its surface touches the Equator throughout its circumference, the meridians of longitude may be projected onto the cylinder as equidistant straight lines perpendicular to the Equator, and the parallels of latitude marked as lines parallel to the Equator, around the circumference of the cylinder and mathematically spaced for certain characteristics. When the cylinder is cut along some meridian and unrolled, a cylindrical projection with straight meridians and straight parallels results (see fig. 1). The Mercator projection is the best-known example.

If a cone is placed over the globe, with its peak or apex along the polar axis of the Earth and with the surface of the cone touching the globe along some particular parallel of latitude, a conic (or conical) projection can be produced. This time the meridians are projected onto the cone as equidistant straight lines radiating from the apex, and the parallels are marked as lines around the circumference of the cone in planes perpendicular to the Earth's axis, spaced for the desired characteristics. When the cone is cut along a meridian, unrolled, and laid flat, the meridians remain straight radiating lines, but the parallels are now circular arcs centered on the apex. The angles between meridians are shown smaller than the true angles.

[1] A developable surface is one that can be transformed to a plane without distortion.

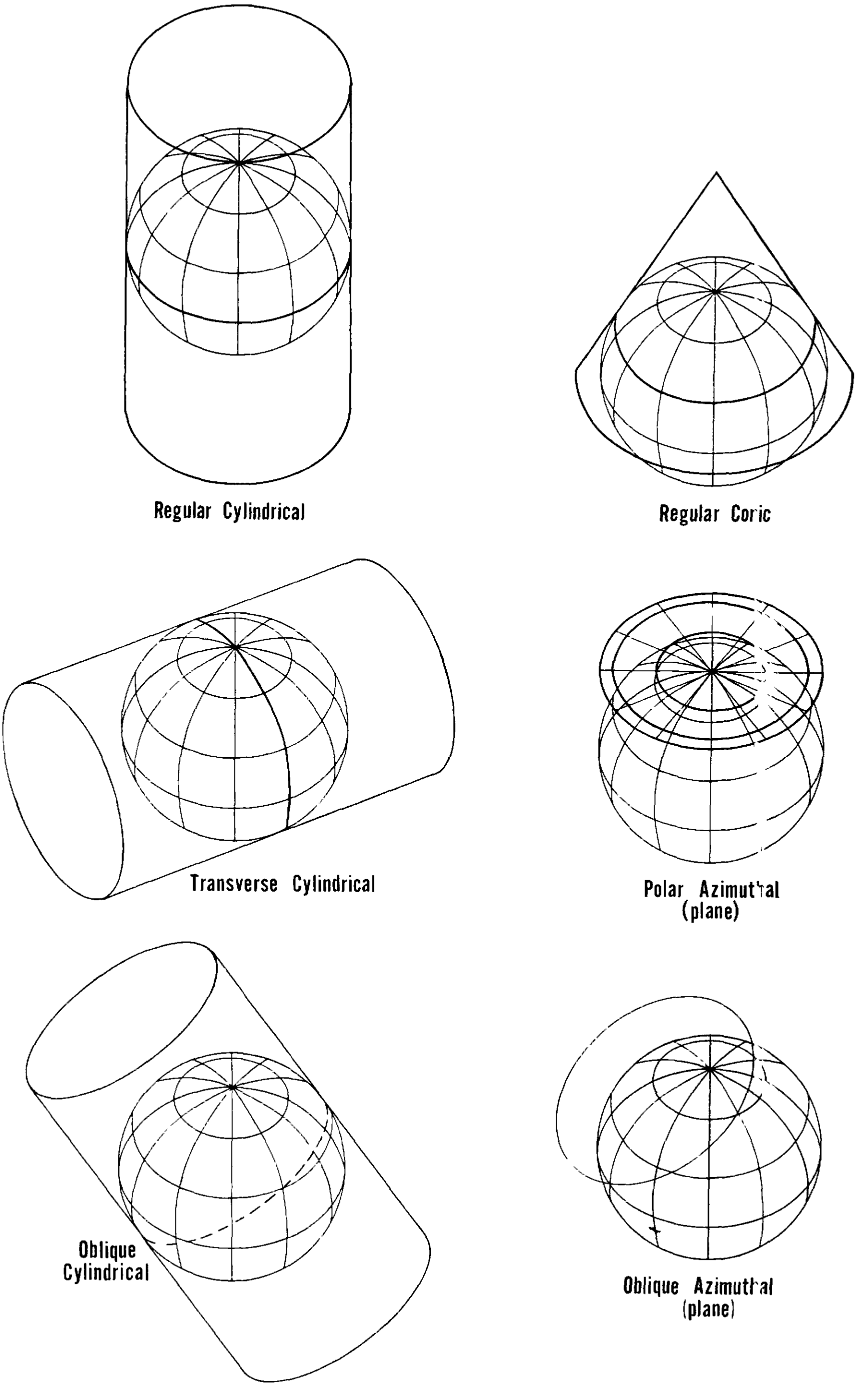


FIGURE 1.–Projection of the Earth onto the three major surfaces. In a few cases, projection is geometric, but in most cases the projection is mathematical to achieve certain features.

A plane tangent to one of the Earth's poles is the basis for polar azimuthal projections. In this case, the group of projections is named for the function, not the plane, since all common tangent-plane projections of the sphere are azimuthal. The meridians are projected as straight lines radiating from a point, but they are spaced at their true angles instead of the smaller angles of the conic projections. The parallels of latitude are complete circles, centered on the pole. On some important azimuthal projections, such as the Stereographic (for the sphere), the parallels are geometrically projected from a common point of perspective; on others, such as the Azimuthal Equidistant, they are nonperspective.

The concepts outlined above may be modified in two ways, which still provide cylindrical, conic, or azimuthal projections (although the azimuthals retain this property precisely only for the sphere).

(1) The cylinder or cone may be secant to or cut the globe at two parallels instead of being tangent to just one. This conceptually provides two standard parallels; but for most conic projections this construction is not geometrically correct. The plane may likewise cut through the globe at any parallel instead of touching a pole.

(2) The axis of the cylinder or cone can have a direction different from that of the Earth's axis, while the plane may be tangent to a point other than a pole (fig. 1). This type of modification leads to important oblique, transverse, and equatorial projections, in which most meridians and parallels are no longer straight lines or arcs of circles. What were standard parallels in the normal orientation now become standard lines not following parallels of latitude.

Some other projections used by the USGS resemble one or another of these categories only in some respects. The Sinusoidal projection is called pseudocylindrical because its latitude lines are parallel and straight, but its meridians are curved. The Polyconic projection is projected onto cones tangent to each parallel of latitude, so the meridians are curved, not straight. Still others are more remotely related to cylindrical, conic, or azimuthal projections, if at all.

2. LONGITUDE AND LATITUDE

To identify the location of points on the Earth, a graticule or network of longitude and latitude lines has been superimposed on the surface. They are commonly referred to as meridians and parallels, respectively. Given the North and South Poles, which are approximately the ends of the axis about which the Earth rotates, and the Equator, an imaginary line halfway between the two poles, the parallels of latitude are formed by circles surrounding the Earth and in planes parallel with that of the Equator. If circles are drawn equally spaced along the surface of the sphere, with 90 spaces from the Equator to each pole, each space is called a degree of latitude. The circles are numbered from 0° at the Equator to 90° North and South at the respective poles. Each degree is subdivided into 60 minutes and each minute into 60 seconds of arc.

Meridians of longitude are formed with a series of imaginary lines, all intersecting at both the North and South Poles, and crossing each parallel of latitude at right angles, but striking the Equator at various points. If the Equator is equally divided into 360 parts, and a meridian passes through each mark, 360 degrees of longitude result. These degrees are also divided into minutes and seconds. While the length of a degree of latitude is always the same on a sphere, the lengths of degrees of longitude vary with the latitude (see fig. 2). At the Equator on the sphere, they are the same length as the degree of latitude, but elsewhere they are shorter.

There is only one location for the Equator and poles which serve as references for counting degrees of latitude, but there is no natural origin from which to count degrees of longitude, since all meridians are identical in shape and size. It, thus, becomes necessary to choose arbitrarily one meridian as the starting point, or prime meridian. There have been many prime meridians in the course of history, swayed by national pride and international influence. Eighteenth-century maps of the American colonies often show longitude from London or Philadelphia. During the 19th century, boundaries of new States were described with longitudes west of a meridian through Washington, D.C., 77°03′02.3″ west of the Greenwich (England) Prime Meridian, which was increasingly referenced on 19th century maps (Van Zandt, 1976, p. 3). In 1884, the International Meridian Conference, meeting in Washington, agreed to adopt the "meridian passing through the center of the transit instrument at the Observatory of Greenwich as the initial meridian for longitude," resolving that "from this meridian longitude shall be counted in two directions up to 180 degrees, east longitude being plus and west longitude minus " (Brown, 1949, p. 297).

When constructing meridians on a map projection, the central meridian, usually a straight line, is frequently taken to be the starting point or 0° longitude for calculation purposes. When the map is completed with labels, the meridians are marked with respect to the Greenwich Prime Meridian. The formulas in this bulletin are arranged so that Greenwich longitude may be used directly.

The concept of latitudes and longitudes was originated early in recorded history by Greek and Egyptian scientists, especially the Greek astronomer Hipparchus (2nd century, B.C.). Claudius Ptolemy further formalized the concept (Brown, 1949, p. 50, 52, 68).

Because calculations relating latitude and longitude to positions of points on a given map can become quite involved, rectangular grids have been developed for the use of surveyors. In this way, each point may be designated merely by its distance from two perpendicular axes on the flat map.

3. THE DATUM AND THE EARTH AS AN ELLIPSOID

For many maps, including nearly all maps in commercial atlases, it may be assumed that the Earth is a sphere. Actually, it is more nearly an oblate ellipsoid of revolution, also called an oblate spheroid. This is an ellipse rotated about its shorter axis. The flattening of the ellipse for the Earth is only about one part in three hundred; but it is sufficient to become a necessary part of calculations in plotting accurate maps at a scale of 1:100,000 or larger, and is significant even for 1:5,000,000-scale maps of the United States, affecting plotted shapes by up to ⅔ percent. On small-scale maps, including single-sheet world maps, the oblateness is negligible. Formulas for both the sphere and ellipsoid will be discussed in this bulletin wherever the projection is used in both forms.

The Earth is not an exact ellipsoid, and deviations from this shape are continually evaluated. For map projections, however, the problem has been confined to selecting constants for the ellipsoidal shape and size and has not generally been extended to incorporating the much smaller deviations from this shape, except that different reference ellipsoids are used for the mapping of different regions of the Earth.

An official shape of the ellipsoid was defined in 1924, when the International Union of Geodesy and Geophysics (IUGG) adopted a flattening of exactly 1 part in 297 and a semimajor axis (or equatorial radius) of exactly 6,378,388 m. The radius of the Earth along the polar axis is then 1/297 less than 6,378,388, or approximately 6,356,911.9 m. This is called the International ellipsoid and is based on John Fillmore Hayford's calculations in 1909 from U.S. Coast and Geodetic Survey measurements made entirely within the United States (Brown, 1949, p. 293; Hayford, 1909). This ellipsoid was not adopted for use in North America.

There are over a dozen other principal ellipsoids, however, which are still used by one or more countries (table 1). The different dimensions do not only result from varying accuracy in the geodetic measurements (the measurements of locations on the Earth), but the curvature of the Earth's surface is not uniform due to irregularities in the gravity field.

Until recently, ellipsoids were only fitted to the Earth's shape over a particular country or continent. The polar axis of the reference ellipsoid for such a region, therefore, normally does not coincide with the axis of the actual Earth, although it is made parallel. The same applies to the two equatorial planes. The discrepancy between centers is usually a few hundred meters at most. Only satellite-determined coordinate

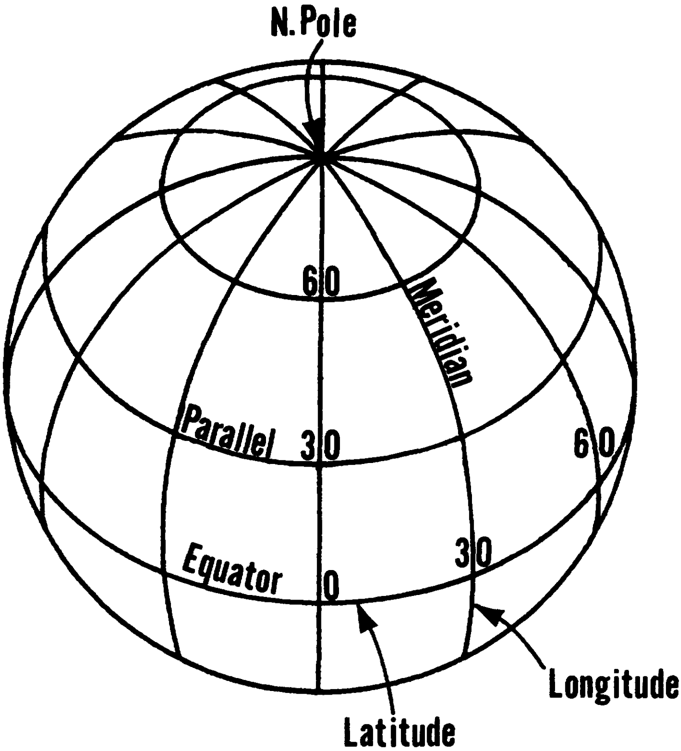


FIGURE 2.—Meridians and parallels on the sphere.

systems, such as the WGS 72 mentioned below, are considered geocentric. Ellipsoids for the latter systems represent the entire Earth more accurately than ellipsoids determined from ground measurements, but they do not generally give the "best fit" for a particular region.

The reference ellipsoid is used with an "initial point" of reference on the surface to produce a datum, the name given to a smooth mathematical surface that closely fits the mean sea-level surface throughout the area of interest. The "initial point" is assigned a latitude, longitude, and elevation above the ellipsoid. Once a datum is adopted, it provides the surface to which ground control measurements are referred. The latitude and longitude of all the control points in a given area are then computed relative to the adopted ellipsoid and the adopted "initial point." The projection equations of large-scale maps must use the same ellipsoid parameters as those used to define the local datum; otherwise, the projections will be inconsistent with the ground control.

TABLE 1.–*Some Official Ellipsoids in use Throughout the World*[1]

Name	Date	Equatorial Radius, a, meters	Polar Radius b, meters	Flattening f	Use
GRS 1980[2]	1980	6,378,137*	6,356,752.3	1/298.257	Newly adopted
WGS 72[3]	1972	6,378,135*	6,356,750.5	1/298.26	NASA
Australian	1965	6,378,160*	6,356,774.7	1/298.25*	Australia
Krasovsky	1940	6,378,245*	6,356,863.0	1/298.3*	Soviet Union
Internat'l Hayford	1924 1909	6,378,388*	6,356,911.9	1/297*	Remainder of the world.†
Clarke	1880	6,378,249.1	6,356,514.9	1/293.46**	Most of Africa; France
Clarke	1866	6,378,206.4*	6,356,583.8*	1/294.98	North America; Philippines.
Airy	1849	6,377,563.4	6,356,256.9	1/299.32**	Great Britain
Bessel	1841	6,377,397.2	6,356,079.0	1/299.15**	Central Europe; Chile; Indonesia.
Everest	1830	6,377,276.3	6,356,075.4	1/300.80**	India; Burma; Pakistan; Afghan.; Thailand; etc.

Values are shown to accuracy in excess significant figures, to reduce computational confusion.

[1] Maling, 1973, p. 7; Thomas, 1970, p. 84; Army, 1973, p. 4, endmap; Colvocoresses, 1969, p. 33; World Geodetic, 1974.

[2] Geodetic Reference System. Ellipsoid derived from adopted model of Earth.

[3] World Geodetic System. Ellipsoid derived from adopted model of Earth.

* Taken as exact values. The third number (where two are asterisked) is derived using the following relationships: $b=a(1-f)$; $f=1-b/a$. Where only one is asterisked (for 1972 and 1980), certain physical constants not shown are taken as exact, but f as shown is the adopted value.

** Derived from a and b, which are rounded off as shown after conversions from lengths in feet.

† Other than regions listed elsewhere in column, or some smaller areas.

"The first official geodetic datum in the United States was the New England Datum, adopted in 1879. It was based on surveys in the eastern and northeastern states and referenced to the Clarke Spheroid of 1866, with triangulation station Principio, in Maryland, as the origin. The first transcontinental arc of triangulation was completed in 1899, connecting independent surveys along the Pacific Coast. In the intervening years, other surveys were extended to the Gulf of Mexico. The New England Datum was thus extended to the south and west without major readjustment of the surveys in the east. In 1901, this expanded network was officially designated the United States Standard Datum, and triangulation station Meades Ranch, in Kansas, was the origin. In 1913, after the geodetic organizations of Canada and Mexico formally agreed to base their triangulation networks on the United States network, the datum was renamed the North American Datum.

"By the mid-1920's, the problems of adjusting new surveys to fit into the existing network were acute. Therefore, during the 5-year period 1927–1932 all available primary data were adjusted into a system now known as the North American 1927 Datum.*** The coordinates of station Meades Ranch were not changed but the revised coordinates of the network comprised the North American 1927 Datum" (National Academy of Sciences, 1971, p. 7).

The ellipsoid adopted for use in North America is the result of the 1866 evaluation by the British geodesist Alexander Ross Clarke using measurements made by others of meridian arcs in western Europe, Russia, India, South Africa, and Peru (Clarke and Helmert, 1911, p. 807–808). This resulted in an adopted equatorial radius of 6,378,206.4 m and a polar radius of 6,356,583.8 m, or an approximate flattening of 1/294.9787. Since Clarke is also known for an 1880 revision used in Africa, the Clarke 1866 ellipsoid is identified with the year.

Satellite tracking data have provided geodesists with new measurements to define the best Earth-fitting ellipsoid and for relating existing coordinate systems to the Earth's center of mass. The Defense Mapping Agency's efforts produced the World Geodetic System 1966 (WGS 66), followed by a more recent evaluation (1972) producing the WGS 72. The polar axis of the Clarke 1866 ellipsoid, as used with the North American 1927 Datum, is calculated to be 159 m from that of WGS 72. The equatorial planes are 176 m apart (World Geodetic System Committee, 1974, p. 30).

Work is underway at the National Geodetic Survey to replace the North American 1927 Datum. The new datum, expected to be called "North American Datum 1983," will be Earth-centered based on satellite tracking data. The IUGG early in 1980 adopted a new model of the Earth called the Geodetic Reference System (GRS) 1980, from which is derived an ellipsoid very similar to that for the WGS 72; it is expected that this ellipsoid will be adopted for the new North American datum.

For the mapping of other planets and natural satellites, only Mars is treated as an ellipsoid. The Moon, Mercury, Venus, and the satellites of Jupiter and Saturn are taken as spheres (table 2).

In most map projection formulas, some form of the eccentricity e is used, rather than the flattening f. The relationship is as follows:

$$e^2 = 2f - f^2, \text{ or } f = 1 - (1 - e^2)^{1/2}$$

For the Clarke 1866, e^2 is 0.006768658.

AUXILIARY LATITUDES

By definition, the geographic or geodetic latitude, which is normally the latitude referred to for a point on the Earth, is the angle which a line perpendicular to the surface of the ellipsoid at the given point makes with the plane of the Equator. It is slightly greater in magnitude than the geocentric latitude, except at the Equator and poles, where it

TABLE 2.–*Official figures for extraterrestrial mapping*

[(From Batson, 1973, p. 4433; 1976, p. 59; 1979; Davies and Batson, 1975, p. 2420; Pettengill, 1980; Batson, Private commun., 1981.) Radius of Moon chosen so that all elevations are positive. Radius of Mars is based on a level of 6.1 millibar atmospheric pressure; Mars has both positive and negative elevations]

Body	Equatorial radius a* (kilometers)
Earth's Moon	1,738.0
Mercury	2,439.0
Venus	6,051.4
Mars	3,393.4*
Galilean satellites of Jupiter	
Io	1,816
Europa	1,563
Ganymede	2,638
Callisto	2,410
Satellites of Saturn	
Mimas	195
Enceladus	250
Tethys	525
Dione	560
Rhea	765
Hyperion	155
Iapetus	720

* Above bodies are taken as spheres except for Mars, an ellipsoid with eccentricity e of 0.101929. Flattening $f=1-(1-e^2)^{1/2}$.

is equal. The geocentric latitude is the angle made by a line to the center of the ellipsoid with the equatorial plane.

Formulas for the spherical form of a given map projection may be adapted for use with the ellipsoid by substitution of one of various "auxiliary latitudes" in place of the geodetic latitude. Oscar S. Adams (1921) derived or presented five substitute latitudes. In using them, the ellipsoidal Earth is, in effect, first transformed to a sphere under certain restraints such as conformality or equal area, and the sphere is then projected onto a plane. If the proper auxiliary latitudes are chosen, the sphere may have either true areas, true distances in certain directions, or conformality, relative to the ellipsoid. Spherical map projection formulas may then be used for the ellipsoid solely with the substitution of the appropriate auxiliary latitudes.

It should be made clear that this substitution will generally not give the projection in its preferred form. For example, using the conformal latitude (defined below) in the spherical Transverse Mercator equations will give a true ellipsoidal, conformal Transverse Mercator, but the central meridian cannot be true to scale. More involved formulas are necessary, since uniform scale on the central meridian is a standard requirement for this projection as commonly used in the ellipsoidal form. For the regular Mercator, on the other hand, simple substitution of the conformal latitude is sufficient to obtain both conformality and an Equator of correct scale for the ellipsoid.

Adams gave formulas for all these auxiliary latitudes in closed or exact form, as well as in series, except for the authalic (equal-area) latitude, which could also have been given in closed form. Both forms are given below. In finding the auxiliary latitude from the geodetic latitude, the closed form may be more useful for computer programs. For the inverse cases, to find geodetic from auxiliary latitudes, most closed forms require iteration, so that the series form is probably preferable. The series form shows more readily the amount of deviation from the geodetic latitude ϕ. The formulas given later for the individual ellipsoidal projections incorporate these formulas as needed, so there is no need to refer back to these for computation, but the various auxiliary latitudes are grouped together here for comparison.

The conformal latitude χ, giving a sphere which is truly conformal in accordance with the ellipsoid (Adams, 1921, p. 18, 84),

$$\chi = 2 \arctan \{\tan (\pi/4 + \phi/2) [(1 - e \sin \phi)/(1 + e \sin \phi)]^{e/2}\} - \pi/2 \quad (3\text{–}1)$$

$$= \phi - (e^2/2 + 5e^4/24 + 3e^6/32 + \ldots) \sin 2\phi + (5e^4/48 + 7e^6/80 + \ldots) \sin 4\phi - (13e^6/480 + \ldots) \sin 6\phi + \ldots \quad (3\text{–}2)$$

with χ and ϕ in radians. In seconds of arc for the Clarke 1866 ellipsoid,

$$\chi = \phi - 700.04'' \sin 2\phi + 0.99'' \sin 4\phi \quad (3\text{–}3)$$

The inverse formula, for ϕ in terms of χ, may be a rapid iteration of an exact rearrangement of (3–1), successively placing the value of ϕ calculated on the left side into the right side of (3–4) for the next calculation, using χ as the first trial ϕ. When ϕ changes by less than a desired convergence value, iteration is stopped.

$$\phi = 2 \arctan \{\tan (\pi/4 + \chi/2)[(1 + e \sin \phi)/(1 - e \sin \phi)]^{e/2}\} - \pi/2 \quad (3\text{–}4)$$

The inverse formula may also be written as a series, without iteration (Adams, 1921, p. 85):

$$\phi = \chi + (e^2/2 + 5e^4/24 + e^6/12 + \ldots) \sin 2\chi + (7e^4/48 + 29e^6/240 + \ldots) \sin 4\chi + (7e^6/120 + \ldots) \sin 6\chi + \ldots \quad (3\text{–}5)$$

or, for the Clarke 1866 ellipsoid, in seconds,

$$\phi = \chi + 700.04'' \sin 2\chi + 1.39'' \sin 4\chi \quad (3\text{–}6)$$

Adams referred to χ as the isometric latitude, but this name is now applied to ψ, a separate very nonlinear function of ϕ, which is directly proportional to the spacing of parallels of latitude from the Equator on the ellipsoidal Mercator projection. It is also useful for other conformal projections:

$$\psi = \ln\{\tan(\pi/4+\phi/2)\,[(1-e\sin\phi)/(1+e\sin\phi)]^{e/2}\} \tag{3-7}$$

Because of the rapid variation from ϕ, ψ is not given here in series form. By comparing equations (3-1) and (3-7), it may be seen, however, that

$$\psi = \ln\tan(\pi/4+\chi/2) \tag{3-8}$$

so that χ may be determined from the series in (3-2) and converted to ψ with (3-8), although there is no particular advantage over using (3-7).

For the inverse of (3-7), to find ϕ in terms of ψ, the choice is between iteration of a closed equation (3-10) and use of series (3-5) with a simple inverse of (3-8):

$$\chi = 2\arctan \mathrm{e}^{\psi} - \pi/2 \tag{3-9}$$

where e is the base of natural logarithms, 2.71828.

For the iteration, apply the principle of successive substitution used in (3-4) to the following, with $(2\arctan \mathrm{e}^{\psi}-\pi/2)$ as the first trial ϕ:

$$\phi = 2\arctan\{\mathrm{e}^{\psi}[(1+e\sin\phi)/(1-e\sin\phi)]^{e/2}\} - \pi/2 \tag{3-10}$$

Note that e and e are not the same.

The authalic latitude β, on a sphere having the same surface area as the ellipsoid, provides a sphere which is truly equal-area (authalic), relative to the ellipsoid:

$$\beta = \arcsin(q/q_p) \tag{3-11}$$

where

$$q = (1-e^2)\{\sin\phi/(1-e^2\sin^2\phi) - (1/(2e))\ln[(1-e\sin\phi)/(1+e\sin\phi)]\} \tag{3-12}$$

and q_p is q evaluated for a ϕ of 90°. The radius R_q of the sphere having the same surface area as the ellipsoid is calculated as follows:

$$R_q = a(q_p/2)^{1/2} \tag{3-13}$$

where a is the semimajor axis of the ellipsoid. For the Clarke 1866, R_q is 6,370,997.2 m.

The equivalent series for β (Adams, 1921, p. 85)

$$\beta = \phi - (e^2/3 + 31e^4/180 + 59e^6/560 + \dots)\sin 2\phi + (17e^4/360 + 61e^6/1260 + \dots)\sin 4\phi - (383e^6/45360 + \dots)\sin 6\phi + \dots \tag{3-14}$$

where β and ϕ are in radians. For the Clarke 1866 ellipsoid, the formula in seconds of arc is:

$$\beta = \phi - 467.01''\sin 2\phi + 0.45''\sin 4\phi \tag{3-15}$$

For ϕ in terms of β, an iterative inverse of (3–12) may be used with the inverse of (3–11):

$$\phi=\phi+\frac{(1-e^2\sin^2\phi)^2}{2\cos\phi}\left[\frac{q}{1-e^2}-\frac{\sin\phi}{1-e^2\sin^2\phi}+\frac{1}{2e}\ln\left(\frac{1-e\sin\phi}{1+e\sin\phi}\right)\right] \quad (3\text{–}16)$$

where $q=q_p\sin\beta$ (3–17)

q_p is found from (3–12) for a ϕ of 90°, and the first trial ϕ is arcsin $(q/2)$, used on the right side of (3–16) for the calculation of ϕ on the left side, which is then used on the right side until the change is less than a preset limit. (Equation (3–16) is derived from equation (3–12) using a standard Newton-Raphson iteration.)

To find ϕ from β with a series:

$$\begin{aligned}\phi=\beta&+(e^2/3+31e^4/180+517e^6/5040+\ldots)\sin 2\beta\\&+(23e^4/360+251e^6/3780+\ldots)\sin 4\beta\\&+(761e^6/45360+\ldots)\sin 6\beta+\ldots\end{aligned} \quad (3\text{–}18)$$

or, for the Clarke 1866 ellipsoid, in seconds,

$$\phi=\beta+467.01''\sin 2\beta+0.61''\sin 4\beta \quad (3\text{–}19)$$

The rectifying latitude μ, giving a sphere with correct distances along the meridians, requires a series in any case (or a numerical integration which is not shown).

$$\mu=\pi M/2M_p \quad (3\text{–}20)$$

where $M=a[(1-e^2/4-3e^4/64-5e^6/256-\ldots)\phi-(3e^2/8+3e^4/32+45e^6/1024+\ldots)\sin 2\phi+(15e^4/256+45e^6/1024+\ldots)\sin 4\phi-(35e^6/3072+\ldots)\sin 6\phi+\ldots]$ (3–21)

and M_p is M evaluated for a ϕ of 90°, for which all sine terms drop out. M is the distance along the meridian from the Equator to latitude ϕ. For the Clarke 1866 ellipsoid, the constants simplify to

$$M=111132.0894\phi^\circ-16216.94\sin 2\phi+17.21\sin 4\phi-0.02\sin 6\phi \quad (3\text{–}22)$$

The first coefficient in (3–21) has been multiplied by $\pi/180$ to use ϕ in degrees. To use μ properly, the radius R_M of the sphere must be $2M_p/\pi$ for correct scale. For the Clarke 1866 ellipsoid, R_M is 6,367,399.7 m. A series combining (3–20) and (3–21) is given by Adams (1921, p. 125):

$$\begin{aligned}\mu=\phi&-(3e_1/2-9e_1^3/16+\ldots)\sin 2\phi+(15e_1^2/16-15e_1^4/32+\ldots)\\&\sin 4\phi-(35e_1^3/48-\ldots)\sin 6\phi+\ldots\end{aligned} \quad (3\text{–}23)$$

where $e_1=[1-(1-e^2)^{1/2}]/[1+(1-e^2)^{1/2}]$ (3–24)

and μ and ϕ are given in radians. For the Clarke 1866 ellipsoid, in seconds,

$$\mu=\phi-525.33''\sin 2\phi+0.56''\sin 4\phi \quad (3\text{–}25)$$

The inverse of equations (3–23) or (3–25), for ϕ in terms of μ, given M, will be found useful for several map projections to avoid iteration, since a series is required in any case (Adams, 1921, p. 128).

$$\phi = \mu + (3e_1/2 - 27e_1^3/32 + \dots) \sin 2\mu + (21e_1^2/16 - 55e_1^4/32 + \dots) \sin 4\mu + (151e_1^3/96 - \dots) \sin 6\mu + \dots \quad (3\text{–}26)$$

where e_1 is found from equation (3–24) and μ from (3–20), but M is given, not calculated from (3–21). For the Clarke 1866 ellipsoid, in seconds of arc,

$$\phi = \mu + 525.33'' \sin 2\mu + 0.78'' \sin 4\mu \quad (3\text{–}27)$$

The remaining auxiliary latitudes listed by Adams (1921, p. 84) are more useful for derivation than in substitutions for projections:

The geocentric latitude ϕ_g referred to in the first paragraph in this section is simply as follows:

$$\phi_g = \arctan [(1 - e^2) \tan \phi] \quad (3\text{–}28)$$

As a series,

$$\phi_g = \phi - e_2 \sin 2\phi + (e_2^2/2) \sin 4\phi - (e_2^3/3) \sin 6\phi + \dots \quad (3\text{–}29)$$

where ϕ_g and ϕ are in radians and $e_2 = e^2/(2 - e^2)$. For the Clarke 1866 ellipsoid, in seconds of arc,

$$\phi_g = \phi - 700.44'' \sin 2\phi + 1.19'' \sin 4\phi \quad (3\text{–}30)$$

The reduced or *parametric latitude* η of a point on the ellipsoid is the latitude on a sphere of radius a for which the parallel has the same radius as the parallel of geodetic latitude ϕ on the ellipsoid through the given point:

$$\eta = \arctan [(1 - e^2)^{1/2} \tan \phi] \quad (3\text{–}31)$$

As a series,

$$\eta = \phi - e_1 \sin 2\phi + (e_1^2/2) \sin 4\phi - (e_1^3/3) \sin 6\phi + \dots \quad (3\text{–}32)$$

where e_1 is found from equation (3–24), and η and ϕ are in radians. For the Clarke 1866 ellipsoid, using seconds of arc,

$$\eta = \phi - 350.22'' \sin 2\phi + 0.30'' \sin 4\phi \quad (3\text{–}33)$$

The inverses of equations (3–28) and (3–31) for ϕ in terms of geocentric or reduced latitudes are relatively easily derived and are noniterative. The inverses of series equations (3–29), (3–30), (3–32), and (3–33) are therefore omitted. Table 3 lists the correction for these auxiliary latitudes for each 5° of geodetic latitude.

TABLE 3.—*Corrections for auxiliary latitudes on the Clarke 1866 ellipsoid*

[Corrections are given, rather than actual values. For example, if the geodetic latitude is 50° N., the conformal latitude is 50° − 11′29.7″ = 49°48′30.3″ N. For southern latitudes, the corrections are the same, disregarding the sign of the latitude. That is, the conformal latitude for a ϕ of lat. 50° S. is 49°48′30.3″ S. From Adams, 1921]

Geodetic (ϕ)	Conformal ($\chi-\phi$)	Authalic ($\beta-\phi$)	Rectifying ($\mu-\phi$)	Geocentric ($\phi_g-\phi$)	Parametric ($\eta-\phi$)
90°	0′00.0″	0′00.0″	0′00.0″	0′00.0″	0′00.0″
85	− 201.9	−121.2	−131.4	− 202.0	−100.9
80	− 400.1	−240.0	−300.0	− 400.3	−200.0
75	− 550.9	−353.9	−423.1	− 551.3	−255.4
70	− 731.0	−500.6	−538.2	− 731.4	−345.4
65	− 857.2	−558.2	−643.0	− 857.7	−428.6
60	−1007.1	−644.8	−735.4	−1007.6	−503.6
55	−1058.5	−719.1	−814.0	−1058.9	−529.3
50	−1129.7	−740.1	−837.5	−1130.2	−545.0
45	−1140.0	−747.0	−845.3	−1140.5	−550.2
40	−1129.1	−739.8	−837.2	−1129.4	−544.8
35	−1057.2	−718.6	−813.3	−1057.4	−528.9
30	−1005.4	−644.1	−734.5	−1005.6	−503.0
25	− 855.3	−557.3	−641.9	− 855.4	−428.0
20	− 729.0	−459.7	−537.1	− 729.1	−344.8
15	− 549.2	−353.1	−422.2	− 549.2	−254.9
10	− 358.8	−239.4	−259.3	− 358.8	−159.6
5	− 201.2	−120.9	−131.0	− 201.2	−100.7
0	000.0	000.0	000.0	000.0	000.0

4. SCALE VARIATION AND ANGULAR DISTORTION

Since no map projection maintains correct scale throughout, it is important to determine the extent to which it varies on a map. On a world map, qualitative distortion is evident to an eye familiar with maps, noting the extent to which landmasses are improperly sized or out of shape, and the extent to which meridians and parallels do not intersect at right angles, or are not spaced uniformly along a given meridian or given parallel. On maps of countries or even of continents, distortion may not be evident to the eye, but becomes apparent upon careful measurement and analysis.

TISSOT'S INDICATRIX

In 1859 and 1881, Tissot published a classic analysis of the distortion which occurs on a map projection (Tissot, 1881; Adams, 1919, p. 153–163; Maling, 1973, p. 64–67). The intersection of any two lines on the Earth is represented on the flat map with an intersection at the same or a different angle. At almost every point on the Earth, there is a right angle intersection of two lines in some direction (not necessarily a meridian and a parallel) which are also shown at right angles on the map. All the other intersections at that point on the Earth will not intersect at the same angle on the map, unless the map is conformal. The greatest deviation from the correct angle is called ω, the maximum angular deformation. For a conformal map, ω is zero.

Tissot showed this relationship graphically with a special ellipse of distortion called an indicatrix. An infinitely small circle on the Earth projects as an infinitely small, but perfect, ellipse on any map projection. If the projection is conformal, the ellipse is a circle, an ellipse of zero eccentricity. Otherwise, the ellipse has a major axis and minor axis which are directly related to the scale distortion and to the maximum angular deformation.

In figure 3, the left-hand drawing shows a circle representing the infinitely small circular element, crossed by a meridian λ and parallel ϕ on the Earth. The right-hand drawing shows this same element as it may appear on a typical map projection. For general purposes, the map is assumed to be neither conformal nor equal-area. The meridian and parallel may no longer intersect at right angles, but there is a pair of axes which intersect at right angles on both Earth (AB and CD) and map ($A'B'$ and $C'D'$). There is also a pair of axes which intersect at right angles on the Earth (EF and GH), but at an angle on the map ($E'F'$ and $G'H'$) farthest from a right angle. The latter case has the maximum

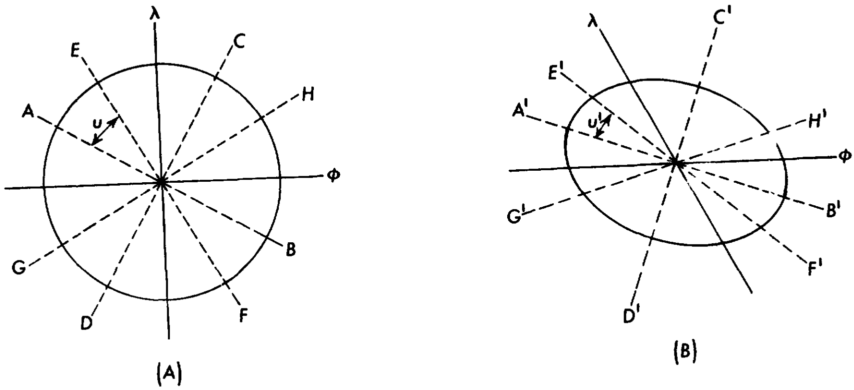


FIGURE 3.—Tissot's Indicatrix. An infinitely small circle on the Earth (*A*) appears as an ellipse on a typical map (*B*). On a conformal map, (*B*) is a circle of the same or of a different size.

angular deformation ω. The orientation of these axes is such that $u+u'=90°$, or, for small distortions, the lines fall about halfway between $A'B'$ and $C'D'$. The orientation is of much less interest than the size of the deformation. If a and b, the major and minor semiaxes of the indicatrix, are known, then

$$\sin(\omega/2)=|a-b|/(a+b) \tag{4-1}$$

If lines λ and ϕ coincide with a and b, in either order, as in cylindrical and conic projections, the calculation is relatively simple, using equations (4–2) through (4–6) given below.

Scale distortion is most often calculated as the ratio of the scale along the meridian or along the parallel at a given point to the scale at a standard point or along a standard line, which is made true to scale. These ratios are called "scale factors." That along the meridian is called h and along the parallel, k. The term "scale error" is frequently applied to $(h-1)$ and $(k-1)$. If the meridians and parallels intersect at right angles, coinciding with a and b in figure 3, the scale factor in any other direction at such a point will fall between h and k. Angle ω may be calculated from equation (4–1), substituting h and k in place of a and b. In general, however, the computation of ω is much more complicated, but is important for knowing the extent of the angular distortion throughout the map.

The formulas are given here to calculate h, k, and ω; but the formulas for h and k are applied specifically to all projections for which they are deemed useful as the projection formulas are given later. Formulas for ω for specific projections have generally been omitted.

Another term occasionally used in practical map projection analysis is "convergence" or "grid declination." This is the angle between true north and grid north (or direction of the Y axis). For regular cylindrical projections this is zero, for regular conic and polar azimuthal projections it is a simple function of longitude, and for other projections it may be determined from the projection formulas by calculus as the slope of the meridian (dy/dx) at a given latitude. It is used primarily by surveyors for fieldwork with topographic maps. It has been decided not to discuss convergence further in this bulletin.

DISTORTION FOR PROJECTIONS OF THE SPHERE

The formulas for distortion are simplest when applied to regular cylindrical, conic (or conical), and polar azimuthal projections of the sphere. On each of these types of projections, scale is solely a function of the latitude.

Given the formulas for rectangular coordinates x and y of any cylindrical projection as functions solely of longitude λ and latitude ϕ, respectively,

$$h = dy/(R d\phi) \tag{4-2}$$

$$k = dx/(R \cos \phi d\lambda) \tag{4-3}$$

Given the formulas for polar coordinates ρ and θ of any conic projection as functions solely of ϕ and λ, respectively, where n is the cone constant or ratio of θ to $(\lambda - \lambda_0)$,

$$h = -d\rho/(R d\phi) \tag{4-4}$$

$$k = n\rho/(R \cos \phi) \tag{4-5}$$

Given the formulas for polar coordinates ρ and θ of any polar azimuthal projection as functions solely of ϕ and λ, respectively, equations (4–4) and (4–5) apply, with n equal to 1.0:

$$h = -d\rho/(R d\phi) \tag{4-4}$$

$$k = \rho/(R \cos \phi) \tag{4-6}$$

Equations (4–4) and (4–6) may be adapted to any azimuthal projection centered on a point other than the pole. In this case h' is the scale factor in the direction of a straight line radiating from the center, and k' is the scale factor in a direction perpendicular to the radiating line, all at an angular distance c from the center:

$$h' = d\rho/(R dc) \tag{4-7}$$

$$k' = \rho/(R \sin c) \tag{4-8}$$

An analogous relationship applies to scale factors on oblique cylindrical and conic projections.

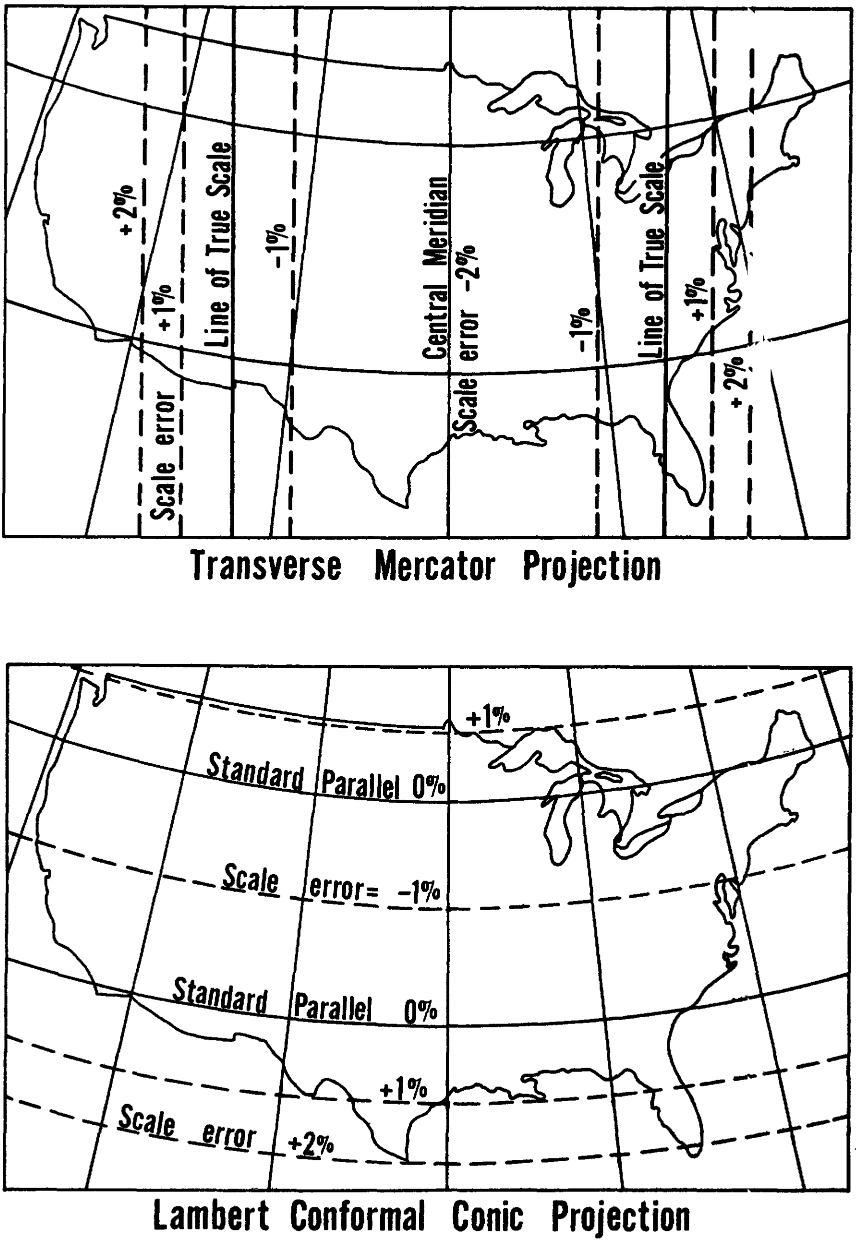


Figure 4.—Distortion patterns on common conformal map projections. The Transverse Mercator and the Stereographic are shown with reduction in scale along the central meridian or at the center of projection, respectively. If there is no reduction, there is a single line of true scale along the central meridian on the Transverse Mercator and only a point of true scale at the center of the Stereographic. The illustrations are conceptual rather than precise, since each base map projection is an identical conic.

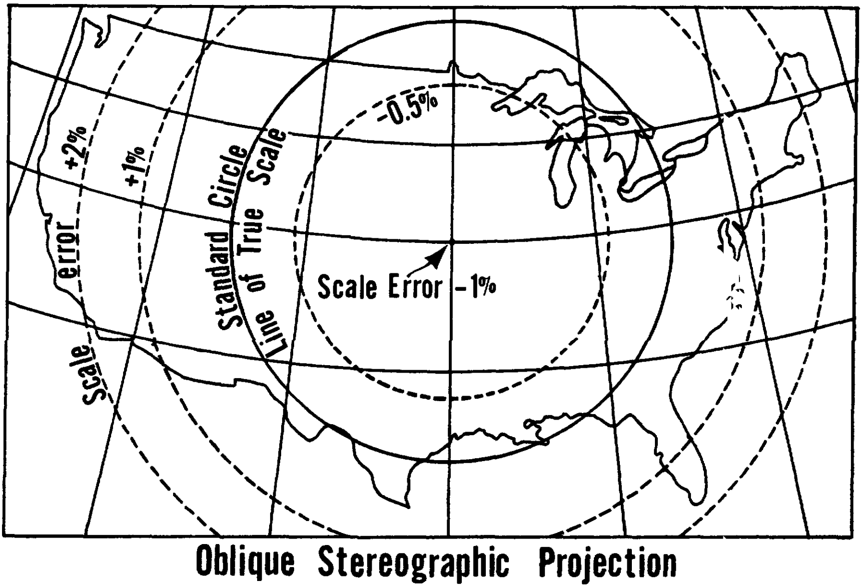


FIGURE 4.—Continued.

For any of the pairs of equations from (4–2) through (4–8), the maximum angular deformation ω at any given point is calculated simply, as stated above,

$$\sin {}^1\!/\!_2\omega = |h-k|/(h+k) \tag{4–9}$$

where $|h-k|$ signifies the absolute value of $(h-k)$, or the positive value without regard to sign. For equations (4–7) and (4–8), h' and k' are used in (4–9) instead of h and k, respectively. In figure 4, distortion patterns are shown for three conformal projections of the United States, choosing arbitrary lines of true scale.

For the general case, including all map projections of the sphere, rectangular coordinates x and y are often both functions of both ϕ and λ, so they must be partially differentiated with respect to both ϕ and λ, holding λ and ϕ, respectively, constant. Then,

$$h = (1/R)\,[(\partial x/\partial\phi)^2 + (\partial y/\partial\phi)^2]^{1/2} \tag{4–10}$$

$$k = [1/(R\cos\phi)]\,[(\partial x/\partial\lambda)^2 + (\partial y/\partial\lambda)^2]^{1/2} \tag{4–11}$$

$$a' = (h^2 + k^2 + 2hk\sin\theta')^{1/2} \tag{4–12}$$

$$b' = (h^2 + k^2 - 2hk\sin\theta')^{1/2} \tag{4–13}$$

where $\cos\theta' = [(\partial y/\partial\phi)\,(\partial y/\partial\lambda) + (\partial x/\partial\phi)\,(\partial x/\partial\lambda)]/(hk\cos\phi)$ (4–14)

θ' is the angle at which a given meridian and parallel intersect, and a' and b' are convenient terms. The maximum and minimum scale factors a and b, at a given point, may be calculated thus:

$$a = (a' + b')/2 \quad (4\text{–}12a)$$
$$b = (a' - b')/2 \quad (4\text{–}13a)$$

Equation (4–1) simplifies as follows for the general case:

$$\sin(\omega/2) = b'/a' \quad (4\text{–}1a)$$

The areal scale factor s:

$$s = hk \sin\theta' \quad (4\text{–}15)$$

For special cases:
(1) $s = hk$ if meridians and parallels intersect at right angles ($\theta' = 90°$);
(2) $h = k$ and $\omega = 0$ if the map is conformal;
(3) $h = 1/k$ on an equal-area map if meridians and parallels intersect at right angles.[2]

DISTORTION FOR PROJECTIONS OF THE ELLIPSOID

The derivation of the above formulas for the sphere utilizes the basic formulas for the length of a given spacing (usually 1° or 1 radian) along a given meridian or a given parallel. The following formulas give the length of a radian of latitude (L_ϕ) and of longitude (L_λ) for the sphere:

$$L_\phi = R \quad (4\text{–}16)$$
$$L_\lambda = R\cos\phi \quad (4\text{–}17)$$

where R is the radius of the sphere. For the length of 1° of latitude or longitude, these values are multiplied by $\pi/180$.

The radius of curvature on a sphere is the same in all directions. On the ellipsoid, the radius of curvature varies at each point and in each direction along a given meridian, except at the poles. The radius of curvature R' in the plane of the meridian is calculated as follows:

$$R' = a(1-e^2)/(1-e^2\sin^2\phi)^{3/2} \quad (4\text{–}18)$$

The length of a radian of latitude is defined as the circumference of a circle of this radius, divided by 2π, or the radius itself. Thus,

$$L_\phi = a(1-e^2)/(1-e^2\sin^2\phi)^{3/2} \quad (4\text{–}19)$$

For the radius of curvature N of the ellipsoid in a plane perpendicular to the meridian and also perpendicular to a plane tangent to the surface,

[2] Maling (1973, p. 49–81) has helpful derivations of these equations in less condensed forms. There are typographical errors in several of the equations in Maling, but these may be detected by following the derivation closely.

$$N = a/(1 - e^2 \sin^2\phi)^{1/2} \tag{4-20}$$

Radius N is also the length of the perpendicular to the surface from the surface to the polar axis. The length of a radian of longitude is found, as in equation (4-17), by multiplying N by $\cos\phi$, or

$$L_\lambda = a \cos\phi/(1 - e^2 \sin^2\phi)^{1/2} \tag{4-21}$$

The lengths of 1° of latitude and 1° of longitude for the Clarke 1866 and the International ellipsoids are given in table 4. They are found from equations (4-19) and (4-21), multiplied by $\pi/180$ to convert to lengths for 1°.

When these formulas are applied to equations (4-2) through (4-6), the values of h and k for the ellipsoidal forms of the projections are found to be as follows:

For cylindrical projections:

$$h = dy/(R'd\phi)$$
$$= (1 - e^2 \sin^2\phi)^{3/2}\, dy/[a(1 - e^2)d\phi] \tag{4-22}$$
$$k = dx/(N \cos\phi d\lambda)$$
$$= (1 - e^2 \sin^2\phi)^{1/2}\, dx/(a \cos\phi\, d\lambda) \tag{4-23}$$

TABLE 4.—*Lengths, in meters, of 1° of latitude and longitude on two ellipsoids of reference*

Latitude (ϕ)	Clarke 1866 ellipsoid		International (Hayford) ellipsoid	
	1° lat.	1° long.	1° lat.	1° long.
90°	111,699.4	0.0	111,700.0	0.0
85	111,690.7	9,735.0	111,691.4	9,735.0
80	111,665.0	19,394.4	111,665.8	19,394.5
75	111,622.9	28,903.3	111,624.0	28,903.5
70	111,565.9	38,188.2	111,567.4	38,188.5
65	111,495.7	47,177.5	111,497.7	47,177.9
60	111,414.5	55,802.2	111,417.1	55,802.8
55	111,324.8	63,996.4	111,327.9	63,997.3
50	111,229.3	71,698.1	111,233.1	71,699.2
45	111,130.9	78,849.2	111,135.4	78,850.5
40	111,032.7	85,396.1	111,037.8	85,397.7
35	110,937.6	91,290.3	110,943.3	91,292.2
30	110,848.5	96,488.2	110,854.8	96,490.4
25	110,768.0	100,951.9	110,774.9	100,954.3
20	110,698.7	104,648.7	110,706.0	104,651.4
15	110,642.5	107,551.9	110,650.2	107,554.8
10	110,601.1	109,640.7	110,609.1	109,643.7
5	110,575.7	110,899.9	110,583.9	110,903.1
0	110,567.2	111,320.7	110,575.5	111,323.9

For conic projections:

$$h = -d\rho/(R'd\phi)$$
$$= -(1-e^2\sin^2\phi)^{3/2}\,d\rho/[a(1-e^2)d\phi] \qquad (4\text{–}24)$$
$$k = n\rho/(N\cos\phi)$$
$$= n\rho(1-e^2\sin^2\phi)^{1/2}/(a\cos\phi) \qquad (4\text{–}25)$$

For polar azimuthal projections:

$$h = -(1-e^2\sin^2\phi)^{3/2}\,d\rho/[a(1-e^2)d\phi] \qquad (4\text{–}24)$$
$$k = \rho(1-e^2\sin^2\phi)^{1/2}/(a\cos\phi) \qquad (4\text{–}26)$$

Equations (4–7) and (4–8) do not have ellipsoidal equivalents. Equation (4–9) remains the same for equations (4–22) through (4–26):

$$\sin{}^1\!/\!_2\omega = |h-k|/(h+k) \qquad (4\text{–}9)$$

For the general projection of the ellipsoid, equations (4–10) and (4–11) are similarly modified:

$$h = [(\partial x/\partial\phi)^2 + (\partial y/\partial\phi)^2]^{1/2}(1-e^2\sin^2\phi)^{3/2}/[a(1-e^2)] \qquad (4\text{–}27)$$
$$k = [(\partial x/\partial\lambda)^2 + (\partial y/\partial\lambda)^2]^{1/2}(1-e^2\sin^2\phi)^{1/2}/(a\cos\phi) \qquad (4\text{–}28)$$

Equations (4–12) through (4–15), (4–12a), (4–13a), and (4–1a), listed for the sphere, apply without change.

Specific calculations are shown during the discussion of individual projections.

The importance of using the ellipsoid instead of the sphere for designing a projection may be quantitatively evaluated by determining the ratio or product of some of the elementary relationships. The ratio of the differential length of a radian of latitude along a meridian on the sphere to that on the ellipsoid is found by dividing equation (4–16) by equation (4–19), or

$$C_m = R(1-e^2\sin^2\phi)^{3/2}/[a(1-e^2)] \qquad (4\text{–}29)$$

A related ratio for the length of a radian of longitude along a parallel on the sphere to that on the ellipsoid is found by dividing equation (4–17) by equation (4–21), or

$$C_p = R(1-e^2\sin^2\phi)^{1/2}/a \qquad (4\text{–}30)$$

From these, the local shape factor C_s may be found as the ratio of (4–29) to (4–30):

$$C_s = C_m/C_p = (1-e^2\sin^2\phi)/(1-e^2) \qquad (4\text{–}31)$$

and the area factor C_a is their product:

$$C_a = C_mC_p = R^2(1-e^2\sin^2\phi)^2/[a^2(1-e^2)] \qquad (4\text{–}32)$$

If h and k are calculated for the spherical version of a map projection, the actual scale factors on the spherical version relative to the ellipsoid may be determined by multiplying h by C_m and k by C_p. For normal cylindrical and conic projections and polar azimuthal projections, the conformality or shape factor may be taken as h/k (not the same as ω) multiplied by C_s, and the area scale factor hk may be multiplied by C_a.

Except for C_s, which is independent of R/a, R must be given an arbitrary value such as R_q (see equation (3–13)), R_M (see second sentence following equation (3–22)), or another reasonable balance between the major and minor semiaxes a and b of the ellipsoid. Using R_q and the Clarke 1866 ellipsoid, table 5 shows the magnitude of these corrections. Thus, a conformal projection based on the sphere has the correct shape at the poles for the ellipsoid, but the shape is about ⅔ of 1 percent (0.00681) in error near the Equator (that is, Tissot's Indicatrix is an ellipse with minor axis about ⅔ of 1 percent shorter than the major axis at the Equator when the spherical form is compared to the ellipsoid).

A map extending over a large area will have a scale variation of several percent, which far outweighs the significance of the less-than-1-percent variation between sphere and ellipsoid. A map of a small area, such as a large-scale detailed topographic map, or even a narrow strip map, has a small-enough intrinsic scale variation to make the ellipsoidal correction a significant factor in accurate mapping; e.g., a 7.5-min quadrangle normally has an intrinsic scale variation of 0.0002 percent or less.

TABLE 5.—*Ellipsoidal correction factors to apply to spherical projections based on Clarke 1866 ellipsoid*

Lat. (N&S)	C_m*	C_p	C_s	C_a*
90°	0.99548	0.99548	1.00000	0.99099
75	.99617	.99571	1.00046	.99189
60	.99803	.99633	1.00170	.99437
45	1.00058	.99718	1.00341	.99775
30	1.00313	.99802	1.00511	1.00114
15	1.00499	.99864	1.00636	1.00363
0	1.00568	.99887	1.00681	1.00454
Multiply by**	h	k	h/k	hk

* $C_m=1.0$ for 48.24° lat. and $C_a=1.0$ for 35.32° lat. Values of C_m, C_p, and C_a are based on a radius of 6,370,997 m for the sphere used in the spherical map projection.

** h = scale factor along meridian.

k = scale factor along parallel of latitude.

For normal cylindrical and conic projections and polar azimuthal projections:

h/k = shape factor.

hk = area scale factor.

For example, if, on a spherical Albers Equal-Area Conic map projection based on sphere of radius 6,370,997 m, $h=1.00132$ and $k=0.99868$ at lat. 45° N., this map has an area scale factor of $1.00132\times0.99868\times0.99775=0.99775$, relative to the correct area scale for the Clarke 1966 ellipsoid. If the ellipsoidal Albers were used, this factor would be 1.0.

5. TRANSFORMATION OF MAP GRATICULES

As discussed later, several map projections have been adapted to showing some part of the Earth for which the lines of true scale have an orientation or location different from that intended by the inventor of the basic projection. This is equivalent to moving or transforming the graticule of meridians and parallels on the Earth so that the "north pole" of the graticule assumes a position different from that of the true North Pole of the Earth. The projection for the sphere may be plotted using the original formulas or graphical construction, but applying them to the new graticule orientation. The actual meridians and parallels may then be plotted by noting their relationship on the sphere to the new graticule, and landforms drawn with respect to the actual geographical coordinates as usual.

In effect, this procedure was used in the past in an often entirely graphical manner. It required considerable care to avoid cumulative errors resulting from the double plotting of graticules. With computers and programmable hand calculators, it now can be a relatively routine matter to calculate directly the rectangular coordinates of the actual graticule in the transformed positions or, with an automatic plotter, to obtain the transformed map directly from the computer.

The transformation most notably has been applied to the azimuthal and cylindrical projections, but in a few cases it has been used with conic, pseudocylindrical, and other projections. While it is fairly straightforward to apply a suitable transformation to the sphere, transformation is much more difficult on the ellipsoid because of the constantly changing curvature. Transformation has been applied to the ellipsoid, however, in important cases under certain limiting conditions.

If either true pole is at the center of an azimuthal map projection, the projection is called the *polar* aspect. If a point on the Equator is made the center, the projection is called the *equatorial* or, less often, *meridian* or *meridional* aspect. If some other point is central, the projection is the *oblique* or, occasionally, *horizon* aspect.

For cylindrical and most other projections, such transformations are called *transverse* or *oblique,* depending on the angle of rotation. In transverse projections, the true poles of the Earth lie on the equator of the basic projection, and the poles of the projection lie on the Equator of the Earth. Therefore, one meridian of the true Earth lies along the equator of the basic projection. The Transverse Mercator projection is the best-known example and is related to the regular Mercator in this manner. For oblique cylindrical projections, the true poles of the Earth lie somewhere between the poles and the equator of the basic projection. Stated another way, the equator of the basic projection is drawn along some great circle route other than the Equator or a meridian of the Earth for the oblique cylindrical aspect. The Oblique Mercator is the most common example. Further subdivisions of these aspects have been made; for example, the transverse aspect may be first transverse, second transverse, or transverse oblique, depending on the positions of the true poles along the equator of the basic projection (Wray, 1974). This has no significance in a transverse cylindrical projection, since the appearance of the map does not change, but for pseudocylindrical projections such as the Sinusoidal, it makes a difference, if the additional nomenclature is desired.

To determine formulas for the transformation of the sphere, two basic laws of spherical trigonometry are used. Referring to the spherical triangle in figure 5, with three points having angles A, B, and C on the sphere, and three great circle arcs a, b, and c connecting them, the Law of Sines declares that

$$\sin A/\sin a = \sin B/\sin b = \sin C/\sin c \tag{5-1}$$

while by the Law of Cosines,

$$\cos c = \cos b \cos a + \sin b \sin a \cos C \tag{5-2}$$

If C is placed at the North Pole, it becomes the angle between two meridians extending to A and B. If A is taken as the starting point on the sphere, and B the second point, c is the great circle distance between them, and angle A is the azimuth Az east of north which point B bears to point A. When latitude ϕ_1 and longitude λ_0 are used for point A, and ϕ and λ are used for point B, equation (5-2) becomes the following for great circle distance:

$$\cos c = \sin \phi_1 \sin \phi + \cos \phi_1 \cos \phi \cos (\lambda - \lambda_0) \tag{5-3}$$

While (5-3) is the standard and simplest form of this equation, it is not accurate for values of c very close to zero. For such cases, the equation may be rearranged as follows:

$$\sin c = \{\cos^2 \phi \sin^2 (\lambda - \lambda_0) + [\cos \phi_1 \sin \phi - \sin \phi_1 \cos \phi \cos (\lambda - \lambda_0)]^2\}^{1/2} \tag{5-3a}$$

This equation is unsatisfactory when c is close to 90° or is greater than 90°. For general purposes, the still longer tangent form is suggested, for which simplification is not very helpful:

$$\tan c = \sin c/\cos c \tag{5-3b}$$

where $\sin c$ and $\cos c$ are found from (5-3a) and (5-3), respectively, and quadrant adjustment is made as described under the list of symbols.

Equation (5–1) becomes the following for the azimuth:

$$\sin Az = \sin(\lambda - \lambda_0) \cos \phi / \sin c \quad (5\text{–}4)$$

or, with some rearrangement,

$$\cos Az = [\cos \phi_1 \sin \phi - \sin \phi_1 \cos \phi \cos(\lambda - \lambda_0)] / \sin c \quad (5\text{–}5)$$

or, eliminating c,

$$\tan Az = \cos \phi \sin(\lambda - \lambda_0) / [\cos \phi_1 \sin \phi - \sin \phi_1 \cos \phi \cos(\lambda - \lambda_0)] \quad (5\text{–}6)$$

Either of the three equations (5–4) through (5–6) may be used for the azimuth, depending on the form of equation preferred. Equation (5–6) is usually preferred, since it avoids the inaccuracies of finding an arcsin near 90° or an arccos near 0°. Quadrant adjustment as described under the list of symbols should be employed.

Applying these relationships to transformations, without showing some intermediate derivations, formulas (5–7) through (5–10) are obtained. To place the North Pole of the sphere at a latitude α on a meridian β east of the central meridian ($\lambda' = 0$) of the basic projection (see fig. 6), the transformed latitude ϕ' and transformed longitude λ' on the basic projection which correspond to latitude ϕ and longitude λ of the spherical Earth may be calculated as follows, letting the central meridian λ_0 correspond with $\lambda' = \beta$:

$$\sin \phi' = \sin \alpha \sin \phi - \cos \alpha \cos \phi \cos(\lambda - \lambda_0) \quad (5\text{–}7)$$

$$\sin(\lambda' - \beta) = \cos \phi \sin(\lambda - \lambda_0) / \cos \phi' \quad (5\text{–}8)$$

or

$$\cos(\lambda' - \beta) = [\sin \alpha \cos \phi \cos(\lambda - \lambda_0) + \cos \alpha \sin \phi] / \cos \phi' \quad (5\text{–}9)$$

or

$$\tan(\lambda' - \beta) = \cos \phi \sin(\lambda - \lambda_0) / [\sin \alpha \cos \phi \cos(\lambda - \lambda_0) + \cos \alpha \sin \phi] \quad (5\text{–}10)$$

Equation (5–10) is generally preferable to (5–8) or (5–9) for the reasons stated after equation (5–6).

These are general formulas for the oblique transformation. (For azimuthal projections, β may always be taken as zero. Other values of β merely have the effect of rotating the X and Y axes without changing the projection.)

The inverse forms of these equations are similar in appearance. To find the geographic coordinates in terms of the transformed coordinates,

$$\sin \phi = \sin \alpha \sin \phi' + \cos \alpha \cos \phi' \cos(\lambda' - \beta) \quad (5\text{–}11)$$

$$\sin(\lambda - \lambda_0) = \cos \phi' \sin(\lambda' - \beta) / \cos \phi \quad (5\text{–}12)$$

or

$$\cos(\lambda - \lambda_0) = [\sin \alpha \cos \phi' \cos(\lambda' - \beta) - \cos \alpha \sin \phi'] / \cos \phi \quad (5\text{–}13)$$

or

$$\tan(\lambda - \lambda_0) = \cos \phi' \sin(\lambda' - \beta) / [\sin \alpha \cos \phi' \cos(\lambda' - \beta) - \cos \alpha \sin \phi'] \quad (5\text{–}14)$$

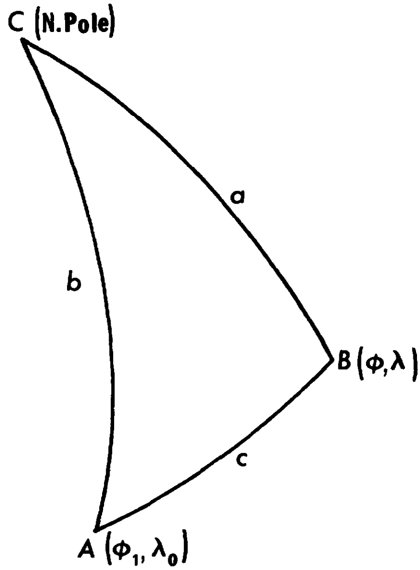


FIGURE 5. – Spherical triangle.

with equation (5–14) usually preferable to (5–12) and (5–13) for the same reasons as those given for (5–6).

If $\alpha=0$, the formulas simplify considerably for the transverse or equatorial aspects. It is then more convenient to have central meridian λ_0 coincide with the equator of the basic projection rather than with its meridian β. This may be accomplished by replacing $(\lambda-\lambda_0)$ with $(\lambda-\lambda_0-90°)$ and simplifying.

If $\beta=0$, so that the true North Pole is placed at $(\lambda'=0, \phi'=0)$:

$$\sin\phi' = -\cos\phi \sin(\lambda-\lambda_0) \quad (5\text{–}15)$$

$$\cos\lambda' = \sin\phi/[1-\cos^2\phi \sin^2(\lambda-\lambda_0)]^{1/2} \quad (5\text{–}16)$$

or

$$\tan\lambda' = -\cos(\lambda-\lambda_0)/\tan\phi \quad (5\text{–}17)$$

If $\beta=90°$, placing the true North Pole at $(\lambda'=90°, \phi'=0)$:

$$\sin\phi' = -\cos\phi \sin(\lambda-\lambda_0) \quad (5\text{–}15)$$

$$\cos\lambda' = \cos\phi \cos(\lambda-\lambda_0)/[1-\cos^2\phi \sin^2(\lambda-\lambda_0)]^{1/2} \quad (5\text{–}18)$$

or

$$\tan\lambda' = \tan\phi/\cos(\lambda-\lambda_0) \quad (5\text{–}19)$$

The inverse equations (5–11) through (5–14) may be similarly altered.

As stated earlier, these formulas may be directly incorporated into the formulas for the rectangular coordinates x and y of the basic map

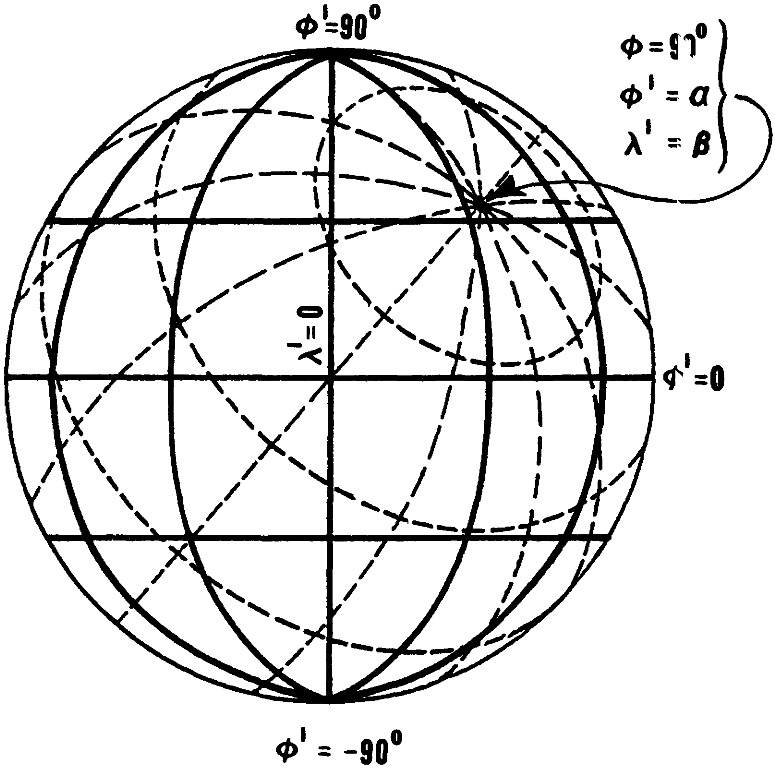


FIGURE 6. – Rotation of a graticule for transformation of projection. Dashed lines show actual longitudes and latitudes (λ and ϕ). Solid lines show the transformed longitudes and latitudes (λ' and ϕ') from which rectangular coordinates (x and y) are determined according to map projection used.

projection for a direct computer or calculator output. In some other cases, especially in the past, it may be easier to calculate the transverse or oblique projection coordinates by first calculating ϕ' and λ' for each point to be plotted (such general tables have been prepared), and then calculating the rectangular coordinates by inserting ϕ' and λ' one by one into the basic projection formulas. In still other cases, a graphical method has been used.

While these formulas, or their equivalents, will be incorporated into the formulas given later for individual oblique and transverse projections, the concept should help interrelate the various aspects or types of centers of a given projection. The extension of these concepts to the ellipsoid is much more involved technically and in some cases requires approximation. General discussion of this is omitted here.

6. CLASSIFICATION OF MAP PROJECTIONS

Because of the hundreds of map projections already published and the seemingly infinite number which are theoretically possible, considerable attention has been given to classification of projections so that the user is not overwhelmed by the numbers and the variety. One obvious type of classification has already been implied in this work: division of map projections into those which are (1) equal-area, (2) conformal, (3) equidistant, (4) azimuthal, and (5) miscellaneous. This is an unsatisfactory approach because of overlapping and because so many then fall into the "miscellaneous" category.

The most popular classification, which is partially used in this bulletin, is division by type of construction: (1) cylindrical, (2) conic, (3) azimuthal, (4) pseudocylindrical, (5) pseudoconical, and (6) miscellaneous. Each of these divisions may be subdivided, especially the latter. This type of classification is often easier to distinguish, but it is far from ideal. Since nearly all projections used by the USGS fall into the first three categories, and a fourth category called "space map projections" is introduced, the "miscellaneous" category is limited to two projections in this bulletin.

Interest has been shown in some other forms of classification which are more suitable for extensive treatises. In 1962, Waldo R. Tobler provided a simple but all-inclusive proposal which has aroused considerable interest (Tobler, 1962; Maling, 1973, p. 98–104; Maurer, 1935, p. v–vii). Tobler's classification involves eight categories, four for rectangular and four for polar coordinates. For the rectangular coordinates, category A includes all projections in which both x and y vary with both latitude ϕ and longitude λ, category B includes all in which y varies with both ϕ and λ while x is only a function of λ, C includes those projections in which x varies with both ϕ and λ while y varies only with ϕ, and D is for those in which x is only a function of λ and y only of ϕ. There are very few published projections in category B, but C is usually called pseudocylindrical, D is cylindrical, and A includes nearly all the rest which do not fit the polar coordinate categories.

Tobler's categories A to D for polar coordinates are respectively the same as those for rectangular, except that radius ρ is read for y and angle θ is read for x. The regular conic and azimuthal projections fall into category D, and the pseudoconical (such as Bonne's) into C. Category A may have a few projections like A (rectangular) for which polar coordinates are more convenient than rectangular. There are no well-known projections in B (polar).

Hans Maurer's detailed map projection treatise of 1935 introduced a "Linnaean" classification with five families ("true-circular", "straight-symmetrical," "curved-symmetrical," "less regular," and "combination grids," to quote a translation) subdivided into branches, subbranches, classes, groups, and orders (Maurer, 1935). As Maling says, Maurer's system "is only useful to the advanced student of the subject," but Maurer attempts for map projections what Linnaeus, the Swedish botanist, accomplished for plants and animals in the eighteenth century (Maling, 1973, p. 98). Other approaches have been taken by Lee (1944) and by Goussinsky (1951).

The individual projections used by the USGS are discussed below.

CYLINDRICAL MAP PROJECTIONS

The map projection best known by name is certainly the Mercator – one of the cylindricals. Perhaps easiest to draw, if simple tables are on hand, the regular cylindrical projections consist of meridians which are equidistant parallel straight lines, crossed at right angles by straight parallel lines of latitude, generally not equidistant. Geometrically, cylindrical projections can be partially developed by unrolling a cylinder which has been wrapped around a globe representing the Earth, touching at the Equator, and on which meridians have been projected from the center of the globe (fig. 1). The latitudes can also be perspectively projected onto the cylinder for some projections (such as the Cylindrical Equal-Area and Gall's), but not on those which are discussed in this bulletin. When the cylinder is wrapped around the globe in a different direction, so that it is no longer tangent along the Equator, an oblique or transverse projection results, and neither the meridians nor the parallels will generally be straight lines.

7. MERCATOR PROJECTION

SUMMARY

- Cylindrical.
- Conformal.
- Meridians are equally spaced straight lines.
- Parallels are unequally spaced straight lines, closest near the Equator, cutting meridians at right angles.
- Scale is true along the Equator, or along two parallels equidistant from the Equator.
- Loxodromes (rhumb lines) are straight lines.
- Not perspective.
- Poles are at infinity; great distortion of area in polar regions.
- Used for navigation.
- Presented by Mercator in 1569.

HISTORY

The well-known Mercator projection was perhaps the first projection to be regularly identified when atlases of over a century ago gradually began to name projections used, a practice now fairly commonplace. While the projection was apparently used by Erhard Etzlaub of Nuremburg (1462–1532) on a small map on the cover of some sundials constructed in 1511 and 1513, the principle remained obscure until Gerhardus Mercator (1512–94) independently developed it and presented it in 1569 on a large world map of 21 sheets totaling about 1.3 by 2 m (Keuning, 1955, p. 17–18).

Mercator, born at Rupelmonde in Flanders, was probably originally named Gerhard Cremer (or Kremer), but he always used the latinized form. To his contemporaries and to later scholars, he is better known for his skills in map and globe making, for being the first to use the term "atlas" to describe a collection of maps in a volume, for his calligraphy, and for first naming North America as such on a map in 1538. To the world at large, his name is identified chiefly with his projection, which he specifically developed to aid navigation. His 1569 map is entitled "Nova et Aucta Orbis Terrae Descriptio ad Usum Navigantium Emendate Accommodata (A new and enlarged description of the Earth with corrections for use in navigation)." He described in Latin the nature of the projection in a large panel covering much of his portrayal of North America:

" * * * In this mapping of the world we have [desired] to spread out the surface of the globe into a plane that the places shall everywhere be properly located, not only with respect to their true direction and distance, one from another, but also in accordance with their due longitude and latitude; and further, that the shape of the lands, as they

FIGURE 7.—Gerhardus Mercator (1512–94). The inventor of the most famous map projection, which is the prototype for conformal mapping.

appear on the globe, shall be preserved as far as possible. For this there was needed a new arrangement and placing of meridians, so that they shall become parallels, for the maps hitherto produced by geographers are, on account of the curving and the bending of the meridians, unsuitable for navigation * * *. Taking all this into consideration, we have somewhat increased the degrees of latitude toward each pole, in proportion to the increase of the parallels beyond the ratio they really have to the equator." (Fite and Freeman, 1926, p. 77–78).

Mercator probably determined the spacing graphically, since tables of secants had not been invented. Edward Wright (ca. 1558–1615) of England later developed the mathematics of the projection and in 1599 published tables of cumulative secants, thereby indicating the spacing from the Equator (Keuning, 1955, p. 18).

FEATURES AND USAGE

The meridians of longitude of the Mercator projection are vertical parallel equally spaced lines, cut at right angles by horizontal straight parallels which are increasingly spaced toward each pole so that conformality exists (fig. 8). The spacing of parallels at a given latitude on the sphere is proportional to the secant of the latitude.

The major navigational feature of the projection is found in the fact that a sailing route between two points is shown as a straight line, if the direction or azimuth of the ship remains constant with respect to north. This kind of route is called a loxodrome or rhumb line and is usually longer than the great circle path (which is the shortest possible route on the sphere). It is the same length as a great circle only if it follows the Equator or a meridian.

The great distortion of area on the Mercator projection of the Earth leads to mistaken concepts when it is the chief basis of world maps seen by students in school. The classic comparison of areas is between Greenland and South America. Greenland appears larger, although it is only one-eighth the size of South America. Furthermore, the North and South Poles cannot be shown, since they are at infinite distance from other parallels on the projection, giving a student an impression they are inaccessible (which of course they seemed to explorers long after the time of Mercator). The last fifty years have seen an increased emphasis on the use of other projections for world maps in published atlases.

Nevertheless, the Mercator projection is fundamental in the development of map projections, especially those which are conformal. It remains a standard navigational tool. It is also especially suitable for conformal maps of equatorial regions. The USGS has recently used it as an inset of the Hawaiian Islands on the 1:500,000-scale base map of Hawaii, for a Bathymetric Map of the Northeast Equatorial Pacific

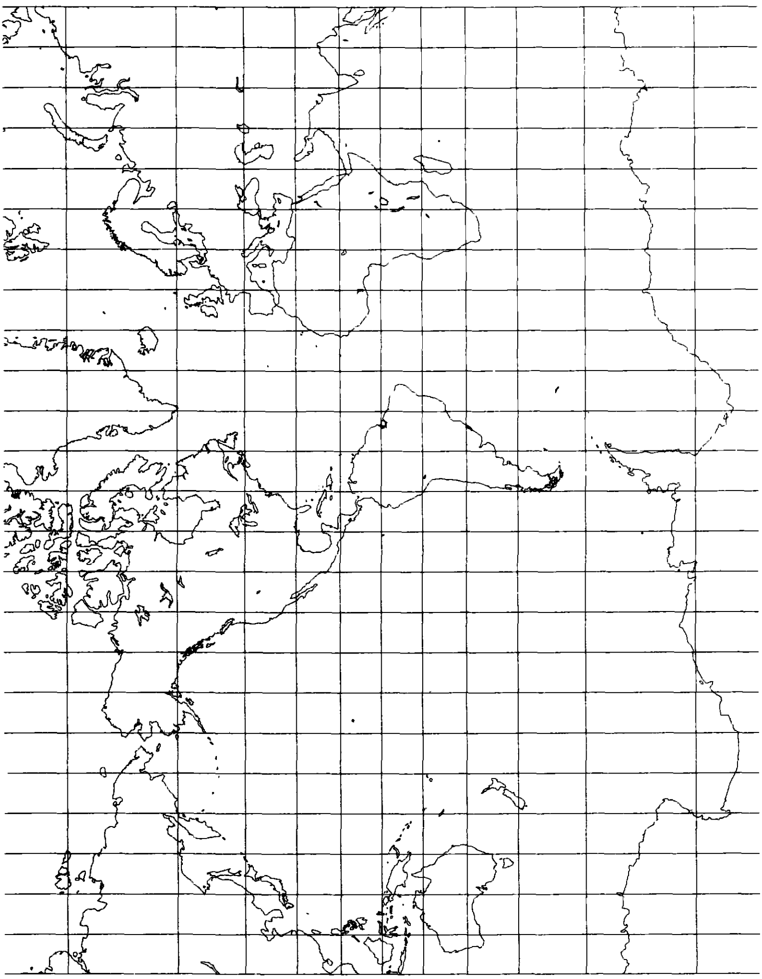


FIGURE 8.—The Mercator projection. The best-known projection. All angles are shown correctly; therefore, small shapes are true, and it is called conformal. Since rhumb lines are shown straight on this projection, it is very useful in navigation. It is commonly used to show equatorial regions of the Earth and other bodies.

Ocean (although the projection is not stated) and for a Tectonic Map of the Indonesia region, the latter two both in 1978 and at a scale of 1:5,000,000.

The first detailed map of an entire planet other than the Earth was issued in 1972 at a scale of 1:25,000,000 by the USGS Center of Astrogeology, Flagstaff, Ariz., following imaging of Mars by Mariner 9. Maps of Mars at other scales have followed. The mapping of the planet Mercury followed the flybys of Mariner 10 in 1974. Beginning in the late 1960's, geology of the visible side of the Moon was mapped by the USGS in quadrangle fashion at a scale of 1:1,000,000. The four Galilean satellites of Jupiter and several satellites of Saturn are being mapped following the Voyager missions of 1979–81. For all these bodies, the Mercator projection has been used to map equatorial portions, but coverage extends in some cases to lats. 65° N. and S. (See table 6.)

The cloudy atmosphere of Venus, circled by the Pioneer Venus Orbiter beginning in late 1978, is delaying more precise mapping of that planet, but the Mercator projection alone has been used to show altitudes based on radar reflectivity over about 93 percent of the surface.

FORMULAS FOR THE SPHERE

There is no suitable geometrical construction of the Mercator projection. For the sphere, the formulas for rectangular coordinates are as follows:

$$x = R(\lambda - \lambda_0) \tag{7-1}$$

$$y = R \ln \tan(\pi/4 + \phi/2) \tag{7-2}$$

or

$$y = R \operatorname{arctanh}(\sin \phi) \tag{7-2a}$$

where R is the radius of the sphere at the scale of the map as drawn, and ϕ and λ are given in radians. The X axis lies along the Equator, x increasing easterly. The Y axis lies along the central meridian λ_0, y increasing northerly. If $(\lambda - \lambda_0)$ lies outside the range $\pm 180°$, 360° should be added or subtracted so it will fall inside the range. To use ϕ and λ in degrees,

$$x = \pi R(\lambda° - \lambda_0°)/180° \tag{7-1a}$$

$$y = R \ln \tan(45° + \phi°/2) \tag{7-2b}$$

or

$$y = R \operatorname{arctanh}(\sin \phi) \tag{7-2c}$$

Equations (7–2a) and (7–2c) may be more convenient to use than (7–2) or (7–2b), if hyperbolic functions are standard to the computer or calculator. Note that if ϕ is $\pm\pi/2$ or $\pm 90°$, y is infinite. For scale factors, application of equations (4–2), (4–3), and (4–9) to (7–1) and (7–2) or

TABLE 6.—*Mercator Projection: Used for extraterrestrial mapping*

[From Batson, 1973; Davies and Batson, 1975; Batson and others, 1980; Pettengill, 1980; Batson, private commun., 1981]

Body[1]	Scale[2]	Range in lat.	Adjacent Projection	Overlap	Matching Parallel with (scale)[3]	Comments
Moon	1:1,000,000 (geologic series)	16°S.–16°N.	Lambert Conformal Conic	0°	16° (1:1,021,000)	Quadrangles 20° long. × 16° lat.
Mercury	1:15,000,000	57°S.–57°N.	Polar Stereographic	2°	56° (1:8,388,000)	--
	1:5,000,000	25°S.–25°N.	Lambert Conformal Conic	5°	22.5° (1:4,619,000)	Quadrangles 72° long. × 50° lat.
Venus	1:50,000,000	65°S.–78°N.	none	--	--	--
Mars	1:25,000,000	65°S.–65°N.	Polar Stereographic	10°	60° (1:12,549,000)	--
	1:15,000,000	57°S.–57°N.	Polar Stereographic	2°	56° (1:8,418,000)	--
	1:5,000,000	30°S.–30°N.	Lambert Conformal Conic	0°	30° (1:4,336,000)	Quadrangles 45° long. × 30° lat.
	1:2,000,000	30°S.–30°N.	Lambert Conformal Conic	0°	30° (1:1,953,000)	Quadrangles 22.5° long. × 15° lat.

Galilean satellites of Jupiter						
Io	1:25,000,000	57°S.-57°N.	Polar Stereographic	2°	56° (1:13,980,000)	--
Europa	1:15,000,000	57°S.-57°N.	Polar Stereographic	2°	56° (1:8,388,000)	--
Ganymede	1:5,000,000 (Io & Europa)	50°S.-50°N.	Polar Stereographic	5°	45° (1:4,268,000)	Quadrangles 180° long. × 100° lat.
Callisto	1:5,000,000 (Ganymede & Callisto)	22°S.-22°N.	Lambert Conformal Conic	1°	21.3° (1:4,780,000)	Quadrangles 72° long. × 44° lat.
Satellites of Saturn						
Mimas, Enceladus, Hyperion	1:5,000,000	57°S.-57°N.	Polar Stereographic	2°	56° (1:2,796,000)	--
Tethys, Dione, Rhea, Iapetus	1:10,000,000	57°S.-57°N.	Polar Stereographic	2°	56° (1:5,592,000)	--

[1] Taken as sphere, except for Mars (ellipsoid). See table 2.

[2] Scale at equator, except for Moon (at 11°00'45" N. & S.), Io and Europa 1:5,000,000 (at 34°04' N. & S. and pole of Stereographic), Ganymede and Callisto 1:5,000,000 (at 13°00' N. & S. and pole of Stereographic), and Mars 1:2,000,000 (at 27°29' N. & S. and standard parallels of Lambert Conformal Conic).

[3] Some scales revised from those previously published, per communications from Batson, 1979. Matching parallels are both N. & S.

(7–2a), gives results consistent with the conformal feature of the Mercator projection:

$$h = k = \sec\phi = 1/\cos\phi \qquad (7\text{–}3)$$
$$\omega = 0$$

Normally, for conformal projections, the use of h (the scale factor along a meridian) is omitted, and k (the scale factor along a parallel) is used for the scale factor in any direction. The areal scale factor for conformal projections is k^2 or $\sec^2\phi$ for the Mercator in spherical form.

The inverse formulas for the sphere, to obtain ϕ and λ from rectangular coordinates, are as follows:

$$\phi = \pi/2 - 2\arctan(e^{-y/R}) \qquad (7\text{–}4)$$
$$\lambda = x/R + \lambda_0 \qquad (7\text{–}5)$$

Here e = 2.7182818, the base of natural logarithms, not eccentricity. These and subsequent formulas are given only in radians, as stated earlier, unless the degree symbol is used. Numerical examples (see Appendix A) are given in degrees, showing conversion.

FORMULAS FOR THE ELLIPSOID

For the ellipsoid, the corresponding equations for the Mercator are only a little more involved:

$$x = a(\lambda - \lambda_0) \qquad (7\text{–}6)$$
$$y = a\ln\left[\tan(\pi/4 + \phi/2)\left(\frac{1 - e\sin\phi}{1 + e\sin\phi}\right)^{e/2}\right] \qquad (7\text{–}7)$$

where a is the equatorial radius of the ellipsoid, and e is its eccentricity. Comparing equation (3–7), it is seen that $y = a\psi$. From equations (4–22) and (4–23), it may be found that

$$h = k = (1 - e^2\sin^2\phi)^{1/2}/\cos\phi \qquad (7\text{–}8)$$

and of course $\omega = 0$. The areal scale factor is k^2. The derivation of these equations is shown in Thomas (1952, p. 1, 2, 85–90).

The X and Y axes are oriented as they are for the spherical formulas, and $(\lambda - \lambda_0)$ should be similarly adjusted. Thomas also provides a series equivalent to equation (7–7), slightly modified here for consistency:

$$y/a = \ln\tan(\pi/4 + \phi/2) - (e^2 + e^4/4 + e^6/8 + \ldots)\sin\phi$$
$$+ (e^4/12 + e^6/16 + \ldots)\sin 3\phi - (e^6/80 + \ldots)\sin 5\phi + \ldots \qquad (7\text{–}7a)$$

The inverse formulas for the ellipsoid require rapidly converging iteration, if the closed forms of the equations for finding ϕ are used:

$$\phi = \pi/2 - 2\arctan\{t[(1 - e\sin\phi)/(1 + e\sin\phi)]^{e/2}\} \qquad (7\text{–}9)$$

where $t = e^{-y/a}$ (7–10)

e is the base of natural logarithms, 2.71828 . . . ,
and the first trial $\phi = \pi/2 - 2 \arctan t$ (7–11)

Inserting the first trial ϕ in the right side of equation (7–9), ϕ on the left side is calculated. This becomes the new trial ϕ, which is used on the right side. The process is repeated until the change in ϕ is less than a chosen convergence factor depending on the accuracy desired. This ϕ is then the final value. For λ,

$$\lambda = x/a + \lambda_0 \tag{7–12}$$

The scale factor is calculated from equation (7–8), using the calculated ϕ.

To avoid the iteration, the series (3–5) may be used with (7–13) in place of (7–9):

$$\phi = \chi + (e^2/2 + 5e^4/24 + e^6/12 + \ldots) \sin 2\chi + (7e^4/48 + 29e^6/240 + \ldots) \sin 4\chi + (7e^6/120 + \ldots) \sin 6\chi + \ldots \tag{3–5}$$

where $\chi = \pi/2 - 2 \arctan t$ (7–13)

Rectangular coordinates for each 5° of latitude are given in table 7, for both the sphere and the Clarke 1866 ellipsoid, assuming R and a are both 1.0. It should be noted that k for the sphere applies only to the sphere. The spherical projection is not conformal with respect to the ellipsoidal Earth, although the variation is negligible for a map with an equatorial scale of 1:15,000,000 or smaller.

MERCATOR PROJECTION WITH ANOTHER STANDARD PARALLEL

The above formulas are based on making the Equator of the Earth true to scale on the map. Thus, the Equator may be called the standard parallel. It is also possible to have, instead, another parallel (actually two) as standard, with true scale. For the Mercator, the map will look exactly the same; only the scale will be different. If latitude ϕ_1 is made standard (the opposite latitude $-\phi_1$ is also standard), the above forward formulas are adapted by multiplying the right side of equations (7–1) through (7–3) for the sphere, including the alternate forms, by cos ϕ_1. For the ellipsoid, the right sides of equations (7–6), (7–7), (7–8), and (7–7a) are multiplied by $\cos \phi_1/(1 - e^2 \sin^2 \phi_1)^{1/2}$. For inverse equations, divide x and y by the same values before use in equations (7–4) and (7–5) or (7–10) and (7–12). Such a projection is most commonly used for a navigational map of part of an ocean, such as the North Atlantic Ocean, but the USGS has used it for equatorial quadrangles of some extraterrestrial bodies as described in table 6.

TABLE 7.—*Mercator projection: Rectangular coordinates*

Latitude (ϕ)	Sphere ($R=1$)		Clarke 1866 ellipsoid ($a=1$)	
	y	k	y	k
90°	Infinite	Infinite	Infinite	Infinite
85	3.13130	11.47371	3.12454	11.43511
80	2.43625	5.75877	2.42957	5.73984
75	2.02759	3.86370	2.02104	3.85148
70	1.73542	2.92380	1.72904	2.91505
65	1.50645	2.36620	1.50031	2.35961
60	1.31696	2.00000	1.31109	1.99492
55	1.15423	1.74345	1.14868	1.73948
50	1.01068	1.55572	1.00549	1.55263
45	.88137	1.41421	.87658	1.41182
40	.76291	1.30541	.75855	1.30358
35	.65284	1.22077	.64895	1.21941
30	.54931	1.15470	.54592	1.15372
25	.45088	1.10338	.44801	1.10271
20	.35638	1.06418	.35406	1.06376
15	.26484	1.03528	.26309	1.03504
10	.17543	1.01543	.17425	1.01532
5	.08738	1.00382	.08679	1.00379
0	.00000	1.00000	.00000	1.00000
x	$0.017453\ (\lambda-\lambda_0)$		$0.017453\ (\lambda-\lambda_0)$	

Note: x, y = rectangular coordinates.
ϕ = geodetic latitude.
$(\lambda-\lambda_0)$ = geodetic longitude, measured east from origin in degrees.
k = scale factor, relative to scale at Equator.
R = radius of sphere at scale of map.
a = equatorial radius of ellipsoid at scale of map.
If latitude is negative (south), reverse sign of y.

8. TRANSVERSE MERCATOR PROJECTION

SUMMARY

- Cylindrical (transverse).
- Conformal.
- Central meridian, each meridian 90° from central meridian, and Equator are straight lines.
- Other meridians and parallels are complex curves.
- Scale is true along central meridian, or along two straight lines equidistant from and parallel to central meridian. (These lines are only approximately straight for the ellipsoid.)
- Scale becomes infinite 90° from central meridian.
- Used extensively for quadrangle maps at scales from 1:24,000 to 1:250,000.
- Presented by Lambert in 1772.

HISTORY

Since the regular Mercator projection has little error close to the Equator (the scale 10° away is only 1.5 percent larger than the scale at the Equator), it has been found very useful in the transverse form, with the equator of the projection rotated 90° to coincide with the desired central meridian. This is equivalent to wrapping the cylinder around a sphere or ellipsoid representing the Earth so that it touches the central meridian throughout its length, instead of following the Equator of the Earth. The central meridian can then be made true to scale, no matter how far north and south the map extends, and regions near it are mapped with low distortion. Like the regular Mercator, the map is conformal.

The Transverse Mercator projection in its spherical form was invented by the prolific Alsatian mathematician and cartographer Johann Heinrich Lambert (1728–77). It was the third of six new projections which he described in 1772 in his classic *Beiträge* (Lambert, 1772). At the same time, he also described what are now called the Lambert Conformal Conic and the Lambert Azimuthal Equal-Area, both of which will be discussed subsequently; others are omitted here. He described the Transverse Mercator as a conformal adaptation of the Sinusoidal projection, then commonly in use (Lambert, 1772, p. 57–58). Lambert's derivation was followed with a table of coordinates and a map of the Americas drawn according to the projection.

Little use has been made of the Transverse Mercator for single maps of continental areas. While Lambert only indirectly discussed its ellipsoidal form, mathematician Carl Friedrich Gauss (1777–1855) analyzed it further in 1822, and L. Krüger published studies in 1912 and 1919 providing formulas suitable for calculation relative to the ellipsoid. It is,

FIGURE 9.–Johann Heinrich Lambert (1728–77). Inventor of the Transverse Mercator, the Conformal Conic, the Azimuthal Equal-Area, and other important projections, as well as outstanding developments in mathematics, astronomy, and physics.

therefore, sometimes called the Gauss conformal or the Gauss-Krüger projection in Europe, but Transverse Mercator, a term first applied by the French map projection compiler Germain, is the name normally used in the United States (Thomas, 1952, p. 91–92; Germain, 1865?, p. 347).

Until recently, the Transverse Mercator projection was not precisely applied to the ellipsoid for the entire Earth. Ellipsoidal formulas were limited to series for relatively narrow bands of about ±4° longitude. In 1945, E. H. Thompson (and in 1962, L. P. Lee) presented exact or closed formulas permitting calculation of coordinates for the full ellipsoid, although elliptic functions, and therefore lengthy series, numerical integrations, and (or) iterations, are involved (Lee, 1976, p. 92–101; Snyder, 1979a, p. 73; Dozier, 1980).

The formulas for the complete ellipsoid are interesting academically, but they are practical only within a band between 4° of longitude and some 10° to 15° of arc distance on either side of the central meridian, because of the much more significant scale errors fundamental to any projection covering a larger area.

FEATURES

The meridians and parallels of the Transverse Mercator are no longer the straight lines they are on the regular Mercator, except for the Earth's Equator, the central meridian, and each meridian 90° away from the central meridian. Other meridians and parallels are complex curves.

The spherical form is conformal, as is the parent projection, and scale error is only a function of the distance from the central meridian, just as it is only a function of the distance from the Equator on the regular Mercator. The ellipsoidal form is also exactly conformal, but its scale error is slightly affected by factors other than the distance alone from the central meridian (Lee, 1976, p. 98).

The scale along the central meridian may be made true to scale, or deliberately reduced to a slightly smaller constant scale so that the mean scale of the entire map is more nearly correct. There are also forms of the ellipsoidal Transverse Mercator on which the central meridian is not held at a constant scale, but these forms are not used in practice (Lee, 1976, p. 100–101). If the central meridian is mapped at a reduced scale, two straight lines parallel to it and equally spaced from it, one on either side, become true to scale on the sphere. These lines are not perfectly straight on the ellipsoidal form.

With the scale along the central meridian remaining constant, the Transverse Mercator is an excellent projection for lands extending predominantly north and south.

USAGE

The Transverse Mercator projection (spherical or ellipsoidal) was not described by Close and Clarke in their generally detailed article in the 1911 *Encyclopaedia Britannica* because it was "seldom used" (Close and Clarke, 1911, p. 663). Deetz and Adams (1934) favorably referred to it several times, but as a slightly used projection.

The spherical form of the Transverse Mercator has been used by the USGS only recently. In 1979, this projection was chosen for a base map of North America at a scale of 1:5,000,000 to replace the Bipolar Oblique Conic Conformal projection previously used for tectonic and other geologic maps. The scale factor along the central meridian, long. 100° W., is reduced to 0.926. The radius of the Earth is taken at 6,371,204 m, with approximately the same surface area as the International ellipsoid, placing the two straight lines of true design scale 2,343 km on each side of the central meridian.

While its use in the spherical form is limited, the ellipsoidal form of the Transverse Mercator is probably used more than any other one projection for geodetic mapping.

In the United States, it is the projection used in the State Plane Coordinate System (SPCS) for States with predominant north-south extent. (The Lambert Conformal Conic is used for the others, except for the panhandle of Alaska, which is prepared on the Oblique Mercator. Alaska, Florida, and New York use both the Transverse Mercator and the Lambert Conformal Conic for different zones.) Except for narrow States, such as Delaware, New Hampshire, and New Jersey, all States using the Transverse Mercator are divided into two to eight zones, each with its own central meridian, along which the scale is slightly reduced to balance the scale throughout the map. Each zone is designed to maintain scale distortion within 1 part in 10,000.

In addition to latitude and longitude as the basic frame of reference, the corresponding rectangular grid coordinates in feet are used to designate locations (Mitchell and Simmons, 1945). The parameters for each State are given in table 8. All are based on the Clarke 1866 ellipsoid. It is important to note that, for the metric conversion to feet using this coordinate system, 1 m equals exactly 39.37 in., not the current standard accepted by the National Bureau of Standards in 1959, in which 1 in. equals exactly 2.54 cm. Surveyors continue to follow the former conversion for consistency. The difference is only two parts in a million, but it is enough to cause confusion, if it is not accounted for.

Beginning with the late 1950's, the Transverse Mercator projection was used by the USGS for nearly all new quadrangles (maps normally bounded by meridians and parallels) covering those States using the TM Plane Coordinates, but the central meridian and scale factor are those of the SPCS zone. Thus, all quadrangles for a given zone may be mosaicked exactly. Beginning in 1977, many USGS maps have been produced on the Universal Transverse Mercator projection (see below). Prior to the late 1950's, the Polyconic projection was used. The change in projection was facilitated by the use of high-precision rectangular-coordinate plotting machines. Some maps produced on the Transverse Mercator projection system during this transition period are identified

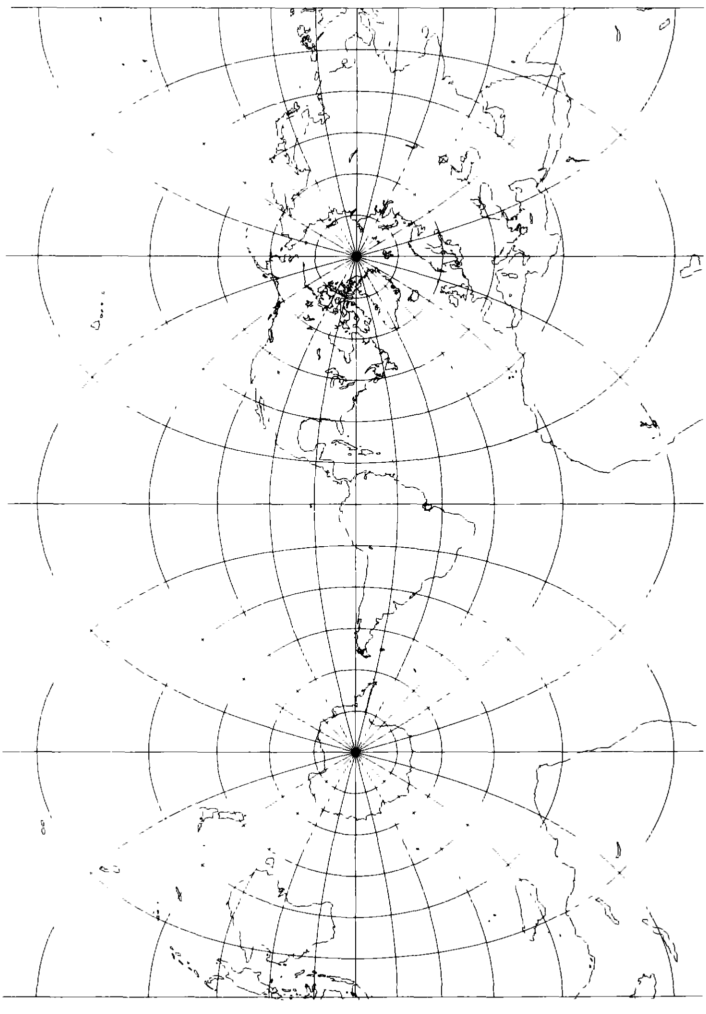


FIGURE 10.—The Transverse Mercator projection. While the regular Mercator has constant scale along the Equator, the Transverse Mercator has constant scale along any chosen central meridian. This projection is conformal and is often used to show regions with greater north-south extent.

as being prepared according to the Polyconic projection. Since most quadrangles cover only 7½ minutes (at a scale of 1:24,000) or 15 minutes (at 1:62,500) of latitude and longitude, the difference between the Polyconic and the Transverse Mercator for such a small area is much more significant due to the change of central meridian than due to the change of projection. The difference is still slight and is detailed later under the discussion of the Polyconic projection. The Transverse

TABLE 8.—*U.S. State plane coordinate systems*

[T indicates Transverse Mercator; L, Lambert Conformal Conic; H, Hotine Oblique Mercator. Modified slightly and updated from Mitchell and Simmons, 1945, p. 45–47]

Area	Projection	Zones	Area	Projection	Zones
Alabama	T	2	Montana	L	3
Alaska	T	8	Nebraska	L	2
	L	1	Nevada	T	3
	H	1	New Hampshire	T	1
Arizona	T	3	New Jersey	T	1
Arkansas	L	2	New Mexico	T	3
California	L	7	New York	T	3
Colorado	L	3		L	1
Connecticut	L	1	North Carolina	L	1
Delaware	T	1	North Dakota	L	2
Florida	T	2	Ohio	L	2
	L	1	Oklahoma	L	2
Georgia	T	2	Oregon	L	2
Hawaii	T	5	Pennsylvania	L	2
Idaho	T	3	Puerto Rico & Virgin Islands	L	2
Illinois	T	2	Rhode Island	T	1
Indiana	T	2	Samoa	L	1
Iowa	L	2	South Carolina	L	2
Kansas	L	2	South Dakota	L	2
Kentucky	L	2	Tennessee	L	1
Louisiana	L	3	Texas	L	5
Maine	T	2	Utah	L	3
Maryland	L	1	Vermont	T	1
Massachusetts	L	2	Virginia	L	2
Michigan[1]			Washington	L	2
obsolete	T	3	West Virginia	L	2
current	L	3	Wisconsin	L	3
Minnesota	L	3	Wyoming	T	4
Mississippi	T	2			
Missouri	T	3			

Transverse Mercator projection

Zone	Central meridian	Scale reduction[2]	Origin[3] (latitude)
Alabama			
East	85°50′ W.	1:25,000	30°30′ N.
West	87 30	1:15,000	30 00
Alaska[4]			
2	142 00	1:10,000	54 00
3	146 00	1:10,000	54 00
4	150 00	1:10,000	54 00
5	154 00	1:10,000	54 00
6	158 00	1:10,000	54 00
7	162 00	1:10,000	54 00
8	166 00	1:10,000	54 00
9	170 00	1:10,000	54 00
Arizona			
East	110 10	1:10,000	31 00
Central	111 55	1:10,000	31 00
West	113 45	1:15,000	31 00
Delaware	75 25	1:200,000	38 00
Florida[4]			
East	81 00	1:17,000	24 20
West	82 00	1:17,000	24 20

TABLE 8. – *U.S. State plane coordinate systems* – Continued

Transverse Mercator projection – Continued			
Zone	**Central meridian**	**Scale reduction**[2]	**Origin**[3] **(latitude)**
Georgia			
East	82°10′ W.	1:10,000	30°00′ N.
West	84 10	1:10,000	30 00
Hawaii			
1	155 30	1:30,000	18 50
2	156 40	1:30,000	20 20
3	158 00	1:100,000	21 10
4	159 30	1:100,000	21 50
5	160 10	0	21 40
Idaho			
East	112 10	1:19,000	41 40
Central	114 00	1:19,000	41 40
West	115 45	1:15,000	41 40
Illinois			
East	88 20	1:40,000	36 40
West	90 10	1:17,000	36 40
Indiana			
East	85 40	1:30,000	37 30
West	87 05	1:30,000	37 30
Maine			
East	68 30	1:10,000	43 50
West	70 10	1:30,000	42 50
Michigan (old)[4]			
East	83 40	1:17,500	41 30
Central	85 45	1:11,000	41 30
West	88 45	1:11,000	41 30
Mississippi			
East	88 50	1:25,000	29 40
West	90 20	1:17,000	30 30
Missouri			
East	90 30	1:15,000	35 50
Central	92 30	1:15,000	35 50
West	94 30	1:17,000	36 10
Nevada			
East	115 35	1:10,000	34 45
Central	116 40	1:10,000	34 45
West	118 35	1:10,000	34 45
New Hampshire	71 40	1:30,000	42 30
New Jersey	74 40	1:40,000	38 50
New Mexico			
East	104 20	1:11,000	31 00
Central	106 15	1:10,000	31 00
West	107 50	1:12,000	31 00
New York[4]			
East	74 20	1:30,000	40 00
Central	76 35	1:16,000	40 00
West	78 35	1:16,000	40 00
Rhode Island	71 30	1:160,000	41 05
Vermont	72 30	1:28,000	42 30

TABLE 8.—*U.S. State plane coordinate systems*—Continued

Transverse Mercator projection—Continued

Zone	Central meridian	Scale reduction[2]	Origin[3] (latitude)
Wyoming			
East	105°10′ W.	1:17,000	40°40′ N.
East Central	107 20	1:17,000	40 40
West Central	108 45	1:17,000	40 40
West	110 05	1:17,000	40 40

Lambert Conformal Conic projection

Zone	Standard parallels		Origin[5] Long.	Origin[5] Lat.
Alaska[4]				
10	51°50′ N.	53°50′ N.	176°00′ W.[5a]	51°00′ N.
Arkansas				
North	34 56	36 14	92 00	34 20
South	33 18	34 46	92 00	32 40
California				
I	40 00	41 40	122 00	39 20
II	38 20	39 50	122 00	37 40
III	37 04	38 26	120 30	36 30
IV	36 00	37 15	119 00	35 20
V	34 02	35 28	118 00	33 30
VI	32 47	33 53	116 15	32 10
VII	33 52	34 25	118 20	34 08[5b]
Colorado				
North	39 43	40 47	105 30	39 20
Central	38 27	39 45	105 30	37 50
South	37 14	38 26	105 30	36 40
Connecticut	41 12	41 52	72 45	40 50[5d]
Florida[4]				
North	29 35	30 45	84 30	29 00
Iowa				
North	42 04	43 16	93 30	41 30
South	40 37	41 47	93 30	40 00
Kansas				
North	38 43	39 47	98 00	38 20
South	37 16	38 34	98 30	36 40
Kentucky				
North	37 58	38 58	84 15	37 30
South	36 44	37 56	85 45	36 20
Louisiana				
North	31 10	32 40	92 30	30 40
South	29 18	30 42	91 20	28 40
Offshore	26 10	27 50	91 20	25 40
Maryland	38 18	39 27	77 00	37 50[5c]
Massachusetts				
Mainland	41 43	42 41	71 30	41 00[5d]
Island	41 17	41 29	70 30	41 00[5e]

TABLE 8.—*U.S. State plane coordinate systems*—Continued

Lambert Conformal Conic projection—Continued

Zone	Standard parallels		Origin[5] Long.	Origin[5] Lat.
Michigan (current)[4]				
North	45°29′ N.	47°05′ N.	87°00′ W.	44°47′ N.
Central	44 11	45 42	84 20	43 19
South	42 06	43 40	84 20	41 30
Minnesota				
North	47 02	48 38	93 06	46 30
Central	45 37	47 03	94 15	45 00
South	43 47	45 13	94 00	43 00
Montana				
North	47 51	48 43	109 30	47 00
Central	46 27	47 53	109 30	45 50
South	44 52	46 24	109 30	44 00
Nebraska				
North	41 51	42 49	100 00	41 20
South	40 17	41 43	99 30	39 40
New York[4]				
Long Island	40 40	41 02	74 00	40 30[5f]
North Carolina	34 20	36 10	79 00	33 45
North Dakota				
North	47 26	48 44	100 30	47 00
South	46 11	47 29	100 30	45 40
Ohio				
North	40 26	41 42	82 30	39 40
South	38 44	40 02	82 30	38 00
Oklahoma				
North	35 34	36 46	98 00	35 00
South	33 56	35 14	98 00	33 20
Oregon				
North	44 20	46 00	120 30	43 40
South	42 20	44 00	120 30	41 40
Pennsylvania				
North	40 53	41 57	77 45	40 10
South	39 56	40 58	77 45	39 20
Puerto Rico and Virgin Islands				
1	18 02	18 26	66 26	17 50[5g]
2 (St. Croix)	18 02	18 26	66 26	17 50[5f, g]
Samoa	14°16′ S.	(single)	170 00[5h]	-- --
South Carolina				
North	33°46′ N.	34 58	81 00	33 00
South	32 20	33 40	81 00	31 50
South Dakota				
North	44 25	45 41	100 00	43 50
South	42 50	44 24	100 20	42 20
Tennessee	35 15	36 25	86 00	34 40[5f]

TABLE 8. – *U.S. State plane coordinate systems* – Continued

Lambert Conformal Conic projection – Continued

Zone	Standard parallels		Origin[5] Long.	Origin[5] Lat.
Texas				
North	34°39′ N.	36°11′ N.	101°30′ W.	34°00′ N.
North central	32 08	33 58	97 30	31 40
Central	30 07	31 53	100 20	29 40
South central	28 23	30 17	99 00	27 50
South	26 10	27 50	98 30	25 40
Utah				
North	40 43	41 47	111 30	40 20
Central	39 01	40 39	111 30	38 20
South	37 13	38 21	111 30	36 40
Virginia				
North	38 02	39 12	78 30	37 40
South	36 46	37 58	78 30	36 20
Washington				
North	47 30	48 44	120 50	47 00
South	45 50	47 20	120 30	45 20
West Virginia				
North	39 00	40 15	79 30	38 30
South	37 29	38 53	81 00	37 00
Wisconsin				
North	45 34	46 46	90 00	45 10
Central	44 15	45 30	90 00	43 50
South	42 44	44 04	90 00	42 00

Hotine Oblique Mercator projection

Zone	Center of projection Long.	Center of projection Lat.	Azimuth of central line	Scale[6] reduction
Alaska[4]				
1	133°40′ W.	57°00′ N.	arctan (−3/4)	1:10,000

Note. – All these systems are based on the Clarke 1866 ellipsoid.

[1]The major and minor axes of the ellipsoid are taken at exactly 1.0000382 times those of the Clarke 1866, for Michigan only. This incorporates an average elevation throughout the State of about 800 ft, with limited variation.

[2]Along the central meridian.

[3]At origin, $x = 500{,}000$ ft, $y = 0$ ft, except for Alaska zone 7, $x = 700{,}000$ ft; Alaska zone 9, $x = 600{,}000$ ft; and New Jersey, $x = 2{,}000{,}000$ ft.

[4]Additional zones listed in this table under other projection(s).

[5]At origin, $x = 2{,}000{,}000$ ft, $y = 0$ ft, except (a) $x = 3{,}000{,}000$ ft, (b) $x = 4{,}186{,}692.58$, $y = 4{,}160{,}926.74$ ft, (c) $x = 800{,}000$ ft, (d) $x = 600{,}000$ ft, (e) $x = 200{,}000$ ft, (f) $y = 100{,}000$ ft, (g) $x = 500{,}000$ ft, (h) $x = 500{,}000$ ft, $y = 0$, but radius to lat. of origin = −82,000,000 ft.

[6]At central point.

Mercator is used in many other countries for official topographic mapping as well. The Ordnance Survey of Great Britain began switching from a Transverse Equidistant Cylindrical (the Cassini-Soldner) to the Transverse Mercator about 1920.

The use of the Transverse Mercator for quadrangle maps has been recently extended by the USGS to include the planets Mercury and Mars. Although other projections are used at smaller scales, quadrangles at scales of 1:1,000,000 and 1:250,000, and covering areas from 200 to 800 km on a side, are drawn to the ellipsoidal Transverse Mercator between lats. 65° N. and S. on Mars, and to the spherical Transverse Mercator for any latitudes on Mercury. The scale factor along the central meridian is made 1.0 in all cases.

In addition to its own series of larger-scale quadrangle maps, the Army Map Service used the Transverse Mercator for two other major mapping operations: (1) a series of 1:250,000-scale quadrangle maps covering the entire country, and (2) as the geometric basis for the Universal Transverse Mercator (UTM) grid.

The entire area of the United States has been mapped since the 1940's in sections 2° of longitude (between even-numbered meridians, but in 3° sections in Alaska) by 1° of latitude (between each full degree) at a scale of 1:250,000, with the UTM grid superimposed and with some variations in map boundaries at coastlines. These maps were drawn with reference to their own central meridians, not the central meridians of the UTM zones (see below), although the 0.9996 central scale factor was employed. The central meridian of about one-third of the maps coincides with the central meridian of the zone, but it does not for about two-thirds, the "wing" sheets, which therefore do not perfectly match the center sheets. The USGS has assumed publication and revision of this series and is casting new maps using the correct central meridians.

Transverse Mercator quadrangle maps fit continuously in a north-south direction, provided they are prepared at the same scale, with the same central meridian, and for the same ellipsoid. They do not fit exactly from east to west, if they have their own central meridians; although quadrangles and other maps properly constructed at the same scale, using the SPCS or UTM projection, fit in all directions within the same zone.

UNIVERSAL TRANSVERSE MERCATOR PROJECTION

The Universal Transverse Mercator (UTM) projection and grid were adopted by the U.S. Army in 1947 for designating rectangular coordinates on large-scale military maps of the entire world. The UTM is the ellipsoidal Transverse Mercator to which specific parameters, such as central meridians, have been applied. The Earth, between lats. 84°

N. and 80° S., is divided into 60 zones each generally 6° wide in longitude. Bounding meridians are evenly divisible by 6°, and zones are numbered from 1 to 60 proceeding east from the 180th meridian from Greenwich with minor exceptions. There are letter designations from south to north (see fig. 11). Thus, Washington, D.C., is in grid zone 18S, a designation covering a quadrangle from long. 72° to 78° W. and from lat. 32° to 40° N. Each of these quadrangles is further subdivided into grid squares 100,000 meters on a side with double-letter designations, including partial squares at the grid boundaries. From lat. 84° N. and 80° S. to the respective poles, the Universal Polar Stereographic (UPS) projection is used instead.

As with the SPCS, each geographic location in the UTM projection is given x and y coordinates, but in meters, not feet, according to the Transverse Mercator projection, using the meridian halfway between the two bounding meridians as the central meridian, and reducing its scale to 0.9996 of true scale (a 1:2,500 reduction). The reduction was chosen to minimize scale variation in a given zone; the variation reaches 1 part in 1,000 from true scale at the Equator. The USGS, for civilian mapping, uses only the zone number and the x and y coordinates, which are sufficient to define a point, if the ellipsoid and the hemisphere (north or south) are known; the 100,000-m square identification is not essential. The lines of true scale are approximately parallel to and approximately 180 km east and west of the central meridian. Between them, the scale is too small; beyond them, it is too great. In the Northern Hemisphere, the Equator at the central meridian is considered the origin, with an x coordinate of 500,000 m and a y of 0. For the Southern Hemisphere, the same point is the origin, but, while x remains 500,000 m, y is 10,000,000 m. In each case, numbers increase toward the east and north. Negative coordinates are thus avoided (Army, 1973, p. 7, endmap). A page of coordinates for the UTM projection is shown in table 9.

The ellipsoidal Earth is used throughout the UTM projection system, but the reference ellipsoid changes with the particular region of the Earth. For all land under United States jurisdiction, the Clarke 1866 ellipsoid is used for the map projection. For the UTM grid superimposed on the map of Hawaii, however, the International ellipsoid is used. The Geological Survey uses the UTM graticule and grid for its 1:250,000- and larger-scale maps of Alaska, and applies the UTM grid lines or tick marks to its quadrangles and State base maps for the other States, although they are generally drawn with different projections or parameters.

FORMULAS FOR THE SPHERE

A partially geometric construction of the Transverse Mercator for the sphere involves constructing a regular Mercator projection and us-

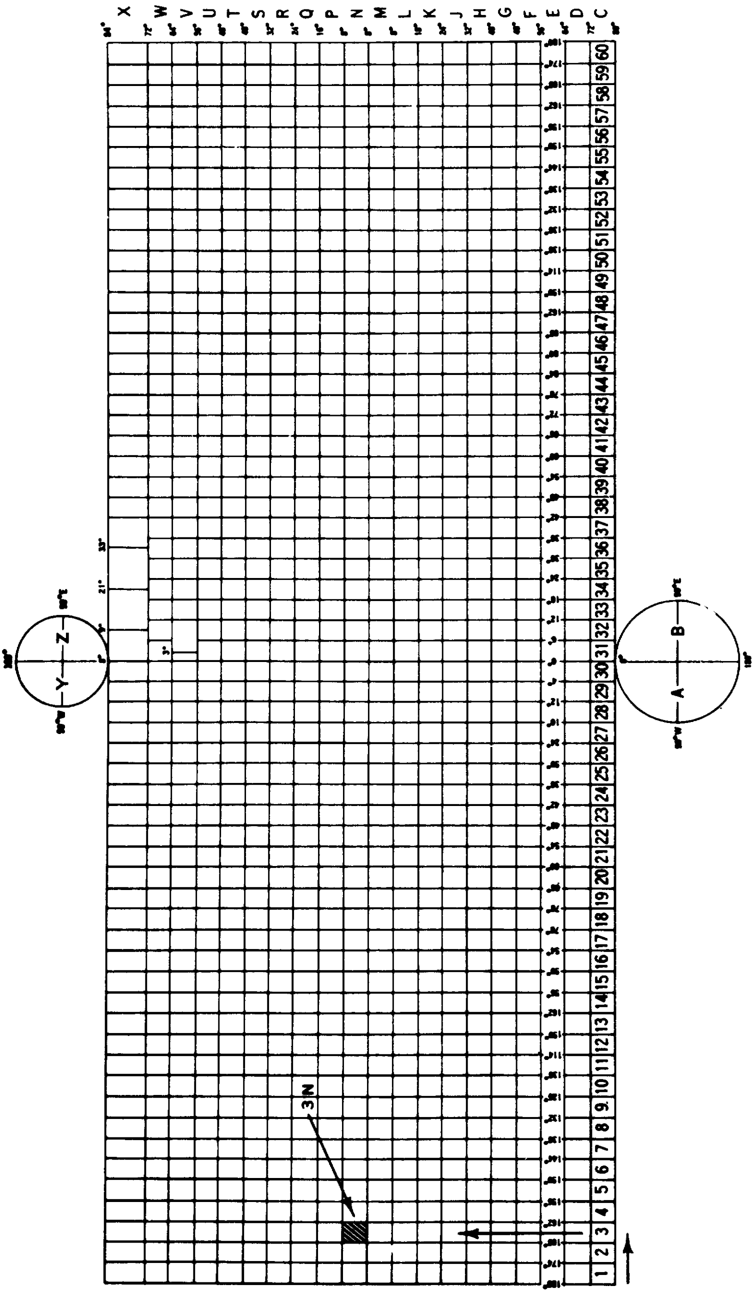


FIGURE 11.—Universal Transverse Mercator (UTM) grid zone designations for the world shown on an Equidistant Cylindrical projection index map.

TABLE 9.–*Universal Transverse Mercator grid coordinates*

U.T.M. GRID COORDINATES • CLARKE 1866 SPHEROID **METERS**

LATITUDE 48°00'00"			
Δλ	West of C.M. E	East of C.M. E	N
0°00'00"	500,000.0	500,000.0	5,316,081.3
07 30	490,675.3	509,324.7	5,316,088.9
15 00	481,350.5	518,649.5	5,316,111.6
22 30	472,025.8	527,974.2	5,316,149.4
30 00	462,701.1	537,298.9	5,316,202.3
37 30	453,376.4	546,623.6	5,316,270.3
45 00	444,051.8	555,948.2	5,316,353.5
52 30	434,727.1	565,272.9	5,316,451.7
1 00 00	425,402.5	574,597.5	5,316,565.1
07 30	416,078.0	583,922.0	5,316,693.6
15 00	406,753.5	593,246.5	5,316,837.3
22 30	397,429.0	602,571.0	5,316,996.1
30 00	388,104.5	611,895.5	5,317,169.9
37 30	378,780.2	621,219.8	5,317,359.0
45 00	369,455.9	630,544.1	5,317,563.1
52 30	360,131.6	639,868.4	5,317,782.4
2 00 00	350,807.4	649,192.6	5,318,016.8
07 30	341,483.3	658,516.7	5,318,266.3
15 00	332,159.3	667,840.7	5,318,531.0
22 30	322,835.4	677,164.6	5,318,810.8
30 00	313,511.5	686,488.5	5,319,105.8
37 30	304,187.7	695,812.3	5,319,415.9
45 00	294,864.1	705,135.9	5,319,741.1
52 30	285,540.5	714,459.5	5,320,081.5
3 00 00	276,217.0	723,783.0	5,320,437.0
07 30	266,893.7	733,106.3	5,320,807.7
15 00	257,570.5	742,429.5	5,321,193.6
22 30	248,247.4	751,752.6	5,321,594.6
30 00	238,924.4	761,075.6	5,322,010.8
37 30	229,601.5	770,398.5	5,322,442.1
45 00	220,278.8	779,721.2	5,322,888.6
52 30	210,956.2	789,043.8	5,323,350.3
4 00 00	201,633.8	798,366.2	5,323,827.1

LATITUDE 48°15'00"			
Δλ	West of C.M. E	East of C.M. E	N
0°00'00"	500,000.0	500,000.0	5,343,868.4
07 30	490,720.4	509,279.6	5,343,875.9
15 00	481,440.8	518,559.2	5,343,898.6
22 30	472,161.2	527,838.8	5,343,936.3
30 00	462,881.7	537,118.3	5,343,979.2
37 30	453,602.1	546,397.9	5,344,057.2
45 00	444,322.6	555,677.4	5,344,140.2
52 30	435,043.1	564,956.9	5,344,238.4
1 00 00	425,763.7	574,236.3	5,344,351.7
07 30	416,484.3	583,515.7	5,344,480.1
15 00	407,204.9	592,795.1	5,344,623.6
22 30	397,925.6	602,074.4	5,344,782.2
30 00	388,646.3	611,353.7	5,344,955.9
37 30	379,367.1	620,632.9	5,345,144.8
45 00	370,088.0	629,912.0	5,345,348.7
52 30	360,808.9	639,191.1	5,345,567.8
2 00 00	351,529.9	648,470.1	5,345,802.0
07 30	342,251.0	657,749.0	5,346,051.3
15 00	332,972.2	667,027.8	5,346,315.7
22 30	323,693.4	676,306.6	5,346,595.3
30 00	314,414.8	685,585.2	5,346,889.9
37 30	305,136.2	694,863.8	5,347,199.7
45 00	295,857.8	704,142.2	5,347,524.7
52 30	286,579.4	713,420.6	5,347,864.7
3 00 00	277,301.2	722,698.8	5,348,219.9
07 30	268,023.1	731,976.9	5,348,590.3
15 00	258,745.1	741,254.9	5,348,975.8
22 30	249,467.3	750,532.7	5,349,376.4
30 00	240,189.6	759,810.4	5,349,792.2
37 30	230,912.0	769,088.0	5,350,223.1
45 00	221,634.6	778,365.4	5,350,669.2
52 30	212,357.3	787,642.7	5,351,130.4
4 00 00	203,080.2	796,919.8	5,351,606.8

LATITUDE 48°07'30"			
Δλ	West of C.M. E	East of C.M. E	N
0°00'00"	500,000.0	500,000.0	5,329,974.7
07 30	490,697.8	509,302.2	5,329,982.3
15 00	481,395.6	518,604.4	5,330,004.9
22 30	472,093.5	527,906.5	5,330,042.7
30 00	462,791.3	537,208.7	5,330,095.6
37 30	453,489.2	546,510.8	5,330,163.6
45 00	444,187.1	555,812.9	5,330,246.7
52 30	434,885.0	565,115.0	5,330,344.9
1 00 00	425,582.9	574,417.1	5,330,458.3
07 30	416,280.9	583,719.1	5,330,586.7
15 00	406,979.0	593,021.0	5,330,730.3
22 30	397,677.0	602,323.0	5,330,889.0
30 00	388,375.2	611,624.8	5,331,062.8
37 30	379,073.4	620,926.6	5,331,251.7
45 00	369,771.6	630,228.4	5,331,455.8
52 30	360,469.9	639,530.1	5,331,675.0
2 00 00	351,168.3	648,831.7	5,331,909.3
07 30	341,866.8	658,133.2	5,332,158.7
15 00	332,565.3	667,434.7	5,332,423.2
22 30	323,264.0	676,736.0	5,332,702.9
30 00	313,962.7	686,037.3	5,332,997.7
37 30	304,661.5	695,338.5	5,333,307.7
45 00	295,360.4	704,639.6	5,333,632.8
52 30	286,059.5	713,940.5	5,333,973.0
3 00 00	276,758.6	723,241.4	5,334,328.4
07 30	267,457.9	732,542.2	5,334,698.9
15 00	258,157.2	741,842.8	5,335,084.6
22 30	248,856.7	751,143.3	5,335,485.4
30 00	239,556.4	760,443.6	5,335,901.4
37 30	230,256.1	769,743.9	5,336,332.5
45 00	220,956.0	779,044.0	5,336,778.8
52 30	211,656.1	788,343.9	5,337,240.3
4 00 00	202,356.3	797,643.7	5,337,716.9

LATITUDE 48°22'30"			
Δλ	West of C.M. E	East of C.M. E	N
0°00'00"	500,000.0	500,000.0	5,357,762.3
07 30	490,743.0	509,257.0	5,357,769.9
15 00	481,486.1	518,513.9	5,357,792.5
22 30	472,229.2	527,770.8	5,357,830.3
30 00	462,972.2	537,027.8	5,357,883.1
37 30	453,715.3	546,284.7	5,357,951.0
45 00	444,458.5	555,541.5	5,358,034.1
52 30	435,201.6	564,798.4	5,358,132.2
1 00 00	425,944.8	574,055.2	5,358,245.4
07 30	416,688.0	583,312.0	5,358,373.8
15 00	407,431.3	592,568.7	5,358,517.2
22 30	398,174.6	601,825.4	5,358,675.7
30 00	388,918.0	611,082.0	5,358,849.4
37 30	379,661.5	620,338.5	5,359,038.1
45 00	370,405.0	629,595.0	5,359,242.0
52 30	361,148.6	638,851.4	5,359,460.9
2 00 00	351,892.2	648,107.8	5,359,695.0
07 30	342,636.0	657,364.0	5,359,944.2
15 00	333,379.8	666,620.2	5,360,208.5
22 30	324,123.7	675,876.3	5,360,487.9
30 00	314,867.7	685,132.3	5,360,782.4
37 30	305,611.9	694,388.1	5,361,092.0
45 00	296,356.1	703,643.9	5,361,416.8
52 30	287,100.4	712,899.6	5,361,756.7
3 00 00	277,844.9	722,155.1	5,362,111.7
07 30	268,589.5	731,410.5	5,362,481.9
15 00	259,334.2	740,665.8	5,362,867.2
22 30	250,079.1	749,920.9	5,363,267.6
30 00	240,824.1	759,175.9	5,363,683.1
37 30	231,569.2	768,430.8	5,364,113.9
45 00	222,314.5	777,685.5	5,364,559.7
52 30	213,060.0	786,940.0	5,365,020.7
4 00 00	203,805.6	796,194.4	5,365,496.8

GRID COORDINATES FOR 7.5 MINUTE INTERSECTIONS ing a transforming map to convert meridians and parallels on one sphere to equivalent meridians and parallels on a sphere rotated to place the equator of one along the chosen central meridian of the other. Such a transforming map may be the equatorial aspect of the Stereographic or other azimuthal projection, drawn twice to the same scale on transparencies. The transparencies may then be superimposed at 90° angles and the points compared.

In an age of computers, it is much more satisfactory to use mathematical formulas. The rectangular coordinates for the Transverse Mercator applied to the sphere (Thomas, 1952, p.6):

$$x = \tfrac{1}{2} R k_0 \ln [(1+B)/(1-B)] \tag{8-1}$$

or

$$x = R k_0 \operatorname{arctanh} B \tag{8-2}$$
$$y = R k_0 \{\arctan [\tan \phi / \cos (\lambda - \lambda_0)] - \phi_0\} \tag{8-3}$$
$$k = k_0 / (1 - B^2)^{1/2} \tag{8-4}$$

where

$$B = \cos \phi \sin (\lambda - \lambda_0) \tag{8-5}$$

(note: If $B = \pm 1$, x is infinite)

and k_0 is the scale factor along the central meridian λ_0. The origin of the coordinates is at (ϕ_0, λ_0). The Y axis lies along the central meridian λ_0, y increasing northerly, and the X axis is perpendicular, through ϕ_0 at λ_0, x increasing easterly.

The inverse formulas for (ϕ, λ) in terms of (x, y):

$$\phi = \arcsin [\sin D / \cosh (x/Rk_0)] \tag{8-6}$$
$$\lambda = \lambda_0 + \arctan [\sinh (x/Rk_0) / \cos D] \tag{8-7}$$

where

$$D = y/(Rk_0) + \phi_0, \text{ using radians} \tag{8-8}$$

Rectangular coordinates for the sphere are shown in table 10. Only one octant (quadrant of a hemisphere) needs to be listed, since all other octants are identical except for sign change.

FORMULAS FOR THE ELLIPSOID

For the ellipsoidal form, the most practical form of the equations is a set of series approximations which converge rapidly in a zone extending 3° to 4° of longitude from the central meridian. Beyond this, the series have insufficient terms for the accuracy required. Coordinate axes are the same as they are for the spherical formulas above. The formulas below are only slightly modified from those presented in standard references (Army, 1973, p. 5–7; Thomas, 1952, p. 2–3).

$$x = k_0N[A + (1 - T + C)A^3/6 + (5 - 18T + T^2 + 72C - 58e'^2)A^5/120] \quad (8\text{–}9)$$

$$y = k_0\{M - M_0 + N \tan \phi\, [A^2/2 + (5 - T + 9C + 4C^2) A^4/24 + (61 - 58T + T^2 + 600C - 330e'^2)A^6/720]\} \quad (8\text{–}10)$$

$$k = k_0[1 + (1 + C)A^2/2 + (5 - 4T + 42C + 13C^2 - 28e'^2)\, A^4/24 + (61 - 148T + 16T^2)A^6/720] \quad (8\text{–}11)$$

where k_0 = scale on central meridian (e.g., 0.9996 for the UTM projection)

$$e'^2 = e^2/(1 - e^2) \quad (8\text{–}12)$$

$$N = a/(1 - e^2 \sin^2 \phi)^{1/2} \quad (4\text{–}20)$$

$$T = \tan^2\phi \quad (8\text{–}13)$$

$$C = e'^2 \cos^2 \phi \quad (8\text{–}14)$$

$$A = \cos \phi\ (\lambda - \lambda_0), \text{ with } \lambda \text{ and } \lambda_0 \text{ in radians} \quad (8\text{–}15)$$

$$M = a[(1 - e^2/4 - 3e^4/64 - 5e^6/256 - \ldots)\, \phi - (3e^2/8 + 3e^4/32 + 45e^6/1024 + \ldots) \sin 2\phi + (15e^4/256 + 45e^6/1024 + \ldots) \sin 4\phi - (35e^6/3072 + \ldots) \sin 6\phi + \ldots] \quad (3\text{–}21)$$

with ϕ in radians. M is the true distance along the central meridian from the Equator to ϕ. See equation (3–22) for a simplification for the Clarke 1866 ellipsoid.

$M_0 = M$ calculated for ϕ_0, the latitude crossing the central meridian λ_0, at the origin of the x, y coordinates.

Note: If $\phi = \pm\pi/2$, all equations should be omitted except (3–21), from which M and M_0 are calculated. Then $x = 0$, $y = k_0(M - M_0)$, $k = k_0$.

Equation (8–11) for k may also be written as a function of x and ϕ:

$$k = k_0[1 + (1 + e'^2 \cos^2 \phi)x^2/(2k_0^2N^2)] \quad (8\text{–}16)$$

These formulas are somewhat more precise than those used to compute the State Plane Coordinate tables, which were adapted to use desk calculators of 30–40 years ago.

For the *inverse formulas* (Army, 1973, p. 6, 7, 46; Thomas, 1952, p. 2–3):

$$\phi = \phi_1 - (N_1 \tan \phi_1/R_1)[D^2/2 - (5 + 3T_1 + 10C_1 - 4C_1^2 - 9e'^2)D^4/24 + (61 + 90T_1 + 298C_1 + 45T_1^2 - 252e'^2 - 3C_1^2)D^6/720] \quad (8\text{–}17)$$

$$\lambda = \lambda_0 + [D - (1 + 2T_1 + C_1)D^3/6 + (5 - 2C_1 + 28T_1 - 3C_1^2 + 8e'^2 + 24T_1^2)D^5/120]/\cos \phi_1 \quad (8\text{–}18)$$

where ϕ_1 is the "footpoint latitude" or the latitude at the central meridian which has the same y coordinate as that of the point (ϕ, λ).

It may be found from equation (3–26):

$$\phi_1 = \mu + (3e_1/2 - 27e_1^3/32 + \ldots) \sin 2\mu + (21e_1^2/16 - 55e_1^4 32 - \ldots) \sin 4\mu + (151e_1^3/96 + \ldots) \sin 6\mu + \ldots \quad (3\text{–}26)$$

where

$$e_1 = [1 - (1 - e^2)^{1/2}]/[1 + (1 - e^2)^{1/2}] \quad (3\text{–}24)$$

and, in a rearrangement of (3–20) and (3–21),

$$\mu = M/[a(1 - e^2/4 - 3e^4/64 - 5e^6/256 - \ldots)] \quad (8\text{–}19)$$

while

$$M = M_0 + y/k_0 \quad (8\text{–}20)$$

with M_0 calculated from equation (3–21) or (3–22) for the given ϕ_0.

From ϕ_1, other terms below are calculated for use in equations (8–17) and (8–18). (If $\phi_1 = \pm\pi/2$, (8–12), (8–21) through (8–25), (8–17) and (8–18) are omitted, but $\phi = \pm 90°$, taking the sign of y, while λ is indeterminate, and may be called λ_0. Also, $k = k_0$.)

$$e'^2 = e^2/(1 - e^2) \quad (8\text{–}12)$$
$$C_1 = e'^2 \cos^2\phi_1 \quad (8\text{–}21)$$
$$T_1 = \tan^2\phi_1 \quad (8\text{–}22)$$
$$N_1 = a/(1 - e^2 \sin^2\phi_1)^{1/2} \quad (8\text{–}23)$$
$$R_1 = a(1 - e^2)/(1 - e^2 \sin^2\phi_1)^{3/2} \quad (8\text{–}24)$$
$$D = x/(N_1 k_0) \quad (8\text{–}25)$$

"MODIFIED TRANSVERSE MERCATOR" PROJECTION

In 1972, the USGS devised a projection specifically for the revision of a 1954 map of Alaska. The projection was drawn to a scale of 1:2,000,000 and published at 1:2,500,000 (map "E") and 1:1,584,000 (map "B"). Graphically prepared by adapting coordinates for the Universal Transverse Mercator projection, it is identified as a "Modified Transverse Mercator" projection. It resembles the Transverse Mercator in a very limited manner and cannot be considered a cylindrical projection. It approximates an Equidistant Conic projection for the ellipsoid in actual construction. Because of the projection name, it is listed here. The projection was also used in 1974 for a base map of the Aleutian-Bering Sea Region published at the 1:2,500,000 scale.

The basis for the name is clear from an unpublished 1972 description of the projection, in which it is also stressed that the "latitudinal lines are parallel" and the "longitudinal lines are straight." The computations

"were taken from the AMS Technical Manual #21 (Universal Transverse Mercator) based on the Clarke 1866 Spheroid.*** The projection was started from a N–S central construction line of the 153° longitude which is also the centerline of Zone 5 from the UTM tables.

TABLE 10.—*Transverse Mercator projection: Rectangular coordinates for the sphere*

[Radius of the Earth is 1.0 unit. Longitude measured from central meridian. y coordinate is in parentheses under x coordinate. Origin of rectangular coordinates at Equator and central meridian. x increases east; y increases north. One octant of globe is given; other octants are symmetrical]

Lat. \ Long.	0°	10°	20°	30°	40°
90°	0.0000 (1.57080)	0.0000 (1.57080)	0.0000 (1.57080)	0.0000 (1.57080)	0.0000 (1.57080)
80	.00000 (1.39626)	.03016 (1.39886)	.05946 (1.40659)	.08704 (1.41926)	.11209 (1.43653)
70	.00000 (1.22173)	.05946 (1.22662)	.11752 (1.24125)	.17271 (1.26545)	.22349 (1.29888)
60	.00000 (1.04720)	.08704 (1.05380)	.17271 (1.07370)	.25541 (1.10715)	.33320 (1.15438)
50	.00000 (.87266)	.11209 (.88019)	.22349 (.90311)	.33320 (.94239)	.43943 (.99951)
40	.00000 (.69813)	.13382 (.70568)	.26826 (.72891)	.40360 (.76961)	.53923 (.83088)
30	.00000 (.52360)	.15153 (.53025)	.30535 (.55094)	.46360 (.58800)	.62800 (.64585)
20	.00000 (.34907)	.16465 (.35401)	.33320 (.36954)	.50987 (.39786)	.69946 (.44355)
10	.00000 (.17453)	.17271 (.17717)	.35051 (.18549)	.53923 (.20086)	.74644 (.22624)
0	.00000 (.00000)	.17543 (.00000)	.35638 (.00000)	.54931 (.00000)	.76291 (.00000)

TABLE 10.—*Transverse Mercator projection: Rectangular coordinates for the sphere*—Continued

Lat. \ Long.	50°	60°	70°	80°	90°
90°	0.0000 (1.57080)	0.0000 (1.57080)	0.0000 (1.57080)	0.0000 (1.57080)	0.0000 (1.57080)
80	.13382 (1.45794)	.15153 (1.48286)	.16465 (1.51056)	.17271 (1.54019)	.17543 (1.57080)
70	.26826 (1.34097)	.30535 (1.39078)	.33320 (1.44695)	.35051 (1.50768)	.35638 (1.57080)
60	.40360 (1.21544)	.46360 (1.28976)	.50987 (1.37584)	.53923 (1.47087)	.54931 (1.57080)
50	.53923 (1.07616)	.62800 (1.17355)	.69946 (1.29132)	.74644 (1.42611)	.76291 (1.57080)
40	.67281 (.91711)	.79889 (1.03341)	.90733 (1.18375)	.98310 (1.36673)	1.01068 (1.57080)
30	.79889 (.73182)	.97296 (.85707)	1.13817 (1.03599)	1.26658 (1.27864)	1.31696 (1.57080)
20	.90733 (.51522)	1.13817 (.62923)	1.38932 (.81648)	1.62549 (1.12564)	1.73542 (1.57080)
10	.98310 (.26773)	1.26658 (.33904)	1.62549 (.47601)	2.08970 (.79305)	2.43625 (1.57080)
0	1.01068 (.00000)	1.31696 (.00000)	1.73542 (.00000)	2.43625 (.00000)	Inf.

TABLE 11.—*Universal Transverse Mercator projection: Location of points with given scale factor*

[x coordinates in meters at various latitudes. Based on inversion of equation (8–16), using Clarke 1866 ellipsoid. Values are on or to right of central meridian (x = 500,000 m). For coordinates left of central meridian, subtract values of x from 1,000,000 m. Latitude is north or south]

Lat.	Scale factor					
	0.9996	**0.9998**	**1.0000**	**1.0002**	**1.0004**	**1.0006**
80°	500,000	627,946	680,943	721,609	755,892	786,096
70	500,000	627,871	680,836	721,478	755,741	785,927
60	500,000	627,755	680,673	721,278	755,510	785,668
50	500,000	627,613	680,472	721,032	755,223	785,352
40	500,000	627,463	680,260	720,772	754,925	785,015
30	500,000	627,322	680,060	720,528	754,643	784,700
20	500,000	627,207	679,898	720,329	754,414	784,443
10	500,000	627,132	679,792	720,199	754,264	784,276
0	500,000	627,106	679,755	720,154	754,212	784,218

Along this line each even degree latitude was plotted from book values. At the plotted point for the 64° latitude, a perpendicular to the construction line (153°) was plotted. From the center construction line for each degree east and west for 4° (the limits of book value of Zone #5) the curvature of latitude was plotted. From this 64° latitude, each 2° latitude north to 70° and south to 54° was constructed parallel to the 64° latitude line. Each degree of longitude was plotted on the 58° and 68° latitude line. Through corresponding degrees of longitude along these two lines of latitude a straight line (line of longitude) was constructed and projected to the limits of the map. This gave a small projection 8° in width and approximately 18° in length. This projection was repeated east and west until a projection of some 72° in width was attained."

For transferring data to and from the Alaska maps, it was necessary to determine projection formulas for computer programing. Since it appeared to be unnecessarily complicated to derive formulas based on the above construction, it was decided to test empirical formulas with actual coordinates. After careful measurements of coordinates for graticule intersections were made in 1979 on the stable-base map, it was determined that the parallels very closely approximate concentric circular arcs, spaced in proportion to their true distances on the ellipsoid, while the meridians are nearly equidistant straight lines radiating from the center of the circular arcs. Two parallels have a scale equal to that along the meridians. The Equidistant Conic projection for the ellipsoid with two standard parallels was then applied to these coordinates as the closest approximation among projections with available formulas. After various trial values for scale and standard parallels were tested, the empirical formulas below (equations (8–26) through (8–32)) were obtained. These agree with measured values within 0.005 inch at mapping scale for 44 out of 58 measurements made on the map and within 0.01 inch for 54 of them.

FORMULAS FOR THE "MODIFIED TRANSVERSE MERCATOR" PROJECTION

The "Modified Transverse Mercator" projection was found to be most closely equivalent to an Equidistant Conic projection for the Clarke 1866 ellipsoid, with the scale along the meridians reduced to 0.9992 of true scale and the standard parallels at lat. 66.09° and 53.50° N. (also at 0.9992 scale factor). For the Alaska Map "E" at 1:2,500,000, using long. 150° W. as the central meridian and lat. 58° N. as the latitude of the origin on the central meridian, the general formulas (Snyder, 1978a, p. 378) reduce with the above parameters to the following, giving x and y in meters at the map scale. The Y axis lies along the central meridian, y increasing northerly, and the X axis is perpendicular at the origin, x increasing easterly.

For the forward formulas:

$$x = \rho \sin \theta \tag{8-26}$$
$$y = 1.5616640 - \rho \cos \theta \tag{8-27}$$

where

$$\theta^\circ = 0.8625111(\lambda^\circ + 150^\circ) \tag{8-28}$$
$$\rho = 4.1320402 - 0.04441727\phi^\circ + 0.0064816 \sin 2\phi \tag{8-29}$$

For the inverse formulas:

$$\lambda^\circ = (1/0.8625111) \arctan [x/(1.5616640 - y)] - 150^\circ \tag{8-30}$$
$$\phi^\circ = (4.1320402 + 0.0064816 \sin 2\phi - \rho)/0.04441727 \tag{8-31}$$

where

$$\rho = [x^2 + (1.5616640 - y)^2]^{1/2} \tag{8-32}$$

For Alaska Map "B" at a scale of 1:1,584,000, the same formulas may be used, except that x and y are (2,500/1,584) times the values obtained from (8–26) and (8–27). For the inverse formulas, the given x and y must be divided by (2,500/1,584) before insertion into (8–30) and (8–32).

The equation for ϕ, (8–31), involves iteration by successive substitution. If an initial ϕ of 60° is inserted into the right side, ϕ on the left may be calculated and substituted into the right in place of the previous trial ϕ. Recalculations continue until the change in ϕ is less than a preset convergence. If λ as calculated is less than −180°, it should be added to 360° and labeled East Longitude.

Formulas to adjust x and y for the map inset of the Aleutian Islands are omitted here, but the coordinates above are rotated counterclockwise 29.79° and transposed +0.798982 m for x and +0,347600 m for y.

9. OBLIQUE MERCATOR PROJECTION

SUMMARY

- Cylindrical (oblique).
- Conformal.
- Two meridians 180° apart are straight lines.
- Other meridians and parallels are complex curves.
- Scale on the spherical form is true along chosen central line, a great circle at an oblique angle, or along two straight lines parallel to central line. The scale on the ellipsoidal form is similar, but varies slightly from this pattern.
- Scale becomes infinite 90° from the central line.
- Used for grids on maps of the Alaska panhandle, for mapping in Switzerland, Madagascar, and Borneo and for atlas maps of areas with greater extent in an oblique direction
- Developed 1900–50 by Rosenmund, Laborde, Hotine, and others.

HISTORY

There are several geographical regions such as the Alaska panhandle centered along lines which are neither meridians nor parallels, but which may be taken as great circle routes passing through the region. If conformality is desired in such cases, the Oblique Mercator is a projection which should be considered.

The historical origin of the Oblique Mercator projection does not appear to be sharply defined, although it is a logical generalization of the regular and Transverse Mercator projections. Apparently, Rosenmund (1903) made the earliest published reference, when he devised an ellipsoidal form which is used for topographic mapping of Switzerland. The projection was not mentioned in the detailed article on "Map Projections" in the 1911 *Encyclopaedia Britannica* (Close and Clarke, 1911) or in Hinks' brief text (1912). Laborde applied the Oblique Mercator to the ellipsoid for the topographic mapping of Madagascar in 1928 (Young, 1930; Laborde, 1928). H. J. Andrews (1935, 1938) proposed the spherical forms for maps of the United States and Eurasia. Hinks presented seven world maps on the Oblique Mercator, with poles located in several different positions, and a consequent variety in the regions shown more satisfactorily (Hinks, 1940, 1941).

A study of conformal projections of the ellipsoid by British geodesist Martin Hotine (1898–1968), published in 1946–47, is the basis of the U.S. use of the ellipsoidal Oblique Mercator, which Hotine called the "rectified skew orthomorphic" (Hotine, 1947, p. 66–67). The Hotine approach has limitations, as discussed below, but it provides closed formulas which have been adapted for U.S. mapping of suitable zones.

One of its limitations is overcome by a recent series form of the ellipsoidal Oblique Mercator (Snyder, 1979a, p. 74), but other limitations result instead. This later form resulted from development of formulas for the continuous mapping of satellite images, using the Space Oblique Mercator projection (to be discussed later).

While Hotine projected the ellipsoid conformally onto an "aposphere" of constant total curvature and thence to a plane, Laborde and also J. H. Cole (1943, p. 16–30) projected the ellipsoid onto a "conformal sphere," using conformal latitudes (described earlier) to make the sphere conformal with respect to the ellipsoid, then plotted the spherical Oblique Mercator from this intermediate sphere. Rosenmund's system for Switzerland is a more complex double projection through a conformal sphere (Rosenmund, 1903; Bolliger, 1967).

FEATURES

The Oblique Mercator for the sphere is equivalent to a regular Mercator projection which has been altered by wrapping a cylinder around the sphere so that it touches the surface along the great circle path chosen for the central line, instead of along the Earth's Equator. A set of transformed meridians and parallels relative to the great circle may be plotted bearing the same relationship to the rectangular coordinates for the Oblique Mercator projection, as the geographic meridians and parallels bear to the regular Mercator. It is, therefore, possible to convert the geographic meridians and parallels to the transformed values and then to use the regular Mercator equations, substituting the transformed values in place of the geographic values. This is the procedure for the sphere, although combined formulas are given below, but it becomes much more complicated for the ellipsoid. The advent of present-day computers and programmable pocket calculators make these calculations feasible for sphere or ellipsoid.

The resulting Oblique Mercator map of the world (fig. 12) thus resembles the regular Mercator with the landmasses rotated so that the poles and Equator are no longer in their usual positions. Instead, two points 90° away from the chosen great circle path through the center of the map are at infinite distance off the map. Normally, the Oblique Mercator is used only to show the region near the central line and for a relatively short portion of the central line. Under these conditions, it looks similar to maps of the same area using other projections, except that careful scale measurements will show differences.

It should be remembered that the regular Mercator is in fact a limiting form of the Oblique Mercator with the Equator as the central line, while the Transverse Mercator is another limiting form of the Oblique with a meridian as the central line. As with these limiting

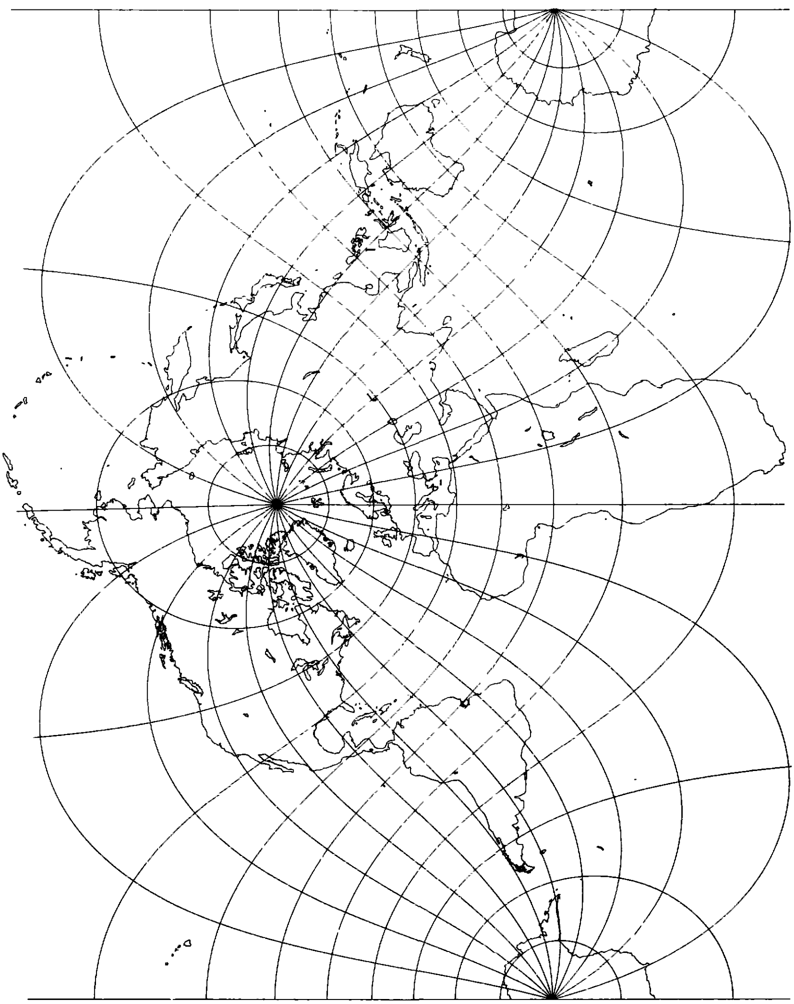


FIGURE 12.—Oblique Mercator projection with the center of projection at lat. 45° N. on the central meridian. A straight line through the point and, in this example, perpendicular to the central meridian is true to scale. The projection is conformal and has been used for regions lying along a line oblique to meridians.

forms, the scale along the central line of the Oblique Mercator may be reduced to balance the scale throughout the map.

USAGE

The Oblique Mercator projection is used in the spherical form for a few atlas maps. For example, the National Geographic Society uses it for atlas and sheet maps of Hawaii, the West Indies, and New Zealand. In the ellipsoidal form it was used, as mentioned above, by Rosenmund for Switzerland and Laborde for Madagascar. Hotine used it for Malaya and Borneo and Cole for Italy. It is used in the Hotine form by the USGS for grid marks on zone 1 (the panhandle) of Alaska, using the State Plane Coordinate System as adapted to this projection by Erwin Schmid of the former Coast and Geodetic Survey.

More recently, the Hotine form was adapted by John B. Rowland (USGS) for mapping Landsat satellite imagery in two sets of five discontinuous zones from north to south (table 12). The central line of the latter is only a close approximation to the satellite groundtrack, which does not follow a great circle route on the Earth; instead, it follows a path of constantly changing curvature. Until the mathematical implementation of the Space Oblique Mercator (SOM) projection, the Hotine Oblique Mercator (HOM) was probably the most suitable projection available for mapping Landsat type data. In addition to Landsat, the HOM projection has been used to cast Heat Capacity Mapping Mission (HCMM) imagery since 1978. NOAA (National Oceanic and Atmospheric Administration) has also cast some weather satellite imagery on the HOM to make it compatible with Landsat in the polar regions which are beyond Landsat coverage (above lat. 82°).

The parameters for a given map according to the Oblique Mercator projection may be selected in various ways. If the projection is to be used for the map of a smaller region, two points located near the limits of the region may be selected to lie upon the central line, and various constants may be calculated from the latitude and longitude of each of the two points. A second approach is to choose a central point for the map and an azimuth for the central line, which is made to pass through the central point. A third approach, more applicable to the map of a large portion of the Earth's surface, treated as spherical, is to choose a location on the original sphere of the pole for a transformed sphere with the central line as the equator. Formulas are given for each of these approaches, for sphere and ellipsoid.

FORMULAS FOR THE SPHERE

Starting with the forward equations, for rectangular coordinates in terms of latitude and longitude:

TABLE 12.—*Hotine Oblique Mercator projection parameters used for Landsat 1, 2, and 3 imagery*

HOM zone	Limiting latitudes	Central latitude	Central longitude[1]	Azimuth of axis
1	48°N–81°N	67.0983°N	81.9700°W	24.7708181°
2	23°N–48°N	36.0000°N	99.2750°W	14.3394883°
3	23°S–23°N	0.0003°N	108.5069°W	13.001443°
4	23°S–48°S	36.0000°S	117.7388°W	14.33948832°
5	48°S–81°S	67.0983°S	135.0438°W	24.7708181°
6	48°S–81°S	67.0983°S	85.1220°E	−24.7708181°
7	23°S–48°S	36.0000°S	67.8170°E	−14.33948832°
8	23°S–23°N	0.0003°N	58.5851°E	−13.001443°
9	23°N–48°N	36.0000°N	49.3532°E	−14.33948832°
10	48°N–81°N	67.0983°N	32.0482°E	−24.7708181°

[1]For path 31. For other path numbers p, the central longitude is decreased (west is negative) by $(360°/251)\times(p-31)$.
Note: These parameters are used with equations given under Alternate B of ellipsoidal Oblique Mercator formulas, with ϕ_0 the central latitude, λ_c the central longitude, and α_c the azimuth of axis east of north. Scale factor k_0 at center is 1.0.

1. Given two points to lie upon the central line, with latitudes and longitudes (ϕ_1,λ_1) and (ϕ_2,λ_2) and longitude increasing easterly and relative to Greenwich. The pole of the oblique transformation at (ϕ_p,λ_p) may be calculated as follows:

$$\lambda_p = \arctan [(\cos \phi_1 \sin \phi_2 \cos \lambda_1 - \sin \phi_1 \cos \phi_2 \cos \lambda_2)/ (\sin \phi_1 \cos \phi_2 \sin \lambda_2 - \cos \phi_1 \sin \phi_2 \sin \lambda_1)] \quad (9\text{–}1)$$

$$\phi_p = \arctan [-\cos (\lambda_p - \lambda_1)/\tan \phi_1] \quad (9\text{–}2)$$

The Fortran ATAN2 function or its equivalent should be used with equation (9–1), but not with (9–2). The other pole is located at $(-\phi_p,\lambda_p \pm \pi)$. Using the positive (northern) value of ϕ_p, the following formulas give the rectangular coordinates for point (ϕ,λ), with k_0 the scale factor along the central line:

$$x = Rk_0 \arctan\{[\tan \phi \cos \phi_p + \sin \phi_p \sin (\lambda - \lambda_0)]/\cos (\lambda - \lambda_0)\} \quad (9\text{–}3)$$

$$y = \tfrac{1}{2}Rk_0 \ln[(1+A)/(1-A)] \quad (9\text{–}4)$$

or

$$y = Rk_0 \operatorname{arctanh} A \quad (9\text{–}4a)$$

$$k = k_0/(1-A^2)^{1/2} \quad (9\text{–}5)$$

where

$$A = \sin \phi_p \sin \phi - \cos \phi_p \cos \phi \sin (\lambda - \lambda_0) \quad (9\text{–}6)$$

With these formulas, the origin of rectangular coordinates lies at

$$\phi_0 = 0$$

$$\lambda_0 = \lambda_p + \pi/2 \quad (9\text{–}6a)$$

and the X axis lies along the central line, x increasing easterly. The transformed poles are y equals infinity.
2. Given a central point (ϕ_c, λ_c) with longitude increasing easterly and relative to Greenwich, and azimuth β east of north of the central line through (ϕ_c,λ_c), the pole of the oblique transformation at (ϕ_p, λ_p) may be calculated as follows:

$$\phi_p = \arcsin(\cos\phi_c \sin\beta) \tag{9-7}$$
$$\lambda_p = \arctan[-\cos\beta/(-\sin\phi_c \sin\beta)] + \lambda_c \tag{9-8}$$

These values of ϕ_p and λ_p may then be used in equations (9–3) through (9–6) as before.
3. For an extensive map, ϕ_p and λ_p may be arbitrarily chosen by eye to give the pole for a central line passing through a desired portion of the globe. These values may then be directly used in equations (9–3) through (9–6) without intermediate calculation.

For the inverse formulas, equations (9–1) and (9–2) or (9–7) and (9–8) must first be used to establish the pole of the oblique transformation, if it is not known already. Then,

$$\phi = \arcsin[\sin\phi_p \tanh(y/Rk_0) + \cos\phi_p \sin(x/Rk_0)/\cosh(y/Rk_0)] \tag{9-9}$$
$$\lambda = \lambda_0 + \arctan\{[\sin\phi_p \sin(x/Rk_0) - \cos\phi_p \sinh(y/Rk_0)]/\cos(x/Rk_0)\} \tag{9-10}$$

FORMULAS FOR THE ELLIPSOID

These are the formulas provided by Hotine, slightly altered to use a positive eastern longitude (he used positive western longitude), to simplify calculations of hyperbolic functions, and to use symbols consistent with those of this bulletin. The central line is a geodesic, or the shortest route on an ellipsoid, corresponding to a great circle route on the sphere.

It is customary to provide rectangular coordinates for the Hotine in terms either of (u,v) or (x,y). The (u,v) coordinates are similar in concept to the (x,y) calculated for the foregoing spherical formulas, with u corresponding to x for the spherical formulas, increasing easterly from the origin along the central line, but v corresponds to $-y$ for the spherical formulas, so that v increases southerly in a direction perpendicular to the central line. For the Hotine, x and y are calculated from (u,v) as "rectified" coordinates with the Y axis following the meridian passing through the center point, and increasing northerly as usual, while the X axis lies east and west through the same point. The X and Y axes thus lie in directions like those of the Transverse Mercator, but the scale-factor relationships remain those of the Oblique Mercator.

The normal origin for (u,v) coordinates in the Hotine Oblique Mercator is approximately at the intersection of the central line with the Earth's Equator. Actually it occurs at the crossing of the central line with the equator of the "aposphere," and is, thus, a rather academic location. The "aposphere" is a surface with a constant "total" curvature based on the curvature along the meridian and perpendicular thereto on the ellipsoid at the chosen central point for the projection. The ellipsoid is conformally projected onto this aposphere, then to a sphere, then to a plane. As a result, the Hotine is perfectly conformal, but the scale along the central line is true only at the chosen central point along that line or along a relatively flat elliptically shaped line approximately centered on that point, if the scale of the central point is arbitrarily reduced to balance scale over the map. The variation in scale along the central line is extremely small for a map extending less than 45° in arc, which includes most existing usage of the Hotine. A longer central line suggests the use of a different set of formulas, available as a limiting form of the Space Oblique Mercator projection. On Rosenmund's (1903), Laborde's (1928), and Cole's (1943) versions of the ellipsoidal Oblique Mercator, the central line is a great circle arc on the intermediate conformal sphere, not a geodesic. As on Hotine's version, this central line is not quite true to scale except at one or two chosen points.

The projection constants may be established for the Hotine in one of two ways, as they were for the spherical form. Two desired points, widely separated on the map, may be made to fall on the central line of the projection, or the central line may be given a desired azimuth through a selected central point. Taking these approaches in order:

Alternate A, with the central line passing through two given points. Given:

a and e for the reference ellipsoid.

k_0 = scale factor at the selected center of the map, lying on the central line.

ϕ_0 = latitude of selected center of the map.

(ϕ_1, λ_1) = latitude and longitude (east of Greenwich is positive) of the first point which is to lie on the central line.

(ϕ_2, λ_2) = latitude and longitude of the second point which is to lie on the central line.

(ϕ, λ) = latitude and longitude of the point for which the coordinates are desired.

There are limitations to the use of variables in these formulas: To avoid indeterminates and division by zero, ϕ_0 or ϕ_1 cannot be $\pm\pi/2$, ϕ_1 cannot be zero or equal to ϕ_2 (although ϕ_2 may be zero), and ϕ_2 cannot be $-\pi/2$. Neither ϕ_0, ϕ_1, nor ϕ_2 should be $\pm\pi/2$ in any case, since this would cause the central line to pass through the pole, for which the Transverse Mercator or polar Stereographic would probably be a more suitable choice. A change of 10^{-7} radian in variables from these special values will permit calculation of an otherwise unsatisfactory condition.

It is also necessary to place both (ϕ_1, λ_1) and (ϕ_2, λ_2) on the ascending portion, or both on the descending portion, of the central line, relative to the Earth's Equator. That is, the central line should not pass through a maximum or minimum between these two points.

If e is zero, the Hotine formulas give coordinates for the spherical Oblique Mercator.

Because of the involved nature of the Hotine formulas, they are given here in an order suitable for calculation, and in a form eliminating the use of hyperbolic functions as given by Hotine in favor of single calculations of exponential functions to save computer time. The corresponding Hotine equations are given later for comparison.

$$B=[1+e^2 \cos^4 \phi_0/(1-e^2)]^{1/2} \quad (9\text{–}11)$$

$$A=aBk_0(1-e^2)^{1/2}/(1-e^2 \sin^2 \phi_0) \quad (9\text{–}12)$$

$$t_0=\tan (\pi/4-\phi_0/2)/[(1-e \sin \phi_0)/(1+e \sin \phi_0)]^{e/2} \quad (9\text{–}13)$$

t_1 = same as (9–13), using ϕ_1 in place of ϕ_0.

t_2 = same as (9–13), using ϕ_2 in place of ϕ_0.

$$D=B(1-e^2)^{1/2}/[\cos \phi_0(1-e^2 \sin^2\phi_0)^{1/2}] \quad (9\text{–}14)$$

If $\phi_0=0$, D may calculate to slightly less than 1.0 and create a problem in the next step. If $D^2<1$, it should be made 1.

$$E=[D\pm(D^2-1)^{1/2}]t_0^B, \text{ taking the sign of } \phi_0 \quad (9\text{–}15)$$

$$H=t_1^B \quad (9\text{–}16)$$

$$L=t_2^B \quad (9\text{–}17)$$

$$F=E/H \quad (9\text{–}18)$$

$$G=(F-1/F)/2 \quad (9\text{–}19)$$

$$J=(E^2-LH)/(E^2+LH) \quad (9\text{–}20)$$

$$P=(L-H)/(L+H) \quad (9\text{–}21)$$

$$\lambda_0=½(\lambda_1+\lambda_2)-\arctan \{J \tan [B(\lambda_1-\lambda_2)/2]/P\}/B \quad (9\text{–}22)$$

$$\gamma_0=\arctan \{\sin [B(\lambda_1-\lambda_0)]/G\} \quad (9\text{–}23)$$

$$\alpha_c=\arcsin [D \sin \gamma_0] \quad (9\text{–}24)$$

To prevent problems when straddling the 180th meridian with λ_1 and λ_2, before calculating (9–22), if $(\lambda_1-\lambda_2)< -180°$, subtract 360° from λ_2. If $(\lambda_1-\lambda_2)>180°$, add 360° to λ_2. Also adjust λ_0 and $(\lambda_1-\lambda_0)$ to fall between $\pm 180°$ by adding or subtracting 360°. The Fortran ATAN2 function is not to be used for equations (9–22) and (9–23). The above equations (9–11) through (9–24) provide constants for a given map and do not involve a specific point (ϕ,λ). Angle α_c is the azimuth of the central line as it crosses latitude ϕ_0, measured east of north. For point (ϕ, λ), calculate the following:

t = same as equation (9–13), but using ϕ in place of ϕ_0.
If $\phi=\pm\pi/2$, do not calculate t, but go instead to (9–30).

$$Q=E/t^B \quad (9\text{–}25)$$

$$S=(Q-1/Q)/2 \quad (9\text{–}26)$$

$$T=(Q+1/Q)/2 \quad (9\text{–}27)$$

$$V=\sin [B(\lambda-\lambda_0)] \quad (9\text{–}28)$$

$$U=(-V\cos\gamma_0+S\sin\gamma_0)/T \qquad (9\text{–}29)$$
$$v=A\ln[(1-U)/(1+U)]/2B \qquad (9\text{–}30)$$

Note: If $U=\pm 1$, v is infinite; if $\phi=\pm\pi/2$, $v=(A/B)\ln\tan(\pi/4\mp\gamma_0/2)$

$$u=A\arctan\{(S\cos\gamma_0+V\sin\gamma_0)/\cos[B(\lambda-\lambda_0)]\}/B \qquad (9\text{–}31)$$

Note: If $\cos[B(\lambda-\lambda_0)]=0$, $u=AB(\lambda-\lambda_0)$. If $\phi=\pm\pi/2$, $u=A\phi/B$.

Care should be taken that $(\lambda-\lambda_0)$ has 360° added or subtracted, if the 180th meridian falls between, since multiplication by B eliminates automatic correction with the sin or cos function.

The scale factor:

$$k=A\cos(Bu/A)(1-e^2\sin^2\phi)^{1/2}/\{a\cos\phi\cos[B(\lambda-\lambda_0)]\} \qquad (9\text{–}32)$$

If "rectified" coordinates (x, y) are desired, with the origin at a distance (x_0,y_0) from the origin of the (u,v) coordinates, relative to the (X,Y) axes (see fig. 13):

$$x=v\cos\alpha_c+u\sin\alpha_c+x_0 \qquad (9\text{–}33)$$
$$y=u\cos\alpha_c-v\sin\alpha_c+y_0 \qquad (9\text{–}34)$$

The formulas given by Hotine and essentially repeated in Thomas (1952, p. 7–9), modified for positive east longitude, u and v increasing in the directions shown in figure 13, and symbols consistent with the above, relate to the foregoing formulas as follows:[3]

Equivalent to (9–11):

$$e'^2=e^2/(1-e^2)$$
$$B=(1+e'^2\cos^4\phi_0)^{1/2}$$

Equivalent to (9–12):

$$R'_0=a(1-e^2)/(1-e^2\sin^2\phi_0)^{3/2}$$
$$N_0=a/(1-e^2\sin^2\phi_0)^{1/2}$$
$$A=Bk_0(R'_0N_0)^{1/2}$$

Other formulas:

$$r_0=N_0\cos\phi_0$$
$$\psi_n=\ln\{\tan(\pi/4+\phi_n/2)[(1-e\sin\phi_n)/(1+e\sin\phi_n)]^{e/2}\}$$

Note: ψ_n is equivalent to $(-\ln t_n)$ using equation (9–13).

$$C=\pm\operatorname{arccosh}(A/r_0)-B\psi_0$$

Note: C is equivalent to $\ln E$, where E is found from equation (9–15); D, from (9–14), is (A/r_0).

$$\tan[\tfrac{1}{2}B(\lambda_1+\lambda_2)-B\lambda_0]=\frac{\tan[\tfrac{1}{2}B(\lambda_1-\lambda_2)]\tanh[\tfrac{1}{2}B(\psi_1+\psi_2)+C]}{\tanh[\tfrac{1}{2}B(\psi_1-\psi_2)]}$$

[3] Hotine uses positive west longitude, x corresponding to u here, and y corresponding to $-v$ here. Thomas uses positive west longitude, y corresponding to u here, and x corresponding to $-v$ here. In calculations of Alaska zone 1, west longitude is positive, but u and v agree with u and v, respectively, here.

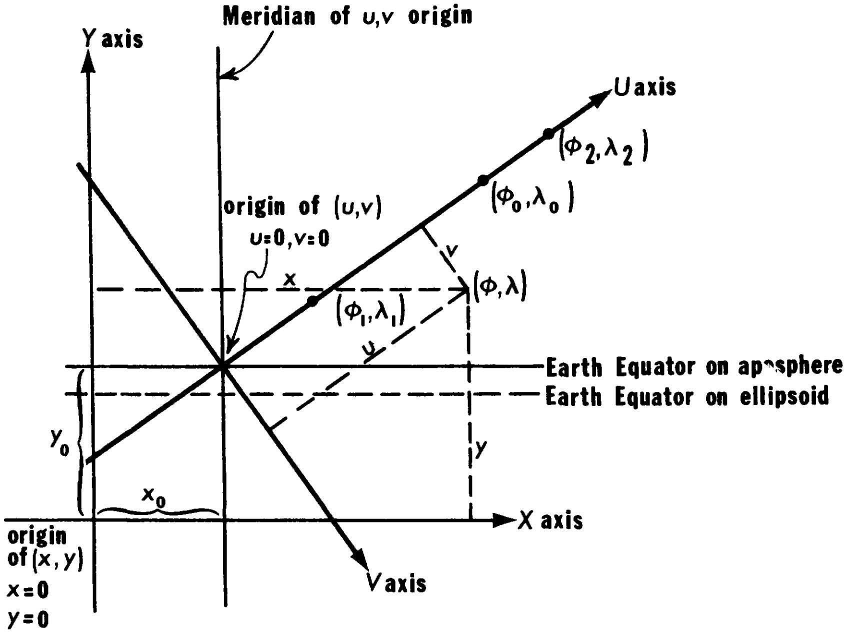


FIGURE 13.—Coordinate system for the Hotine Oblique Mercator projection.

The tanh in the numerator is J from equation (9–20), while the tanh in the denominator is P from (9–21). The entire equation is equivalent to (9–22).

$$\tan \gamma_0 = \sin [B(\lambda_1 - \lambda_0)]/\sinh (B\psi_1 + C)$$

This equation is equivalent to (9–23), the sinh being equivalent to G from (9–19).

$$\tanh (Bv/Ak_0) = \{\cos \gamma_0 \sin [B(\lambda - \lambda_0)] - \sin \gamma_0 \sinh (B\psi + C)\}/\cosh (B\psi + C)$$

This equation is equivalent to (9–30), with S the sinh function and T the cosh function.

$$\tan (Bu/Ak_0) = \{\cos\gamma_0 \sinh (B_\psi + C) + \sin \gamma_0 \sin [B(\lambda - \lambda_0)]\}/\cos [B(\lambda - \lambda_0)]$$

This equation is equivalent to (9–31).

Alternate B. The following equations provide constants for the Hotine Oblique Mercator projection to fit a given central point and azimuth of the central line through the central point. Given: a, e, k_0, ϕ_0, and (ϕ, λ) as for alternate A, but instead of (ϕ_1, λ_1) and (ϕ_2, λ_2), λ_c and α_c are given,

where

(ϕ_0, λ_c) = latitude and longitude (east of Greenwich is positive), respectively, of the selected center of the map, falling on the central line.

α_c = angle of azimuth east of north, for the central line as it passes through the center of the map (ϕ_0, λ_c).

Limitations: ϕ_0 cannot be zero or $\pm\pi/2$, and the central line cannot be at a maximum or minimum latitude at ϕ_0. If $e=0$, these formulas also give coordinates for the spherical Oblique Mercator. As with alternate A, these formulas are given in the order of calculation and are modified to minimize exponential computations. Several of these equations are the same as some of the equations for alternate A:

$$B=[1+e^2\cos^4\phi_0/(1-e^2)]^{1/2} \quad (9\text{–}11)$$

$$A=aBk_0(1-e^2)^{1/2}/(1-e\sin^2\phi_0) \quad (9\text{–}12)$$

$$t_0=\tan(\pi/4-\phi_0/2)/[(1-e\sin\phi_0)/(1+e\sin\phi_0)]^{e/2} \quad (9\text{–}13)$$

$$D=B(1-e^2)^{1/2}/[\cos\phi_0(1-e^2\sin^2\phi_0)^{1/2}] \quad (9\text{–}14)$$

If $\phi_0=0$, D may calculate to slightly less than 1.0 and create a problem in the next step. If $D^2<1$, it should be made 1.

$$F=D\pm(D^2-1)^{1/2},\text{ taking the sign of }\phi_0 \quad (9\text{–}35)$$

$$E=Ft_0^B \quad (9\text{–}36)$$

$$G=(F-1/F)/2 \quad (9\text{–}19)$$

$$\gamma_0=\arcsin(\sin\alpha_c/D) \quad (9\text{–}37)$$

$$\lambda_0=\lambda_c-[\arcsin(G\tan\gamma_0)]/B \quad (9\text{–}38)$$

The values of u and v for center point (ϕ_0, λ_c) may be calculated directly at this point:

$$u_{(\phi_0,\lambda_c)}=\pm(A/B)\arctan[(D^2-1)^{1/2}/\cos\alpha_c],\text{ taking the sign of }\phi_0. \quad (9\text{–}39)$$

$$v_{(\phi_0,\lambda_c)}=0$$

These are the constants for a given map. Equations (9–25) through (9–32) for alternate A may now be used in order, following calculation of the above constants.

The inverse equations for the Hotine Oblique Mercator projection on the ellipsoid may be shown with few additional formulas. To determine ϕ and λ from x and y, or from u and v, the same parameters of the map must be given, except for ϕ and λ, and the constants of the map are found from the above equations (9–11) through (9–24) for alternate A or (9–11) through (9–38) for alternate B. Then, if x and y are given in accordance with the definitions for the forward equations, they must first be converted to (u, v):

$$v=(x-x_0)\cos\alpha_c-(y-y_0)\sin\alpha_c \qquad (9\text{–}40)$$
$$u=(y-y_0)\cos\alpha_c+(x-x_0)\sin\alpha_c \qquad (9\text{–}41)$$

If (u, v) are given, or calculated as just above, the following steps are performed in order:

$$Q'=\mathrm{e}^{-(Bv/A)} \qquad (9\text{–}42)$$

where e = 2.71828. . . , the base of natural logarithms

$$S'=(Q'-1/Q')/2 \qquad (9\text{–}43)$$
$$T'=(Q'+1/Q')/2 \qquad (9\text{–}44)$$
$$V'=\sin(Bu/A) \qquad (9\text{–}45)$$
$$U'=(V'\cos\gamma_0+S'\sin\gamma_0)/T' \qquad (9\text{–}46)$$
$$t=\{E/[(1+U')/(1-U')]^{1/2}\}^{1/B} \qquad (9\text{–}47)$$

But if $U'=\pm 1$, $\phi=\pm 90°$, taking the sign of U', λ may be called λ_0, and equations (7–9) and (9–48) below are omitted.

$$\phi=\pi/2-2\arctan\{t[(1-e\sin\phi)/(1+e\sin\phi)]^{e/2}\} \qquad (7\text{–}9)$$

Equation (7–9) is solved by iteration, using $\phi=(\pi/2-2\arctan t)$ as the first trial ϕ on the right side, and using the successive calculations of ϕ on the left side as successive values of ϕ on the right side, until the change in ϕ is less than a chosen convergence value.

$$\lambda=\lambda_0-\arctan[(S'\cos\gamma_0-V'\sin\gamma_0)/\cos(Bu/A)]/B \qquad (9\text{–}48)$$

Since the arctan (found as the ATAN2 function) is divided by B, it is necessary to add or subtract 360° properly, before the division.

To avoid the iteration, the series (3–5) may be used with (7–13) in place of (7–9):

$$\phi=\chi+(e^2/2+5e^4/24+e^6/12+\ldots)\sin 2\chi+(7e^4/48+29e^6/240+\ldots)\sin 4\chi+(7e^6/120+\ldots)\sin 6\chi+\ldots \qquad (3\text{–}5)$$

where

$$\chi=\pi/2-2\arctan t \qquad (7\text{–}13)$$

The equivalent inverse equations as given by Hotine are as follows, following the calculation of constants using the same formulas as those given in his forward equations:

$$\tan[B(\lambda-\lambda_0)]=[\sin\gamma_0\sin(Bu/A)+\cos\gamma_0\sinh(Bv/A)]/\cos(Bu/A)$$
$$\tanh(B\psi+C)=[\cos\gamma_0\sin(Bu/A)-\sin\gamma_0\sinh(Bv/A)]/\cosh(Bv/A)$$

10. MILLER CYLINDRICAL PROJECTION

SUMMARY

- Neither equal-area nor conformal.
- Used only in spherical form.
- Cylindrical.
- Meridians and parallels are straight lines, intersecting at right angles.
- Meridians are equidistant; parallels spaced farther apart away from Equator.
- Poles shown as lines.
- Compromise between Mercator and other cylindrical projections.
- Used for world maps.
- Presented by Miller in 1942.

HISTORY AND FEATURES

The need for a world map which avoids some of the scale exaggeration of the Mercator projection has led to some commonly used cylindrical modifications, as well as to other modifications which are not cylindrical. The earliest common cylindrical example was developed by Rev. James Gall of Edinburgh about 1855 (Gall, 1885, p. 119–123). His meridians are equally spaced, but the parallels are spaced at increasing intervals away from the Equator. The parallels of latitude are actually projected onto a cylinder wrapped about the sphere, but cutting it at lats. 45° N. and S., the point of perspective being a point on the Equator opposite the meridian being projected. It is used in several British atlases, but seldom in the United States. Gall's projection is neither conformal nor equal-area, but has a blend of various features. Unlike the Mercator, Gall's shows the poles as lines running across the top and bottom of the map.

What might be called the American version of Gall's projection is the Miller Cylindrical projection (fig. 14), presented in 1942 by Osborn Maitland Miller (1897–1979) of the American Geographical Society, New York (Miller, 1942). Born in Perth, Scotland, and educated in Scotland and England, Miller came to the Society in 1922. There he developed several improved surveying and mapping techniques. An expert in aerial photography, he developed techniques for converting high-altitude photographs into maps. He led or joined several expeditions of explorers and advised leaders of others. He retired in 1968, having been best known to cartographers for several map projections, including the Bipolar Oblique Conic Conformal, the Prolated Stereographic, and especially his Cylindrical projection.

Miller had been asked by S. Whittemore Boggs, Geographer of the U.S. Department of State, to study further alternatives to the

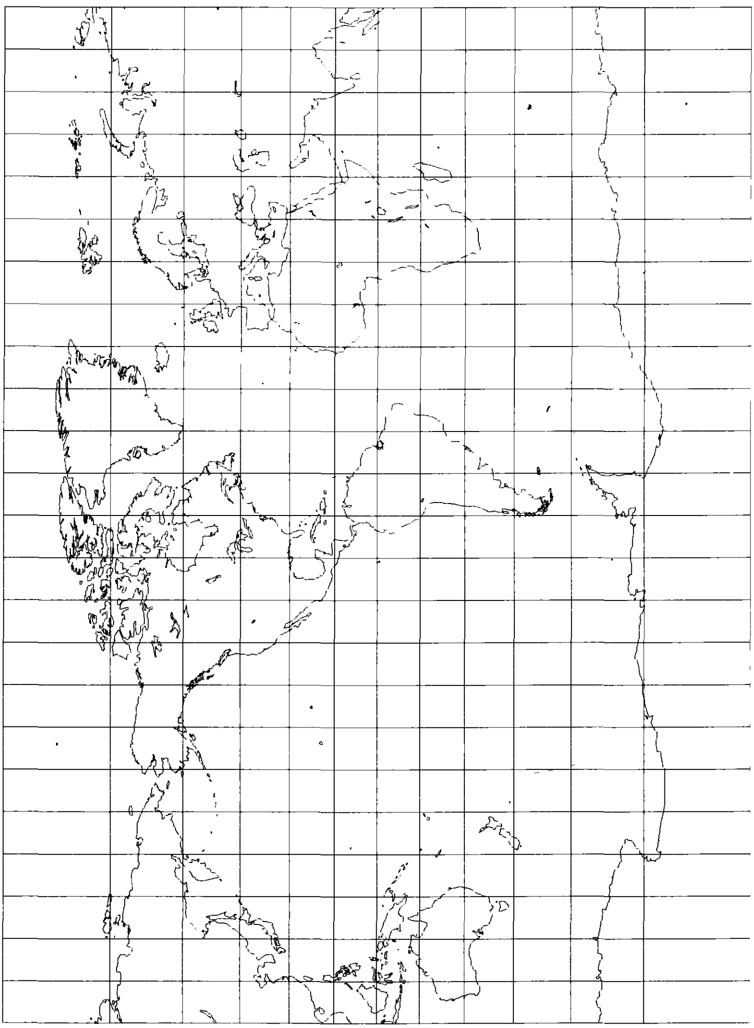


FIGURE 14.—The Miller Cylindrical projection. A projection resembling the Mercator, but having less relative area distortion in polar regions. Neither conformal nor equal-area.

Mercator, Gall's, and other cylindrical world maps. In his presentation, Miller listed four proposals, but the one he preferred, and the one used, is a fairly simple mathematical modification of the Mercator projection. Like Gall's, it shows visible straight lines for the poles, increasingly spaced parallels away from the Equator, equidistant meridians, and is not equal-area, equidistant along meridians, nor conformal. While the standard parallels, or lines true to scale and free of distortion, on Gall's are at lats. 45° N. and S., on the Miller only the Equator is standard. Unlike Gall's, Miller's is not a perspective projection.

The Miller Cylindrical projection is used for world maps and in several atlases, including the *National Atlas of the United States* (USGS, 1970, p. 330–331).

As Miller (1942) stated, "the practical problem considered here is to find a system of spacing the parallels of latitude such that an acceptable balance is reached between shape and area distortion. By an 'acceptable' balance is meant one which to the uncritical eye does not obviously depart from the familiar shapes of the land areas as depicted by the Mercator projection but which reduces areal distortion as far as possible under these conditions * * *. After some experimenting, the [Modified Mercator (b)] was judged to be the most suitable for Mr. Boggs's purpose * * *."

FORMULAS FOR THE SPHERE

Miller's spacings of parallels from the Equator are the same as if the Mercator spacings were calculated for 0.8 times the respective latitudes, with the result divided by 0.8. As a result, the spacing of parallels near the Equator is very close to the Mercator arrangement.

The forward formulas, then, are as follows:

$$x = R(\lambda - \lambda_0) \tag{10-1}$$

$$y = R[\ln \tan(\pi/4 + 0.4\phi)]/0.8 \tag{10-2}$$

or

$$y = R[\operatorname{arctanh}(\sin 0.8\phi)]/0.8 \tag{10-2a}$$

The scale factor, using equations (4–2) and (4–3),

$$h = \sec 0.8\phi \tag{10-3}$$

$$k = \sec \phi \tag{10-4}$$

The maximum angular deformation ω, from equation (4–9),

$$\sin \tfrac{1}{2}\omega = (\cos 0.8\phi - \cos \phi)/(\cos 0.8\phi + \cos \phi) \tag{10-5}$$

The X axis lies along the Equator, x increasing easterly. The Y axis lies along the central meridian λ_0, y increasing northerly. If $(\lambda - \lambda_0)$ lies outside the range of ±180°, 360° should be added or subtracted so that it will fall inside the range. The inverse equations are easily derived from equations (10–1) through (10–2a):

$$\phi = 2.5 \arctan e^{(0.8y/R)} - 5\pi/8 \qquad (10\text{–}6)$$

or

$$\phi = \arcsin [\tanh (0.8y/R)]/0.8 \qquad (10\text{–}6a)$$

where e is 2.71828 . . . , the base of natural logarithms.

$$\lambda = \lambda_0 + x/R \qquad (10\text{–}7)$$

Rectangular coordinates are given in table 13. There is no basis for an ellipsoidal equivalent, since the projection is used for maps of the entire Earth and not for local areas at large scale.

TABLE 13.—*Miller Cylindrical projection: Rectangular coordinates*

[Radius of sphere = 1.0]

ϕ	y	h	k	ω
90°	2.30341	3.23607	Infinite	180.00°
85	2.04742	2.66947	11.47371	77.00
80	1.83239	2.28117	5.75877	51.26
75	1.64620	2.00000	3.86370	37.06
70	1.48131	1.78829	2.92380	27.89
65	1.33270	1.62427	2.36620	21.43
60	1.19683	1.49448	2.00000	16.64
55	1.07113	1.39016	1.74345	12.95
50	.95364	1.30541	1.55572	10.04
45	.84284	1.23607	1.41421	7.71
40	.73754	1.17918	1.30541	5.82
35	.63674	1.13257	1.22077	4.30
30	.53962	1.09464	1.15470	3.06
25	.44547	1.06418	1.10338	2.07
20	.35369	1.04030	1.06418	1.30
15	.26373	1.02234	1.03528	.72
10	.17510	1.00983	1.01543	.32
5	.08734	1.00244	1.00382	.08
0	.00000	1.00000	1.00000	.00
x	0.017453 $(\lambda° - \lambda_0°)$			

Note: x, y = rectangular coordinates.
ϕ = geodetic latitude.
$(\lambda° - \lambda_0°)$ = geodetic longitude, measured east from origin in degrees.
h = scale factor along meridian.
k = scale factor along parallel.
ω = maximum angular deformation, degrees.

Origin of coordinates at intersection of Equator with λ_0. X axis increases east, Y axis increases north. For southern (negative) ϕ, reverse sign of y.

11. EQUIDISTANT CYLINDRICAL PROJECTION

SUMMARY

- Cylindrical.
- Neither equal-area nor conformal.
- Meridians and parallels are equidistant straight lines, intersecting at right angles.
- Poles shown as lines.
- Used for world or regional maps.
- Very simple construction.
- Used only in spherical form.
- Presented by Eratosthenes (B.C.) or Marinus (A.D. 100).

HISTORY AND FEATURES

While the Equidistant Cylindrical projection is listed last among the cylindricals because of its limited use by the USGS and generally limited value, it is probably the simplest of all map projections to construct and one of the oldest. The meridians and parallels are all equidistant straight parallel lines, the two sets crossing at right angles.

The projection originated probably with Eratosthenes (275?–195? B.C.), the scientist and geographer noted for his fairly accurate measure of the size of the Earth. Claudius Ptolemy credited Marinus of Tyre with the invention about A.D. 100 stating that, while Marinus had previously evaluated existing projections, the latter had chosen "a manner of representing the distances which gives the worst results of all." Only the parallel of Rhodes (lat. 36° N.) was made true to scale on the world map, which meant that the meridians were spaced at about four-fifths of the spacing of the parallels for the same degree interval (Keuning, 1955, p. 13).

Ptolemy approved the use of the projection for maps of smaller areas, however, with spacing of meridians to provide correct scale along the central parallel. All the Greek manuscript maps for the *Geographia,* dating from the 13th century, use the Ptolemy modification. It was used for some maps until the eighteenth century, but is now used primarily for a few maps on which distortion is considered less important than the ease of displaying special information. The projection is given a variety of names such as Equidistant Cylindrical, Rectangular, La Carte Parallélogrammatique, Die Rechteckige Plattkarte, and Equirectangular (Steers, 1970, p. 135–136). It was called the projection of Marinus by Nordenskiöld (1889).

If the Equator is made the standard parallel, true to scale and free of distortion, the meridians are spaced at the same distances as the parallels, and the graticule appears square. This form is often called the Plate Carrée or the Simple Cylindrical projection.

The USGS uses the Equidistant Cylindrical projection for index maps of the conterminous United States, with insets of Alaska, Hawaii, and various islands on the same projection. One is entitled "Topographic Mapping Status and Progress of Operations (7½- and 15-minute series)," at an approximate scale of 1:5,000,000. Another shows the status of intermediate-scale quadrangle mapping. Neither the scale nor the projection is marked, to avoid implying that the maps are suitable for normal geographic information. Meridian spacing is about four-fifths of the spacing of parallels, by coincidence the same as that chosen by Marinus. The Alaska inset is shown at about half the scale and with a change in spacing ratios. Individual States are shown by the USGS on other index maps using the same projection and spacing ratio to indicate the status of aerial photography.

The projection was chosen largely for ease in computerized plotting. While the boundaries on the base map may be as difficult to plot on this projection as on the others, the base map needs to be prepared only once. Overlays of digital information, which may then be printed in straight lines, may be easily updated without the use of cartographic and photographic skills. The 4:5 spacing ratio is a convenience based on computer line and character spacing and is not an attempt to achieve a particular standard parallel, which happens to fall near lat. 37° N.

FORMULAS FOR THE SPHERE

The formulas for rectangular coordinates are almost as simple to use as the geometric construction. Given R, λ_0, ϕ_1, λ, and ϕ for the forward solution, x and y are found thus:

$$x = R(\lambda - \lambda_0)\cos\phi_1 \qquad (11\text{-}1)$$

$$y = R\phi \qquad (11\text{-}2)$$

$$h = 1 \qquad (11\text{-}3)$$

$$k = \cos\phi_1/\cos\phi \qquad (11\text{-}4)$$

The X axis coincides with the Equator, with x increasing easterly, while the Y axis follows the central meridian λ_0, y increasing northerly. It is necessary to adjust $(\lambda - \lambda_0)$, if it is beyond the range $\pm 180°$, by adding or subtracting 360°. The standard parallel is ϕ_1 (also $-\phi_1$). For the inverse formulas, given R, λ_0, ϕ_1, x, and y, to find ϕ and λ:

$$\phi = y/R \qquad (11\text{-}5)$$

$$\lambda = \lambda_0 + x/(R\cos\phi_1) \qquad (11\text{-}6)$$

Numerical examples are omitted in the appendix, due to simplicity. It must be remembered, as usual, that angles above are given in radians.

CONIC MAP PROJECTIONS

Cylindrical projections are used primarily for complete world maps, or for maps along narrow strips of a great circle arc, such as the Equator, a meridian, or an oblique great circle. To show a region for which the greatest extent is from east to west in the temperate zones, conic projections are usually preferable to cylindrical projections.

Normal conic projections are distinguished by the use of arcs of concentric circles for parallels of latitude and equally spaced straight radii of these circles for meridians. The angles between the meridians on the map are smaller than the actual differences in longitude. The circular arcs may or may not be equally spaced, depending on the projection. The Polyconic projection and oblique conic projections have characteristics different from these.

The name "conic" originates from the fact that the more elementary conic projections may be derived by placing a cone on the top of a globe representing the Earth, the apex or tip in line with the axis of the globe, and the sides of the cone touching or tangent to the globe along a specified "standard" latitude which is true to scale and without distortion (see fig. 1). Meridians are drawn on the cone from the apex to the points at which the corresponding meridians on the globe cross the standard parallel. Other parallels are then drawn as arcs centered on the apex in a manner depending on the projection. If the cone is cut along one meridian and unrolled, a conic projection results. A secant cone results if the cone cuts the globe at two specified parallels. Meridians and parallels can be marked on the secant cone somewhat as above, but this will not result in any of the common conic projections with two standard parallels. They are derived from various desired scale relationships instead, and the spacing of the meridians as well as the parallels is not the same as the projection onto a secant cone.

There are three important classes of conic projections: the equidistant (or simple), the conformal, and the equal-area. The Equidistant Conic, with parallels equidistantly spaced, originated in a rudimentary form with Claudius Ptolemy. It eventually developed into commonly used present-day forms which have one or two standard parallels selected for the area being shown. It is neither conformal nor equal-area, but north-south scale along all meridians is correct, and the projection can be a satisfactory compromise for errors in shape, scale, and area, especially when the map covers a small area. It is primarily used in the spherical form, although the ellipsoidal form is available and useful. The USGS uses the Equidistant Conic in an approximate form for a map of Alaska, identified as a "Modified Transverse Mercator" projection, and also in the limiting equatorial form: the Equidistant Cylindrical. Both are described earlier.

The Lambert Conformal Conic projection with two standard parallels is used frequently for large- and small-scale maps. The parallels are more closely spaced near the center of the map. The Lambert has also been used slightly in the oblique form. The Albers Equal-Area Conic with two standard parallels is used for sectional maps of the U.S. and for maps of the conterminous United States. The Albers parallels are spaced more closely near the north and south edges of the map. There are some conic projections, such as perspective conics, which do not fall into any of these three categories, but they are rarely used.

The useful conic projections may be geometrically constructed only in a limited sense, using polar coordinates which must be calculated. After a location is chosen, usually off the final map, for the center of the circular arcs which will represent parallels of latitude, meridians are constructed as straight lines radiating from this center and spaced from each other at an angle equal to the product of the cone constant times the difference in longitude. For example, if a 10° graticule is planned, and the cone constant is 0.65, the meridian lines are spaced at 10° times 0.65 or 6.5°. Each parallel of latitude may then be drawn as a circular arc with a radius previously calculated from formulas for the particular conic projection.

12. ALBERS EQUAL-AREA CONIC PROJECTION

SUMMARY

- Conic.
- Equal-Area.
- Parallels are unequally spaced arcs of concentric circles, more closely spaced at the north and south edges of the map.
- Meridians are equally spaced radii of the same circles, cutting parallels at right angles.
- There is no distortion in scale or shape along two standard parallels, normally, or along just one.
- Poles are arcs of circles.
- Used for equal-area maps of regions with predominant east-west expanse, especially the conterminous United States.
- Presented by Albers in 1805.

HISTORY

One of the most commonly used projections for maps of the conterminous United States is the equal-area form of the conic projection, using two standard parallels. This projection was first presented by Heinrich Christian Albers (1773–1833), a native of Lüneburg, Germany, in a German periodical of 1805 (Albers, 1805; Bonacker and Anliker, 1930). The Albers projection was used for a German map of Europe in 1817, but it was promoted for maps of the United States in the early part of the twentieth century by Oscar S. Adams of the Coast and Geodetic Survey as "an equal-area representation that is as good as any other and in many respects superior to all others" (Adams, 1927, p. 1).

FEATURES AND USAGE

The Albers is the projection exclusively used by the USGS for sectional maps of all 50 States of the United States in the *National Atlas* of 1970, and for other U.S. maps at scales of 1:2,500,000 and smaller. The latter maps include the base maps of the United States issued in 1961, 1967, and 1972, the Tectonic Map of the United States (1962), and the Geologic Map of the United States (1974), all at 1:2,500,000. The USGS has also prepared a U.S. base map at 1:3,168,000 (1 inch = 50 miles).

Like other normal conics, the Albers Equal-Area Conic projection (fig. 15) has concentric arcs of circles for parallels and equally spaced radii as meridians. The parallels are not equally spaced, but they are farthest apart in the latitudes between the standard parallels and closer together to the north and south. The pole is not the center of the circles, but is normally an arc itself.

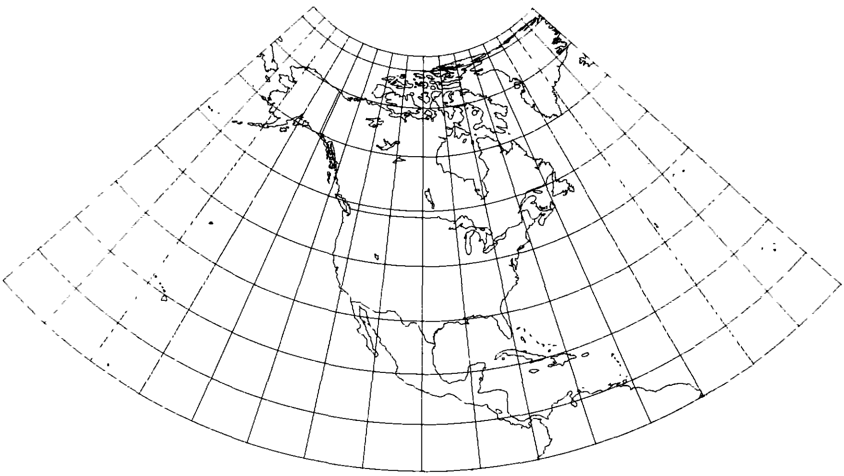


FIGURE 15. – Albers Equal-Area Conic projection, with standard parallels 20° and 60° N. This illustration includes all of North America to show the change in spacing of the parallels. When used for maps of the 48 conterminous States standard parallels are 29.5° and 45.5° N.

If the pole is taken as one of the two standard parallels, the Albers formulas reduce to a limiting form of the projection called Lambert's Equal-Area Conic (not discussed here, and not to be confused with his Conformal Conic, to be discussed later). If the pole is the only standard parallel, the Albers formulas simplify to provide the polar aspect of the Lambert Azimuthal Equal-Area (discussed later). In both of these limiting cases, the pole is a point. If the Equator is the one standard parallel, the projection becomes Lambert's Cylindrical Equal-Area (not discussed), but the formulas must be modified. None of these extreme cases applies to the normal use of the Albers, with standard parallels in the temperate zones, such as usage for the United States.

Scale along the parallels is too small between the standard parallels and too large beyond them. The scale along the meridians is just the opposite, and in fact the scale factor along meridians is the reciprocal of the scale factor along parallels, to maintain equal area.

To map a given region, standard parallels should be selected to minimize variations in scale. Not only are standard parallels correct in scale along the parallel; they are correct in every direction. Thus, there is no angular distortion, and conformality exists along these standard parallels, even on an equal-area projection. They may be on opposite sides of, but not equidistant from the Equator. Deetz and Adams (1934, p. 79, 91) recommended in general that standard parallels be placed one-sixth of the displayed length of the central meridian from the northern and southern limits of the map. Hinks (1912, p. 87) suggested one-seventh instead of one-sixth. Others have suggested selecting standard parallels of conics so that the maximum scale error (1 minus the scale factor) in the region between them is equal and opposite in sign to the error at the upper and lower parallels, or so that the scale factor at the middle parallel is the reciprocal of that at the limiting parallels. Zinger in 1916 and Kavraisky in 1934 chose standard parallels so that least-square errors in linear scale were minimal for the actual land or country being displayed on the map. This involved weighting each latitude in accordance with the land it contains (Maling, 1960, p. 263–266).

The standard parallels chosen by Adams for Albers maps of the conterminous United States are lats. 29.5° and 45.5° N. These parallels provide "for a scale error slightly less than 1 per cent in the center of the map, with a maximum of 1¼ per cent along the northern and southern borders." (Deetz and Adams, 1934, p. 91). For maps of Alaska, the chosen standard parallels are lats. 55° and 65° N., and for Hawaii, lats. 8° and 18° N. In the latter case, both parallels are south of the islands, but they were chosen to include maps of the more southerly Canal Zone and especially the Philippine Islands. These parallels apply to all maps prepared by the USGS on the Albers projection, originally using Adams's published tables of coordinates for the Clarke 1866 ellipsoid (Adams, 1927).

Without measuring the spacing of parallels along a meridian, it is almost impossible to distinguish an unlabeled Albers map of the United States from other conic forms. It is only when the projection is extended considerably north and south, well beyond the standard parallels, that the difference is apparent without scaling.

Since meridians intersect parallels at right angles, it may at first seem that there is no angular distortion. However, scale variations along the meridians cause some angular distortion for any angle other than that between the meridian and parallel, except at the standard parallels.

FORMULAS FOR THE SPHERE

The Albers Equal-Area Conic projection may be constructed with only one standard parallel, but it is nearly always used with two. The forward formulas for the sphere are as follows, to obtain rectangular or polar coordinates, given R, ϕ_1, ϕ_2, ϕ_0, λ_0, ϕ, and λ:

$$x = \rho \sin \theta \tag{12-1}$$

$$y = \rho_0 - \rho \cos \theta \tag{12-2}$$

where

$$\rho = R(C - 2n \sin \phi)^{1/2}/n \tag{12-3}$$

$$\theta = n(\lambda - \lambda_0) \tag{12-4}$$

$$\rho_0 = R(C - 2n \sin \phi_0)^{1/2}/n \tag{12-3a}$$
$$C = \cos^2 \phi_1 + 2n \sin \phi_1 \tag{12-5}$$
$$n = (\sin \phi_1 + \sin \phi_2)/2 \tag{12-6}$$

ϕ_0, λ_0 = the latitude and longitude, respectively, for the origin of the rectangular coordinates.

ϕ_1, ϕ_2 = standard parallels.

The Y axis lies along the central meridian λ_0, y increasing northerly. The X axis intersects perpendicularly at ϕ_0, x increasing easterly. If $(\lambda - \lambda_0)$ exceeds the range $\pm 180°$, $360°$ should be added or subtracted to place it within the range. Constants n, C, and ρ_0 apply to the entire map, and thus need to be calculated only once. If only one standard parallel ϕ_1 is desired (or if $\phi_1 = \phi_2$), $n = \sin \phi_1$. By contrast, a geometrically secant cone requires a cone constant n of $\sin \frac{1}{2}(\phi_1 + \phi_2)$, slightly but distinctly different from equation (12–6). If the projection is designed primarily for the Northern Hemisphere, n and ρ are positive. For the Southen Hemisphere, they are negative. The scale along the meridians, using equation (4–4),

$$h = \cos \phi/(C - 2n \sin \phi)^{1/2} \tag{12-7}$$

If equation (4–5) is used, k will be found to be the reciprocal of h, satisfying the requirement for an equal-area projection when meridians and parallels intersect at right angles. The maximum angular deformation may be calculated from equation (4–9). It may be seen from equation (12–7), and indeed from equations (4–4) and (4–5), that distortion is strictly a function of latitude, and not of longitude. This is true of any regular conic projection.

For the inverse formulas for the sphere, given R, ϕ_1, ϕ_2, ϕ_0, λ_0, x, and y: n, C and ρ_0 are calculated from equations (12–6), (12–5), and (12–3a), respectively. Then,

$$\phi = \arcsin \{[C - (\rho n/R)^2]/(2n)\} \tag{12-8}$$
$$\lambda = \lambda_0 + \theta/n \tag{12-9}$$

where

$$\rho = [x^2 + (\rho_0 - y)^2]^{1/2} \tag{12-10}$$
$$\theta = \arctan [x/(\rho_0 - y)] \tag{12-11}$$

Note: to use the ATAN2 Fortran function, if n is negative, reverse the signs of x, y, and ρ_0 (given a negative sign by equation (12–3a)) before inserting them in equation (12–11).

FORMULAS FOR THE ELLIPSOID

The formulas displayed by Adams and most other writers describing the ellipsoidal form include series, but the equations may be expressed in closed forms which are suitable for programing, and involve no numerical integration or iteration in the forward form. Nearly all published maps of the United States based on the Albers use the ellipsoidal form because of the use of tables for the original base maps. (Adams, 1927, p. 1–7; Deetz and Adams, 1934, p. 93–99; Snyder, 1979a, p. 71). Given a, e, ϕ_1, ϕ_2, ϕ_0, λ_0, ϕ, and λ:

$$x = \rho \sin \theta \tag{12–1}$$
$$y = \rho_0 - \rho \cos \theta \tag{12–2}$$

where

$$\rho = a(C - nq)^{1/2}/n \tag{12–12}$$
$$\theta = n(\lambda - \lambda_0) \tag{12–4}$$
$$\rho_0 = a(C - nq_0)^{1/2}/n \tag{12–12a}$$
$$C = m_1^2 + nq_1 \tag{12–13}$$
$$n = (m_1^2 - m_2^2)/(q_2 - q_1) \tag{12–14}$$
$$m = \cos \phi/(1 - e^2 \sin^2 \phi)^{1/2} \tag{12–15}$$
$$q = (1 - e^2)\{\sin \phi/(1 - e^2 \sin^2\phi) - [1/(2e)] \ln[(1 - e \sin \phi)/(1 + e \sin \phi)]\} \tag{3–12}$$

with the same subscripts 1, 2, or none applied to m and ϕ in equation (12–15), and 0, 1, 2, or none applied to q and ϕ in equation (3–12), as required by equations (12–12), (12–12a), (12–13), (12–14), and (12–17). As with the spherical case, ρ and n are negative, if the projection is centered in the Southern Hemisphere. For the scale factor, modifying (4–25):

$$k = \rho n/am \tag{12–16}$$
$$= (C - nq)^{1/2}/m \tag{12–17}$$
$$h = 1/k \tag{12–18}$$

While many ellipsoidal equations apply to the sphere if e is made zero, equation (3–12) becomes indeterminate. Actually, if $e = 0$, $q = 2 \sin \phi$. The axes and limitations on $(\lambda - \lambda_0)$ are the same as those stated for the spherical formulas. Here, too, constants n, C, and ρ_0 need to be determined just once for the entire map.

For the inverse formulas for the ellipsoid, given, a, e, ϕ_1, ϕ_2, ϕ_0, λ_0, x, and y: n, C, and ρ_0 are calculated from equations (12–14), (12–13), and (12–12a), respectively. Then,

$$\phi = \phi + \frac{(1 - e^2 \sin^2 \phi)^2}{2 \cos \phi} \left[\frac{q}{1 - e^2} - \frac{\sin \phi}{1 - e^2 \sin^2 \phi} + \frac{1}{2e} \ln\left(\frac{1 - e \sin \phi}{1 + e \sin \phi}\right)\right] \tag{3–16}$$
$$\lambda = \lambda_0 + \theta/n \tag{12–9}$$

where

$$q = (C - \rho^2 n^2/a^2)/n \tag{12–19}$$
$$\rho = [x^2 + (\rho_0 - y)^2]^{1/2} \tag{12–10}$$
$$\theta = \arctan [x/(\rho_0 - y)] \tag{12–11}$$

To use the Fortran ATAN2 function, if n is negative, reverse the signs of x, y, and ρ_0 before insertion into equation (12–11). Equation (3–16) involves iteration by first trying $\phi = \arcsin(q/2)$ on the right side, calculating ϕ on the left side, substituting this new ϕ on the right side, etc., until the change in ϕ is negligible. If

$$q = \pm\{1 - [(1 - e^2)/2e] \ln [(1 - e)/(1 + e)]\} \qquad (12\text{–}20)$$

iteration does not converge, but $\phi = \pm 90°$, taking the sign of q.

Instead of the iteration, a series may be used for the inverse ellipsoidal formulas:

$$\phi = \beta + (e^2/3 + 31e^4/180 + 517e^6/5040 + \ldots) \sin 2\beta + (23e^4/360 + 251e^6/3780 + \ldots) \sin 4\beta + (761e^6/45360 + \ldots) \sin 6\beta + \ldots \qquad (3\text{–}18)$$

where β, the authalic latitude, adapting equations (3–11) and (3–12), is found thus:

$$\beta = \arcsin (q/\{1 - [(1 - e^2)/2e] \ln [(1 - e)/(1 + e)]\}) \qquad (12\text{–}21)$$

but q is still found from equation (12–19). Equations (12–9), (12–10), and (12–11) also apply unchanged.

Polar coordinates for the Albers Equal-Area Conic are given for both the spherical and ellipsoidal forms, using standard parallels of lat. 29.5° and 45.5° N. (table 14). A graticule extended to the North Pole is shown in figure 15.

TABLE 14.—*Albers Equal-Area Conic projection: Polar coordinates*

[Standard parallels: 29.5°, 45.5° N]

Lat.	Projection for sphere (R = 6,370,997 m) (n = 0.6028370) ρ	h	k	Projection for Clarke 1866 ellipsoid (a = 6,378,206.4 m) (n = 0.6029035) ρ	h	k
52°	6,693,511	0.97207	1.02874	6,713,781	0.97217	1.02863
51	6,801,923	.97779	1.02271	6,822,266	.97788	1.02263
50	6,910,941	.98296	1.01733	6,931,335	.98303	1.01727
49	7,020,505	.98760	1.01255	7,040,929	.98765	1.01251
48	7,130,555	.99173	1.00834	7,150,989	.99177	1.00830
47	7,241,038	.99538	1.00464	7,261,460	.99540	1.00462
46	7,351,901	.99857	1.00143	7,372,290	.99858	1.00143
45.5	7,407,459	1.00000	1.00000	7,427,824	1.00000	1.00000
45	7,463,094	1.00132	.99868	7,483,429	1.00132	.99869
44	7,574,570	1.00365	.99636	7,594,829	1.00364	.99637
43	7,686,282	1.00558	.99445	7,706,445	1.00556	.99447
42	7,798,186	1.00713	.99292	7,818,233	1.00710	.99295
41	7,910,244	1.00832	.99175	7,930,153	1.00828	.99178
40	8,022,413	1.00915	.99093	8,042,164	1.00911	.99097
39	8,134,656	1.00965	.99044	8,154,230	1.00961	.99048
38	8,246,937	1.00983	.99027	8,266,313	1.00978	.99031
37	8,359,220	1.00970	.99040	8,378,379	1.00965	.99044
36	8,471,472	1.00927	.99082	8,490,394	1.00923	.99086
35	8,583,660	1.00855	.99152	8,602,328	1.00852	.99155
34	8,695,753	1.00757	.99249	8,714,149	1.00753	.99252
33	8,807,723	1.00632	.99372	8,825,828	1.00629	.99375
32	8,919,539	1.00481	.99521	8,937,337	1.00479	.99523
31	9,031,175	1.00306	.99694	9,048,649	1.00305	.99696
30	9,142,602	1.00108	.99892	9,159,737	1.00107	.99893
29.5	9,198,229	1.00000	1.00000	9,215,189	1.00000	1.00000
29	9,253,796	.99887	1.00114	9,270,575	.99887	1.00113
28	9,364,731	.99643	1.00358	9,381,141	.99645	1.00357
27	9,475,383	.99378	1.00626	9,491,411	.99381	1.00623
26	9,585,731	.99093	1.00915	9,601,361	.99097	1.00911
25	9,695,749	.98787	1.01227	9,710,969	.98793	1.01222
24	9,805,417	.98463	1.01561	9,820,216	.98470	1.01554
23	9,914,713	.98119	1.01917	9,929,080	.98128	1.01908
22	10,023,616	.97757	1.02294	10,037,541	.97768	1.02283

Note: ρ = radius of latitude circle, meters.
h = scale factor along meridians.
k = scale factor along parallels.
R = assumed radius of sphere.
a = assumed semimajor axis of ellipsoid.
n = cone constant, or ratio of angle between meridians on map to true angle.

13. LAMBERT CONFORMAL CONIC PROJECTION

SUMMARY

- Conic.
- Conformal.
- Parallels are unequally spaced arcs of concentric circles, more closely spaced near the center of the map.
- Meridians are equally spaced radii of the same circles, thereby cutting parallels at right angles.
- Scale is true along two standard parallels, normally, or along just one.
- Pole in same hemisphere as standard parallels is a point; other pole is at infinity.
- Used for maps of countries and regions with predominant east-west expanse.
- Presented by Lambert in 1772.

HISTORY

The Lambert Conformal Conic projection (fig. 16) was almost completely overlooked between its introduction and its revival by France for battle maps of the First World War. It was the first new projection which Johann Heinrich Lambert presented in his *Beiträge* (Lambert, 1772), the publication which contained his Transverse Mercator described previously. In some atlases, particularly British, the Lambert Conformal Conic is called the "Conical Orthomorphic" projection.

Lambert developed the regular Conformal Conic as the oblique aspect of a family containing the previously known polar Stereographic and regular Mercator projections. As he stated, "Stereographic representations of the spherical surface, as well as Mercator's nautical charts, have the peculiarity that all angles maintain the sizes that they have on the surface of the globe. This yields the greatest similarity that any plane figure can have with one drawn on the surface of a sphere. The question has not been asked whether this property occurs only in the two methods of representation mentioned or whether these two representations, so different in appearances, can be made to approach each other through intermediate stages. * * * if there are stages intermediate to these two representations, they must be sought by allowing the angle of intersection of the meridians to be arbitrarily larger or smaller than its value on the surface of the sphere. This is the way in which I shall now proceed" (Lambert, 1772, p. 28, translation by Tobler). Lambert then developed the mathematics for both the spherical and ellipsoidal forms for two standard parallels and included a small map of Europe as an example (Lambert, 1772, p. 28–38, 87–89).

FEATURES

Many of the comments concerning the appearance of the Albers and the selection of its standard parallels apply to the Lambert Conformal

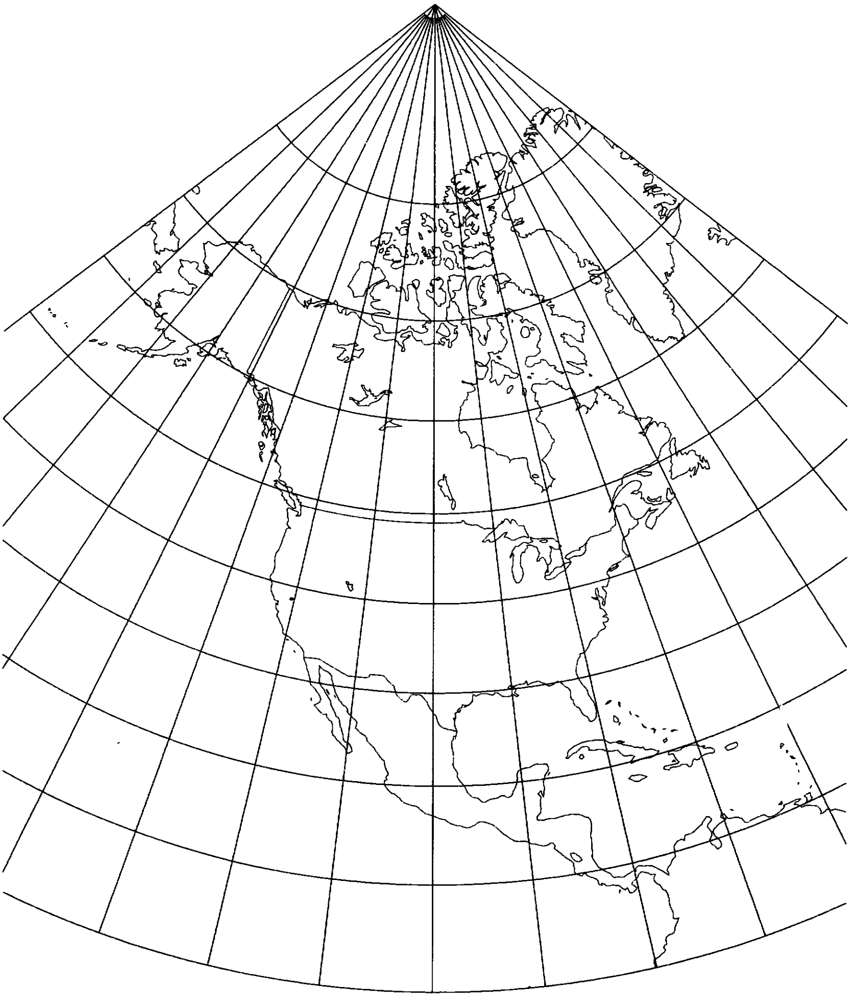


FIGURE 16.—Lambert Conformal Conic projection, with standard parallels 20° and 60° N. North America is illustrated here to show the change in spacing of the parallels. When used for maps of the conterminous United States or individual States, standard parallels are 33° and 45° N.

Conic when an area the size of the conterminous United States or smaller is considered. As stated before, the spacing of the parallels must be measured to distinguish among the various conic projections for such an area. If the projection is extended toward either pole and the Equator, as on a map of North America, the differences become more obvious. Although meridians are equally spaced radii of the concentric circular arcs representing parallels of latitude, the parallels become further apart as the distance from the central parallels increases. Conformality fails at each pole, as in the case of the regular Mercator. The pole in the same hemisphere as the standard parallels is shown on the Lambert Conformal Conic as a point. The other pole is at infinity. Straight lines between points approximate great circle arcs for maps of moderate coverage, but only the Gnomonic projection rigorously has this feature and then only for the sphere. (The Gnomonic is not discussed in detail.)

Two parallels may be made standard or true to scale, as well as conformal. It is also possible to have just one standard parallel. Since there is no angular distortion at any parallel (except at the poles), it is possible to change the standard parallels to just one, or to another pair, just by changing the scale applied to the existing map and calculating a pair of standard parallels fitting the new scale. This is not true of the Albers, on which only the original standard parallels are free from angular distortion.

The scale is too small between the standard parallels and too large beyond them. This applies to the scale along meridians, as well as along parallels, or in any other direction, since they are equal at any given point. Thus, in the State Plane Coordinate Systems (SPCS) for States using the Lambert, the choice of standard parallels has the effect of reducing the scale of the central parallel by an amount which cannot be expressed simply in exact form, while the scale for the central meridian of a map using the Transverse Mercator is normally reduced by a simple fraction.

USAGE

After the reappearance of the Lambert Conformal Conic in France during the First World War, the Coast and Geodetic Survey immediately began publishing tables for the projection (Deetz, 1918a, 1918b). It was only a couple of decades before the Lambert Conformal Conic was adopted as the official projection for the SPCS for States of predominantly east-west expanse. The prototype was the North Carolina Coordinate System, established in 1933. Within a year or so, similar systems were devised for many other States, while a Transverse Mercator system was prepared for the remaining States.

One or more zones is involved in the system for each State (see table 8) (Mitchell and Simmons, 1945, p. vi). In addition, the Lambert is used for the Aleutian Islands of Alaska, Long Island in New York, and northwestern Florida, although the Transverse Mercator (and Oblique Mercator in one case) is used for the rest of each of these States.

The Lambert Conformal Conic is used for the 1:1,000,000-scale regional world aeronautical charts, the 1:500,000-scale sectional aeronautical charts, and 1:500,000-scale State base maps (all 48 contiguous States[4] have the same standard parallels of lat. 33° and 45° N., and thus match). Also cast on the Lambert are most of the 1:24,000-scale 7½-minute quadrangles prepared after 1957 which lie in zones for which the Lambert is the base for the SPCS. In the latter case, the standard parallels for the zone are used, rather than parameters designed for the individual quadrangles. Thus, all quadrangles for a given zone may be mosaicked exactly. (The projection used previously was the Polyconic, and some recent quadrangles are being produced to the Universal Transverse Mercator projection.)

The Lambert Conformal Conic has also been adopted as the official topographic projection for some other countries. It appears in *The National Atlas* (USGS, 1970, p. 116) for a map of hurricane patterns in the North Atlantic, and the Lambert is used by the USGS for a map of the United States showing all 50 States in their true relative positions. The latter map is at scales of both 1:6,000,000 and 1:10,000,000, with standard parallels 37° and 65° N.

In 1962, the projection for the International Map of the World at a scale of 1:1,000,000 was changed from a modified Polyconic to the Lambert Conformal Conic between lats. 84° N. and 80° S. The polar Stereographic projection is used in the remaining areas. The sheets are generally 6° of longitude wide by 4° of latitude high. The standard parallels are placed at one-sixth and five-sixths of the latitude spacing for each zone of 4° latitude, and the reference ellipsoid is the International (United Nations, 1963, p. 9–27). This specification has been subsequently used by the USGS in constructing several maps for the IMW series.

Perhaps the most recent new topographic use for the Lambert Conformal Conic projection by the USGS is for middle latitudes of the 1:1,000,000-scale geologic series of the Moon and for some of the maps of Mercury, Mars, and Jupiter's satellites Ganymede and Callisto (see table 15).

[4]For Hawaii, the standard parallels are lats. 20°40′ and 23°20′ N.; the corresponding base map was not prepared for Alaska.

FORMULAS FOR THE SPHERE

For the projection as normally used, with two standard parallels, the equations for the sphere may be written as follows: Given R, ϕ_1, ϕ_2, ϕ_0, λ_0, ϕ, and λ:

$$x = \rho \sin\theta \quad (12\text{–}1)$$
$$y = \rho_0 - \rho \cos\theta \quad (12\text{–}2)$$

where

$$\rho = RF/\tan^n(\pi/4 + \phi/2) \quad (13\text{–}1)$$
$$\theta = n(\lambda - \lambda_0) \quad (12\text{–}4)$$
$$\rho_0 = RF/\tan^n(\pi/4 + \phi_0/2) \quad (13\text{–}1a)$$
$$F = \cos\phi_1 \tan^n(\pi/4 + \phi_1/2)/n \quad (13\text{–}2)$$
$$n = \ln(\cos\phi_1/\cos\phi_2)/\ln[\tan(\pi/4 + \phi_2/2)/\tan(\pi/4 + \phi_1/2)] \quad (13\text{–}3)$$

ϕ_0, λ_0 = the latitude and longitude for the origin of the rectangular coordinates.
ϕ_1, ϕ_2 = standard parallels.

The Y axis lies along the central meridian λ_0, y increasing northerly; the X axis intersects perpendicularly at ϕ_0, x increasing easterly. If $(\lambda - \lambda_0)$ exceeds the range $\pm 180°$, $360°$ should be added or subtracted. Constants n, F, and ρ_0 need to be determined only once for the entire map.

If only one standard parallel ϕ_1 is desired, $n = \sin\phi_1$. The scale along meridians or parallels, using equations (4–4) or (4–5),

$$k = h = \cos\phi_1 \tan^n(\pi/4 + \phi_1/2)/[\cos\phi \tan^n(\pi/4 + \phi/2)] \quad (13\text{–}4)$$

The maximum angular deformation $\omega = 0$, since the projection is conformal. As with the other regular conics, k is strictly a function of latitude. For a projection centered in the Southern Hemisphere, n and ρ are negative.

For the inverse formulas for the sphere, given R, ϕ_1, ϕ_2, ϕ_0, λ_0, x, and y: n, F, and ρ_0 are calculated from equations (13–3), (13–2), and (13–1a), respectively. Then,

$$\phi = 2 \arctan(RF/\rho)^{1/n} - \pi/2 \quad (13\text{–}5)$$
$$\lambda = \theta/n + \lambda_0 \quad (12\text{–}9)$$

where

$$\rho = \pm[x^2 + (\rho_0 - y)^2]^{1/2}, \text{ taking the sign of } n \quad (12\text{–}10)$$
$$\theta = \arctan[x/(\rho_0 - y)] \quad (12\text{–}11)$$

The Fortran ATAN2 function does not apply to equation (13–5), but when it is used for equation (12–11), and n is negative, the signs of x, y, and ρ_0 (negative from equation (13–1a)) must be reversed before inser-

TABLE 15.—*Lambert Conformal Conic Projection: Used for extraterrestrial mapping*

[From Batson, 1973; Davies and Batson, 1975; Batson and others, 1980; Batson, private commun., 1981]

Body[1]	Scale[2]	Range in Lat. (Standard Parallels)[3]	Adjacent Projections[4]	Overlap	Matching Parallels with (scale)[5]	Comments
Moon	1:1,000,000	16° to 48° N. & S. (21°20′, 42°40′)	Mercator	0°	16° (1:1,021,000)	Quadrangles 20° to 30° long. × 16° lat.
			Lambert Conformal Conic	0°	none	
		48° to 64° N. & S. (53°20′, 74°40′)	Lambert Conformal Conic	0°	none	Do.
			----------	--	--	
Mercury	1:5,000,000	20° to 70° N. & S. (28°, 62°) (1:4,765,000)	Mercator	5°	22.5° (1:4,619,000)	Quadrangles 90° long. × 50° lat.
			Polar Stereographic	5°	67.5° (1:4,568,000)	
Mars	1:5,000,000	30° to 65° N. & S. (35.83°, 59.17°) (1:4,441,000)	Mercator	0°	30° (1:4,336,000)	Quadrangles 60° long. × 35° lat.
			Polar Stereographic	0°	65° (1:4,306,000)	
	1:2,000,000	30° to 65° N. & S. (35.83°, 59.17°)	Mercator	0°	30° (1:1,953,000)	Quadrangles 22.5° long. × 17.5° lat. (between 30° & 47.5° lat).
			Polar Stereographic	0°	65° (1:1,939,000)	30° long. × 17.5° lat. (between 47.5° & 65° lat.).
			Galilean satellites of Jupiter			
Ganymede } Callisto	1:5,000,000	21° to 66° N. & S. (30°, 58°)	Mercator	1°	21.3° (1:4,780,000)	Quadrangles 90° long. × 45° lat.
			Polar Stereographic	1°	65.2° (1:4,769,000)	

[1]Taken as sphere, except for Mars (ellipsoid). See table 2.
[2]Scale at equator of Mercator zones (Mercury and Mars), at standard parallels (Moon, Ganymede, and Callisto and 1:2,000,000 Mars), also at pole of polar Stereographic (Ganymede and Callisto).
[3]Scale also given if other than that in second column.
[4]First projection named is toward equator, second is toward pole.
[5]Matching parallels are both N. & S.

tion into the equation. If $\rho = 0$, equation (13–5) involves division by zero, but ϕ is $\pm 90°$, taking the sign of n.

The standard parallels normally used for maps of the conterminous United States are lats. 33° and 45° N., which give approximately the least overall error within those boundaries. The ellipsoidal form is used for such maps, based on the Clarke 1866 ellipsoid (Adams, 1918).

The standard parallels of 33° and 45° were selected by the USGS because of the existing tables by Adams (1918), but Adams chose them to provide a maximum scale error between latitudes 30.5° and 47.5° of one-half of 1 percent. A maximum scale error of 2.5 percent occurs in southernmost Florida (Deetz and Adams, 1934, p. 80). Other standard parallels would reduce the maximum scale error for the United States, but at the expense of accuracy in the center of the map.

FORMULAS FOR THE ELLIPSOID

The ellipsoidal formulas are essential when applying the Lambert Conformal Conic to mapping at a scale of 1:100,000 or larger and important at scales of 1:5,000,000. Given a, e, ϕ_1, ϕ_2, ϕ_0, λ_0, ϕ, and λ:

$$x = \rho \sin \theta \tag{12-1}$$
$$y = \rho_0 - \rho \cos \theta \tag{12-2}$$
$$k = \rho n/(am) \tag{12-16}$$
$$= m_1 t^n/(m t_1^n) \tag{13-6}$$

where

$$\rho = aFt^n \tag{13-7}$$
$$\theta = n(\lambda - \lambda_0) \tag{12-4}$$
$$\rho_0 = aFt_0^n \tag{13-7a}$$
$$n = (\ln m_1 - \ln m_2)/(\ln t_1 - \ln t_2) \tag{13-8}$$
$$m = \cos\phi/(1 - e^2 \sin^2 \phi)^{1/2} \tag{12-15}$$
$$t = \tan(\pi/4 - \phi/2)/[(1 - e\sin\phi)/(1 + e\sin\phi)]^{e/2} \tag{13-9}$$
$$F = m_1/(n t_1^n) \tag{13-10}$$

with the same subscripts 1, 2, or none applied to m and ϕ in equation (12–15), and 0, 1, 2, or none applied to t and ϕ in equation (13–9), as required by equations (13–6), (13–7), and (13–8). As with other conics, a negative n and ρ result for projections centered in the Southern Hemisphere. If $\phi = \pm 90°$, ρ is zero for the same sign as n and infinite for the opposite sign. If $\phi_1 = \phi_2$, for the Lambert with a single standard parallel, equation (13–8) is indeterminate, but $n = \sin \phi_1$. Origin and orientation of axes for x and y are the same as those for the spherical form. Constants n, F, and ρ_0 may be determined just once for the entire map.

When the above equations for the ellipsoidal form are used, they give values of n and ρ slightly different from those in the accepted tables of coordinates for a map of the United States, according to the Lambert Conformal Conic projection. The discrepancy is 35–50 m in the radius and 0.0000035 in n. The rectangular coordinates are correspondingly affected. The discrepancy is less significant when it is realized that the radius is measured to the pole, and that the distance from the 50th parallel to the 25th parallel on the map at full scale is only 12 m out of 2,800,000 or 0.0004 percent. For calculating convenience 60 years ago, the tables were, in effect, calculated using instead of equation (13–9),

$$t = \tan(\pi/4 - \phi_g/2) \tag{13–9a}$$

where ϕ_g is the geocentric latitude, or, as shown earlier,

$$\phi_g = \arctan[(1-e^2)\tan\phi] \tag{3–28}$$

In conventional terminology, the t of equation (13–9) is usually written as $\tan \frac{1}{2}Z$, where Z is the colatitude of the conformal latitude χ (see equation (3–1)).

For the existing tables, then, ϕ_g, the geocentric latitude, was used for convenience in place of χ, the conformal latitude (Adams, 1918, p. 6–9, 34). A comparison of series equations (3–3) and (3–30), or of the corresponding columns in table 3, shows that the two auxiliary latitudes χ and ϕ_g are numerically very nearly the same.

There may be much smaller discrepancies found between coordinates as calculated on modern computers and those listed in tables for the SPCS. This is due to the slightly reduced (but sufficient) accuracy of the desk calculators of 30–40 years ago and the adaptation of formulas to be more easily utilized by them.

The inverse formulas for ellipsoidal coordinates, given a, e, ϕ_1, ϕ_2, ϕ_0, λ_0, ϕ, and λ: n, F, and ρ_0 are calculated from equations (13–8), (13–10), (13–7a), respectively. Then,

$$\phi = \pi/2 - 2\arctan\{t[(1-e\sin\phi)/(1+e\sin\phi)]^{e/2}\} \tag{7–9}$$

where

$$t = (\rho/aF)^{1/n} \tag{13–11}$$

$$\rho = \pm[x^2 - (\rho_0 - y)^2]^{1/2}, \text{ taking the sign of } n. \tag{12–10}$$

$$\lambda = \theta/n + \lambda_0 \tag{12–9}$$

$$\theta = \arctan[x/(\rho_0 - y)] \tag{12–11}$$

As with the spherical formulas, the Fortran ATAN2 function does not apply to equation (7–9), but for equation (12–11), if n is negative, the signs of x, y, and ρ_0 must be reversed.

Equation (7–9) involves rapidly converging iteration: Calculate t from (13–11). Then, assuming an initial trial ϕ equal to $(\pi/2 - 2\arctan t)$ in the right side of equation (7–9), calculate ϕ on the left side. Substitute the calculated ϕ into the right side, calculate a new ϕ, etc., until ϕ does not change significantly from the preceding trial value of ϕ.

To avoid iteration, series (3–5) may be used with (7–13) in place of (7–9):

$$\phi = \chi + (e^2/2 + 5e^4/24 + e^6/12 + \ldots) \sin 2\chi + (7e^4/48 + 29e^6/240 + \ldots) \sin 4\chi + (7e^6/120 + \ldots) \sin 6\chi + \ldots \quad (3\text{–}5)$$

where

$$\chi = \pi/2 - 2 \arctan t \quad (7\text{–}13)$$

If rectangular coordinates for maps based on the tables using geocentric latitude are to be converted to latitude and longitude, the inverse formulas are the same as those above, except that equation (13–9a) is used instead of (13–9) for calculations leading to n, F, and ρ_0, and equation (7–9), or (3–5) and (7–13), is replaced with the following which does not involve iteration:

$$\phi = \arctan [\tan \phi_g/(1 - e^2)] \quad (13\text{–}13)$$

where

$$\phi_g = \pi/2 - 2 \arctan t \quad (13\text{–}14)$$

and t is calculated from equation (13–11).

Polar coordinates for the Lambert Conformal Conic are given for both the spherical and ellipsoidal forms, using standard parallels of 33° and 45° N. (table 16). The data based on the geocentric latitude are given for comparison. A graticule extended to the North Pole is shown in figure 16.

14. BIPOLAR OBLIQUE CONIC CONFORMAL PROJECTION

SUMMARY

- Two oblique conic projections, side-by-side, but with poles 104° apart.
- Conformal.
- Meridians and parallels are complex curves, intersecting at right angles.
- Scale is true along two standard transformed parallels on each conic projection, neither of these lines following any geographical meridian or parallel.
- Very small deviation from conformality, where the two conic projections join.
- Specially developed for a map of the Americas.
- Used only in spherical form.
- Presented by Miller and Briesemeister in 1941.

HISTORY

A "tailor-made" projection is one designed for a certain geographical area. O. M. Miller used the term for some projections which he developed for the American Geographical Society (AGS) or for their clients. The Bipolar Oblique Conic Conformal projection, developed with William A. Briesemeister, was presented in 1941 and designed specifically for a map of North and South America constructed in several sheets by the AGS at a scale of 1:5,000,000 (Miller, 1941).

It is an adaptation of the Lambert Conformal Conic projection to minimize scale error over the two continents by accommodating the fact that North America tends to curve toward the east as one proceeds from north to south, while South America tends to curve in the opposite direction. Because of the relatively small scale of the map, the Earth was treated as a sphere. To construct the map, a great circle arc 104° long was selected to cut through Central America from southwest to northeast, beginning at lat. 20° S. and long. 110° W. and terminating at lat. 45° N. and the resulting longitude of about 19°59′36″ W.

The former point is used as the pole and as the center of transformed parallels of latitude for an Oblique Conformal Conic projection with two standard parallels (at polar distances of 31° and 73°) for all the land in the Americas southeast of the 104° great circle arc. The latter point serves as the pole and center of parallels for an identical projection for all land northwest of the same arc. The inner and outer standard parallels of the northwest portion of the map, thus, are tangent to the outer and inner standard parallels, respectively, of the southeast portion, touching at the dividing line (104° − 31° = 73°).

TABLE 16.—*Lambert Conformal Conic projection: Polar coordinates*

[Standard parallels: 33° and 45° N.]

Lat.	Projection for sphere (R=6,370,997 m) (n=0.6304777)			Projection for Clarke 1866 ellipsoid (a=6,378,206.4 m) Conformal lat.[1] (n=0.6304965)			Geocentric lat.[2] (n=0.6305000)
	ρ	k	k^2	ρ	k	k^2	ρ
52°	6,359,534	1.02222	1.04494	6,379,530	1.02215	1.04480	--
51	6,472,954	1.01787	1.03606	6,493,008	1.01781	1.03595	6,492,973
50	6,585,914	1.01394	1.02807	6,606,007	1.01389	1.02798	6,605,970
49	6,698,458	1.01040	1.02091	6,718,571	1.01037	1.02084	6,718,537
48	6,810,631	1.00725	1.01456	6,830,746	1.00723	1.01451	6,830,708
47	6,922,475	1.00448	1.00898	6,942,573	1.00446	1.00894	6,942,534
46	7,034,030	1.00206	1.00413	7,054,092	1.00206	1.00412	7,054,052
45	7,145,336	1.00000	1.00000	7,165,344	1.00000	1.00000	7,165,303
44	7,256,432	.99828	.99656	7,276,367	.99828	.99657	7,276,330
43	7,367,355	.99689	.99379	7,387,198	.99690	.99381	7,387,158
42	7,478,142	.99582	.99167	7,497,873	.99584	.99170	7,497,833
41	7,588,828	.99508	.99018	7,608,429	.99510	.99022	7,608,384
40	7,699,449	.99464	.98932	7,718,900	.99467	.98936	7,718,857
39	7,810,038	.99452	.98907	7,829,321	.99454	.98911	7,829,278
38	7,920,631	.99470	.98942	7,939,726	.99472	.98946	7,939,680
37	8,031,259	.99517	.99036	8,050,148	.99519	.99040	8,050,107
36	8,141,957	.99594	.99190	8,160,619	.99596	.99193	8,160,581
35	8,252,757	.99700	.99402	8,271,174	.99702	.99404	8,271,129
34	8,363,692	.99836	.99672	8,381,843	.99836	.99673	8,381,798
33	8,474,793	1.00000	1.00000	8,492,660	1.00000	1.00000	8,492,614
32	8,586,092	1.00193	1.00386	8,603,656	1.00192	1.00385	8,603,610
31	8,697,622	1.00415	1.00831	8,714,863	1.00413	1.00827	8,714,820
30	8,809,415	1.00665	1.01335	8,826,313	1.00662	1.01328	8,826,267
29	8,921,502	1.00944	1.01897	8,938,038	1.00940	1.01888	8,937,986
28	9,033,915	1.01252	1.02520	9,050,070	1.01246	1.02507	9,050,021
27	9,146,686	1.01589	1.03203	9,162,440	1.01581	1.03186	9,162,396
26	9,259,848	1.01954	1.03947	9,275,181	1.01944	1.03927	9,275,132
25	9,373,433	1.02349	1.04754	9,388,326	1.02337	1.04729	9,388,277
24	9,487,474	1.02774	1.05625	9,501,906	1.02759	1.05595	9,501,859
23	9,602,003	1.03228	1.06560	9,615,955	1.03211	1.06525	9,615,911
22	9,717,054	1.03712	1.07563	9,730,506	1.03692	1.07521	9,730,456

[1]Based on rigorous equations using conformal latitude.
[2]Based on geocentric latitude as given in Adams (1918, p. 34) and Deetz and Adams (1934, p. 84).

Notes: ρ = radius of latitude circles, meters.
k = scale factor (linear).
k^2 = scale factor (areal).
a = assumed semimajor axis of ellipsoid.
R = assumed radius of sphere.
n = cone constant, or ratio of angle between meridians on map to true angle.

The scale of the map was then increased by about 3.5 percent, so that the linear scale error along the central parallels (at a polar distance of 52°, halfway between 31° and 73°) is equal and opposite in sign (−3.5 percent) to the scale error along the two standard parallels (now +3.5 percent) which are at the normal map limits. Under these conditions, transformed parallels at polar distances of about 36.34° and 66.58° are true to scale and are actually the standard transformed parallels.

The use of the Oblique Conformal Conic projection was not original with Miller and Briesemeister. The concept involves the shifting of the graticule of meridians and parallels for the regular Lambert Conformal Conic so that the pole of the projection is no longer at the pole of the Earth. This is the same principle as the transformation for the Oblique Mercator projection. The bipolar concept is unique, however, and it has apparently not been used for any other maps.

FEATURES AND USAGE

The Geological Survey has used the North American portion of the map for the Geologic Map (1965), the Basement Map (1967), the Geothermal Map, and the Metallogenic Map, all retaining the original scale of 1:5,000,000. The Tectonic Map of North America (1969) is generally based on the Bipolar Oblique Conic Conformal, but there are modifications near the edges. An oblique conic projection about a single transformed pole would suffice for either one of the continents alone, but the AGS map served as an available base map at an appropriate scale. In 1979, the USGS decided to replace this projection with the Transverse Mercator for a map of North America.

The projection is conformal, and each of the two conic projections has all the characteristics of the Lambert Conformal Conic projection, except for the important difference in location of the pole, and a very narrow band near the center. While meridians and parallels on the oblique projection intersect at right angles because the map is conformal, the parallels are not arcs of circles, and the meridians are not straight, except for the peripheral meridian from each transformed pole to the nearest normal pole.

The scale is constant along each circular arc centered on the transformed pole for the conic projection of the particular portion of the map. Thus, the two lines at a scale factor of 1.035, that is, both pairs of the official standard transformed parallels, are mapped as circular arcs forming the letter "S." The 104° great circle arc separating the two oblique conic projections is a straight line on the map, and all other straight lines radiating from the poles for the respective conic projections are transformed meridians and are therefore great circle routes. These straight lines are not normally shown on the finished map.

At the juncture of the two conic projections, along the 104° axis, there is actually a slight mathematical discontinuity at every point except for the two points at which the transformed parallels of polar distance 31° and 73° meet. If the conic projections are strictly followed, there is a maximum discrepancy of 1.6 mm at the 1:5,000,000 scale at the midpoint of this axis, halfway between the poles or between the intersections of the axis with the 31° and 73° transformed parallels. In other words, a meridian approaching the axis from the south is shifted up to 1.6 mm along the axis as it crosses. Along the axis, but beyond the portion between the lines of true scale, the discrepancy increases markedly, until it is over 240 mm at the transformed poles. These latter areas are beyond the needed range of the map and are not shown, just as the polar areas of the regular Lambert Conformal Conic are normally omitted. This would not happen if the Oblique Equidistant Conic projection were used.

The discontinuity was resolved by connecting the two arcs with a straight line tangent to both, a convenience which leaves the small intermediate area slightly nonconformal. This adjustment is included in the formulas below.

FORMULAS FOR THE SPHERE

The original map was prepared by the American Geographical Society, in an era when automatic plotters and easy computation of coordinates were not yet present. Map coordinates were determined by converting the geographical coordinates of a given graticule intersection to the transformed latitude and longitude based on the poles of the projection, then to polar coordinates according to the conformal projection, and finally to rectangular coordinates relative to the selected origin.

The following formulas combine these steps in a form which may be programed for the computer. First, various constants are calculated from the above parameters, applying to the entire map. Since only one map is involved, the numerical values are inserted in formulas, except where the numbers are transcendental and are referred to by symbols.

If the southwest pole is at point A, the northeast pole is at point B, and the center point on the axis is C,

$$\lambda_B = -110° + \arccos \{[\cos 104° - \sin(-20°) \sin 45°]/ [\cos(-20°) \cos 45°]\} \quad (14\text{–}1)$$

$= -19°59'36''$ long., the longitude of B (negative is west long.)

$$n = (\ln \sin 31° - \ln \sin 73°)/[\ln \tan (31°/2) - \ln \tan (73°/2)] \quad (14\text{–}2)$$

$= 0.63056$, the cone constant for both conic projections

$F_0 = R \sin 31°/[n \tan^n (31°/2)]$ (14–3)
$= 1.83376\ R$, where R is the radius of the globe at the scale of the map. For the 1:5,000,000 map, R was taken as 6,371,221 m, the radius of a sphere having a volume equal to that of the International ellipsoid.

$k_0 = 2/[1 + nF_0 \tan^n 26°/(R \sin 52°)]$ (14–4)
$= 1.03462$, the scale factor by which the coordinates are multiplied to balance the errors

$F = k_0 F_0$ (14–5)
$= 1.89725\ R$, a convenient constant

$Az_{AB} = \arccos \{[\cos(-20°) \sin 45° - \sin(-20°) \cos 45° \cos (\lambda_B + 110°)]/\sin 104°\}$ (14–6)
$= 46.78203°$, the azimuth east of north of B from A

$Az_{BA} = \arccos \{[\cos 45° \sin(-20°) - \sin 45° \cos(-20°) \cos (\lambda_B + 110°)]/\sin 104°\}$ (14–7)
$= 104.42834°$, the azimuth west of north of A from B

$T = \tan^n (31°/2) + \tan^n (73°/2)$ (14–8)
$= 1.27247$, a convenient constant

$\rho_c = \frac{1}{2} FT$ (14–9)
$= 1.20709\ R$, the radius of the center point of the axis from either pole

$z_c = 2 \arctan (T/2)^{1/n}$ (14–10)
$= 52.03888°$, the polar distance of the center point from either pole

Note that z_c would be exactly 52°, if there were no discontinuity at the axis. The values of ϕ_c, λ_c, and Az_c are calculated as if no adjustment were made at the axis due to the discontinuity. Their use is completely arbitrary and only affects positions of the arbitrary X and Y axes, not the map itself. The adjustment is included in formulas for a given point.

$\phi_c = \arcsin [\sin(-20°) \cos z_c + \cos(-20°) \sin z_c \cos Az_{AB}]$ (14–11)
$= 17°16'28''$ N. lat., the latitude of the center point, on the southern-cone side of the axis

$\lambda_c = \arcsin (\sin z_c \sin Az_{AB}/\cos \phi_c) - 110°$ (14–12)
$= -73°00'27''$ long., the longitude of the center point, on the southern-cone side of the axis

$Az_c = \arcsin [\cos(-20°) \sin Az_{AB}/\cos \phi_c]$ (14–13)
$= 45.81997°$, the azimuth east of north of the axis at the center point, relative to meridian λ_c on the southern-cone side of the axis

The remaining equations are given in the order used, for calculating rectangular coordinates for various values of latitude ϕ and longitude λ (measured east from Greenwich, or with a minus sign for the western values used here). It must be established first whether point (ϕ, λ) is north or south of the axis, to determine which conic projection is involved. With these formulas, it is done by comparing the azimuth of point (ϕ, λ) with the azimuth of the axis, all as viewed from B:

$$z_B = \arccos [\sin 45° \sin \phi + \cos 45° \cos \phi \cos (\lambda_B - \lambda)] \quad (14\text{–}14)$$

= polar distance of (ϕ, λ) from pole B

$$Az_B = \arctan \{\sin (\lambda_B - \lambda)/[\cos 45° \tan \phi - \sin 45° \cos (\lambda_B - \lambda)]\} \quad (14\text{–}15)$$

= azimuth of (ϕ, λ) west of north, viewed from B

If Az_B is greater than Az_{BA} (from equation (14–7)), go to equation (14–23). Otherwise proceed to equation (14–16) for the projection from pole B.

$$\rho_B = F \tan^n \tfrac{1}{2} z_B \quad (14\text{–}16)$$

$$k = \rho_B n/(R \sin z_B) \quad (14\text{–}17)$$

= scale factor at point (ϕ, λ), disregarding small adjustment near axis

$$\alpha = \arccos \{[\tan^n \tfrac{1}{2} z_B + \tan^n \tfrac{1}{2}(104° - z_B)]/T\} \quad (14\text{–}18)$$

If $|n\,(Az_{BA} - Az_B)|$ is less than α,

$$\rho_B' = \rho_B/\cos [\alpha - n\,(Az_{BA} - Az_B)] \quad (14\text{–}19)$$

If the above expression is equal to or greater than α,

$$\rho_B' = \rho_B. \quad (14\text{–}20)$$

Then

$$x' = \rho_B' \sin [n\,(Az_{BA} - Az_B)] \quad (14\text{–}21)$$

$$y' = \rho_B' \cos [n\,(Az_{BA} - Az_B)] - \rho_c \quad (14\text{–}22)$$

using constants from equations (14–2), (14–3), (14–7), and (14–9) for rectangular coordinates relative to the axis. To change to nonskewed rectangular coordinates, go to equations (14–32) and (14–33). The following formulas give coordinates for the projection from pole A.

$$z_A = \arccos [\sin (-20°) \sin \phi + \cos (-20°) \cos \phi \cos (\lambda + 110°)] \quad (14\text{–}23)$$

= polar distance of (ϕ, λ) from pole A

$$Az_A = \arctan \{\sin (\lambda + 110°)/[\cos (-20°) \tan \phi - \sin (-20°) \cos (\lambda + 110°)]\} \quad (14\text{–}24)$$

= azimuth of (ϕ, λ) east of north, viewed from A

$$\rho_A = F \tan^n \tfrac{1}{2} z_A \quad (14\text{–}25)$$

$$k = \rho_A n/R \sin z_A = \text{scale factor at point } (\phi, \lambda) \quad (14\text{–}26)$$

$$\alpha = \arccos \{[\tan^n \tfrac{1}{2} z_A + \tan^n \tfrac{1}{2}(104° - z_A)]/T\} \quad (14\text{–}27)$$

If $|n\,(Az_{AB} - Az_A)|$ is less than α,

$$\rho_A' = \rho_A/\cos [\alpha + n\,(Az_{AB} - Az_A)] \quad (14\text{–}28)$$

If the above expression is equal to or greater than α,

$$\rho_A' = \rho_A \tag{14-29}$$

Then

$$x' = \rho_A' \sin [n(Az_{AB} - Az_A)] \tag{14-30}$$
$$y' = -\rho_A' \cos [n(Az_{AB} - Az_A)] + \rho_c \tag{14-31}$$
$$x = -x' \cos Az_c - y' \sin Az_c \tag{14-32}$$
$$y = -y' \cos Az_c + x' \sin Az_c \tag{14-33}$$

where the center point at (ϕ_c, λ_c) is approximately the origin of (x, y) coordinates, the Y axis increasing due north and the X axis due east from the origin. (The meridian and parallel actually crossing the origin are shifted by about 3′ of arc, due to the adjustment at the axis, but their actual values do not affect the calculations here.)

For the inverse formulas for the Bipolar Oblique Conic Conformal, the constants for the map must first be calculated from equations (14–1)–(14–13). Given x and y coordinates based on the above axes, they are then converted to the skew coordinates:

$$x' = -x \cos Az_c + y \sin Az_c \tag{14-34}$$
$$y' = -x \sin Az_c - y \cos Az_c \tag{14-35}$$

If x' is equal to or greater than zero, go to equation (14–36). If x' is negative, go to equation (14–45).

$$\rho_B' = [x'^2 + (\rho_c + y')^2]^{1/2} \tag{14-36}$$
$$Az_B' = \arctan [x'/(\rho_c + y')] \tag{14-37}$$

Let

$$\rho_B = \rho_B' \tag{14-38}$$
$$z_B = 2 \arctan (\rho_B/F)^{1/n} \tag{14-39}$$
$$\alpha = \arccos \{[\tan^n \tfrac{1}{2} z_B + \tan^n \tfrac{1}{2}(104° - z_B)]/T\} \tag{14-40}$$

If $|Az_B'|$ is equal to or greater than α, go to equation (14–42). If $|Az_B'|$ is less than α, calculate

$$\rho_B = \rho_B' \cos (\alpha - Az_B') \tag{14-41}$$

and use this value to recalculate equations (14–39), (14–40), and (14–41), repeating until ρ_B found in (14–41) changes by less than a predetermined convergence. Then,

$$Az_B = Az_{BA} - Az_B'/n \tag{14-42}$$

Using Az_B and the final value of z_B,

$$\phi = \arcsin (\sin 45° \cos z_B + \cos 45° \sin z_B \cos Az_B) \tag{14-43}$$
$$\lambda = \lambda_B - \arctan \{\sin Az_B/[\cos 45°/\tan z_B - \sin 45° \cos Az_B]\} \tag{14-44}$$

The remaining equations are for the southern cone only (negative x'):

$$\rho_A' = [x'^2 + (\rho_c - y')^2]^{1/2} \quad (14\text{-}45)$$
$$Az_A' = \arctan [x'/(\rho_c - y')] \quad (14\text{-}46)$$

Let

$$\rho_A = \rho_A' \quad (14\text{-}47)$$
$$z_A = 2 \arctan (\rho_A/F)^{1/n} \quad (14\text{-}48)$$
$$\alpha = \arccos \{[\tan^n \tfrac{1}{2} z_A + \tan^n \tfrac{1}{2}(104° - z_A)]/T\} \quad (14\text{-}49)$$

If $|Az_A'|$ is equal to or greater than α, go to equation (14–51). If $|Az_A'|$ is less than α, calculate

$$\rho_A = \rho_A' \cos (\alpha + Az_A') \quad (14\text{-}50)$$

and use this value to recalculate equations (14–48), (14–49), and (14–50), repeating until ρ_A found in equation (14–50) changes by less than a predetermined convergence. Then,

$$Az_A = Az_{AB} - Az_A'/n \quad (14\text{-}51)$$

Using Az_A and the final value of z_A,

$$\phi = \arcsin [\sin (-20°) \cos z_A + \cos 20° \sin z_A \cos Az_A] \quad (14\text{-}52)$$
$$\lambda = \arctan \{\sin Az_A/[\cos (-20°)/\tan z_A - \sin (-20°) \cos Az_A]\} - 110° \quad (14\text{-}53)$$

Equations (14–17) or (14–26) may be used for calculating k after ϕ and λ are determined.

A table of rectangular coordinates is given in table 17, based on a radius R of 1.0, while a graticule is shown in figure 17.

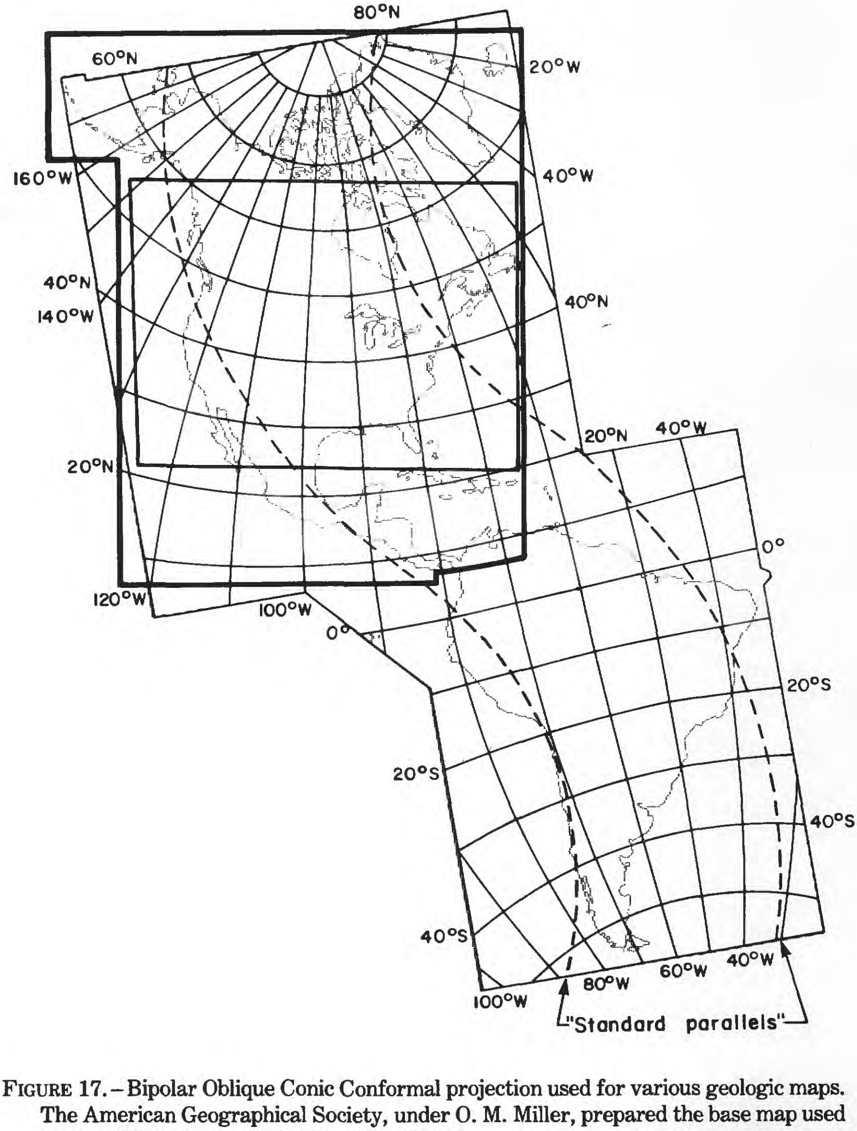


FIGURE 17. – Bipolar Oblique Conic Conformal projection used for various geologic maps. The American Geographical Society, under O. M. Miller, prepared the base map used by the USGS. (Prepared by Tau Rho Alpha.)

TABLE 17.—*Bipolar Oblique Conic Conformal projection: Rectangular coordinates*

[R = 1.0. y coordinates in parentheses below x coordinates. Solid line separates the portions formed from the two transformed poles. Origin at approximately lat. 17°15′ N., long. 73°02′ W., with Y axis due north at that point only]

Lat. W. Long.	170°	160°	150°	140°	130°	120°	110°	100°
90°	-0.14576	--	--	--	--	--	--	--
	(1.24309)	--	--	--	--	--	--	--
80	- .31273	-0.30608	-0.29464	-0.27876	-0.25892	-0.23569	-0.20973	-0.18175
	(1.21904)	(1.19057)	(1.16367)	(1.13914)	(1.11769)	(1.09992)	(1.08634)	(1.07737)
70	- .48092	- .46705	- .44372	- .41182	- .37243	- .32673	- .27593	- .22126
	(1.19421)	(1.13725)	(1.08381)	(1.03535)	(.99311)	(.95806)	(.93098)	(.91246)
60	- .65416	- .63138	- .59444	- .54518	- .48559	- .41763	- .34310	- .26359
	(1.16623)	(1.07999)	(1.00006)	(.92849)	(.86677)	(.81589)	(.77644)	(.74880)
50	- .83656	- .80200	- .74851	- .67943	- .59806	- .50739	- .40985	- .30732
	(1.13260)	(1.01551)	(.90886)	(.81512)	(.73570)	(.67112)	(.62133)	(.58603)
40	--	--	--	- .81523	- .70964	- .59515	- .47485	- .35078
	--	--	--	(.69140)	(.59654)	(.52100)	(.46360)	(.42294)
30	--	--	--	--	- .81990	- .67999	- .53678	- .39231
	--	--	--	--	(.44545)	(.36240)	(.30074)	(.25766)
20	--	--	--	--	--	- .76061	- .59421	- .43026
	--	--	--	--	--	(.19177)	(.12988)	(.08782)
10	--	--	--	--	--	- .83496	- .64522	- .46280
	--	--	--	--	--	(.00499)	(-.05222)	(-.08930)
0	beyond arbitrary map limits		--	--	--	--	- .68704	- .48758
	--	--	--	--	--	--	(-.24918)	(-.27670)
-10	--	--	--	--	--	--	- .72338	- .50751*
	--	--	--	--	--	--	(-.47150)	(-.48360)
-20	--	--	--	--	--	--	- .86567*	- .48812
	--	--	--	--	--	--	(-.84124)	(-.73406)
-30	--	--	--	--	--	--	- .55209	- .38781
	--	--	--	--	--	--	(-1.10271)	(-.96476)
-40	--	--	--	--	--	--	- .37784	- .26583
	--	--	--	--	--	--	(-1.24800)	(-1.14111)
-50	--	--	--	--	--	--	- .23054	- .14798
	--	--	--	--	--	--	(-1.37082)	(-1.28862)
-60	--	--	--	--	--	--	- .09524	- .03499
	--	--	--	--	--	--	(-1.48363)	(-1.42268)
-70	beyond arbitrary map limits		--	--	--	--	.03504	.07542
	--	--	--	--	--	--	(-1.59227)	(-1.55124)
-80	--	--	--	--	--	--	.16491	.18569
	--	--	--	--	--	--	(-1.70055)	(-1.67949)
-90	--	--	--	--	--	--	.29823	--
	--	--	--	--	--	--	(-1.81171)	--

*Adjustment to x and y made for discontinuity near axis of conic projections.

TABLE 17.–*Bipolar Oblique Conic Conformal projection: Rectangular coordinates*–Continued

Lat. W. Long.	90°	80°	70°	60°	50°	40°	30°	20°	10°
90°	-0.14576	--	--	--	--	--	--	--	--
	(1.24309)	--	--	--	--	--	--	--	--
80	– .15254	-0.12293	-0.09378	-0.06599	-0.04047	-0.01809	0.00033	0.01411	0.02275
	(1.07330)	(1.07432)	(1.08048)	(1.09170)	(1.10774)	(1.12816)	(1.15236)	(1.17955)	(1.20877)
70	– .16395	– .10525	.04651	.01074	.06470	.11317	.15365	.18369	.20152
	(.90301)	(.90303)	(.91292)	(.93301)	(.96349)	(1.00421)	(1.05436)	(1.11215)	(1.17478)
60	– .18043	– .09477	– .00767	.07976	.16594	.24806	.32065	.37468	.40201
	(.73324)	(.73013)	(.74005)	(.76403)	(.80369)	(.86133)	(.93920)	(1.03623)	(1.14388)
50	– .20109	– .09192	.01990	.13461	.25295	.37631	.50548	.62083	.64638
	(.56481)	(.55749)	(.56421)	(.58582)	(.62443)	(.68480)	(.77907)	(.93836)	(1.13342)
40	– .22411	– .09519	.03637	.17183	.31377	.46682 *	.64259 *	--	--
	(.39765)	(.38660)	(.38903)	(.40460)	(.43354)	(.47595)	(.54614)	--	--
30	– .24741	– .10203	.04468	.19431	.34922	.51120	.68326	--	--
	(.23065)	(.21759)	(.21675)	(.22664)	(.24602)	(.27522)	(.31537)	--	--
20	– .26899	– .10979	.04816 *	.20770	.37167	.54280	.72518	--	--
	(.06192)	(.04921)	(.04683)	(.05280)	(.06603)	(.08551)	(.11131)	--	--
10	– .28689	– .11634 *	.05000	.21614	.38494	.56021	.74645	--	--
	(– .11090)	(– .12083)	(– .12223)	(– .11773)	(– .10970)	(– .09944)	(– .08790)	--	--
0	– .29905 *	– .11920	.05292	.22166	.39129	.56601	.75029	--	--
	(– .29059)	(– .29390)	(– .29122)	(– .28661)	(– .28234)	(– .28009)	(– .28151)	--	--
-10	– .29984	– .11376	.05921	.22626	.39254	.56225	.73941	--	--
	(– .48267)	(– .47202)	(– .46189)	(– .45503)	(– .45295)	(– .45710)	(– .46938)	--	--
-20	– .27575	– .09495	.07161	.23171	.39016	.55057	.71601	--	--
	(– .68590)	(– .65440)	(– .63424)	(– .62366)	(– .62240)	(– .63119)	(– .65175)	--	--
-30	– .21865	– .05954	.09194	.23925	.38524	.53215	.68181	--	--
	(– .88430)	(– .83575)	(– .80677)	(– .79252)	(– .79127)	(– .80304)	(– .82907)	--	--
-40	– .13981	– .00990	.12002	.24931	.37838	.50784	.63813	--	--
	(–1.06299)	(–1.01016)	(– .97740)	(– .96122)	(– .95992)	(– .97320)	(–1.00184)	--	--
-50	– .05346	.04829	.15387	.26134	.36964	.47806	.58591	--	--
	(–1.22345)	(–1.17590)	(–1.14498)	(–1.12947)	(–1.12858)	(–1.14214)	(–1.17057)	--	--
-60	.03430	.11029	.19081	.27404	.35849	.44283	.52574	--	--
	(–1.37283)	(–1.33514)	(–1.31002)	(–1.29749)	(–1.29753)	(–1.31019)	(–1.33568)	--	--
-70	.12196	.17341	.22844	.28571	.34391	.40173	.45785	beyond	--
	(–1.51739)	(–1.49156)	(–1.47435)	(–1.46615)	(–1.46721)	(–1.47764)	(–1.49748)	arbitrary	--
-80	.20970	.23631	.26481	.29445	.32443	.35394	.38215	map limits	--
	(–1.66218)	(–1.64908)	(–1.64057)	(–1.63693)	(–1.63831)	(–1.64474)	(–1.65615)	--	--
-90	.29823	--	--	--	--	--	--	--	--
	(–1.81171)	--	--	--	--	--	--	--	--

15. POLYCONIC PROJECTION

SUMMARY

- Neither conformal nor equal-area.
- Parallels of latitude (except for Equator) are arcs of circles, but are not concentric.
- Central meridian and Equator are straight lines; all other meridians are complex curves.
- Scale is true along each parallel and along the central meridian, but no parallel is "standard."
- Free of distortion only along the central meridian.
- Used almost exclusively in slightly modified form for large-scale mapping in the U.S. until the 1950's.
- Was apparently originated about 1820 by Hassler.

HISTORY

Shortly before 1820, Ferdinand Rudolph Hassler (fig. 18) began to promote the Polyconic projection, which was to become a standard for much of the official mapping of the United States (Deetz and Adams, 1934, p. 58–60).

Born in Switzerland in 1770, Hassler arrived in the United States in 1805 and was hired 2 years later as the first head of the Survey of the Coast. He was forced to wait until 1811 for funds and equipment, meanwhile teaching to maintain income. After funds were granted, he spent 4 years in Europe securing equipment. Surveying began in 1816, but Congress, dissatisfied with the progress, took the Survey from his control in 1818. The work only foundered. It was returned to Hassler, now superintendent, in 1832. Hassler died in Philadelphia in 1843 as a result of exposure after a fall, trying to save his instruments in a severe wind and hailstorm, but he had firmly established what later became the U.S. Coast and Geodetic Survey (Wraight and Roberts, 1957) and is now the National Ocean Survey.

The Polyconic projection, usually called the American Polyconic in Europe, achieved its name because the curvature of the circular arc for each parallel on the map is the same as it would be following the unrolling of a cone which had been wrapped around the globe tangent to the particular parallel of latitude, with the parallel traced onto the cone. Thus, there are many ("poly-") cones involved, rather than the single cone of each regular conic projection. As Hassler himself described the principles, "[t]his distribution of the projection, in an assemblage of sections of surfaces of successive cones, tangents to or cutting a regular succession of parallels, and upon regularly changing central meridians, appeared to me the only one applicable to the coast of the United States" (Hassler, 1825, p. 407–408).

FIGURE 18.–Ferdinand Rudolph Hassler (1770–1843), first Superintendent of the U.S. Coast Survey and presumed inventor of the Polyconic projection. As a result of his promotion of its use, it became the projection exclusively used for USGS topographic quadrangles for about 70 years.

The term "polyconic" is also applied generically by some writers to other projections on which parallels are shown as circular arcs. Most commonly, the term applies to the specific projection described here.

FEATURES

The Polyconic projection (fig. 19) is neither equal-area nor conformal. Along the central meridian, however, it is both distortion free and true to scale. Each parallel is true to scale, but the meridians are lengthened

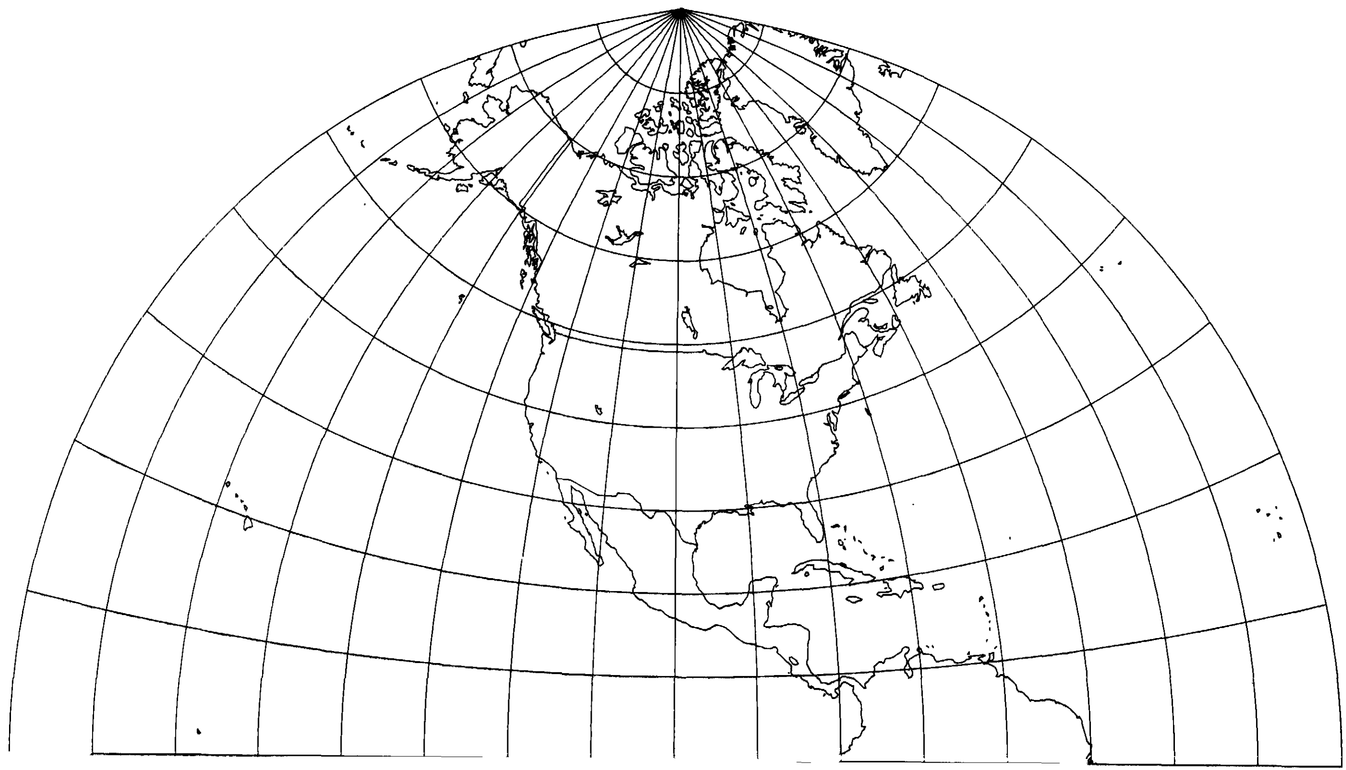


FIGURE 19.—North America on a Polyconic projection grid, central meridian long. 100° W., using a 10° interval. The parallels are arcs of circles which are not concentric, but have radii equal to the radius of curvature of the parallel at the Earth's surface. The meridians are complex curves formed by connecting points marked off along the parallels at their true distances. Used by the USGS for topographic quadrangle maps.

by various amounts to cross each parallel at the correct position along the parallel, so that no parallel is standard in the sense of having conformality (or correct angles), except at the central meridian. Near the central meridian, which is the case with 7½-minute quadrangles, distortion is extremely small. The Polyconic projection is universal in that tables of rectangular coordinates may be used for any Polyconic projection of the same ellipsoid by merely applying the proper scale and central meridian. U.S. Coast and Geodetic Survey Special Publication No. 5 (1900) replaced tables published in 1884 and was often reprinted because of the universality of the projection (the Clarke 1866 is the reference ellipsoid). Polyconic quadrangle maps prepared to the same scale and for the same central meridian and ellipsoid will fit exactly from north to south. Since they are drawn in practice with straight meridians, they also fit east to west, but discrepancies will accumulate if mosaicking is attempted in both directions.

The parallels are all circular arcs, with the centers of the arcs lying along an extension of the straight central meridian, but these arcs are not concentric. Instead, as noted earlier, the radius of each arc is that of the circle developed from a cone tangent to the sphere or ellipsoid at the latitude. For the sphere, each parallel has a radius proportional to the cotangent of the latitude. For the ellipsoid, the radius is slightly different. The Equator is a straight line in either case. Along the central meridian, the parallels are spaced at their true intervals. For the sphere, they are therefore equidistant. Each parallel is marked off for meridians equidistantly and true to scale. The points so marked are connected by the curved meridians.

USAGE

As geodetic and coastal surveying began in earnest during the 19th century, the Polyconic projection became a standard, especially for quadrangles. The name of the projection appears on a later reprint of one of the first published USGS topographic quadrangles, which appeared in 1886. In 1904, the USGS published tables of rectangular coordinates extracted from an 1884 Coast and Geodetic Survey report. They were called "coordinates of curvature," but were actually coordinates for the Polyconic projection, although the latter term was not used (Gannett, 1904, p. 37–48).

As a 1928 USGS bulletin of topographic instructions stated (Beaman, 1928, p. 163):

> "The topographic engineer needs a projection which is simple in construction, which can be used to represent small areas on any part of the globe, and which, for each small area to which it is applied, preserves shapes, areas, distances, and azimuths in their true relation to the surface of the earth. The polyconic projection meets all these needs and was adopted for the standard topographic map of the United States, in which the 1°

quadrangle is the largest unit * * * and the 15′ quadrangle is the average unit. * * * Misuse of this projection in attempts to spread it over large areas—that is, to construct a single map of a large area—has developed serious errors and gross exaggeration of details. For example, the polyconic projection is not at all suitable for a single-sheet map of the United States or of a large State, although it has been so employed."

When coordinate plotters and published tables for the State Plane Coordinate System (SPCS) became available in the late 1950's, the USGS ceased using the Polyconic for new maps, in favor of the Transverse Mercator or Lambert Conformal Conic projections used with the SPCS for the area involved. Some of the quadrangles prepared on one or the other of these projections have continued to carry the Polyconic designation, however.

The Polyconic projection was also used for the Progressive Military Grid for military mapping of the U.S., until its replacement by the Universal Transverse Mercator grid. There were seven zones, A–G, with central meridians every 8° west from long. 73° W. (zone A), each zone having an origin at lat. 40°30′ N. on the central meridian with coordinates $x=1{,}000{,}000$ yards, $y=2{,}000{,}000$ yards (Deetz and Adams, 1934, p. 87–90). Some USGS quadrangles of the 1930's and 1940's display tick marks according to this grid in yards, and many quadrangles then prepared by the Army Map Service and sold by the USGS show a complete grid pattern.

While quadrangles based on the Polyconic provide low-distortion mapping of the local areas, the inability to mosaic these quadrangles in all directions without gaps makes them less satisfactory for a larger region. Quadrangles based on the SPCS may be mosaicked over an entire zone, at the expense of increased distortion.

For an individual quadrangle 7½ or 15 minutes of latitude or longitude on a side, the distance of the quadrangle from the central meridian of a Transverse Mercator zone or from the standard parallels of a Lambert Conformal Conic zone of the SPCS has much more effect than the type of projection upon the variation in measurement of distances on quadrangles based on the various projections. If the central meridians or standard parallels of the SPCS zones fall on the quadrangle, a change of projection from Polyconic to Transverse Mercator or Lambert Conformal Conic results in a difference of less than 0.001 mm in the measurement of the 700–800 mm diagonals of a 7½-minute quadrangle. If the quadrangle is near the edge of a zone, the discrepancy between measurements of diagonals on two maps of the same quadrangle, one using the Transverse Mercator or Lambert Conformal Conic projection and the other using the Polyconic, can reach about 0.05 mm. These differences are exceeded by variations in expansion and contraction of paper maps, so that these mathematical discrepancies apply only to comparisons of stable-base maps.

Before the Lambert became the projection for the 1:500,000 State base map series, the Polyconic was used, but the details are unclear. The Polyconic has also been used for maps of the United States; but, as stated above, the distortion is excessive at the east and west coasts, and most current maps are drawn to either the Lambert or Albers Conic projections.

GEOMETRIC CONSTRUCTION

Because of the simplicity of construction using universal tables with which the central meridian and each parallel may be marked off at true distances, the Polyconic projection was favored long after theoretically better projections became known in geodetic circles.

The Polyconic projection must be constructed with curved meridians and parallels if it is used for single-sheet maps of areas with east-west extent of several degrees. Then, however, the inherent distortion is excessive, and a different projection should be considered. For accurate topographic work, the coverage must remain so small that the meridians and parallels may ironically but satisfactorily be drawn as straight-line segments. Official USGS instructions of 1928 declared that

"* * * in actual practice on projections of small quadrangles, the parallels are not drawn as arcs of circles, but their intersections with the meridians are plotted from the computed x and y values, and the sections of the parallels between adjacent meridians are drawn as straight lines. In polyconic projections of quadrangles of 1° or smaller meridians may be drawn as straight lines, and in large-scale projections of small quadrangles in low latitudes both meridians and parallels may be drawn as straight lines. For example, the curvature of the parallels of a projection of a 15′ quadrangle on a scale of 1:48,000 in latitudes from 0° to 30° is so small that it can not be plotted, and for a 7½′ quadrangle on a scale of 1:31,680 or larger the curvature can not be plotted at any latitude"

(Beaman, 1928, p. 167). This instruction is essentially repeated in the 1964 edition (USGS, 1964, p. 12–13). The formulas given below are based on curved meridians.

FORMULAS FOR THE SPHERE

The principles stated above lead to the following forward formulas for rectangular coordinates for the spherical form of the Polyconic projection, using radians:

If ϕ is 0,

$$x = R(\lambda - \lambda_0) \qquad (7\text{–}1)$$

$$y = -R\phi_0 \qquad (15\text{–}1)$$

If ϕ is not 0,

$$E = (\lambda - \lambda_0) \sin \phi \qquad (15\text{–}2)$$

$$x = R \cot \phi \sin E \qquad (15\text{–}3)$$

$$y = R\,[\phi - \phi_0 + \cot \phi\,(1 - \cos E)] \qquad (15\text{–}4)$$

where ϕ_0 is an arbitrary latitude (frequently the Equator) chosen for the origin of the rectangular coordinates at its intersection with λ_0, the central meridian. As with other conics and the Transverse Mercator, the Y axis coincides with the central meridian, y increasing northerly, and the X axis intersects perpendicularly at ϕ_0, x increasing easterly. If $(\lambda - \lambda_0)$ exceeds the range $\pm 180°$, $360°$ must be added or subtracted to place it within the range. For the scale factor h along the meridians, (Adams, 1919, p. 144–147):

$$h = (1 - \cos^2 \phi \cos E)/(\sin^2 \phi \cos D) \quad (15\text{–}5)$$

where

$$D = \arctan\,[(E - \sin E)/(\sec^2 \phi - \cos E)] \quad (15\text{–}6)$$

If ϕ is 0, this is indeterminate, but h is then $[1 + (\lambda - \lambda_0)^2/2]$. In all cases, the scale factor k along any parallel is 1.0.

The inverse formulas for the sphere are given here in the form of a Newton-Raphson approximation, which converges to any desired accuracy after several iterations, except that if $|\lambda - \lambda_0| > 90°$, a rarely used range, this iteration does not converge, and if $y = -R\phi_0$, it is indeterminate. In the latter case, however,

$$\phi = 0$$
$$\lambda = x/R + \lambda_0 \quad (7\text{–}5)$$

Otherwise, if $y \neq -R\phi_0$, calculations are made in this order:

$$A = \phi_0 + y/R \quad (15\text{–}7)$$
$$B = x^2/R^2 + A^2 \quad (15\text{–}8)$$

Using an initial value of $\phi_n = A$, ϕ_{n+1} is found from equation (15–9),

$$\phi_{n+1} = \phi_n - [A(\phi_n \tan \phi_n + 1) - \phi_n - \tfrac{1}{2}(\phi_n^2 + B) \tan \phi_n]/ [(\phi_n - A)/\tan \phi_n - 1] \quad (15\text{–}9)$$

The new trial value of ϕ_{n+1} is successively substituted in place of ϕ_n, until ϕ_{n+1} differs from ϕ_n by less than a predetermined convergence limit. Then $\phi = \phi_{n+1}$ as finally determined.

$$\lambda = [\arcsin\,(x \tan \phi/R)]/\sin \phi + \lambda_0 \quad (15\text{–}10)$$

If $\phi = \pm 90°$, equation (15–10) is indeterminate, but λ may be given any value, such as λ_0.

FORMULAS FOR THE ELLIPSOID

The forward formulas for the ellipsoidal form of the Polyconic projection are only a little more complicated than those for the sphere. These formulas are theoretically exact. They are adapted from formulas given by the Coast and Geodetic Survey (1946, p. 4):

If ϕ is zero:

$$x = a(\lambda - \lambda_0) \qquad (7\text{–}6)$$
$$y = -M_0 \qquad (15\text{–}11)$$

If ϕ is not zero:

$$E = (\lambda - \lambda_0)\sin\phi \qquad (15\text{–}2)$$
$$x = N\cot\phi\sin E \qquad (15\text{–}12)$$
$$y = M - M_0 + N\cot\phi(1 - \cos E) \qquad (15\text{–}13)$$

where

$$M = a[(1 - e^2/4 - 3e^4/64 - 5e^6/256 - \ldots)\phi - (3e^2/8 + 3e^4/32 + 45e^6/1024 + \ldots)\sin 2\phi + (15e^4/256 + 45e^6/1024 + \ldots)\sin 4\phi - (35e^6/3072 + \ldots)\sin 6\phi + \ldots] \qquad (3\text{–}21)$$
$$N = a/(1 - e^2\sin^2\phi)^{1/2} \qquad (4\text{–}20)$$

and M_0 is found from equation (3–21) by using ϕ_0 for ϕ and M_0 for M, with ϕ_0 the latitude of the origin of rectangular coordinates at its intersection with central meridian λ_0. See the spherical formulas for the orientation of axes. The value of $(\lambda - \lambda_0)$ must be adjusted by adding or subtracting 360°, if necessary to fall within the range of ±180°. For scale factor h along the meridians ($k = 1.0$ along the parallels):

If ϕ is zero,

$$h = [M' + \tfrac{1}{2}(\lambda - \lambda_0)^2]/(1 - e^2) \qquad (15\text{–}14)$$

If ϕ is not zero (Adams, 1919, p. 144–146),

$$h = [1 - e^2 + 2(1 - e^2\sin^2\phi)\sin^2 \tfrac{1}{2}E/\tan^2\phi]/[(1 - e^2)\cos D] \qquad (15\text{–}15)$$

where

$$D = \arctan\{(E - \sin E)/[\sec^2\phi - \cos E - e^2\sin^2\phi/(1 - e^2\sin^2\phi)]\} \qquad (15\text{–}16)$$
$$M' = 1 - e^2/4 - 3e^4/64 - 5e^6/256 - \ldots - 2(3e^2/8 + 3e^4/32 + 45e^6/1024 + \ldots)\cos 2\phi + 4(15e^4/256 + 45e^6/1024 + \ldots)\cos 4\phi - 6(35e^6/3072 + \ldots)\cos 6\phi + \ldots \qquad (15\text{–}17)$$

As with the inverse spherical formulas, the *inverse ellipsoidal formulas* are given in a Newton-Raphson form, converging to any desired degree of accuracy after several iterations. As before, if $|\lambda - \lambda_0| > 90°$, this iteration does not converge, but the projection should not be used in that range in any case. The formulas may be calculated in the following order, given a, e, ϕ_0, λ_0, x, and y. First M_0 is calculated from equation (3–21) above, as in the forward case, with ϕ_0 for ϕ and M_0 for M.

If $y = -M_0$, the iteration is not applicable, but

$$\phi = 0$$
$$\lambda = x/a + \lambda_0 \qquad (7\text{–}12)$$

If $y \neq -M_0$, the calculation is as follows:

$$A = (M_0 + y)/a \qquad (15\text{–}18)$$
$$B = x^2/a^2 + A^2 \qquad (15\text{–}19)$$

Using an initial value of $\phi_n = A$, the following calculations are made:

$$C = (1 - e^2 \sin^2 \phi_n)^{1/2} \tan \phi_n \qquad (15\text{–}20)$$

Then M_n and M_n' are found from equations (3–21) and (15–17) above, using ϕ_n for ϕ, M_n for M, and M_n' for M'. Let $M_a = M_n/a$.

$$\phi_{n+1} = \phi_n - [A(CM_a + 1) - M_a - \tfrac{1}{2}(M_a^2 + B)C]/[e^2 \sin 2\phi_n\,(M_a^2 + B - 2AM_a)/4C + (A - M_a)(CM_n' - 2/\sin 2\phi_n) - M_n'] \qquad (15\text{–}21)$$

Each value of ϕ_{n+1} is substituted in place of ϕ_n, and C, M_n, M_n', and ϕ_{n+1} are recalculated from equations (15–20), (3–21), (15–17), and (15–21), respectively. This process is repeated until ϕ_{n+1} varies from ϕ_n by less than a predetermined convergence value. Then ϕ equals the final ϕ_{n+1}.

$$\lambda = [\arcsin (xC/a)]/\sin \phi + \lambda_0 \qquad (15\text{–}22)$$

using the C calculated for the last ϕ_n from equation (15–20). If $\phi = \pm 90°$, λ is indeterminate, but may be given any value.

Table 18 lists rectangular coordinates for a band 3° on either side of the central meridian for the ellipsoid extending from lat. 23° to 50° N. Figure 19 shows the graticule applied to a map of North America.

TABLE 18.—*Polyconic Projection: Rectangular coordinates for the Clarke 1866 ellipsoid*

[y coordinates in parentheses under x coordinates. Italic indicates h]

Long. λ / Lat. φ	0°	1°	2°	3°
50°	0	71,696	143,379	215,037
	(5,540,628)	(5,541,107)	(5,542,545)	(5,544,941)
	1.000000	*1.000063*	*1.000252*	*1.000568*
49	0	73,172	146,331	219,465
	(5,429,409)	(5,429,890)	(5,431,336)	(5,433,745)
	1.000000	*1.000066*	*1.000263*	*1.000592*
48	0	74,626	149,239	223,827
	(5,318,209)	(5,318,693)	(5,320,144)	(5,322,564)
	1.000000	*1.000068*	*1.000274*	*1.000616*
47	0	76,056	152,100	228,119
	(5,207,028)	(5,207,514)	(5,208,970)	(5,211,397)
	1.000000	*1.000071*	*1.000284*	*1.000640*
46	0	77,464	154,915	232,342
	(5,095,868)	(5,096,354)	(5,097,813)	(5,100,244)
	1.000000	*1.000074*	*1.000295*	*1.000664*
45	0	78,847	157,682	236,493
	(4,984,727)	(4,985,214)	(4,986,673)	(4,989,106)
	1.000000	*1.000076*	*1.000306*	*1.000688*
44	0	80,207	160,401	240,572
	(4,873,606)	(4,874,092)	(4,875,551)	(4,877,982)
	1.000000	*1.000079*	*1.000316*	*1.000712*

TABLE 18.—*Polyconic Projection: Rectangular coordinates for the Clarke 1866 ellipsoid*—Continued

Long. λ / Lat. φ	0°	1°	2°	3°
43°	0	81,541	163,071	244,578
	(4,762,505)	(4,762,990)	(4,764,446)	(4,766,872)
	1.000000	*1.000082*	*1.000327*	*1.000736*
42	0	82,851	165,691	248,508
	(4,651,423)	(4,651,907)	(4,653,358)	(4,655,777)
	1.000000	*1.000084*	*1.000338*	*1.000760*
41	0	84,136	168,260	252,363
	(4,540,361)	(4,540,843)	(4,542,288)	(4,544,696)
	1.000000	*1.000087*	*1.000348*	*1.000784*
40	0	85,394	170,778	256,140
	(4,429,319)	(4,429,798)	(4,431,235)	(4,433,630)
	1.000000	*1.000090*	*1.000359*	*1.000808*
39	0	86,627	173,243	259,839
	(4,318,296)	(4,318,772)	(4,320,199)	(4,322,577)
	1.000000	*1.000092*	*1.000369*	*1.000831*
38	0	87,833	175,656	263,458
	(4,207,292)	(4,207,764)	(4,209,180)	(4,211,539)
	1.000000	*1.000095*	*1.000380*	*1.000855*
37	0	89,012	178,015	266,997
	(4,096,308)	(4,096,775)	(4,098,178)	(4,100,515)
	1.000000	*1.000098*	*1.000390*	*1.000878*
36	0	90,164	180,319	270,455
	(3,985,342)	(3,985,805)	(3,987,192)	(3,989,504)
	1.000000	*1.000100*	*1.000400*	*1.000901*
35	0	91,289	182,568	273,830
	(3,874,395)	(3,874,852)	(3,876,223)	(3,878.507)
	1.000000	*1.000103*	*1.000411*	*1.000924*
34	0	92,385	184,762	277,121
	(3,763,467)	(3,763,918)	(3,765,270)	(3,767,524)
	1.000000	*1.000105*	*1.000421*	*1.000946*
33	0	93,454	186,899	280,328
	(3,652,557)	(3,653,001)	(3,654,333)	(3,656.554)
	1.000000	*1.000108*	*1.000431*	*1.000969*
32	0	94,494	188,980	283,449
	(3,541,665)	(3,542,102)	(3,543,413)	(3,545.597)
	1.000000	*1.000110*	*1.000440*	*1.000991*
31	0	95,505	191,002	286,484
	(3,430,790)	(3,431,220)	(3,432,507)	(3,434.653)
	1.000000	*1.000112*	*1.000450*	*1.001012*
30	0	96,487	192,967	289,432
	(3,319,933)	(3,320,354)	(3,321,617)	(3,323.722)
	1.000000	*1.000115*	*1.000459*	*1.001033*
29	0	97,440	194,872	292,291
	(3,209,093)	(3,209,506)	(3,210,742)	(3,212,803)
	1.000000	*1.000117*	*1.000468*	*1.001054*
28	0	98,363	196,719	295,062
	(3,098,270)	(3,098,673)	(3,099,882)	(3,101,897)
	1.000000	*1.000119*	*1.000477*	*1.001074*

TABLE 18.—*Polyconic Projection: Rectangular coordinates for the Clarke 1866 ellipsoid*—Continued

Long. λ / Lat. φ	0°	1°	2°	3°
27°	0	99,256	198,505	297,742
	(2,987,463)	(2,987,856)	(2,989,036)	(2,991,002)
	1.000000	*1.000122*	*1.000486*	*1.001094*
26	0	100,119	200,231	300,332
	(2,876,672)	(2,877,055)	(2,878,204)	(2,880,119)
	1.000000	*1.000124*	*1.000495*	*1.001113*
25	0	100,951	201,896	302,831
	(2,765,896)	(2,766,269)	(2,767,386)	(2,769,247)
	1.000000	*1.000126*	*1.000503*	*1.001132*
24	0	101,753	203,500	305,237
	(2,655,136)	(2,655,497)	(2,656,580)	(2,658,386)
	1.000000	*1.000128*	*1.000511*	*1.001150*
23	0	102,523	205,042	307,551
	(2,544,390)	(2,544,739)	(2,545,788)	(2,547,536)
	1.000000	*1.000130*	*1.000519*	*1.001168*

Note: x, y = rectangular coordinates, meters; origin at $\phi=0$, $\lambda=0$. Y axis increasing north.
h = scale factor along meridian.
k = scale factor along parallel = 1.0.
λ = longitude east of central meridian. For longitude west of central meridian reverse sign of x.

MODIFIED POLYCONIC FOR THE INTERNATIONAL MAP OF THE WORLD

A modified Polyconic projection was devised by Lallemand of France and in 1909 adopted by the International Map Committee (IMC) in London as the basis for the 1:1,000,000-scale International Map of the World (IMW) series. Used for sheets 6° of longitude by 4° of latitude between lats. 60° N. and 60° S., 12° of longitude by 4° of latitude between lats. 60° and 76° N. or S., and 24° by 4° between lats. 76° and 84° N. or S., the projection differs from the ordinary Polyconic in two principal features: All meridians are straight, and there are two meridians (2° east and west of the central meridian on sheets between lats. 60° N. & S.) that are made true to scale. Between lats. 60° & 76° N. and S., the meridians 4° east and west are true to scale, and between 76° & 84°, the true-scale meridians are 8° from the central meridian (United Nations, 1963, p. 22–23; Lallemand, 1911, p. 559).

The top and bottom parallels of each sheet are nonconcentric circular arcs constructed with radii of $N \cot \phi$, where $N = a/(1-e^2 \sin^2 \phi)^{1/2}$. These radii are the same as the radii on the regular Polyconic for the ellipsoid, and the arcs of these two parallels are marked off true to scale for the straight meridians. The two parallels, however, are spaced from each other according to the true scale along the two standard meridians, not according to the scale along the central meridian, which is slightly reduced. The approximately 440 mm true length of the central meridian at the map scale is thereby reduced by 0.270 to 0.076 mm, depending on the latitude of the sheet. Other parallels of lat. ϕ are circular arcs with radii $N \cot \phi$, intersecting the meridians which are true to scale at the correct points. The parallels strike other meridians at geometrically fixed locations which slightly deviate from the true scale on meridians as well as parallels.

With this modified Polyconic, as with USGS quadrangles based on the rectified Polyconic, adjacent sheets exactly fit together not only north to south, but east to west. There is still a gap when mosaicking in all directions, in that there is a gap between each diagonal sheet and either one or the other adjacent sheet.

In 1962, a U.N. conference on the IMW adopted the Lambert Conformal Conic and Polar Stereographic projections to replace the modified Polyconic (United Nations, 1963, p. 9–10). The USGS has prepared a number of sheets for the IMW series over the years according to the projection officially in use at the time.

AZIMUTHAL MAP PROJECTIONS

A third very important group of map projections, some of which have been known for 2,000 years, consists of five major azimuthal (or zenithal) projections and various less-common forms. While cylindrical and conic projections are related to cylinders and cones wrapped around the globe representing the Earth, the azimuthal projections are formed onto a plane which is usually tangent to the globe at either pole, the Equator, or any intermediate point. These variations are called the polar, equatorial (or meridian or meridional), and oblique (or horizon) aspects, respectively. Some azimuthals are true perspective projections; others are not. Although perspective cylindrical and conic projections are much less used than those which are not perspective, the perspective azimuthals are frequently used and have valuable properties. Complications arise when the ellipsoid is involved, but it is used only in special applications that are discussed below.

As stated earlier, azimuthal projections are characterized by the fact that the direction, or azimuth, from the center of the projection to every other point on the map is shown correctly. In addition, on the spherical forms, all great circles passing through the center of the projection are shown as straight lines. Therefore, the shortest route from this center to any other point is shown as a straight line. This fact made some of these projections especially popular for maps as flight and radio transmission became commonplace.

The five principal azimuthals are as follows:

(1) **Orthographic.** A true perspective, in which the Earth is projected from an infinite distance onto a plane. The map looks like a globe, thus stressing the roundness of the Earth.

(2) **Stereographic.** A true perspective in the spherical form, with the point of perspective on the surface of the sphere at a point exactly opposite the point of tangency for the plane, or opposite the center of the projection, even if the plane is secant. This projection is conformal for sphere or ellipsoid, but the ellipsoidal form is not truly perspective.

(3) **Gnomonic.** A true perspective, with the Earth projected from the center onto the tangent plane. All great circles, not merely those passing through the center, are shown as straight lines on the spherical form. Since the Gnomonic projection has not been used by the USGS, it is not discussed in detail.

(4) **Lambert Azimuthal Equal-Area.** Not a true perspective. Areas are correct, and the overall scale variation is less than that found on the major perspective azimuthals.

(5) **Azimuthal Equidistant.** Not a true perspective. Distances from the center of the projection to any other point are shown correctly. Overall scale variation is moderate compared to the perspective azimuthals.

A sixth azimuthal projection of increasing interest in the space age is the general vertical perspective (resembling the Orthographic), projecting the Earth from any point in space, such as a satellite, onto a tangent or secant plane. It is used primarily in derivations and pictorial representations and has not been used by the Geological Survey for published maps. Therefore, it is not discussed in this bulletin.

As a group, the azimuthals have unique esthetic qualities while remaining functional. There is a unity and roundness of the Earth on each (except perhaps the Gnomonic) which is not as apparent on cylindrical and conic projections.

The simplest forms of the azimuthal projections are the polar aspects, in which all meridians are shown as straight lines radiating at their true angles from the center, while parallels of latitude are circles, concentric about the pole. The difference is in the spacing of the parallels. Table 19 lists for all five of the above azimuthals the radius of every 10° of latitude on a sphere of radius 1.0 unit, centered on the North Pole. Scale factors and maximum angular deformation are also shown. The distortion is the same for the oblique and equatorial aspects at the same angular distance from the center of the projection, except that h and k are along and perpendicular to, respectively, radii from the center, not necessarily along meridians or parallels.

There are two principal drawbacks to the azimuthals. First, they are more difficult to construct than the cylindricals and the conics, except for the polar aspects. This drawback was more applicable, however, in the days before computers and plotters, but it is still more difficult to prepare a map having complex curves between plotted coordinates than it is to draw the entire graticule with circles and straight lines. Nevertheless, an increased use of azimuthal projections in atlases and for other published maps may be expected.

Secondly, most azimuthal maps do not have standard parallels or standard meridians. Each map has only one standard point: the center (except for the Stereographic, which may have a standard circle). Thus, the azimuthals are suitable for minimizing distortion in a somewhat circular region such as Antarctica, but not for an area with predominant length in one direction.

TABLE 19.—*Comparison of major azimuthal projections: Radius, scale factors, maximum angular distortion for projection of sphere with radius 1.0, North Polar aspect*

Lat.	Orthographic			
	Radius	h	k	ω
90°	0.00000	1.00000	1.0	0.000°
80	.17365	.98481	1.0	.877
70	.34202	.93969	1.0	3.563
60	.50000	.86603	1.0	8.234
50	.64279	.76604	1.0	15.23
40	.76604	.64279	1.0	25.12
30	.86603	.50000	1.0	38.94
20	.93969	.34202	1.0	58.72
10	.98481	.17365	1.0	89.51
0	1.00000	.00000	1.0	180.0
−10	—	—	—	—
−20	—	—	—	—
−30	—	—	—	—
−40	(beyond limits of map)		—	—
−50	—	—	—	—
−60	—	—	—	—
−70	—	—	—	—
−80	—	—	—	—
−90	—	—	—	—

Lat.	Stereographic	
	Radius	k*
90°	0.00000	1.00000
80	.17498	1.00765
70	.35263	1.03109
60	.53590	1.07180
50	.72794	1.13247
40	.93262	1.21744
30	1.15470	1.33333
20	1.40042	1.49029
10	1.67820	1.70409
0	2.00000	2.00000
−10	2.38351	2.42028
−20	2.85630	3.03961
−30	3.46410	4.00000
−40	4.28901	5.59891
−50	5.49495	8.54863
−60	7.46410	14.9282
−70	11.3426	33.1634
−80	22.8601	131.646
−90	∞	∞

TABLE 19.—*Comparison of major azimuthal projections: Radius, scale factors, maximum angular distortion for projection of sphere with radius 1.0, North Polar aspect*—Continued

Lat.	Gnomonic			
	Radius	*h*	*k*	ω
90°	0.00000	1.00000	1.00000	0.000°
80	.17633	1.03109	1.01543	.877
70	.36397	1.13247	1.06418	3.563
60	.57735	1.33333	1.15470	8.234
50	.83910	1.70409	1.30541	15.23
40	1.19175	2.42028	1.55572	25.12
30	1.73205	4.00000	2.00000	38.94
20	2.74748	8.54863	2.92380	58.72
10	5.67128	33.1634	5.75877	89.51
0	∞	∞	∞	--
-10	--	--	--	--
-20	--	--	--	--
-30	--	--	--	--
-40	(beyond limits of map)		--	--
-50	--	--	--	--
-60	--	--	--	--
-70	--	--	--	--
-80	--	--	--	--
-90	--	--	--	--

Lat.	Lambert Azimuthal Equal-Area			
	Radius	*h*	*k*	ω
90°	0.00000	1.00000	1.00000	0.000°
80	.17431	.99619	1.00382	.437
70	.34730	.98481	1.01543	1.754
60	.51764	.96593	1.03528	3.972
50	.68404	.93969	1.06418	7.123
40	.84524	.90631	1.10338	11.25
30	1.00000	.86603	1.15470	16.43
20	1.14715	.81915	1.22077	22.71
10	1.28558	.76604	1.30541	30.19
0	1.41421	.70711	1.41421	38.94
-10	1.53209	.64279	1.55572	49.07
-20	1.63830	.57358	1.74345	60.65
-30	1.73205	.50000	2.00000	73.74
-40	1.81262	.42262	2.36620	88.36
-50	1.87939	.34202	2.92380	104.5
-60	1.93185	.25882	3.86370	122.0
-70	1.96962	.17365	5.75877	140.6
-80	1.99239	.08716	11.4737	160.1
-90	2.00000	.00000	∞	180.0

TABLE 19.—*Comparison of major azimuthal projections: Radius, scale factors, maximum angular distortion for projection of sphere with radius 1.0, North Polar aspect*—Continued

Lat.	Azimuthal Equidistant			
	Radius	***h***	***k***	ω
90°	0.00000	1.0	1.00000	0.000°
80	.17453	1.0	1.00510	.291
70	.34907	1.0	1.02060	1.168
60	.52360	1.0	1.04720	2.642
50	.69813	1.0	1.08610	4.731
40	.87266	1.0	1.13918	7.461
30	1.04720	1.0	1.20920	10.87
20	1.22173	1.0	1.30014	15.00
10	1.39626	1.0	1.41780	19.90
0	1.57080	1.0	1.57080	25.66
-10	1.74533	1.0	1.77225	32.35
-20	1.91986	1.0	2.04307	40.09
-30	2.09440	1.0	2.41840	49.03
-40	2.26893	1.0	2.96188	59.36
-50	2.44346	1.0	3.80135	71.39
-60	2.61799	1.0	5.23599	85.57
-70	2.79253	1.0	8.16480	102.8
-80	2.96706	1.0	17.0866	125.6
-90	3.14159	1.0	∞	180.0

Radius = radius of circle showing given latitude.

ω = maximum angular deformation.
h = scale factor along meridian of longitude.
k = scale factor along parallel of latitude.
* For Stereographic, $h = k$ and $\omega = 0$.

16. ORTHOGRAPHIC PROJECTION

SUMMARY

- Azimuthal.
- All meridians and parallels are ellipses, circles, or straight lines.
- Neither conformal nor equal-area.
- Closely resembles a globe in appearance, since it is a perspective projection from infinite distance.
- Only one hemisphere can be shown at a time.
- Much distortion near the edge of the hemisphere shown.
- No distortion at the center only.
- Directions from the center are true.
- Radial scale factor decreases as distance increases from the center.
- Scale in the direction of the lines of latitude is true in the polar aspect.
- Used chiefly for pictorial views.
- Used only in the spherical form.
- Known by Egyptians and Greeks 2,000 years ago.

HISTORY

To the layman, the best known perspective azimuthal projection is the Orthographic, although it is the least useful for measurements. While its distortion in shape and area is quite severe near the edges, and only one hemisphere may be shown on a single map, the eye is much more willing to forgive this distortion than to forgive that of the Mercator projection because the Orthographic projection makes the map look very much like a globe appears, especially in the oblique aspect.

The Egyptians were probably aware of the Orthographic projection, and Hipparchus of Greece (2nd century B.C.) used the equatorial aspect for astronomical calculations. Its early name was "analemma," a name also used by Ptolemy, but it was replaced by "orthographic" in 1613 by Francois d' Aiguillon of Antwerp. While it was also used by Indians and Arabs for astronomical purposes, it is not known to have been used for world maps older than 16th-century works by Albrecht Dürer (1471–1528), the German artist and cartographer, who prepared polar and equatorial versions (Keuning, 1955, p. 6).

FEATURES

The point of perspective for the Orthographic projection is at an infinite distance, so that the meridians and parallels are projected onto the tangent plane with parallel projection lines. All meridians and parallels are shown as ellipses, circles, or straight lines.

As on all polar azimuthal projections, the meridians of the polar Orthographic projection appear as straight lines radiating from the pole at their true angles, while the parallels of latitude are complete circles centered about the pole. On the Orthographic, the parallels are spaced most widely near the pole, and the spacing decreases to zero at the Equator, which is the circle marking the edge of the map (figs. 20, 21*A*). As a result, the land shapes near the pole are prominent, while lands near the Equator are compressed so that they can hardly be recognized. In spite of the fact that the scale along the meridians varies from the correct value at the pole to zero at the Equator, the scale along every parallel is true.

The equatorial aspect of the Orthographic projection has as its center some point on the Earth's Equator. Here, all the parallels of latitude including the Equator are seen edge-on; thus, they appear as straight parallel lines (fig. 21*B*). The meridians, which are shaped like circles on the sphere, are projected onto the map at various inclinations to the lines of perspective. The central meridian, seen edge-on, is a straight line. The meridian 90° from the central meridian is shown as a circle marking the limit of the equatorial aspect. This circle is equidistantly marked with parallels of latitude. Other meridians are ellipses of eccentricities ranging from zero (the bounding circle) to 1.0 (the central meridian).

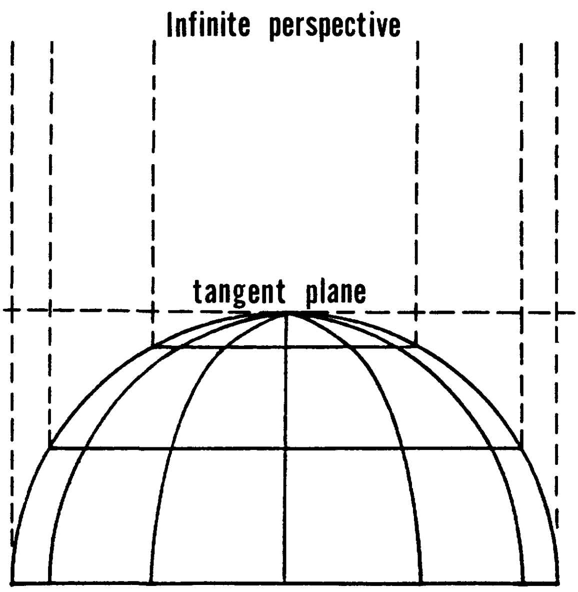
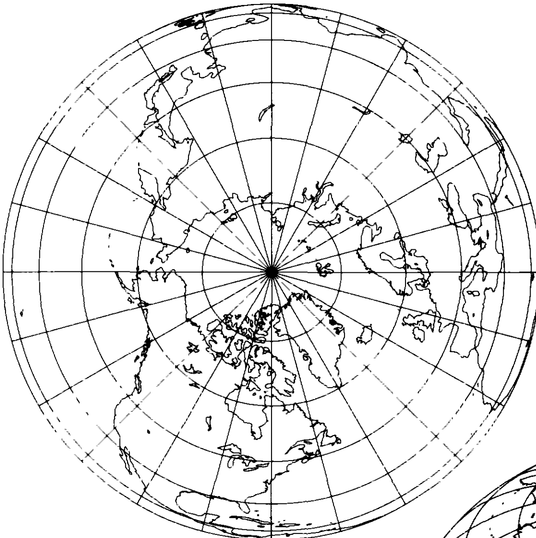
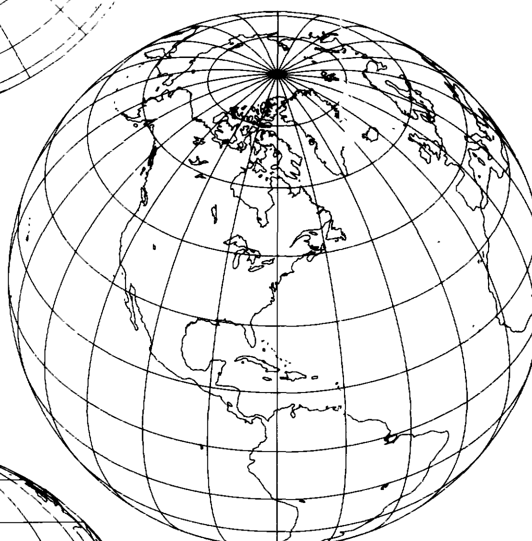


FIGURE 20.–Geometric projection of the parallels of the polar Orthographic projection.

FIGURE 21.–Orthographic projection. (*A*) Polar aspect. (*B*) Equatorial aspect, approximately the view of the Moon, Mars, and other outer planets as seen from the Earth. (*C*) Oblique aspect, centered at lat. 40° N., giving the classic globelike view.



A



C

B

The oblique Orthographic projection, with its center somewhere between the Equator and a pole, gives the classic globelike appearance; and in fact an oblique view, with its center near but not on the Equator or pole, is often preferred to the equatorial or polar aspect for pictorial purposes. On the oblique Orthographic, the only straight line is the central meridian, if it is actually portrayed. All parallels of latitude are ellipses with the same eccentricity (fig. 21*C*). Some of these ellipses are shown completely and some only partially, while some cannot be shown at all. All other meridians are also ellipses of varying eccentricities. No meridian appears as a circle on the oblique aspect.

The intersection of any given meridian and parallel is shown on an Orthographic projection at the same distance from the central meridian, regardless of whether the aspect is oblique, polar, or equatorial, provided the same central meridian and the same scale are maintained. Scale and distortion, as on all azimuthal projections, change only with the distance from the center. The center of projection has no distortion, but the outer regions are compressed, even though the scale is true along all circles drawn about the center. (These circles are not "standard" lines because the scale is true only in the direction followed by the line.)

USAGE

The Orthographic projection seldom appears in atlases, except as a globe in relief without meridians and parallels. When it does appear, it provides a striking view. Richard Edes Harrison has used the Orthographic for several maps in an atlas of the 1940's partially based on this projection. Frank Debenham (1958) used photographed relief globes extensively in *The Global Atlas,* and Rand McNally has done likewise in their world atlases since 1960. The USGS has used it occasionally as a frontispiece or end map (USGS, 1970; Thompson, 1979), but it also provided a base for definitive maps of voyages of discovery across the North Atlantic (USGS, 1970, p. 133).

It became especially popular during the Second World War when there was stress on the global nature of the conflict. With some space flights of the 1960's, the first photographs of the Earth from space renewed consciousness of the Orthographic concept.

GEOMETRIC CONSTRUCTION

The three aspects of the Orthographic projection may be graphically constructed with an adaptation of the draftsman's technique shown by Raisz (1962, p. 180). Referring to figure 22, circle *A* is drawn for the polar aspect, with meridians marked at true angles. Perpendiculars are

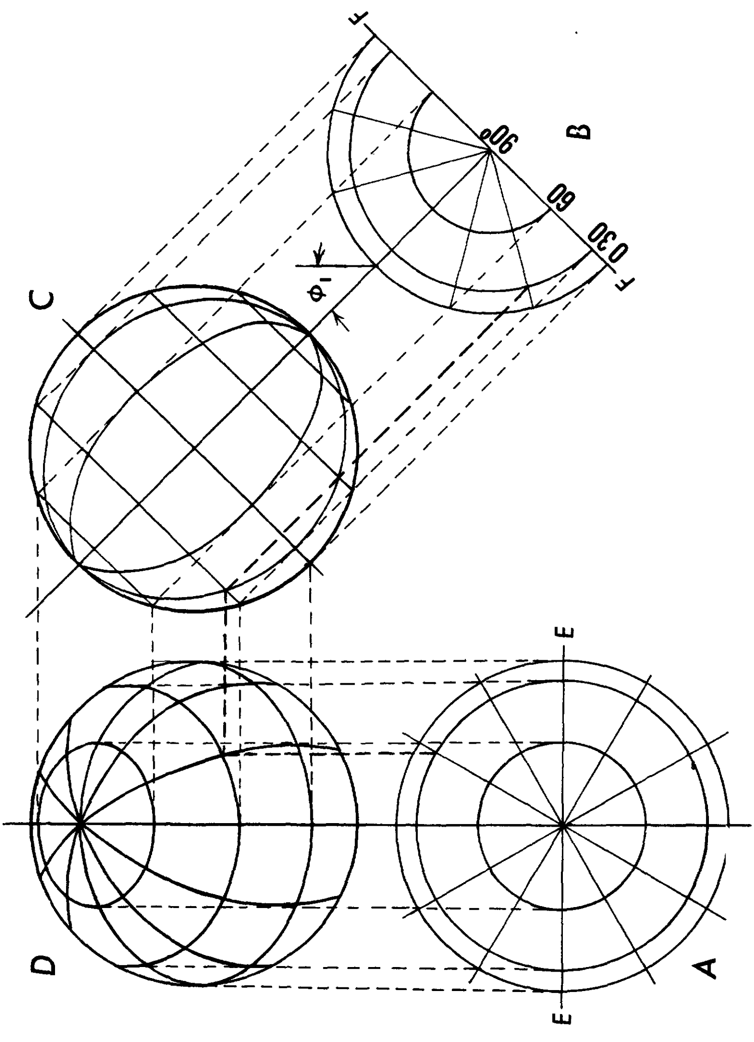


FIGURE 22.—Geometric construction of polar, equatorial, and oblique Orthographic projections.

dropped from the intersections of the outer circle with the meridians onto the horizontal meridian *EE*. This determines the radii of the parallels of latitude, which may then be drawn about the center.

For the equatorial aspect, circle *C* is drawn with the same radius as *A*, circle *B* is drawn like half of circle *A*, and the outer circle of *C* is equidistantly marked to locate intersections of parallels with that circle. Parallels of latitude are drawn as straight lines, with the Equator midway. Parallels are shown tilted merely for use with oblique projection circle *D*. Points at intersections of parallels with other meridians of *B* are then projected onto the corresponding parallels of latitude on *C*, and the new points connected for the meridians of *C*. By tilting graticule *C* at an angle ϕ_1 equal to the central latitude of the desired oblique aspect, the corresponding points of circles *A* and *C* may be projected vertically and horizontally, respectively, onto circle *D* to provide intersections for meridians and parallels.

FORMULAS FOR THE SPHERE

To understand the mathematical concept of the Orthographic projection, it is helpful to think in terms of polar coordinates ρ and θ:

$$\rho = R \sin c \qquad (16\text{–}1)$$

$$\theta = \pi - Az = 180° - Az \qquad (16\text{–}2)$$

where c is the angular distance of the given point from the center of projection. Az is the azimuth east of north, and θ is the polar coordinate east of south. The distance from the center of a point on an Orthographic map projection is thus proportional to the sine of the angular distance from the center on the sphere. Applying equations (5–3), (5–4), and (5–5) for great circle distance c and azimuth Az in terms of latitude and longitude, and equations for rectangular coordinates in terms of polar coordinates, the equations for rectangular coordinates for the oblique Orthographic projection reduce to the following, given R, ϕ_1, λ_0, ϕ, and λ:

$$x = R \cos\phi \sin(\lambda - \lambda_0) \qquad (16\text{–}3)$$

$$y = R[\cos\phi_1 \sin\phi - \sin\phi_1 \cos\phi \cos(\lambda - \lambda_0)] \qquad (16\text{–}4)$$

$$h' = \cos c$$

$$= \sin\phi_1 \sin\phi + \cos\phi_1 \cos\phi \cos(\lambda - \lambda_0) \qquad (16\text{–}5)$$

$$k' = 1.0$$

where ϕ_1 and λ_0 are the latitude and longitude, respectively, of the center point and origin of the projection, h' is the scale factor along a line radiating from the center, and k' is the scale factor in a direction perpendicular to a line radiating from the center. The *Y* axis coincides with the central meridian λ_0, y increasing northerly. All the parallels are ellipses of eccentricity $\cos\phi_1$.

For the north polar Orthographic, letting $\phi_1 = 90°$, x is still found from (16–3), but

$$y = -R \cos \phi \cos (\lambda - \lambda_0) \quad (16\text{–}6)$$
$$h = \sin \phi \quad (16\text{–}7)$$

In polar coordinates,

$$\rho = R \cos \phi \quad (16\text{–}8)$$
$$\theta = \lambda - \lambda_0 \quad (16\text{–}9)$$

For the south polar Orthographic, with $\phi_1 = -90°$, x does not change, but

$$y = R \cos \phi \cos (\lambda - \lambda_0) \quad (16\text{–}10)$$
$$h = -\sin \phi \quad (16\text{–}11)$$

For polar coordinates, ρ is found from (16–8), but

$$\theta = \pi - \lambda + \lambda_0 \quad (16\text{–}12)$$

For the equatorial Orthographic, letting $\phi_1 = 0$, x still does not change from (16–3), but

$$y = R \sin \phi \quad (16\text{–}13)$$

In automatically computing a general set of coordinates for a complete Orthographic map, the distance c from the center should be calculated for each intersection of latitude and longitude to determine whether it exceeds 90° and therefore whether the point is beyond the range of the map. More directly, using equation (5–3),

$$\cos c = \sin \phi_1 \sin \phi + \cos \phi_1 \cos \phi \cos (\lambda - \lambda_0) \quad (5\text{–}3)$$

if $\cos c$ is zero or positive, the point is to be plotted. If $\cos c$ is negative, the point is not to be plotted.

For the inverse formulas for the sphere, to find ϕ and λ, given R, ϕ_1, λ_0, x, and y:

$$\phi = \arcsin [\cos c \sin \phi_1 + (y \sin c \cos \phi_1 / \rho)] \quad (16\text{–}14)$$

But if $\rho = 0$, $\phi = \phi_1$.
If ϕ_1 is not $\pm 90°$,

$$\lambda = \lambda_0 + \arctan [x \sin c / (\rho \cos \phi_1 \cos c - y \sin \phi_1 \sin c)] \quad (16\text{–}15)$$

If ϕ_1 is 90°,

$$\lambda = \lambda_0 + \arctan [x / (-y)] \quad (16\text{–}16)$$

If ϕ_1 is −90°,

$$\lambda = \lambda_0 + \arctan (x/y) \quad (16\text{–}17)$$

Note that, while the ratio $[x/(-y)]$ in (16–16) is numerically the same as $(-x/y)$, the necessary quadrant adjustment is different when using the Fortran ATAN2 function or its equivalent.

In equations (16–14) and (16–15),

$$\rho = (x^2 + y^2)^{1/2} \tag{16–18}$$

$$c = \arcsin(\rho/R) \tag{16–19}$$

Simplification for inverse equations for the polar and equatorial aspects is obtained by giving ϕ_1 values of $\pm 90°$ and $0°$, respectively. They are not given in detail here.

Tables 20 and 21 list rectangular coordinates for the equatorial and oblique aspects, respectively, for a 10° graticule with a sphere of radius $R = 1.0$. For the oblique example, $\phi_1 = 40°$.

TABLE 20.—*Orthographic projection: Rectangular coordinates for equatorial aspect*

Long.		0°	10°	20°	30°	40°
Lat.	*y*			*x*		
90°	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000
80	.9848	.0000	.0302	.0594	.0868	.1116
70	.9397	.0000	.0594	.1170	.1710	.2198
60	.8660	.0000	.0868	.1710	.2500	.3214
50	.7660	.0000	.1116	.2198	.3214	.4132
40	.6428	.0000	.1330	.2620	.3830	.4924
30	.5000	.0000	.1504	.2962	.4330	.5567
20	.3420	.0000	.1632	.3214	.4698	.6040
10	.1736	.0000	.1710	.3368	.4924	.6330
0	.0000	.0000	.1736	.3420	.5000	.6428

Long.	50°	60°	70°	80°	90°
Lat.			*x*		
90°	0.0000	0.0000	0.0000	0.0000	0.0000
80	.1330	.1504	.1632	.1710	.1736
70	.2620	.2962	.3214	.3368	.3420
60	.3830	.4330	.4698	.4924	.5000
50	.4924	.5567	.6040	.6330	.6428
40	.5868	.6634	.7198	.7544	.7660
30	.6634	.7500	.8138	.8529	.8660
20	.7198	.8138	.8830	.9254	.9397
10	.7544	.8529	.9254	.9698	.9848
0	.7660	.8660	.9397	.9848	1.0000

Radius of sphere = 1.0

Origin: $(x, y) = 0$ at (lat., long.) = 0. Y axis increases north. Other quadrants of hemisphere are symmetrical.

TABLE 21.—*Orthographic projection: Rectangular coordinates for oblique aspect centered at lat. 40° N.*

[The circle bounding the hemisphere map has the same coordinates as the $\lambda = 90°$ circle on the equatorial Orthographic projection. The radius of the sphere = 1.0. y coordinate in parentheses under x coordinate]

Lat. \ Long.	0°	10°	20°	30°	40°
90°	0.0000 (.7660)	0.0000 (.7660)	0.0000 (.7660)	0.0000 (.7660)	0.0000 (.7660)
80	.0000 (.6428)	.0302 (.6445)	.0594 (.6495)	.0868 (.6577)	.1116 (.6689)
70	.0000 (.5000)	.0594 (.5033)	.1170 (.5133)	.1710 (.5295)	.2198 (.5514)
60	.0000 (.3420)	.0868 (.3469)	.1710 (.3614)	.2500 (.3851)	.3214 (.4172)
50	.0000 (.1736)	.1116 (.1799)	.2198 (.1986)	.3214 (.2290)	.4132 (.2703)
40	.0000 (.0000)	.1330 (.0075)	.2620 (.0297)	.3830 (.0660)	.4924 (.1152)
30	.0000 (–.1736)	.1504 (–.1652)	.2962 (–.1401)	.4330 (–.0991)	.5567 (–.0434)
20	.0000 (–.3420)	.1632 (–.3328)	.3214 (–.3056)	.4698 (–.2611)	.6040 (–.2007)
10	.0000 (–.5000)	.1710 (–.4904)	.3368 (–.4618)	.4924 (–.4152)	.6330 (–.3519)
0	.0000 (–.6428)	.1736 (–.6330)	.3420 (–.6040)	.5000 (–.5567)	.6428 (–.4924)
–10	.0000 (–.7660)	.1710 (–.7564)	.3368 (–.7279)	.4924 (–.6812)	.6330 (–.6179)
–20	.0000 (–.8660)	.1632 (–.8568)	.3214 (–.8296)	.4698 (–.7851)	.6040 (–.7247)
–30	.0000 (–.9397)	.1504 (–.9312)	.2962 (–.9061)	.4330 (–.8651)	.5567 (–.8095)
–40	.0000 (–.9848)	.1330 (–.9773)	.2620 (–.9551)	.3830 (–.9188)	.4924 (–.8696)
–50	.0000 (–1.0000)	— —	— —	— —	— —

Origin: $(x, y) = 0$ at (lat., long.) = (40°, 0). Y axis increases north. Coordinates shown for central meridian ($\lambda = 0$) and meridians east of central meridian. For meridians west (negative), reverse signs of meridians and of x.

TABLE 21.—*Orthographic projection: Rectangular coordinates for oblique aspect centered at lat. 40° N.*—Continued

Lat. \ Long.	50°	60°	70°	80°	90°
90°	0.0000	0.0000	0.0000	0.0000	0.0000
	(.7660)	(.7660)	(.7660)	(.7660)	(.7660)
80	.1330	.1504	.1632	.1710	.1736
	(.6827)	(.6986)	(.7162)	(.7350)	(.7544)
70	.2620	.2962	.3214	.3368	.3420
	(.5785)	(.6099)	(.6447)	(.6817)	(.7198)
60	.3830	.4330	.4698	.4924	.5000
	(.4568)	(.5027)	(.5535)	(.6076)	(.6634)
50	.4924	.5567	.6040	.6330	.6428
	(.3212)	(.3802)	(.4455)	(.5151)	(.5868)
40	.5868	.6634	.7198	.7544	.7660
	(.1759)	(.2462)	(.3240)	(.4069)	(.4924)
30	.6634	.7500	.8138	.8529	.8660
	(.0252)	(.1047)	(.1926)	(.2864)	(.3830)
20	.7198	.8138	.8830	.9254	.9397
	(−.1263)	(−.0400)	(.0554)	(.1571)	(.2620)
10	.7544	.8529	.9254	.9698	.9848
	(−.2739)	(−.1835)	(−.0835)	(.0231)	(.1330)
0	.7660	.8660	.9397	.9848	1.0000
	(−.4132)	(−.3214)	(−.2198)	(−.1116)	(.0000)
−10	.7544	.8529	.9254	.9698	—
	(−.5399)	(−.4495)	(−.3495)	(−.2429)	—
−20	.7198	.8138	.8830	—	—
	(−.6503)	(−.5640)	(−.4686)	—	—
−30	.6634	.7500	—	—	—
	(−.7408)	(−.6614)	—	—	—
−40	—	—	—	—	—

TABLE 21.—*Orthographic projection: Rectangular coordinates for oblique aspect centered at lat. 40° N.*—Continued

Lat. \ Long.	100°	110°	120°	130°	140°
90°	0.0000	0.0000	0.0000	0.0000	0.0000
	(.7660)	(.7660)	(.7660)	(.7660)	(.7660)
80	.1710	.1632	.1504	1330	.1116
	(.7738)	(.7926)	(.8102)	(.8262)	(.8399)
70	.3368	.3214	.2962	.2620	.2198
	(.7580)	(.7950)	(.8298)	(.8612)	(.8883)
60	.4924	.4698	.4330	.3830	.3214
	(.7192)	(.7733)	(.8241)	(.8700)	(.9096)
50	.6330	.6040	.5567	.4924	.4132
	(.6586)	(.7281)	(.7934)	(.8524)	(.9033)
40	.7544	.7198	.6634	.5868	—
	(.5779)	(.6608)	(.7386)	(.8089)	—
30	.8529	.8138	—	—	—
	(.4797)	(.5734)	—	—	—
20	.9254	—	—	—	—
	(.3669)	—	—	—	—

TABLE 21.—*Orthographic projection: Rectangular coordinates for oblique aspect centered at lat. 40° N.*—Continued

Lat. \ Long.	150°	160°	170°	180°
90°	0.0000 (.7660)	0.0000 (.7660)	0.0000 (.7660)	0.0000 (.7660)
80	.0868 (.8511)	.0594 (.8593)	.0302 (.8643)	.0000 (.8660)
70	.1710 (.9102)	.1170 (.9264)	.0594 (.9364)	.0000 (.9397)
60	.2500 (.9417)	.1710 (.9654)	.0868 (.9799)	.0000 (.9848)
50	.3214 (.9446)	.2198 (.9751)	.1116 (.9937)	.0000 (1.0000)
40	--	--	--	--

17. STEREOGRAPHIC PROJECTION

SUMMARY

- Azimuthal.
- Conformal.
- The central meridian and a particular parallel (if shown) are straight lines.
- All meridians on the polar aspect and the Equator on the equatorial aspect are straight lines.
- All other meridians and parallels are shown as arcs of circles.
- A perspective projection for the sphere.
- Directions from the center of the projection are true (except on ellipsoidal oblique and equatorial aspects).
- Scale increases away from the center of the projection.
- Point opposite the center of the projection cannot be plotted.
- Used for polar maps and miscellaneous special maps.
- Apparently invented by Hipparchus (2nd century B.C.).

HISTORY

The Stereographic projection was probably known in its polar form to the Egyptians, while Hipparchus was apparently the first Greek to use it. He is generally considered its inventor. Ptolemy referred to it as "Planisphaerum," a name used into the 16th century. The name "Stereographic" was assigned to it by François d' Aiguillon in 1613. The polar Stereographic was exclusively used for star maps until perhaps 1507, when the earliest-known use for a map of the world was made by Walther Ludd (Gaultier Lud) of St. Dié, Lorraine.

The oblique aspect was used by Theon of Alexandria in the 4th century for maps of the sky, but it was not proposed for geographical maps until Stabius and Werner discussed it together with their cordiform (heart-shaped) projections in the early 16th century. The earliest-known world maps were included in a 1583 atlas by Jacques de Vaulx (c. 1555–97). The two hemispheres were centered on Paris and its opposite point, respectively.

The equatorial Stereographic originated with the Arabs, and was used by the Arab astronomer Ibn-el-Zarkali (1029–87) of Toledo for an astrolabe. It became a basis for world maps in the early 16th century, with the earliest known examples by Jean Roze (or Rotz), a Norman, in 1542. After Rumold (the son of Gerhardus) Mercator's use of the equatorial Stereographic for the world maps of the atlas of 1595, it became very popular among cartographers (Keuning, 1955, p. 7–9; Nordenskiöld, 1889, p. 90, 92–93).

FEATURES

Like the Orthographic, the Stereographic projection is a true perspective in its spherical form. It is the only known true perspective projection of any kind that is also conformal. Its point of projection is on the surface of the sphere at a point just opposite the point of tangency of the plane or the center point of the projection (fig. 23). Thus, if the North Pole is the center of the map, the projection is from the South Pole. All of one hemisphere can be comfortably shown, but it is impossible to show both hemispheres in their entirety from one center. The point on the sphere opposite the center of the map projects at an infinite distance in the plane of the map.

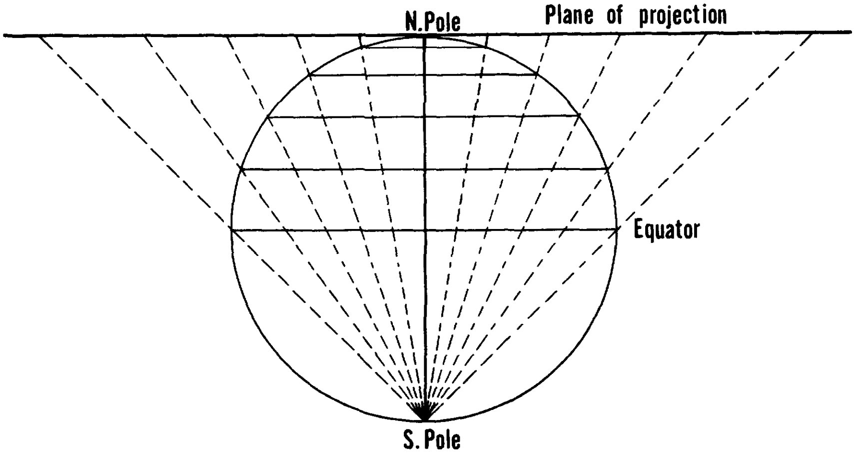
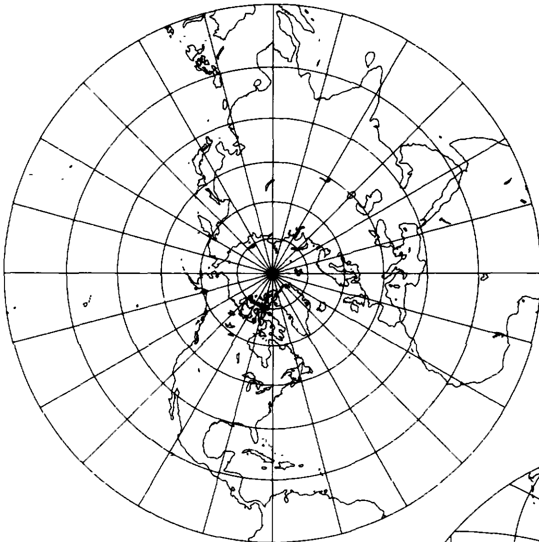


FIGURE 23.–Geometric projection of the polar Stereographic projection.

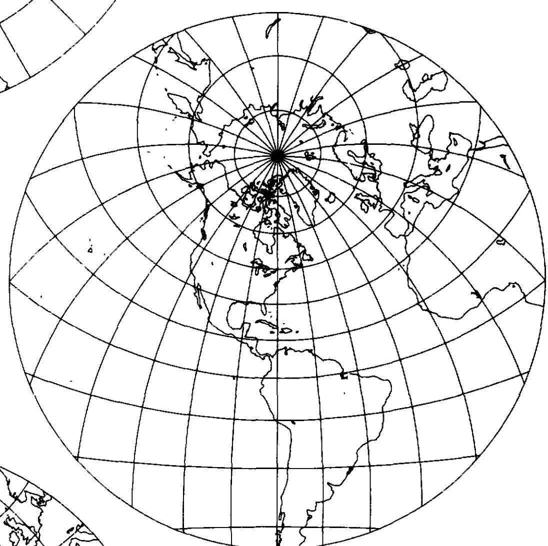
The polar aspect somewhat resembles other polar azimuthals, with straight radiating meridians and concentric circles for parallels (fig. 24*A*). The parallels are spaced at increasingly wide distances, the farther the latitude is from the pole (the Orthographic has the opposite feature).

In the equatorial and oblique aspects, the distinctive appearance of the Stereographic becomes more evident: All meridians and parallels, except for two, are shown as circles, and the meridians intersect the parallels at right angles (figs. 24*B*, *C*). The central meridian is shown straight, as is the parallel of the same numerical value, but opposite in sign to the central parallel. For example, if lat. 40° N. is the central parallel, then lat. 40° S. is shown as a straight line. For the equatorial aspect with lat. 0° as the central parallel, the Equator, which is of course also its own negative counterpart, is shown straight. (For the polar aspect, this has no meaning since the opposite pole cannot be shown.) Circles for parallels are centered along the central meridian;

FIGURE 24.–Stereographic projection. (*A*) polar aspect; the most common scientific projection for polar areas of Earth, Moon, and the planets, since it is conformal. (*B*) equatorial aspect; often used in the 16th and 17th centuries for maps of hemispheres. (*C*) oblique aspect; centered on lat. 40° N. The Stereographic is the only geometric projection of the sphere which is conformal.

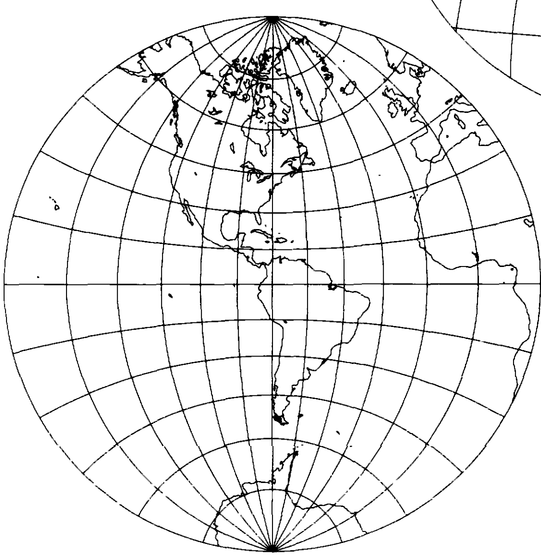


A



C

B



circles for meridians are centered along the straight parallel. The meridian 90° from the central meridian on the equatorial aspect is shown as a circle bounding the hemisphere. This circle is centered on the projection center and is equidistantly marked for parallels of latitude.

As an azimuthal projection, directions from the center are shown correctly in the spherical form. In the ellipsoidal form, only the polar aspect is truly azimuthal, but it is not perspective, in order to retain conformality. The oblique and equatorial aspects of the ellipsoidal Stereographic, in order to be conformal, are neither azimuthal nor perspective. As with other azimuthal projections, there is no distortion at the center, which may be made the "standard point" true to scale in all directions. Because of the conformality of the projection, a Stereographic map may be given, instead of a "standard point," a "standard circle" (or "standard parallel" in the polar aspect) with an appropriate radius from the center, balancing the scale error throughout the map. (On the ellipsoidal oblique or equatorial aspects, the lines of constant scale are not perfect circles.) This cannot be done with nonconformal azimuthal projections. In fact, O. M. Miller (1953) took the standard circle a step further and modified the spherical Stereographic to produce a standard oval better suited for a combined map of Europe and Africa. This projection is called Miller's Prolated Stereographic.

USAGE

While the oblique aspect of the Stereographic projection has been recently used in the spherical form by the USGS for circular maps of portions of the Moon, Mars, and Mercury, generally centered on a basin, the USGS has most often used the Stereographic in the polar aspect and ellipsoidal form for maps of Antarctica. For 1:500,000 sketch maps, the standard parallel is 71° S.; for its 1:250,000-scale series between 80° and the South Pole, the standard parallel is 80°14′ S. The Universal Transverse Mercator (UTM) grid employs the UPS (Universal Polar Stereographic) projection from the North Pole to lat. 84° N., and from the South Pole to lat. 80° S. For the UPS, the scale at each pole is reduced to 0.994, resulting in a standard parallel of about 81°07′ N. or S.

In 1962, a United Nations conference changed the polar portion of the International Map of the World (at a scale of 1:1,000,000) from a modified Polyconic to the polar Stereographic. This has consequently affected IMW sheets drawn by the USGS. North of lat. 84° N. or south of lat. 80° S., it is used "with scale matching that of the Modified Polyconic Projection or the Lambert Conformal Conic Projection at Latitudes 84° N. and 80° S." (United Nations, 1963, p. 10). The reference ellipsoid for all these polar Stereographic projections is the International of 1924.

TABLE 22.—*Polar Stereographic projection: Used for extraterrestial mapping*

[From Batson, 1973; Davies and Batson, 1975; Batson and others, 1980; Batson, private commun., 1981]

Body[1]	Scale[2]	Range in lat. N. or S. (scale at pole)	Adjacent projections	Overlap	Matching parallel with (scale)[3]	Comments
Mercury	1:15,000,000 at Equator	55° to pole (1:9,172,000)	Mercator	2°	56° (1:8,388,000)	--
	1:5,000,000 at Equator.	65° to pole (1:4,749,000)	Lambert Conformal Conic.	5°	67.5° (1:4,568,000)	--
Mars	1:25,000,000 at Equator.	55° to pole (1:13,449,000)	Mercator	10°	60° (1:12,549,000)	--
	1:15,000,000 at Equator.	55° to pole (1:9,203,000)	Mercator	2°	56° (1:8,418,000)	--
	1:5,000,000 at Equator.	65° to pole (1:4,518,000)	Lambert Conformal Conic.	0°	65° (1:4,306,000)	--
	1:1,000,000 } 1:250,000 }	65° to pole (varies)	--	--	--	Quadrangles 200-800 km on a side.
	1:2,000,000	65° to pole (1:2,035,000)	Lambert Conformal Conic.	0°	65° (1:1,939,000)	Quadrangles 45° long. x 13° lat. (between 65° & 78° lat.). Semicircles 180° long. x 12° lat. (between 78° lat & pole).
Galilean satellites of Jupiter						
Io	1:25,000,000 at Equator.	55° to pole (1:15,287,000)	Mercator	2°	56° (1:13,980,000)	
Europa	1:15,000,000 at Equator.	55° to pole (1:9,172,000)	Mercator	2°	56° (1:8,388,000)	
Ganymede	1:5,000,000 (Io & Europa).	45° to pole (1:5,000,000)	Mercator	5°	45° (1:4,268,000)	
Callisto	1:5,000,000 (Ganymede & Callisto).	65° to pole (1:5,000,000)	Lambert Conformal Conic.	1°	65.2° (1:4,769,000)	
Satellites of Saturn						
Mimas, Enceladus, Hyperion	1:5,000,000 at Equator.	55° to pole (1:3,057,000)	Mercator	2°	56° (1:2,796,000)	
Tethys, Dione, Rhea, Iapetus	1:10,000,000 at Equator.	55° to pole (1:6,115,000)	Mercator	2°	56° (1:5,592,000)	

[1] Taken as sphere, except for Mars (ellipsoid). See table 2.

[2] Equator refers to Mercator zone. Scales of 1:1,000,000 and 1:250,000 (Mars) occur at central parallel of quadrangle. Scale of 1:5,000,000 for satellites occurs at pole of Stereographic projection. Scale of 1:2,000,000 occurs at standard parallels of Lambert Conformal Conic projection.

[3] Matching parallels are both N. & S.

The Astrogeology Center of the Geological Survey at Flagstaff, Ariz., has been using the polar Stereographic for the mapping of polar areas of Mars, Mercury, and satellites of Jupiter and Saturn at various scales (see table 22).

The USGS is preparing a geologic map of the Arctic regions, using as a base an American Geographical Society map of the Arctic at a scale of 1:5,000,000. Drawn to the Stereographic projection, the map is based on a sphere having a radius which gives it the same volume as the International ellipsoid, and lat. 71° N. is made the standard parallel.

FORMULAS FOR THE SPHERE

Mathematically, a point at a given angular distance from the chosen center point on the sphere is plotted on the Stereographic projection at a distance from the center proportional to the trigonometric tangent of half that angular distance, and at its true azimuth, or, if the central scale factor is 1,

$$\rho = 2R \tan {}^1\!/_2\, c \qquad (17\text{–}1)$$

$$\theta = \pi - Az = 180° - Az \qquad (16\text{–}2)$$

$$k = \sec^2 {}^1\!/_2\, c \qquad (17\text{–}1a)$$

where c is the angular distance from the center, Az is the azimuth east of north (see equations (5–3) through (5–6)), and θ is the polar coordinate east of south. Combining with standard equations, the formulas for rectangular coordinates of the oblique Stereographic projection are found to be as follows, given R, k_0, ϕ_1, λ_0, ϕ, and λ:

$$x = Rk \cos\phi \sin(\lambda - \lambda_0) \qquad (17\text{–}2)$$

$$y = Rk [\cos\phi_1 \sin\phi - \sin\phi_1 \cos\phi \cos(\lambda - \lambda_0)] \qquad (17\text{–}3)$$

where

$$k = 2k_0/[1 + \sin\phi_1 \sin\phi + \cos\phi_1 \cos\phi \cos(\lambda - \lambda_0)] \qquad (17\text{–}4)$$

and (ϕ_1, λ_0) are the latitude and longitude of the center, which is also the origin. Since this is a conformal projection, k is the scale factor in all directions, based on a central scale factor of k_0, normally 1.0, but which may be reduced. The Y axis coincides with the central meridian λ_0, y increasing northerly and x, easterly.

If $\phi = -\phi_1$, and $\lambda = \lambda_0 \pm 180°$, the point cannot be plotted. Geometrically, it is the point from which projection takes place.

For the north polar Stereographic, with $\phi_1 = 90°$, these simplify to

$$x = 2R\,k_0 \tan(\pi/4 - \phi/2) \sin(\lambda - \lambda_0) \quad (17\text{–}5)$$
$$y = -2R\,k_0 \tan(\pi/4 - \phi/2) \cos(\lambda - \lambda_0) \quad (17\text{–}6)$$
$$k = 2k_0/(1 + \sin\phi) \quad (17\text{–}7)$$
$$\rho = 2R\,k_0 \tan(\pi/4 - \phi/2) \quad (17\text{–}8)$$
$$\theta = \lambda - \lambda_0 \quad (16\text{–}9)$$

For the south polar Stereographic with $\phi_1 = -90°$,

$$x = 2R\,k_0 \tan(\pi/4 + \phi/2) \sin(\lambda - \lambda_0) \quad (17\text{–}9)$$
$$y = 2R\,k_0 \tan(\pi/4 + \phi/2) \cos(\lambda - \lambda_0) \quad (17\text{–}10)$$
$$k = 2k_0/(1 - \sin\phi) \quad (17\text{–}11)$$
$$\rho = 2R\,k_0 \tan(\pi/4 + \phi/2) \quad (17\text{–}12)$$
$$\theta = \pi - \lambda + \lambda_0 \quad (16\text{–}12)$$

For the equatorial aspect, letting $\phi_1 = 0$, x is found from (17–2), but

$$y = R\,k \sin\phi \quad (17\text{–}13)$$
$$k = 2\,k_0/[1 + \cos\phi \cos(\lambda - \lambda_0)] \quad (17\text{–}14)$$

For the inverse formulas for the sphere, given R, k_0, ϕ_1, λ_0, x, and y:

$$\phi = \arcsin[\cos c \sin\phi_1 + (y \sin c \cos\phi_1/\rho)] \quad (16\text{–}14)$$

but if $\rho = 0$, $\phi = \phi_1$.
If ϕ_1 is not $\pm 90°$:

$$\lambda = \lambda_0 + \arctan[x \sin c/(\rho \cos\phi_1 \cos c - y \sin\phi_1 \sin c)] \quad (16\text{–}15)$$

If ϕ_1 is 90°:

$$\lambda = \lambda_0 + \arctan[x/(-y)] \quad (16\text{–}16)$$

If ϕ_1 is $-90°$:

$$\lambda = \lambda_0 + \arctan(x/y) \quad (16\text{–}17)$$

In equations (16–14) and (16–15),

$$\rho = (x^2 + y^2)^{1/2} \quad (16\text{–}18)$$
$$c = 2\arctan[\rho/(2Rk_0)] \quad (17\text{–}15)$$

The similarity of formulas for Orthographic, Stereographic, and other azimuthals may be noted. The equations for k' (k for the Stereographic, $k' = 1.0$ for the Orthographic) and the inverse c are the only differences in forward or inverse formulas for the sphere. The formulas are repeated for convenience, unless shown only a few lines earlier.

Table 23 lists rectangular coordinates for the equatorial aspect for a 10° graticule with a sphere of radius $R = 1.0$.

Following are equations for the centers and radii of the circles representing the meridians and parallels of the oblique Stereographic in the spherical form:

Circles for meridians:

Centers: $x = -2R\, k_0/[\cos\phi_1 \tan(\lambda - \lambda_0)]$ (17–16)

$y = -2R\, k_0 \tan\phi_1$ (17–17)

Radii: $\rho = 2R\, k_0/[\cos\phi_1 \sin(\lambda - \lambda_0)]$ (17–18)

Circles for parallels of latitude:

Centers: $x = 0$

$y = 2R\, k_0 \cos\phi_1/(\sin\phi_1 + \sin\phi)$ (17–19)

Radii: $\rho = 2R\, k_0 \cos\phi/(\sin\phi_1 + \sin\phi)$ (17–20)

Reduction to the polar and equatorial aspects may be made by letting $\phi_1 = \pm 90°$ or $0°$, respectively.

To use a "standard circle" for the spherical Stereographic projection, such that the scale error is a minimum (based on least squares) over the apparent area of the map, the circle has an angular distance c from the center, where

$$c = 2 \arccos (1/\bar{k})^{1/2} \quad (17\text{–}21)$$

$$\bar{k} = \tan^2 \tfrac{1}{2}\beta/(-\ln \cos^2 \tfrac{1}{2}\beta) \quad (17\text{–}22)$$

and β is the great circle distance of the circular limit of the region being mapped stereographically. The calculation is only slightly different if minimum error is based on the true area of the map:

$$\bar{k} = -\ln \cos^2 \tfrac{1}{2}\beta/\sin^2 \tfrac{1}{2}\beta \quad (17\text{–}23)$$

In either case, c of the standard circle is approximately $\beta/\sqrt{2}$.

FORMULAS FOR THE ELLIPSOID

As noted above, the ellipsoidal forms of the Stereographic projection are nonperspective, in order to preserve conformality. The oblique and equatorial aspects are also slightly nonazimuthal for the same reason. The formulas result from replacing geodetic latitude ϕ in the spherical equations with conformal latitude χ (see equation (3–1)), followed by a small adjustment to the scale at the center of projection (Thomas, 1952,

TABLE 23.—*Stereographic projection: Rectangular coordinates for equatorial aspect (sphere)*

[One hemisphere; y coordinate in parentheses under x coordinate]

Lat. \ Long.	0°	10°	20°	30°	40°
90°	0.00000 (2.00000)	0.00000 (2.00000)	0.00000 (2.00000)	0.00000 (2.00000)	0.00000 (2.00000)
80	.00000 (1.67820)	.05150 (1.68198)	.10212 (1.69331)	.15095 (1.71214)	.19703 (1.73837)
70	.00000 (1.40042)	.08885 (1.40586)	.17705 (1.42227)	.26386 (1.44992)	.34841 (1.48921)
60	.00000 (1.15470)	.11635 (1.16058)	.23269 (1.17839)	.34892 (1.20868)	.46477 (1.25237)
50	.00000 (.93262)	.13670 (.93819)	.27412 (.95515)	.41292 (.98421)	.55371 (1.02659)
40	.00000 (.72794)	.15164 (.73277)	.30468 (.74749)	.46053 (.77285)	.62062 (.81016)
30	.00000 (.53590)	.16233 (.53970)	.32661 (.55133)	.49487 (.57143)	.66931 (.60117)
20	.00000 (.35265)	.16950 (.35527)	.34136 (.36327)	.51808 (.37713)	.70241 (.39773)
10	.00000 (.17498)	.17363 (.17631)	.34987 (.18037)	.53150 (.18744)	.72164 (.19796)
0	.00000 (.00000)	.17498 (.00000)	.35265 (.00000)	.53590 (.00000)	.72794 (.00000)

TABLE 23.—*Stereographic projection: Rectangular coordinates for equatorial aspect (sphere)*—Continued

Lat. \ Long.	50°	60°	70°	80°	90°
90°	0.00000 (2.00000)	0.00000 (2.00000)	0.00000 (2.00000)	0.00000 (2.00000)	0.00000 (2.00000)
80	.23933 (1.77184)	.27674 (1.81227)	.30806 (1.85920)	.33201 (1.91196)	.34730 (1.96962)
70	.42957 (1.54067)	.50588 (1.60493)	.57547 (1.68256)	.63588 (1.77402)	.68404 (1.87939)
60	.57972 (1.31078)	.69282 (1.38564)	.80246 (1.47911)	.90613 (1.59368)	1.00000 (1.73205)
50	.69688 (1.08415)	.84255 (1.15945)	.99033 (1.25597)	1.13892 (1.37825)	1.28558 (1.53209)
40	.78641 (.86141)	.95937 (.92954)	1.14080 (1.01868)	1.33167 (1.13464)	1.53209 (1.28558)
30	.85235 (.64240)	1.04675 (.69783)	1.25567 (.77149)	1.48275 (.86928)	1.73205 (1.00000)
20	.89755 (.42645)	1.10732 (.46538)	1.33650 (.51767)	1.59119 (.58808)	1.87939 (.68404)
10	.92394 (.21267)	1.14295 (.23271)	1.38450 (.25979)	1.65643 (.29658)	1.96962 (.34730)
0	.93262 (.00000)	1.15470 (.00000)	1.40042 (.00000)	1.67820 (.00000)	2.00000 (.00000)

Radius of sphere = 1.0.
Origin: $(x, y) = 0$ at (lat., long.) = 0. Y axis increases north. Other quadrants of hemisphere are symmetrical.

p. 14–15, 128–139). The general forward formulas for the oblique aspect are as follows; given a, e, k_0, ϕ_1, λ_0, ϕ, and λ:

$$x = A \cos\chi \sin(\lambda - \lambda_0) \qquad (17\text{–}24)$$
$$y = A [\cos\chi_1 \sin\chi - \sin\chi_1 \cos\chi \cos(\lambda - \lambda_0)] \qquad (17\text{–}25)$$
$$k = A \cos\chi/(am) \qquad (17\text{–}26)$$

where

$$A = 2\,a\,k_0 m_1/\{\cos\chi_1\,[1 + \sin\chi_1 \sin\chi + \cos\chi_1 \cos\chi \cos(\lambda - \lambda_0)]\} \qquad (17\text{–}27)$$
$$\chi = 2 \arctan\{\tan(\pi/4 + \phi/2)[(1 - e \sin\phi)/(1 + e \sin\phi)]^{e/2}\} - \pi/2 \qquad (3\text{–}1)$$
$$m = \cos\phi/(1 - e^2 \sin^2\phi)^{1/2} \qquad (12\text{–}15)$$

and χ_1 and m_1 are χ and m, respectively, calculated using ϕ_1, the central latitude, in place of ϕ, while k_0 is the scale factor at the center (normally 1.0). The origin of x and y coordinates occurs at the center (ϕ_1, λ_0), the Y axis coinciding with the central meridian λ_0, and y increasing northerly and x, easterly. The scale factor is actually k_0 along a near-circle passing through the origin, except for polar and equatorial aspects, where it occurs only at the central point. The radius of this near-circle is almost 0.4° at midlatitudes, and its center is along the central meridian, approaching the Equator from ϕ_1. The scale factor at the center of the circle is within 0.00001 less than k_0.

In the equatorial aspect, with the substitution of $\phi_1 = 0$ (therefore $\chi_1 = 0$), x is still found from (17–24) and k from (17–26),

but

$$y = A \sin\chi \qquad (17\text{–}28)$$
$$A = 2ak_0/[1 + \cos\chi \cos(\lambda - \lambda_0)] \qquad (17\text{–}29)$$

For the north polar aspect, substitution of $\phi_1 = 90°$ (therefore $\chi_1 = 90°$) into equations (17–27) and (12–15) leads to an indeterminate A. To avoid this problem, the polar equations may take the form

$$x = \rho \sin(\lambda - \lambda_0) \qquad (17\text{–}30)$$
$$y = -\rho \cos(\lambda - \lambda_0) \qquad (17\text{–}31)$$
$$k = \rho/(a\,m) \qquad (17\text{–}32)$$

where

$$\rho = 2\,a\,k_0\,t/[(1 + e)^{(1+e)}(1 - e)^{(1-e)}]^{1/2} \qquad (17\text{–}33)$$
$$t = \tan(\pi/4 - \phi/2)/[(1 - e \sin\phi)/(1 + e \sin\phi)]^{e/2} \qquad (13\text{–}9)$$

Equation (17–33) applies only if true scale or known scale factor k_0 is to occur at the pole. For true scale along the circle representing latitude ϕ_c,

$$\rho = a m_c\, t/t_c \tag{17–34}$$

Then the scale at the pole is

$$k_p = \tfrac{1}{2}\, m_c\, [(1+e)^{(1+e)}\,(1-e)^{(1-e)}]^{1/2}/(a\, t_c) \tag{17–35}$$

In equations (17–34) and (17–35), m_c and t_c are found from equations (12–15) and (13–9), respectively, substituting ϕ_c in place of ϕ.

For the south polar aspect, the equations for the north polar aspect may be used, but the signs of x, y, ϕ_c, ϕ, λ, and λ_0 must be reversed to be used in the equations.

For the inverse formulas for the ellipsoid, the oblique and equatorial aspects (where ϕ_1 is not $\pm 90°$) may be solved as follows, given a, e, k_0, ϕ_1, λ_0, x, and y:

$$\phi = 2 \arctan \{\tan (\pi/4 + \chi/2)[(1 + e \sin \phi)/(1 - e \sin \phi)]^{e/2}\} - \pi/2 \tag{3–4}$$

$$\lambda = \lambda_0 + \arctan [x \sin c_e/(\rho \cos \chi_1 \cos c_e - y \sin \chi_1 \sin c_e)] \tag{17–36}$$

where

$$\chi = \arcsin [\cos c_e \sin \chi_1 + (y \sin c_e \cos \chi_1/\rho)] \tag{17–37}$$

but if $\rho = 0$, $\chi = \chi_1$.

$$\rho = (x^2 + y^2)^{1/2} \tag{16–18}$$

$$c_e = 2 \arctan [\rho \cos \chi_1/(2\, a\, k_0\, m_1)] \tag{17–38}$$

and m_1 is found from equation (12–15) above, using ϕ_1 in place of ϕ. Equation (3–4) involves iteration, using χ as the first trial ϕ in the right-hand side, solving for a new trial ϕ on the left side, substituting into the right side, etc., until ϕ changes by less than a preset convergence (such as 10^{-9} radians). Conformal latitude χ_1 is found from (3–1), using ϕ_1 for ϕ. The factor c_e is not the true angular distance, as it is in the spherical case, but it is a convenient expression similar in nature to c, used to find ϕ and λ.

To avoid the iteration of (3–4), this series may be used instead:

$$\phi = \chi + (e^2/2 + 5e^4/24 + e^6/12 + \dots) \sin 2\chi + (7e^4/48 + 29e^6/240 + \dots) \sin 4\chi + (7e^6/120 + \dots) \sin 6\chi + \dots \tag{3–5}$$

The inverse equations for the north polar ellipsoidal Stereographic are as follows; given a, e, ϕ_c, k_0 (if $\phi_c = 90°$), λ_0, x, and y:

$$\phi = \pi/2 - 2 \arctan \{t[(1 - e \sin \phi)/(1 + e \sin \phi)]^{e/2}\} \tag{7–9}$$

$$\lambda = \lambda_0 + \arctan [x/(-y)] \tag{16–16}$$

Equation (7–9) for ϕ also involves iteration. For the first trial, $(\pi/2-2 \arctan t)$ is substituted for ϕ in the right side, and the procedure for solving equation (3–4) just above is followed.

If ϕ_c (the latitude of true scale) is 90°,

$$t=\rho[(1+e)^{(1+e)}(1-e)^{(1-e)}]^{1/2}/(2a\,k_0) \tag{17-39}$$

If ϕ_c is not 90°,

$$t=\rho t_c/(a\,m_c) \tag{17-40}$$

In either case,

$$\rho=(x^2+y^2)^{1/2} \tag{16-18}$$

and t_c and m_c are found from equations (13–9) and (12–15), respectively, listed with the forward equations, using ϕ_c in place of ϕ. Scale factor k is found from equation (17–26) or (17–32) above, for the ϕ found from equation (3–4), (3–5), or (7–9), depending on the aspect.

To avoid iteration, series (3–5) above may be used in place of (7–9), where

$$\chi=\pi/2-2\arctan t \tag{7-13}$$

Inverse equations for the south polar aspect are the same as those for the north polar aspect, but the signs of x, y, λ_0, ϕ_c, ϕ, and λ must be reversed.

Polar coordinates for the ellipsoidal form of the polar Stereographic are given in table 24, using the International ellipsoid and a central scale factor of 1.0.

TABLE 24.—*Ellipsoidal polar Stereographic projection: Polar coordinates*

[International ellipsoid; central scale factor = 1.0]

Latitude	Radius, meters	k, scale factor
90°	0.0	1.000000
89	111,702.7	1.000076
88	223,421.7	1.000305
87	335,173.4	1.000686
86	446,974.1	1.001219
85	558,840.1	1.001906
84	670,788.1	1.002746
83	782,834.3	1.003741
82	894,995.4	1.004889
81	1,007,287.9	1.006193
80	1,119,728.7	1.007653
79	1,232,334.4	1.009270
78	1,345,122.0	1.011045
77	1,458,108.4	1.012979
76	1,571,310.9	1.015073
75	1,684,746.8	1.017328
74	1,798,433.4	1.019746
73	1,912,388.4	1.022329
72	2,026,629.5	1.025077
71	2,141,174.8	1.027993
70	2,256,042.3	1.031078
69	2,371,250.5	1.034335
68	2,486,818.0	1.037765
67	2,602,763.6	1.041370
66	2,719,106.4	1.045154
65	2,835,865.8	1.049117
64	2,953,061.4	1.053264
63	3,070,713.2	1.057595
62	3,188,841.4	1.062115
61	3,307,466.7	1.066826
60	3,426,609.9	1.071732

18. LAMBERT AZIMUTHAL EQUAL-AREA PROJECTION

SUMMARY

- Azimuthal.
- Equal-Area.
- All meridians in the polar aspect, the central meridian in other aspects, and the Equator in the equatorial aspect are straight lines.
- The outer meridian of a hemisphere in the equatorial aspect (for the sphere) and the parallels in the polar aspect (sphere or ellipsoid) are circles.
- All other meridians and parallels are complex curves.
- Not a perspective projection.
- Scale decreases radially as the distance increases from the center, the only point without distortion.
- Scale increases in the direction perpendicular to radii as the distance increases from the center.
- Directions from the center are true for the sphere and the polar ellipsoidal forms.
- Point opposite the center is shown as a circle surrounding the map (for the sphere).
- Used for maps of continents and hemispheres.
- Presented by Lambert in 1772.

HISTORY

The last major projection presented by Johann Heinrich Lambert in his 1772 *Beiträge* was his azimuthal equal-area projection (Lambert, 1772, p. 75–78). His name is usually applied to the projection in modern references, but it is occasionally called merely the Azimuthal (or Zenithal) Equal-Area projection. Not only is it equal-area, with, of course, the azimuthal property showing true directions from the center of the projection, but its scale at a given distance from the center varies less from the scale at the center than the scale of any of the other major azimuthals (see table 19).

Lambert discussed the polar and equatorial aspects of the Azimuthal Equal-Area projection, but the oblique aspect is just as popular now. The polar aspect was apparently independently derived by De Lorgna in Italy in 1789, and the latter was called the originator in a publication a century later (USC&GS, 1882, p. 290). G. A. Ginsburg proposed two modifications of the general Lambert Azimuthal projection in 1949 to reduce the angular distortion at the expense of creating a slight distortion in area (Maling, 1960, p. 206).

FEATURES

The Lambert Azimuthal Equal-Area projection is not a perspective projection. It may be called a "synthetic" azimuthal in that it was derived for the specific purpose of maintaining equal area. The ellipsoidal form maintains equal area, but it is not quite azimuthal except in the polar aspect, so the name for the general ellipsoidal form is a slight misnomer, although it looks like the spherical azimuthal form and has most of its other characteristics.

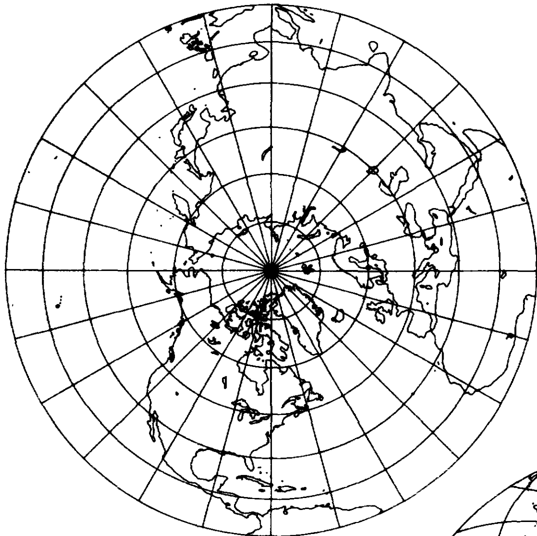
The polar aspect (fig. 25*A*), like that of the Orthographic and Stereographic, has circles for parallels of latitude, all centered about the North or South Pole, and straight equally spaced radii of these circles for meridians. The difference is, once again, in the spacing of the parallels. For the Lambert, the spacing between the parallels gradually decreases with increasing distance from the pole. The opposite pole, not visible on either the Orthographic or Stereographic, may be shown on the Lambert as a large circle surrounding the map, almost half again as far as the Equator from the center. Normally, the projection is not shown beyond one hemisphere (or beyond the Equator in the polar aspect).

The equatorial aspect (fig. 25*B*) has, like the other azimuthals, a straight Equator and straight central meridian, with a circle representing the 90th meridian east and west of the central meridian. Unlike those for the Orthographic and Stereographic, the remaining meridians and parallels are uncommon complex curves. The chief visual distinguishing characteristic is that the spacing of the meridians near the 90th meridian and of the parallels near the poles is about 0.7 of the spacing at the center of the projection, or moderately less to the eye. The parallels of latitude look considerably like circular arcs, except near the 90th meridians, where they exhibit a noticeable turn toward the nearest pole.

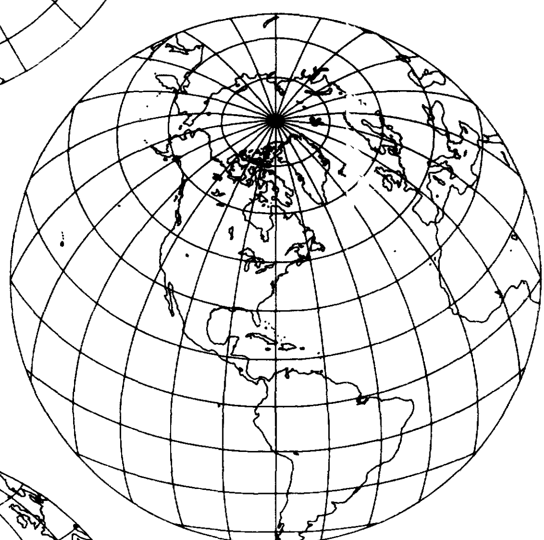
The oblique aspect (fig. 25*C*) of the Lambert Azimuthal Equal-Area resembles the Orthographic to some extent, until it is seen that crowding is far less pronounced as the distance from the center increases. Aside from the straight central meridian, all meridians and parallels are complex curves, not ellipses.

In both the equatorial and oblique aspects, the point opposite the center may be shown as a circle surrounding the map, corresponding to the opposite pole in the polar aspect. Except for the advantage of showing the entire Earth in an equal-area projection from one point, the distortion is so great beyond the inner hemisphere that for world maps the Earth should be shown as two separate hemispherical maps, the second map centered on the point opposite the center of the first map.

FIGURE 25.–Lambert Azimuthal Equal-Area projection. (*A*) polar aspect showing one hemisphere; the entire globe may be included in a circle of 1.41 times the diameter of the Equator. (*B*) equatorial aspect; frequently used in atlases for maps of the Eastern and Western hemispheres. (*C*) oblique aspect; centered on lat. 40° N.



A



C

B

USAGE

The spherical form in all three aspects of the Lambert Azimuthal Equal-Area projection has appeared in recent commercial atlases for Eastern and Western Hemispheres (replacing the long-used Globular projection) and for maps of oceans and most of the continents and polar regions.

The polar aspect appears in the *National Atlas* (USGS, 1970, p. 148–149) for maps delineating north and south polar expeditions, at a scale of 1:39,000,000. It is used at a scale of 1:20,000,000 for the Arctic Region as an inset on the 1978 USGS Map of Prospective Hydrocarbon Provinces of the World.

The USGS has prepared six base maps of the Pacific Ocean on the spherical form of the Lambert Azimuthal Equal-Area. Four sections, at 1:10,000,000, have centers at 35° N., 150° E.; 35° N., 135° W.; 35° S., 135° E.; and 40° S., 100° W. The Pacific-Antarctic region, at a larger scale, is centered at 20° S. and 165° W., while a Pacific Basin map at 1:20,000,000 is centered at the Equator and 160° W. The base maps have been used for individual geographic, geologic, tectonic, minerals, and energy maps. The USGS has also cooperated with the National Geographic Society in revising maps of the entire Moon drawn to the spherical form of the equatorial Lambert Azimuthal Equal-Area.

GEOMETRIC CONSTRUCTION

The polar aspect (for the sphere) may be drawn with a simple geometric construction: In figure 26, if angle *AOR* is the latitude ϕ and *P* is the pole at the center, *PA* is the radius of that latitude on the polar map. The oblique and equatorial aspects have no direct geometric construction. They may be prepared indirectly by using other azimuthal projections (Harrison, 1943), but it is now simpler to plot automatically or manually from rectangular coordinates which are generated by a relatively simple computer program. The formulas are given below.

FORMULAS FOR THE SPHERE

On the Lambert Azimuthal Equal-Area projection for the sphere, a point at a given angular distance from the center of projection is plotted at a distance from the center proportional to the sine of half that angular distance, and at its true azimuth, or

$$\rho = 2R \sin \tfrac{1}{2} c \tag{18-1}$$

$$\theta = \pi - Az = 180° - Az \tag{16-2}$$

$$h' = \cos \tfrac{1}{2} c \tag{18-1a}$$

$$k' = \sec \tfrac{1}{2} c \tag{18-1b}$$

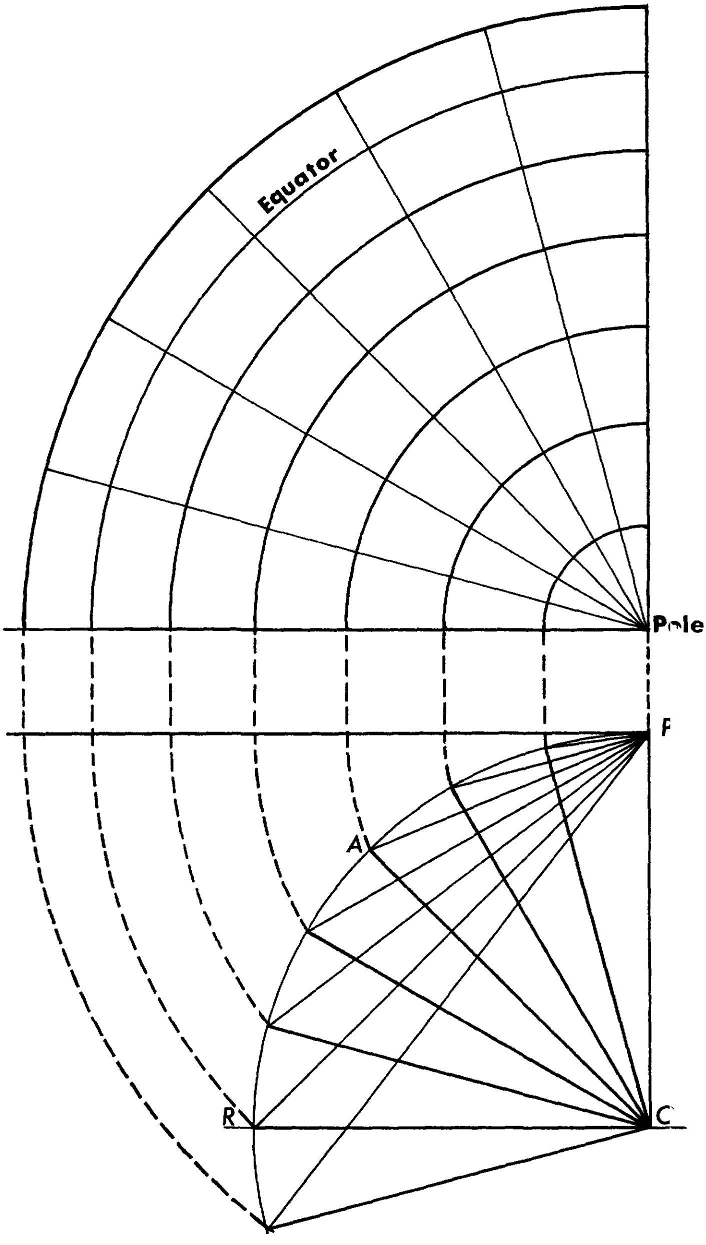


FIGURE 26.—Geometric construction of polar Lambert Azimuthal Equal-Area projection.

where c is the angular distance from the center, Az is the azimuth east of north (see equations (5–3) through (5–6)), and θ is the polar coordinate east of south. The term k' is the scale factor in a direction perpendicular to the radius from the center of the map, not along the parallel, except in the polar case. The scale factor h' in the direction of the radius equals $1/k'$. After combining with standard equations, the formulas for rectangular coordinates for the oblique Lambert Azimuthal Equal-Area projection may be written as follows, given R, ϕ_1, λ_0, ϕ, and λ:

$$x = R\,k' \cos\phi \sin(\lambda - \lambda_0) \tag{18–2}$$
$$y = R\,k' [\cos\phi_1 \sin\phi - \sin\phi_1 \cos\phi \cos(\lambda - \lambda_0)] \tag{18–3}$$

where

$$k' = \{2/[1 + \sin\phi_1 \sin\phi + \cos\phi_1 \cos\phi \cos(\lambda - \lambda_0)]\}^{1/2} \tag{18–4}$$

and (ϕ_1, λ_0) are latitude and longitude of the projection center and origin. The Y axis coincides with the central meridian λ_0, y increasing northerly. For the point opposite the center, at latitude $-\phi_1$ and longitude $\lambda_0 \pm 180°$, these formulas give indeterminants. This point, if the map is to cover the entire sphere, is plotted as a circle of radius $2R$.

For the north polar Lambert Azimuthal Equal-Area, with $\phi_1 = 90°$, since k' is k for the polar aspect, these formulas simplify to

$$x = 2R \sin(\pi/4 - \phi/2) \sin(\lambda - \lambda_0) \tag{18–5}$$
$$y = -2R \sin(\pi/4 - \phi/2) \cos(\lambda - \lambda_0) \tag{18–6}$$
$$k = \sec(\pi/4 - \phi/2) \tag{18–7}$$
$$h = 1/k = \cos(\pi/4 - \phi/2) \tag{18–8}$$

or, using polar coordinates,

$$\rho = 2R \sin(\pi/4 - \phi/2) \tag{18–9}$$
$$\theta = \lambda - \lambda_0 \tag{16–9}$$

For the south polar aspect, with $\phi_1 = -90°$,

$$x = 2R \cos(\pi/4 - \phi/2) \sin(\lambda - \lambda_0) \tag{18–10}$$
$$y = 2R \cos(\pi/4 - \phi/2) \cos(\lambda - \lambda_0) \tag{18–11}$$
$$k = 1/\sin(\pi/4 - \phi/2) \tag{18–12}$$
$$h = \sin(\pi/4 - \phi/2) \tag{18–13}$$

or

$$\rho = 2R \cos(\pi/4 - \phi/2) \tag{18–14}$$
$$\theta = \pi - \lambda + \lambda_0 \tag{16–12}$$

For the equatorial aspect, letting $\phi_1 = 0$, x is found from (18–2), but

$$y = R\,k' \sin\phi \tag{18–15}$$

and

$$k' = \{2/[1 + \cos\phi \cos(\lambda - \lambda_0)]\}^{1/2} \tag{18–16}$$

The maximum angular deformation ω for any of these aspects, derived from equations (4–7) through (4–9), and from the fact that $h' = 1/k'$ for equal-area maps:

$$\sin \tfrac{1}{2}\omega = (k'^2 - 1)/(1 + k'^2) \qquad (18\text{–}17)$$

For the inverse formulas for the sphere, given R, ϕ_1, λ_0, x, and y:

$$\phi = \arcsin [\cos c \sin \phi_1 + (y \sin c \cos \phi_1/\rho)] \qquad (16\text{–}14)$$

But if $\rho = 0$, $\phi = \phi_1$.

If ϕ_1 is not $\pm 90°$:

$$\lambda = \lambda_0 + \arctan [x \sin c/(\rho \cos \phi_1 \cos c - y \sin \phi_1 \sin c)] \qquad (16\text{–}15)$$

If ϕ_1 is 90°:

$$\lambda = \lambda_0 + \arctan [x/(-y)] \qquad (16\text{–}16)$$

If ϕ_1 is $-90°$:

$$\lambda = \lambda_0 + \arctan (x/y) \qquad (16\text{–}17)$$

In equations (16–14) and (16–15),

$$\rho = (x^2 + y^2)^{1/2} \qquad (16\text{–}18)$$

$$c = 2 \arcsin [\rho/(2R)] \qquad (18\text{–}18)$$

It may again be noted that several of the above forward and inverse equations apply to the other azimuthals.

Table 25 lists rectangular coordinates for the equatorial aspect for a 10° graticule with a sphere of radius $R = 1.0$.

FORMULAS FOR THE ELLIPSOID

As noted above, the ellipsoidal oblique aspect of the Lambert Azimuthal Equal-Area projection is slightly nonazimuthal in order to preserve equality of area. To date, the USGS has not used the ellipsoidal form in any aspect. The formulas are analogous to the spherical equations, but involve replacing the geodetic latitude ϕ with authalic latitude β (see equation (3–11)). In order to achieve correct scale in all directions at the center of projection, that is, to make the center a "standard point," a slight adjustment using D is also necessary. The general forward formulas for the oblique aspect are as follows, given a, e, ϕ_1, λ_0, ϕ, and λ:

$$x = B\,D \cos \beta \sin (\lambda - \lambda_0) \qquad (18\text{–}19)$$

$$y = (B/D) [\cos \beta_1 \sin \beta - \sin \beta_1 \cos \beta \cos (\lambda - \lambda_0)] \qquad (18\text{–}20)$$

TABLE 25.—*Lambert Azimuthal Equal-Area projection: Rectangular coordinates for equatorial aspect (sphere)*

[One hemisphere; y coordinate in parentheses under x coordinate]

Lat. \ Long.	0°	10°	20°	30°	40°
90°	0.00000 (1.41421)	0.00000 (1.41421)	0.00000 (1.41421)	0.00000 (1.41421)	0.00000 (1.41421)
80	.00000 (1.28558)	.03941 (1.28702)	.07788 (1.29135)	.11448 (1.29851)	.14830 (1.30842)
70	.00000 (1.14715)	.07264 (1.14938)	.14391 (1.15607)	.21242 (1.16725)	.27676 (1.18296)
60	.00000 (1.00000)	.10051 (1.00254)	.19948 (1.01021)	.29535 (1.02311)	.38649 (1.04143)
50	.00000 (.84524)	.12353 (.84776)	.24549 (.85539)	.36430 (.86830)	.47831 (.88680)
40	.00000 (.68404)	.14203 (.68631)	.28254 (.69317)	.41999 (.70483)	.55281 (.72164)
30	.00000 (.51764)	.15624 (.51947)	.31103 (.52504)	.46291 (.53452)	.61040 (.54826)
20	.00000 (.34730)	.16631 (.34858)	.33123 (.35248)	.49337 (.35915)	.65136 (.36883)
10	.00000 (.17431)	.17231 (.17497)	.34329 (.17698)	.51158 (.18041)	.67588 (.18540)
0	.00000 (.00000)	.17431 (.00000)	.34730 (.00000)	.51764 (.00000)	.68404 (.00000)

Radius of sphere = 1.0.
Origin: $(x, y) = 0$ at (lat., long.) = 0. Y axis increases north. Other quadrants of hemisphere are symmetrical.

where

$$B = R_q \{2/[1 + \sin\beta_1 \sin\beta + \cos\beta_1 \cos\beta \cos(\lambda - \lambda_0)]\}^{1/2} \quad (18\text{–}21)$$
$$D = a\, m_1 /(R_q \cos\beta_1) \quad (18\text{–}22)$$
$$R_q = a\,(q_p/2)^{1/2} \quad (3\text{–}13)$$
$$\beta = \arcsin(q/q_p) \quad (3\text{–}11)$$
$$q = (1 - e^2)\,\{\sin\phi/(1 - e^2\sin^2\phi) - [1/(2e)]\ln [(1 - e\sin\phi)/(1 + e\sin\phi)]\} \quad (3\text{–}12)$$
$$m = \cos\phi/(1 - e^2\sin^2\phi)^{1/2} \quad (12\text{–}15)$$

and β_1 is found from (3–11), using q_1 for q, while q_1 and q_p are found from (3–12) using ϕ_1 and 90°, respectively, for ϕ, and m_1 is found from (12–15), calculated for ϕ_1. The origin occurs at (ϕ_1, λ_0), the Y axis coinciding with the central meridian λ_0, and y increasing northerly. For the equatorial aspect, the equations simplify as follows:

$$x = a\cos\beta\sin(\lambda - \lambda_0)\{2/[1 + \cos\beta\cos(\lambda - \lambda_0)]\}^{1/2} \quad (18\text{–}23)$$
$$y = (R_q^2/a)\sin\beta\,\{2/[1 + \cos\beta\cos(\lambda - \lambda_0)]\}^{1/2} \quad (18\text{–}24)$$

For the polar aspects, D is indeterminate using equations above, but the following equations may be used instead. For the north polar aspect, $\phi_1 = 90°$,

TABLE 25.—*Lambert Azimuthal Equal-Area projection: Rectangular coordinates for equatorial aspect (sphere)*—Continued

Lat. \ Long.	50°	60°	70°	80°	90°
90°	0.00000 (1.41421)	0.00000 (1.41421)	0.00000 (1.41421)	0.00000 (1.41421)	0.00000 (1.41421)
80	.17843 (1.32096)	.20400 (1.33594)	.22420 (1.35313)	.23828 (1.37219)	.24558 (1.39273)
70	.33548 (1.20323)	.38709 (1.22806)	.43006 (1.25741)	.46280 (1.29114)	.48369 (1.32893)
60	.47122 (1.06544)	.54772 (1.09545)	.61403 (1.13179)	.66797 (1.17481)	.70711 (1.22474)
50	.58579 (.91132)	.68485 (.94244)	.77342 (.98088)	.84909 (1.02752)	.90904 (1.08335)
40	.67933 (.74411)	.79778 (.77298)	.90620 (.80919)	1.00231 (.85401)	1.08335 (.90904)
30	.75197 (.56674)	.88604 (.59069)	1.01087 (.62108)	1.12454 (.65927)	1.22474 (.70711)
20	.80380 (.38191)	.94928 (.39896)	1.08635 (.42078)	1.21347 (.44848)	1.32893 (.48369)
10	.83488 (.19217)	.98731 (.20102)	1.13192 (.21240)	1.26747 (.22694)	1.39273 (.24558)
0	.84524 (.00000)	1.00000 (.00000)	1.14715 (.00000)	1.28558 (.00000)	1.41421 (.00000)

$$x = \rho \sin(\lambda - \lambda_0) \tag{17-30}$$
$$y = -\rho \cos(\lambda - \lambda_0) \tag{17-31}$$
$$k = \rho/(a\, m) \tag{17-32}$$

where

$$\rho = a(q_p - q)^{1/2} \tag{18-25}$$

and q_p and q are found from (3–12) as before and m from (12–15) above. Since the meridians and parallels intersect at right angles, and this is an equal-area projection, $h = 1/k$.

For the south polar aspect, ($\phi_1 = -90°$), equations (17–30) and (17–32) remain the same, but

$$y = \rho \cos(\lambda - \lambda_0) \tag{18-26}$$

and

$$\rho = a(q_p + q)^{1/2} \tag{18-27}$$

For the inverse formulas for the ellipsoid, the oblique and equatorial aspects (where ϕ_1 is not $\pm 90°$) may be solved as follows, given a, e, ϕ_1, λ_0, x, and y.

$$\phi = \phi + \frac{(1 - e^2 \sin^2 \phi)^2}{2 \cos \phi} \left[\frac{q}{1-e^2} - \frac{\sin \phi}{1 - e^2 \sin^2 \phi} + \frac{1}{2e} \ln \left(\frac{1 - e \sin \phi}{1 + e \sin \phi} \right) \right] \tag{3-16}$$

$$\lambda = \lambda_0 + \arctan [x \sin c_e / (D\, \rho \cos \beta_1 \cos c_e - D^2 y \sin \beta_1 \sin c_e)] \tag{18-28}$$

where

$$q = q_p\,[\cos c_e \sin \beta_1 + (Dy \sin c_e \cos \beta_1/\rho)] \tag{18–29}$$

but if $\rho = 0$, $q = q_p \sin \beta_1$

$$\rho = [(x/D)^2 + (Dy)^2]^{1/2} \tag{18–30}$$

$$c_e = 2 \arcsin (\rho/2\,R_q) \tag{18–31}$$

and D, R_q, q_p, and β_1 are found from equations (18–22), (3–13), (3–12), (3–11), and (12–15), as in the forward equations above. The factor c_e is not the true angular distance, as c is in the spherical case, but it is a convenient number similar in nature to c, used to find ϕ and λ. Equation (3–16) requires iteration by successive substitution, using arcsin $(q/2)$ as the first trial ϕ on the right side, calculating ϕ on the left side, substituting this new ϕ on the right side, etc., until the change in ϕ is negligible. If, in equation (18–29),

$$q = \pm\{1 - [(1-e^2)/(2\,e)] \ln [(1-e)/(1+e)]\} \tag{12–20}$$

the iteration does not converge, but $\phi = \pm 90°$, taking the sign of q.

To avoid the iteration, equations (3–16), (18–29), and (12–20) may be replaced with the series

$$\phi = \beta + (e^2/3 + 31e^4/180 + 517e^6/5040 + \ldots) \sin 2\beta + (23e^4/360 + 251e^6/3780 + \ldots) \sin 4\beta + (761e^6/45360 + \ldots) \sin 6\beta + \ldots \tag{3–18}$$

where β, the authalic latitude, is found thus:

$$\beta = \arcsin [\cos c_e \sin \beta_1 + (Dy \sin c_e \cos \beta_1/\rho)] \tag{18–32}$$

Equations (18–28), (18–30), and (18–31) still apply. In (18–32), if $\rho = 0$, $\beta = \beta_1$.

The inverse formulas for the polar aspects involve relatively simple transformations of above equations (17–30), (17–31), and (18–25), except that ϕ is found from the iterative equation (3–16), listed just above, in which q is calculated as follows:

$$q = \pm[q_p - (\rho/a)^2] \tag{18–33}$$

taking the sign of ϕ_1. The series (3–18) may be used instead for ϕ, where

$$\beta = \pm \arcsin \{1 - \rho^2/[a^2[1 - ((1-e^2)/(2\,e)) \ln ((1-e)/(1+e))]]\} \tag{18–34}$$

taking the sign of ϕ_1. In any case,

$$\rho = (x^2 + y^2)^{1/2} \tag{16–18}$$

while

$$\lambda = \lambda_0 + \arctan [x/(-y)] \qquad (16\text{–}16)$$

for the north polar case, and

$$\lambda = \lambda_0 + \arctan (x/y) \qquad (16\text{–}17)$$

for the south polar case.

Table 26 lists polar coordinates for the ellipsoidal polar aspect of the Lambert Azimuthal Equal-Area, using the International ellipsoid.

TABLE 26.—*Ellipsoidal polar Lambert Azimuthal Equal-Area projection (International ellipsoid)*

Latitude	Radius, meters	*h*	*k*
90°	0.0	1.000000	1.000000
89	111,698.4	.999962	1.000038
88	223,387.7	.999848	1.000152
87	335,058.5	.999657	1.000343
86	446,701.8	.999391	1.000610
85	558,308.3	.999048	1.000953
84	669,868.8	.998630	1.001372
83	781,374.2	.998135	1.001869
82	892,815.4	.997564	1.002442
81	1,004,183.1	.996918	1.003092
80	1,115,468.3	.996195	1.003820
79	1,226,661.9	.995397	1.004625
78	1,337,754.7	.994522	1.005508
77	1,448,737.6	.993573	1.006469
76	1,559,601.7	.992547	1.007509
75	1,670,337.9	.991446	1.008628
74	1,780,937.2	.990270	1.009826
73	1,891,390.6	.989018	1.011104
72	2,001,689.2	.987691	1.012462
71	2,111,824.0	.986289	1.013902
70	2,221,786.2	.984812	1.015422

h = scale factor along meridian.
k = scale factor along parallel.

19. AZIMUTHAL EQUIDISTANT PROJECTION

SUMMARY

- Azimuthal.
- Distances measured from the center are true.
- Distances not measured along radii from the center are not correct.
- The center of projection is the only point without distortion.
- Directions from the center are true (except on some oblique and equatorial ellipsoidal forms).
- Neither equal-area nor conformal.
- All meridians on the polar aspect, the central meridian on other aspects, and the Equator on the equatorial aspect are straight lines.
- Parallels on the polar projection are circles spaced at true intervals (equidistant for the sphere).
- The outer meridian of a hemisphere on the equatorial aspect (for the sphere) is a circle.
- All other meridians and parallels are complex curves.
- Not a perspective projection.
- Point opposite the center is shown as a circle (for the sphere) surrounding the map.
- Used in the polar aspect for world maps and maps of polar hemispheres.
- Used in the oblique aspect for atlas maps of continents and world maps for aviation and radio use.
- Known for many centuries in the polar aspect.

HISTORY

While the Orthographic is probably the most familiar azimuthal projection, the Azimuthal Equidistant, especially in its polar form, has found its way into many atlases with the coming of the air age for maps of the Northern and Southern Hemispheres or for world maps. The simplicity of the polar aspect for the sphere, with equally spaced meridians and equidistant concentric circles for parallels of latitude, has made it easier to understand than most other projections. The primary feature, showing distances and directions correctly from one point on the Earth's surface, is also easily accepted. In addition, its linear scale distortion is moderate and falls between that of equal-area and conformal projections.

Like the Orthographic, Stereographic, and Gnomonic projections, the Azimuthal Equidistant was apparently used centuries before the 15th-century surge in scientific mapmaking. It is believed that Egyptians used the polar aspect for star charts, but the oldest existing celestial map on the projection was prepared in 1426 by Conrad of Dyffenbach. It was also used in principle for small areas by mariners from earliest times in order to chart coasts, using distances and directions obtained at sea.

The first clear examples of the use of the Azimuthal Equidistant for polar maps of the Earth are those included by Gerhardus Mercator as insets on his 1569 world map, which introduced his famous cylindrical projection. As Northern and Southern Hemispheres, the projection appeared in a manuscript of about 1510 by the Swiss Henricus Loritus, usually called Glareanus (1488–1563), and by several others in the next few decades (Keuning, 1955, p. 4–5). Guillaume Postel is given credit in France for its origin, although he did not use it until 1581. Antonio Cagnoli even gave the projection his name as originator in 1799 (Deetz and Adams, 1934, p. 163; Steers, 1970, p. 234). Philippe Hatt developed ellipsoidal versions of the oblique aspect which are used by the French and the Greeks for coastal or topographic mapping.

Two projections with similar names are called the Two-Point Azimuthal and the Two-Point Equidistant projections. Both were developed about 1920 independently by Maurer (1919) of Germany and Close (1921) of England. The first projection (rarely used) is geometrically a tilting of the Gnomonic projection to provide true azimuths from either of two chosen points instead of from just one. Like the Gnomonic, it shows all great circle arcs as straight lines and is limited to one hemisphere. The Two-Point Equidistant has received moderate use and interest, and shows true distances, but not true azimuths, from either of two chosen points to any other point on the map, which may be extended to show the entire world (Close, 1934).

The Chamberlin Trimetric projection is an approximate "three-point equidistant" projection, constructed so that distances from three chosen points to any other point on the map are approximately correct. The latter distances cannot be exactly true, but the projection is a compromise which the National Geographic Society uses as a standard projection for maps of most continents. This projection was geometrically constructed by the Society, of which Wellman Chamberlin (1908–76) was chief cartographer for many years.

An ellipsoidal adaptation of the Two-Point Equidistant was made by Jay K. Donald of American Telephone and Telegraph Company in 1956 to develop a grid still used by the Bell Telephone system for establishing the distance component of long distance rates. Still another approach is Bomford's modification of the Azimuthal Equidistant, in which the usual circles of constant scale factor perpendicular to the radius from the center are made ovals to give a better average scale factor on a map with a rectangular border (Lewis and Campbell, 1951, p. 7, 12–15).

FEATURES

The Azimuthal Equidistant projection, like the Lambert Azimuthal Equal-Area, is not a perspective projection, but in the spherical form, and in some of the ellipsoidal forms, it has the azimuthal characteristic that all directions or azimuths are correct when measured from the center of the projection. As its special feature, all distances are at true scale when measured between this center and any other point on the map.

The polar aspect (fig. 27*A*), like other polar azimuthals, has circles for parallels of latitude, all centered about the North or South Pole, and equally spaced radii of these circles for meridians. The parallels are, however, spaced equidistantly on the spherical form (or according to actual parallel spacings on the ellipsoid). A world map can extend to the opposite pole, but distortion becomes infinite. Even though the map is finite, the point for the opposite pole is shown as a circle twice the radius of the mapped Equator, thus giving an infinite scale factor along that circle. Likewise, the countries of the outer hemisphere are visibly increasingly distorted as the distance from the center increases, while the inner hemisphere has little enough distortion to appear rather satisfactory to the eye, although the east-west scale along the Equator is almost 60 percent greater than the scale at the center.

As on other azimuthals, there is no distortion at the center of the projection and, as on azimuthals other than the Stereographic, the scale cannot be reduced at the center to provide a standard circle of no distortion elsewhere. It is possible to use an average scale over the map involved to minimize variations in scale error in any direction, but this defeats the main purpose of the projection, that of providing true distance from the center. Therefore, the scale at the projection center should be used for any Azimuthal Equidistant map.

The equatorial aspect (fig. 27*B*) is least used of the three Azimuthal Equidistant aspects, primarily because there are no cities along the Equator from which distances in all directions have been of much interest to map users. Its potential use as a map of the Eastern or Western Hemisphere was usually supplanted first by the equatorial Stereographic projection, later by the Globular projection (both graticules drawn entirely with arcs of circles and straight lines), and now by the equatorial Lambert Azimuthal Equal-Area.

For the equatorial Azimuthal Equidistant projection of the sphere, the only straight lines are the central meridian and the Equator. The outer circle for one hemisphere (the meridian 90° east and west of the central meridian) is equidistantly marked off for the parallels, as it is on other azimuthals. The other meridians and parallels are complex curves constructed to maintain the correct distances and azimuths from the center. The parallels cross the central meridian at their true equidistant spacings, and the meridians cross the Equator equidistantly. The map can be extended, like the polar aspect, to include a much-distorted second hemisphere on the same center.

The oblique Azimuthal Equidistant projection (fig. 27*C*) rather resembles the oblique Lambert Azimuthal Equal-Area when confined to the inner hemisphere centered on any chosen point between Equator and pole. Except for the straight central meridian, the graticule consists of complex curves, positioned to maintain true distance and azimuth from the center. When the outer hemisphere is included, the difference between the Equidistant and the Lambert becomes more pronounced, and while distortion is as extreme as in other aspects, the distances and directions of the features from the center now outweigh the distortion for many applications.

USAGE

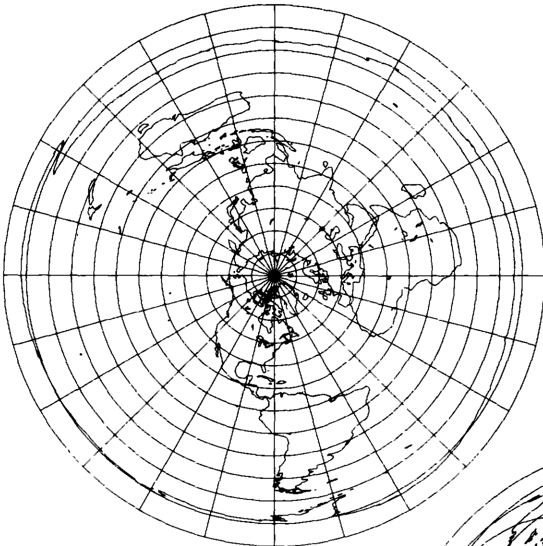
The polar aspect of the Azimuthal Equidistant has regularly appeared in commercial atlases issued during the past century as the most common projection for maps of the north and south polar areas. It is used for polar insets on Van der Grinten-projection world maps published by the National Geographic Society and used as base maps (including the insets) by USGS. The polar Azimuthal Equidistant projection is also normally used when a hemisphere or complete sphere centered on the North or South Pole is to be shown. The oblique aspect has been used for maps of the world centered on important cities or sites and occasionally for maps of continents. Nearly all these maps use the spherical form of the projection.

The USGS has used the Azimuthal Equidistant projection in both spherical and ellipsoidal form. An oblique spherical version of the Earth centered at lat. 40° N., long. 100° W., appears in the *National Atlas* (USGS, 1970, p. 329). At a scale of 1:175,000,000, it does not show meridians and parallels, but shows circles at 1,000-mile intervals from the center. The ellipsoidal oblique aspect is used for the plane coordinate projection system in approximate form for Guam and in nearly rigorous form for islands in Micronesia.

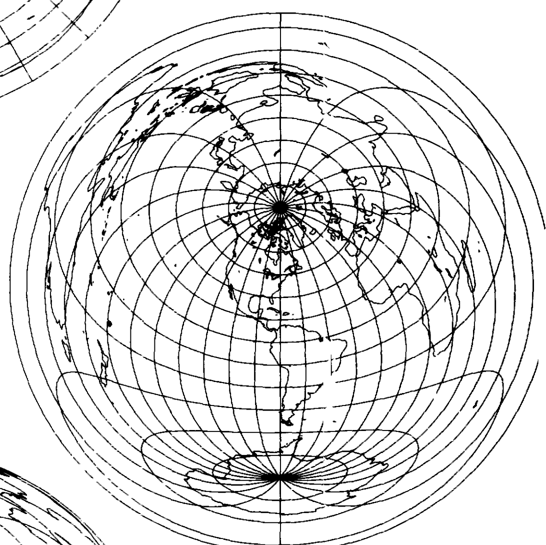
GEOMETRIC CONSTRUCTION

The polar Azimuthal Equidistant is among the easiest projections to construct geometrically, since the parallels of latitude are equally spaced in the spherical case and the meridians are drawn at their true angles. There are no direct geometric constructions for the oblique and equatorial aspects. Like the Lambert Azimuthal Equal-Area, they may be prepared indirectly by using other azimuthal projections (Harrison, 1943), but automatic computer plotting or manual plotting of

FIGURE 27.—Azimuthal Equidistant projection. (*A*) Polar aspect extending to the South Pole; commonly used in atlases for polar maps. (*B*) Equatorial aspect. (*C*) Oblique aspect centered on lat. 40° N. Distance from the center is true to scale.

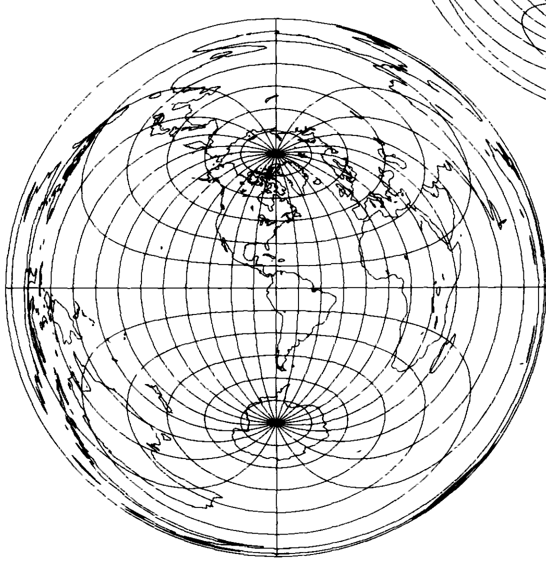


A



C

B



calculated rectangular coordinates is the most suitable means now available.

FORMULAS FOR THE SPHERE

On the Azimuthal Equidistant projection for the sphere, a given point is plotted at a distance from the center of the map proportional to the distance on the sphere and at its true azimuth, or

$$\rho = R\,c \qquad (19\text{–}1)$$
$$\theta = \pi - Az = 180° - Az \qquad (16\text{–}2)$$

where c is the angular distance from the center, Az is the azimuth east of north (see equations (5–3) through (5–6)), and θ is the polar coordinate east of south. For k' and h', see equation (19–2) and the statement below. Combining various equations, the rectangular coordinates for the oblique Azimuthal Equidistant projection are found as follows, given R, ϕ_1, λ_0, ϕ, and λ:

$$x = R\,k' \cos\phi \sin(\lambda - \lambda_0) \qquad (18\text{–}2)$$
$$y = R\,k' [\cos\phi_1 \sin\phi - \sin\phi_1 \cos\phi \cos(\lambda - \lambda_0)] \qquad (18\text{–}3)$$

where

$$k' = c/\sin c \qquad (19\text{–}2)$$
$$\cos c = \sin\phi_1 \sin\phi + \cos\phi_1 \cos\phi \cos(\lambda - \lambda_0) \qquad (5\text{–}3)$$

and (ϕ_1, λ_0) are latitude and longitude of the center of projection and origin. The Y axis coincides with the central meridian λ_0, and y increases northerly. If $\cos c = 1$, equation (19–2) is indeterminate, but $k' = 1$, and $x = y = 0$. If $\cos c = -1$, the point opposite the center $(-\phi_1, \lambda_0 \pm 180°)$ is indicated; it is plotted as a circle of radius πR. The term k' is the scale factor in a direction perpendicular to the radius from the center of the map, not along the parallel, except in the polar case. The scale factor h' in the direction of the radius is 1.0.

For the north polar aspect, with $\phi_1 = 90°$,

$$x = R(\pi/2 - \phi) \sin(\lambda - \lambda_0) \qquad (19\text{–}3)$$
$$y = -R(\pi/2 - \phi) \cos(\lambda - \lambda_0) \qquad (19\text{–}4)$$
$$k = (\pi/2 - \phi)/\cos\phi \qquad (19\text{–}5)$$
$$h = 1.0$$
$$\rho = R(\pi/2 - \phi) \qquad (19\text{–}6)$$
$$\theta = \lambda - \lambda_0 \qquad (16\text{–}9)$$

For the south polar aspect, with $\phi_1 = -90°$,

$$x = R(\pi/2 + \phi) \sin(\lambda - \lambda_0) \qquad (19\text{–}7)$$
$$y = R(\pi/2 + \phi) \cos(\lambda - \lambda_0) \qquad (19\text{–}8)$$
$$k = (\pi/2 + \phi)/\cos\phi \qquad (19\text{–}9)$$
$$h = 1.0$$
$$\rho = R(\pi/2 + \phi) \qquad (19\text{–}10)$$
$$\theta = \pi - \lambda + \lambda_0 \qquad (16\text{–}12)$$

For the equatorial aspect, with $\phi_1 = 0$, x is found from (18–2) and k' from (19–2), but

$$y = R\,k' \sin\phi \tag{19–11}$$

and

$$\cos c = \cos\phi \cos(\lambda - \lambda_0) \tag{19–12}$$

The maximum angular deformation ω for any of these aspects, using equations (4–7) through (4–9), since $h' = 1.0$:

$$\sin \tfrac{1}{2}\omega = (k' - 1)/(k' + 1) \tag{19–13}$$

$$= (c - \sin c)/(c + \sin c) \tag{19–14}$$

For the inverse formulas for the sphere, given R, ϕ_1, λ_0, x, and y:

$$\phi = \arcsin\,[\cos c \sin\phi_1 + (y \sin c \cos\phi_1/\rho)] \tag{16–14}$$

But if $\rho = 0$, $\phi = \phi_1$.
If ϕ_1 is not $\pm 90°$:

$$\lambda = \lambda_0 + \arctan\,[x \sin c/(\rho \cos\phi_1 \cos c - y \sin\phi_1 \sin c)] \tag{16–15}$$

If ϕ_1 is 90°:

$$\lambda = \lambda_0 + \arctan\,[x/(-y)] \tag{16–16}$$

If ϕ_1 is $-90°$:

$$\lambda = \lambda_0 + \arctan\,(x/y) \tag{16–17}$$

In equations (16–14) and (16–15),

$$\rho = (x^2 + y^2)^{1/2} \tag{16–18}$$

$$c = \rho/R \tag{19–15}$$

Except for (19–15), the above inverse formulas are the same as those for the other azimuthals, and (19–2) is the only change from previous azimuthals among the general (oblique) formulas (18–2) through (5–3) for the forward calculations as listed above.

Table 27 shows rectangular coordinates for the equatorial aspect for a 10° graticule with a sphere of radius $R = 1.0$.

FORMULAS FOR THE ELLIPSOID

The formulas for the polar aspect of the ellipsoidal Azimuthal Equidistant projection are relatively simple and are theoretically accurate for a map of the entire world. However, such a use is unnecessary because the errors of the sphere versus the ellipsoid become insignificant when compared to the basic errors of projection. The

TABLE 27.—*Azimuthal Equidistant projection: Rectangular coordinates for equatorial aspect (sphere)*

[One hemisphere; $R=1$. y coordinates in parentheses under x coordinates]

Lat. \ Long.	0°	10°	20°	30°	40°
90°	0.00000	0.00000	0.00000	0.00000	0.00000
	(1.57080)	(1.57080)	(1.57080)	(1.57080)	(1.57080)
80	.00000	.04281	.08469	.12469	.16188
	(1.39626)	(1.39829)	(1.40434)	(1.41435)	(1.42823)
70	.00000	.07741	.15362	.22740	.29744
	(1.22173)	(1.22481)	(1.23407)	(1.24956)	(1.27137)
60	.00000	.10534	.20955	.31145	.40976
	(1.04720)	(1.05068)	(1.06119)	(1.07891)	(1.10415)
50	.00000	.12765	.25441	.37931	.50127
	(.87266)	(.87609)	(.88647)	(.90408)	(.92938)
40	.00000	.14511	.28959	.43276	.57386
	(.69813)	(.70119)	(.71046)	(.72626)	(.74912)
30	.00000	.15822	.31607	.47314	.62896
	(.52360)	(.52606)	(.53355)	(.54634)	(.56493)
20	.00000	.16736	.33454	.50137	.66762
	(.34907)	(.35079)	(.35601)	(.36497)	(.37803)
10	.00000	.17275	.34546	.51807	.69054
	(.17453)	(.17541)	(.17810)	(.18270)	(.18943)
0	.00000	.17453	.34907	.52360	.69813
	(.00000)	(.00000)	(.00000)	(.00000)	(.00000)

TABLE 27.—*Azimuthal Equidistant projection: Rectangular coordinates for equatorial aspect (sphere)*—Continued

Lat. \ Long.	50°	60°	70°	80°	90°
90°	0.00000	0.00000	0.00000	0.00000	0.00000
	(1.57080)	(1.57080)	(1.57080)	(1.57080)	(1.57080)
80	.19529	.22399	.24706	.26358	.27277
	(1.44581)	(1.46686)	(1.49104)	(1.51792)	(1.54693)
70	.36234	.42056	.47039	.50997	.53724
	(1.29957)	(1.33423)	(1.37533)	(1.42273)	(1.47607)
60	.50301	.58948	.66711	.73343	.78540
	(1.13733)	(1.17896)	(1.22963)	(1.28993)	(1.36035)
50	.61904	.73106	.83535	.92935	1.00969
	(.96306)	(1.00602)	(1.05942)	(1.12464)	(1.20330)
40	.71195	.84583	.97392	1.09409	1.20330
	(.77984)	(.81953)	(.86967)	(.93221)	(1.00969)
30	.78296	.93436	1.08215	1.22487	1.36035
	(.59010)	(.62291)	(.66488)	(.71809)	(.78540)
20	.83301	.99719	1.15965	1.31964	1.47607
	(.39579)	(.41910)	(.44916)	(.48772)	(.53724)
10	.86278	1.03472	1.20620	1.37704	1.54693
	(.19859)	(.21067)	(.22634)	(.24656)	(.27277)
0	.87266	1.04720	1.22173	1.39626	1.57080
	(.00000)	(.00000)	(.00000)	(.00000)	(.00000)

Radius of sphere = 1.0.
Origin: $(x, y)=0$ at (lat., long.) $=0$. Y axis increases north. Other quadrants of hemisphere are symmetrical.

polar form is truly azimuthal as well as equidistant. Given a, e, ϕ_1, λ_0, ϕ, and λ, for the north polar aspect, $\phi_1 = 90°$:

$$x = \rho \sin(\lambda - \lambda_0) \quad (17\text{–}30)$$
$$y = -\rho \cos(\lambda - \lambda_0) \quad (17\text{–}31)$$
$$k = \rho/(a\, m) \quad (17\text{–}32)$$

where

$$\rho = M_p - M \quad (19\text{–}16)$$
$$M = a\,[(1 - e^2/4 - 3e^4/64 - 5e^6/256 - \ldots)\phi - (3e^2/8 + 3e^4/32 + 45e^6/1024 + \ldots)\sin 2\phi + (15e^4/256 + 45e^6/1024 + \ldots)\sin 4\phi - (35e^6/3072 + \ldots)\sin 6\phi + \ldots] \quad (3\text{–}21)$$

with M_p the value of M for a ϕ of 90°,
and $m = \cos\phi/(1 - e^2 \sin^2\phi)^{1/2}$ (12–15)

For the south polar aspect, the equations for the north polar aspect apply, except that equations (17–31) and (19–16) become

$$y = \rho \cos(\lambda - \lambda_0) \quad (18\text{–}23)$$
$$\rho = M_p + M \quad (19\text{–}17)$$

The origin falls at the pole in either case, and the Y axis follows the central meridian λ_0. For the north polar aspect, λ_0 is shown below the pole, and y increases along λ_0 toward the pole. For the south polar aspect, λ_0 is shown above the pole, and y increases along λ_0 away from the pole.

Table 28 lists polar coordinates for the ellipsoidal aspect of the Azimuthal Equidistant, using the International ellipsoid.

TABLE 28.—*Ellipsoidal Azimuthal Equidistant projection (International ellipsoid)—Polar Aspect*

Latitude	Radius, meters	h	k
90°	0.0	1.0	1.000000
89	111,699.8	1.0	1.000051
88	223,399.0	1.0	1.000203
87	335,096.8	1.0	1.000457
86	446,792.5	1.0	1.000813
85	558,485.4	1.0	1.001270
84	670,175.0	1.0	1.001830
83	781,860.4	1.0	1.002492
82	893,541.0	1.0	1.003256
81	1,005,216.2	1.0	1.004124
80	1,116,885.2	1.0	1.005095
79	1,228,547.5	1.0	1.006169
78	1,340,202.4	1.0	1.007348
77	1,451,849.2	1.0	1.008631
76	1,563,487.4	1.0	1.010019
75	1,675,116.3	1.0	1.011513
74	1,786,735.3	1.0	1.013113
73	1,898,343.8	1.0	1.014821
72	2,009,941.3	1.0	1.016636
71	2,121,527.1	1.0	1.018560
70	2,233,100.9	1.0	1.020594

h = scale factor along meridian.
k = scale factor along parallel.

For the oblique and equatorial aspects of the ellipsoidal Azimuthal Equidistant, both nearly rigorous and approximate sets of formulas have been derived. For mapping of Guam, the National Geodetic Survey and the USGS use an approximation to the ellipsoidal oblique Azimuthal Equidistant called the "Guam projection." It is described by Claire (1968, p. 52–53) as follows (changing his symbols to match those in this publication):

> "The plane coordinates of the geodetic stations on Guam were obtained by first computing the geodetic distances [c] and azimuths [Az] of all points from the origin by inverse computations. The coordinates were then computed by the equations: [$x=c \sin Az$ and $y=c \cos Az$]. This really gives a true azimuthal equidistant projection. The equations given here are simpler, however, than those for a geodetic inverse computation, and the resulting coordinates computed using them will not be significantly different from those computed rigidly by inverse computation. This is the reason it is called an approximate azimuthal equidistant projection."

The formulas for the Guam projection are equivalent to the following:

$$x = a(\lambda - \lambda_0) \cos\phi / (1 - e^2 \sin^2\phi)^{1/2} \qquad (19\text{–}18)$$

$$y = M - M_1 + x^2 \tan\phi\,(1 - e^2 \sin^2\phi)^{1/2}/(2a) \qquad (19\text{–}19)$$

where M and M_1 are found from equation (3–21) for ϕ and ϕ_1. Actually, the original formulas are given in terms of seconds of rectifying latitude and geodetic latitude and longitude, but they may be written as above. The x coordinate is thus taken as the distance along the parallel, and y is the distance along the central meridian λ_0 with adjustment for curvature of the parallel. The origin occurs at (ϕ_1, λ_0), the Y axis coincides with the central meridian, and y increases northerly.

For Guam, $\phi_1 = 13°28'20.87887''$ N. lat. and $\lambda_0 = 144°44'55.50254''$ E. long., with 50,000 m added to both x and y to eliminate negative numbers. The Clarke 1866 ellipsoid is used.

A more complicated and more accurate approach to the oblique ellipsoidal Azimuthal Equidistant projection is used for plane coordinates of various individual islands of Micronesia. In this form, the true distance and azimuth of any point on the island or in nearby waters are measured from the origin chosen for the island and along the normal section or plane containing the perpendicular to the surface of the ellipsoid at the origin. This is not exactly the same as the shortest or geodesic distance between the points, but the difference is negligible (Bomford, 1971, p. 125). This distance and azimuth are plotted on the map. The projection is, therefore, equidistant and azimuthal with respect to the center and appears to satisfy all the requirements for an ellipsoidal Azimuthal Equidistant projection, although it is described as a "modified" form. The origin is assigned large-enough values of x and y to prevent negative readings.

The formulas for calculation of this distance and azimuth have been published in various forms, depending on the maximum distance involved. The projection system for Micronesia makes use of "Clarke's best formula" and Robbins's inverse of this. These are considered suitable for lines up to 800 km in length. The formulas below, rearranged slightly from Robbins's formulas as given in Bomford (1971, p. 136–137), are extended to produce rectangular coordinates. No iteration is required. They are listed in the order of use, given a central point at lat. ϕ_1, long. λ_0, coordinates x_0 and y_0 of the central point, the Y axis along the central meridian λ_0, y increasing northerly, ellipsoidal parameters a and e, and ϕ and λ.

To find x and y:

$$N_1 = a/(1-e^2\sin^2\phi_1)^{1/2} \tag{4-20a}$$
$$N = a/(1-e^2\sin^2\phi)^{1/2} \tag{4-20}$$
$$\psi = \arctan[(1-e^2)\tan\phi + e^2 N_1 \sin\phi_1/(N\cos\phi)] \tag{19-20}$$
$$Az = \arctan\{\sin(\lambda-\lambda_0)/[\cos\phi_1\tan\psi - \sin\phi_1\cos(\lambda-\lambda_0)]\} \tag{19-21}$$

The ATAN2 Fortran function should be used with equation (19–21), but it is not applicable to (19–20).

If $\sin Az = 0$,

$$s = \pm\arcsin(\cos\phi_1\sin\psi - \sin\phi_1\cos\psi) \tag{19-22}$$

taking the sign of $\cos Az$.

If $\sin Az \neq 0$,

$$s = \arcsin[\sin(\lambda-\lambda_0)\cos\psi/\sin Az] \tag{19-22a}$$

In either case,

$$G = e\sin\phi_1/(1-e^2)^{1/2} \tag{19-23}$$
$$H = e\cos\phi_1\cos Az/(1-e^2)^{1/2} \tag{19-24}$$
$$c = N_1\, s\{1 - s^2H^2(1-H^2)/6 + (s^3/8)GH(1-2H^2) + (s^4/120)[H^2(4-7H^2) - 3G^2(1-7H^2)] - (s^5/48)GH\} \tag{19-25}$$
$$x = c\sin Az + x_0 \tag{19-26}$$
$$y = c\cos Az + y_0 \tag{19-27}$$

where c is the geodetic distance, and Az is azimuth east of north.

Table 29 shows the parameters for the various islands mapped with this projection.

Inverse formulas for the polar ellipsoidal aspect, given a, e, ϕ_1, λ_0, x, and y:

$$\phi = \mu + (3e_1/2 - 27e_1^3/32)\sin 2\mu + (21e_1^2/16 - 55e_1^4/32)\sin 4\mu + (151e_1^3/96)\sin 6\mu + \ldots \tag{3-26}$$

TABLE 29.—*Plane coordinate systems for Micronesia: Clarke 1866 ellipsoid*

Group	Islands	Station at Origin	ϕ_1 Lat N.	λ_0 Long E.	Meters x_0	Meters y_0
Caroline Islands	Yap	Yap Secor	9°32′48.898″	138°10′07.084″	39,987.92	60,022.98
	Palau	Arakabesan Is.	7°21′04.3996″	134°27′01.6015″	50,000.00	150,000.00
	Ponape	Distad (USE)	6°57′54.2725″	158°12′33.4772″	80,122.82	80,747.24
	Truk Atoll	Truk Secor RM 1	7°27′22.3600″	151°50′17.8530″	60,000.00	70,000.00
Mariana Islands	Saipan	Saipan	15°11′05.6830″	145°44′29.9720″	28,657.52	67,199.99
	Rota	Astro	14°07′58.8608″	145°08′03.2275″	5,000.00	5,000.00
Marshall Islands	Majuro Atoll	Dalap	7°05′14.0″	171°22′34.5″	85,000.00	40,000.00

x_0, y_0 = rectangular coordinates of center of projection.
ϕ_1, λ_0 = geodetic coordinates of center of projection.

where

$$e_1 = [1-(1-e^2)^{1/2}]/[1+(1-e^2)^{1/2}] \tag{3-24}$$
$$\mu = M/[a(1-e^2/4-3e^4/64-5e^6/256-\ldots)] \tag{8-19}$$
$$M = M_p - \rho \text{ for the north polar case,} \tag{19-28}$$

and

$$M = \rho - M_p \text{ for the south polar case.} \tag{19-29}$$

Equation (3-21), listed with the forward equations, is used to find M_p for $\phi = 90°$. For either case,

$$\rho = (x^2+y^2)^{1/2} \tag{16-18}$$

For longitude, for the north polar case,

$$\lambda = \lambda_0 + \arctan[x/(-y)] \tag{16-16}$$

For the south polar case,

$$\lambda = \lambda_0 + \arctan(x/y) \tag{16-17}$$

Inverse formulas for the Guam projection (Claire, 1968, p. 53) involve an iteration of two equations, which may be rearranged and rewritten in the following form consistent with the above formulas. Given a, e, ϕ_1, λ_0, x, and y, M_1 is calculated for ϕ_1 from (3-21), given with forward equations. (If false northings and eastings are included in x and y, they must be subtracted first.)

Then, first assuming $\phi = \phi_1$,

$$M = M_1 + y - x^2 \tan\phi\,(1-e^2\sin^2\phi)^{1/2}/(2a) \tag{19-30}$$

Using this M, μ is calculated from (8-19) and inserted into the right side of (3-26) to solve for a new ϕ on the left side. This is inserted into (19-30), a new M is found, and it is resubstituted into (8-19), μ into (3-26), etc., until ϕ on the left side of (3-26) changes by less than a chosen convergence figure, for a final ϕ. Then

$$\lambda = \lambda_0 + x(1-e^2\sin^2\phi)^{1/2}/(a\cos\phi) \tag{19-31}$$

The inverse Guam formulas arbitrarily stop at three iterations, which are sufficient for the small area.

For the Micronesia version of the ellipsoidal Azimuthal Equidistant projection, the inverse formulas given below are "Clarke's best formula," as given in Bomford (1971, p. 133) and do not involve iteration. They have also been rearranged to begin with rectangular coordinates, but they are also suitable for finding latitude and longitude accurately for a point at any given distance c (up to about 800 km) and azimuth Az (east of north) from the center, if equations (19-32) and (19-33) are deleted. In order of use, given a, e, central point at lat. ϕ_1, long. λ_0, rectangular coordinates of center x_0, y_0, and x and y for another point, to find ϕ and λ:

$$c = [(x - x_0)^2 + (y - y_0)^2]^{1/2} \tag{19-32}$$
$$Az = \arctan [(x - x_0)/(y - y_0)] \tag{19-33}$$
$$N_1 = a/(1 - e^2 \sin^2 \phi_1)^{1/2} \tag{4-20a}$$
$$A = -e^2 \cos^2 \phi_1 \cos^2 Az/(1 - e^2) \tag{19-34}$$
$$B = 3e^2 (1 - A) \sin \phi_1 \cos \phi_1 \cos Az/(1 - e^2) \tag{19-35}$$
$$D = c/N_1 \tag{19-36}$$
$$E = D - A (1 + A) D^3/6 - B(1 + 3A) D^4/24 \tag{19-37}$$
$$F = 1 - AE^2/2 - BE^3/6 \tag{19-38}$$
$$\psi = \arcsin (\sin \phi_1 \cos E + \cos \phi_1 \sin E \cos Az) \tag{19-39}$$
$$\lambda = \lambda_0 + \arcsin (\sin Az \sin E/\cos \psi) \tag{19-40}$$
$$\phi = \arctan [(1 - e^2 F \sin \phi_1/\sin \psi) \tan \psi/(1 - e^2)] \tag{19-41}$$

The ATAN2 function of Fortran, or equivalent, should be used in equation (19-33), but not (19-41).

SPACE MAP PROJECTIONS

One of the most recent developments in map projections has been that involving a time factor, relating a mapping satellite revolving in an orbit about a rotating Earth. With the advent of automated continuous mapping in the near future, the static projections previously available are not sufficient to provide the accuracy merited by the imagery, without frequent readjustment of projection parameters and discontinuity at each adjustment. Projections appropriate for such satellite mapping are much more complicated mathematically, but, once derived, can be handled by computer.

Several such space map projections have been conceived, and all but one have been mathematically developed. The Space Oblique Mercator projection, suitable for mapping imagery from Landsat and other vertically scanning satellites, is described below. The Space Oblique Conformal Conic is a still more complex projection, currently only in conception, but for which mathematical development will be required when satellite side-looking imagery has been developed to an extent sufficient to encourage its use. Satellite-tracking projections are simpler, but are less important and are not discussed here (Snyder, 1981a).

20. SPACE OBLIQUE MERCATOR PROJECTION

SUMMARY

- Modified cylindrical projection with map surface defined by satellite orbit.
- Designed especially for continuous mapping of satellite imagery.
- Basically conformal, especially in region of satellite scanning.
- Groundtrack of satellite, a curved line on the globe, is shown as a curved line on the map and is continuously true to scale as orbiting continues.
- All meridians and parallels are curved lines, except meridian at each polar approach.
- Recommended only for a relatively narrow band along the groundtrack.
- Developed 1973–79 by Colvocoresses, Snyder, and Junkins.

HISTORY

The launching of an Earth-sensing satellite by the National Aeronautics and Space Administration in 1972 led to a new era of mapping on a continuous basis from space. This satellite, first called ERTS-1 and renamed Landsat-1 in 1975, was followed by two others, all of which circled the Earth in a nearly circular orbit inclined about 99° to the Equator and scanning a swath about 185 km (officially 100 nautical miles) wide from an altitude of about 919 km. A fourth Landsat has somewhat different orbital parameters.

Continuous mapping of this band required a new map projection. Although conformal mapping was desired, the normal choice, the Oblique Mercator projection, is unsatisfactory for two reasons. First, the Earth is rotating at the same time the satellite is moving in an orbit which lies in a plane almost at a right angle to the plane of the Equator, with the double-motion effect producing a curved groundtrack, rather than one formed by the intersection of the Earth's surface with a plane passing through the center of the Earth. Second, the only available Oblique Mercator projections for the ellipsoid are for limited coverage near the chosen central point.

What was needed was a map projection on which the groundtrack remained true-to-scale throughout its course. This course does not, in the case of Landsat, return to the same point for 251 revolutions. It was also felt necessary that conformality be closely maintained within the range of the swath mapped by the satellite.

Alden P. Colvocoresses of the Geological Survey was the first to realize not only that such a projection was needed, but also that it was mathematically feasible. He defined it geometrically (Colvocoresses, 1974) and immediately began to appeal for the development of formulas. The following formulas resulted from the writer's response to Colvocoresses' appeal made at a geodetic conference at The Ohio State University in 1976. While the formulas were derived (1977–79) for Landsat, they are applicable to any satellite orbiting the Earth in a circular or elliptical orbit and at any inclination. Less complete formulas were also developed in 1977 by John L. Junkins, then of the University of Virginia. The following formulas are limited to nearly circular orbits. A complete derivation for orbits of any ellipticity is given by Snyder (1981) and another summary by Snyder (1978b).

FEATURES AND USAGE

The Space Oblique Mercator (SOM) projection visually differs from the Oblique Mercator projection in that the central line (the groundtrack of the orbiting satellite) is slightly curved, rather than straight. For Landsat, this groundtrack appears as a nearly sinusoidal curve crossing the X axis at an angle of about 8°. The scanlines, perpendicular to the orbit in space, are slightly skewed with respect to the perpendicular to the groundtrack when plotted on the sphere or ellipsoid. Due to Earth rotation, the scanlines on the Earth (or map) intersect the groundtrack at about 86° near the Equator, but at 90° when the groundtrack makes its closest approach to either pole. With the curved groundtrack, the scanlines generally are skewed with respect to the X and Y axes, inclined about 4° to the Y axis at the Equator, and not at all at the polar approaches.

The Landsat orbit intersects the plane of the Equator at an inclination of about 99°, measured as the angle between the direction of satellite revolution and the direction of Earth rotation. Thus the groundtrack reaches limits of about lat. 81° N. and S. (180° minus 99°). The 185-km swath scanned by Landsat, about 0.83° on either side of the groundtrack, leads to complete coverage of the Earth from about lat. 82° N. to 82° S. in the course of the 251 revolutions. With a nominal altitude of about 919 km, the time of one revolution is 103.267 minutes, and the orbit is designed to complete the 251 revolutions in exactly 18 days.

As on the normal Oblique Mercator, all meridians and parallels are curved lines, except for the meridian crossed by the groundtrack at each polar approach. While the straight meridians are 180° apart on the normal Oblique Mercator, they are about 167° apart on the SOM for Landsat, since the Earth advances about 26° in longitude for each revolution of the satellite.

As developed, the SOM is not perfectly conformal for either the sphere or ellipsoid, although the error is negligible within the scanning range for either. Along the groundtrack, scale in the direction of the groundtrack is correct for sphere or ellipsoid, while conformality is correct for the sphere and within 0.0005 percent of correct for the ellipsoid. At 1° away from the groundtrack, the Tissot Indicatrix (the ellipse of distortion) is flattened a maximum of 0.001 percent for the sphere and a maximum of 0.006 percent for the ellipsoid (this would be zero if conformal). The scale 1° away from the groundtrack averages 0.015 percent greater than that at the groundtrack, a value which is fundamental to projection. As a result of the slight nonconformality, the scale 1° away from the groundtrack on the ellipsoid then varies from 0.012 to 0.018 percent more than the scale along the groundtrack.

A prototype version of the SOM was used by NASA with a geometric analogy proposed by Colvocoresses (1974) while he was seeking the more rigorous mathematical development. This consisted basically of moving an obliquely tangent cylinder back and forth on the sphere so that a circle around it which would normally be tangent shifted to follow the groundtrack. This is suitable near the Equator but leads to errors of about 0.1 percent near the poles, even for the sphere. In 1977, John B. Rowland of the USGS applied the Hotine Oblique Mercator (described previously) to Landsat 1, 2, and 3 orbits in five stationary zones, with smaller but significant errors (up to twice the scale variation of the SOM) resulting from the fact that the groundtrack cannot follow the straight central line of the HOM. In addition, there are discontinuities at the zone changes. This was done to fill the void resulting from the lack of SOM formulas.

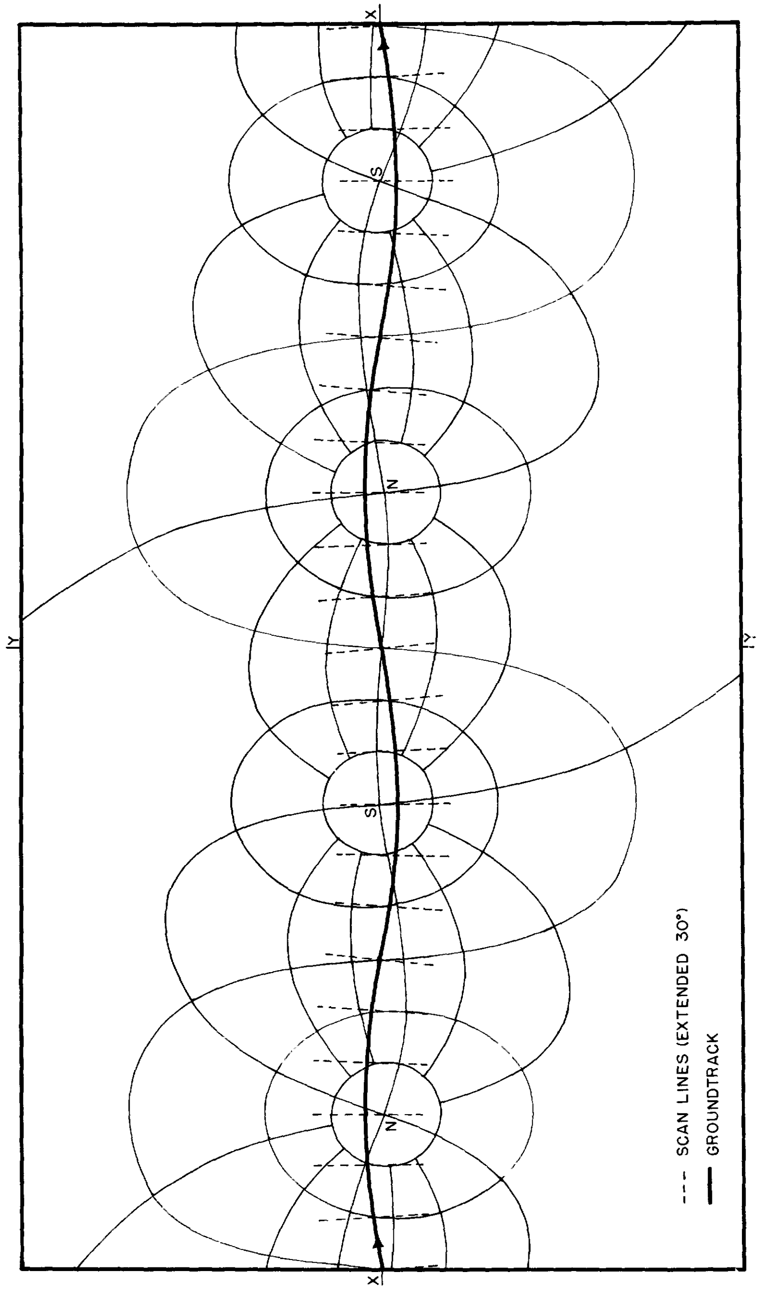


FIGURE 28.—Two orbits of the Space Oblique Mercator projection, shown for Landsat. Designed for a narrow band along groundtrack, which remains true to scale. Note the change in longitude at a given latitude along the groundtrack, with successive orbits.

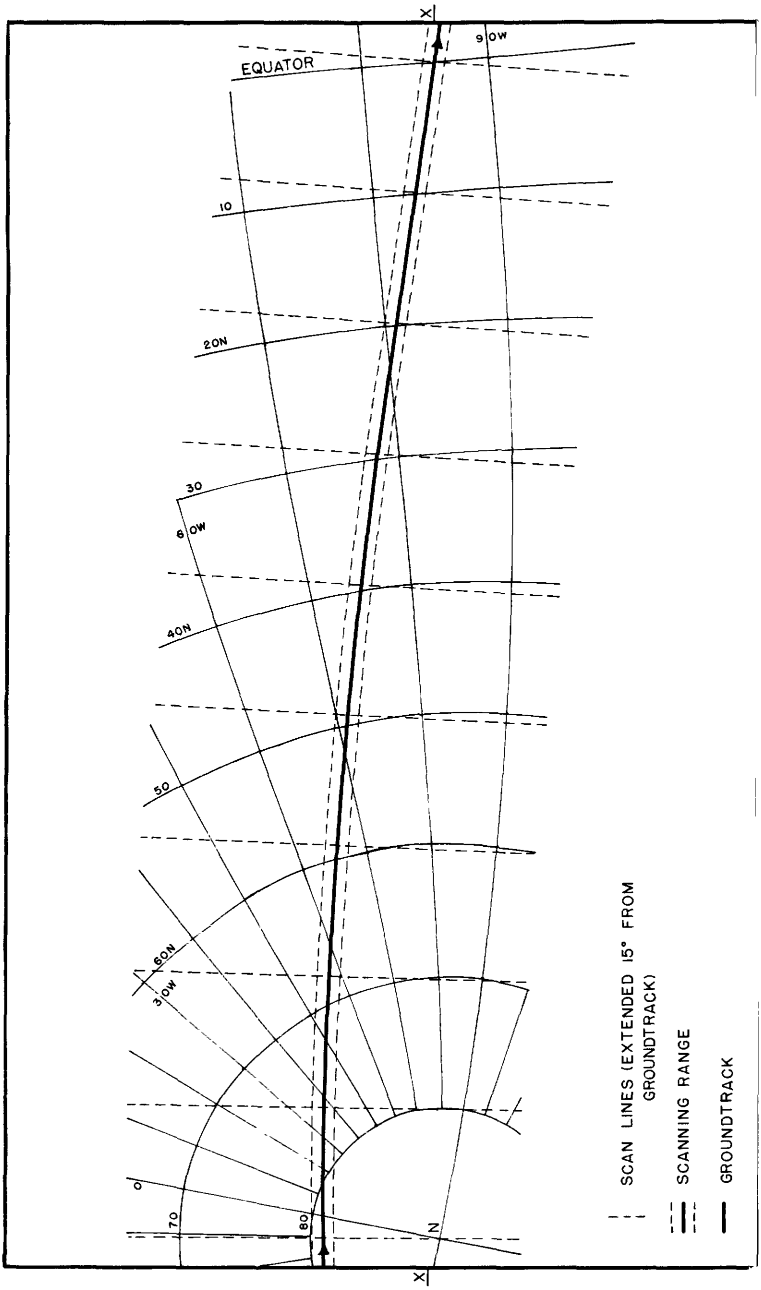


FIGURE 29.—One quadrant of the Space Oblique Mercator projection. An "enlargement" of part of figure 28, beginning at the North Pole.

As of this writing, the final SOM development has not replaced the HOM programing for Landsat mapping, but this is expected in the near future. The projection is included here because of its potential use and the fact that it was developed wholly within or under the supervision of the USGS. Figures 28 and 29 show the SOM extended to two orbits with a 30° graticule and for one-fourth of an orbit with a 10° graticule, respectively. The progressive advance of meridians may be seen in figure 28. Both views are for Landsat constants.

FORMULAS FOR THE SPHERE

Both iteration and numerical integration are involved in the formulas as presented for sphere or ellipsoid. The iteration is quite rapid (three to five iterations required for ten-place accuracy), and the numerical integration is greatly simplified by the use of rapidly converging Fourier series. The coefficients for the Fourier series may be calculated once for a given satellite orbit. [Some formulas below are slightly simplified from those first published (Snyder, 1978b).]

For the forward equations, for the sphere and circular orbit, to find (x, y) for a given (ϕ, λ), it is necessary to be given R, i, P_2, P_1, λ_0, ϕ, and λ, where

R = the radius of the globe at the scale of the map.

i = angle of inclination between the plane of the Earth's Equator and the plane of the satellite orbit, measured counterclockwise from the Equator to the orbital plane at the ascending node (99.092° for Landsat).

P_2 = time required for revolution of the satellite (103.267 min for Landsat).

P_1 = length of Earth's rotation with respect to the precessed ascending node. For Landsat, the satellite orbit is Sun-synchronous; that is, it is always the same with respect to the Sun, equating P_1 to the solar day (1,440 min). The ascending node is the point on the satellite orbit at which the satellite crosses the Earth's equatorial plane in a northerly direction.

λ_0 = the geodetic longitude of the ascending node at time $t=0$.

(ϕ, λ) = geodetic latitude and longitude of point to be plotted on map.

t = time elapsed since the satellite crossed the ascending node for the orbit considered to be the initial one. This may be the current orbit or any earlier one, as long as the proper λ_0 is used.

First, various constants applying to the entire map for all the satellite's orbits should be calculated a single time:

$$B=(2/\pi)\int_0^{\pi/2} [(H-S^2)/(1+S^2)^{1/2}]d\lambda' \qquad (20\text{–}1)$$

$$A_n=[4/(\pi n)]\int_0^{\pi/2}[(H-S^2)/(1+S^2)^{1/2}] \cos n\lambda' \, d\lambda' \qquad (20\text{–}2)$$

for $n=2$ and 4 only.

$$C_n = [4(H+1)/(\pi n)] \int_0^{\pi/2} [S/(1+S^2)^{1/2}] \cos n\lambda' \, d\lambda' \qquad (20\text{–}3)$$

for $n = 1$ and 3 only.

For calculating A_n, B, and C_n, numerical integration using Simpson's rule is recommended, with 9° intervals in λ' (sufficient for ten-place accuracy). The terms shown are sufficient for seven-place accuracy, ample for the sphere. For H and S in equations (20–1) through (20–3):

$$H = 1 - (P_2/P_1) \cos i \qquad (20\text{–}4)$$
$$S = (P_2/P_1) \sin i \cos \lambda' \qquad (20\text{–}5)$$

To find x and y, with the X axis passing through each ascending and descending node (wherever the groundtrack crosses the Equator), x increasing in the direction of satellite motion, and the Y axis passing through the ascending node for time $t = 0$:

$$x/R = B\lambda' + A_2 \sin 2\lambda' + A_4 \sin 4\lambda' + \ldots - [S/(1+S^2)^{1/2}] \ln \tan(\pi/4 + \phi'/2) \qquad (20\text{–}6)$$
$$y/R = C_1 \sin \lambda' + C_3 \sin 3\lambda' + \ldots + [1/(1+S^2)^{1/2}] \ln \tan(\pi/4 + \phi'/2) \qquad (20\text{–}7)$$

where B, A_n, and C_n and constants calculated above, S is calculated from (20–5) for each point, and

$$\lambda' = \arctan(\cos i \tan \lambda_t + \sin i \tan \phi / \cos \lambda_t) \qquad (20\text{–}8)$$
$$\lambda_t = \lambda - \lambda_0 + (P_2/P_1)\lambda' \qquad (20\text{–}9)$$
$$\phi' = \arcsin(\cos i \sin \phi - \sin i \cos \phi \sin \lambda_t) \qquad (20\text{–}10)$$
$$\lambda_0 = 128.87° - (360°/251)p \text{ (Landsat 1, 2, 3 only)} \qquad (20\text{–}11)$$

p = path number of Landsat orbit for which the ascending node occurs at time $t = 0$. This ascending node is prior to the start of the path, so that the path extends from ¼ orbit past this node to $^5/_4$ orbit past it.

λ' = "transformed longitude," the angular distance along the groundtrack, measured from the initial ascending node ($t = 0$), and directly proportional to t for a circular orbit, or $\lambda' = 360°\, t/P_2$.

λ_t = a "satellite-apparent" longitude, the longitude relative to λ_0 seen by the satellite if the Earth were stationary.

ϕ' = "transformed latitude," the angular distance from the groundtrack, positive to the left of the satellite as it proceeds along the orbit.

Finding λ' from equations (20–8) and (20–9) involves iteration performed in the following manner: After selecting ϕ and λ, the λ' of the nearest polar approach, λ_p', is used as the first trial λ' on the right side of (20–9); λ_t is calculated and substituted into (20–8) to find a new λ'. A quadrant adjustment (see below) is applied to λ', since the computer normally calculates arctan as an angle between $-90°$ and $90°$, and this λ' is used as the next trial λ' in (20–9), etc., until λ' changes by less than a chosen convergence factor. The value of λ_p' may be determined as follows, for any number of revolutions:

$$\lambda_p' = 90° \times (4N + 2 \pm 1) \tag{20–12}$$

where N is the number of orbits completed at the last ascending node before the satellite passes the nearest pole, and the $\pm$ takes minus in the Northern Hemisphere and plus in the Southern (either for the Equator). Thus, if only the first path number past the ascending node is involved, λ_p' is 90° for the first quadrant (North Pole to Equator), 270° for the second and third quadrants (Equator to South Pole to Equator), and 450° for the fourth quadrant (Equator to North Pole). For quadrant adjustment to λ' calculated from (20–8), the Fortran ATAN2 or its equivalent should not be used. Instead, λ' should be increased by λ_p' minus the following factor: 90° times sin λ_p' times ± 1 (taking the sign of cos λ_{tp}, where $\lambda_{tp} = \lambda - \lambda_0 + (P_2/P_1)\lambda_p'$). If cos λ_{tp} is zero, the final λ' is λ_p'. Thus, the adder to the arctan is 0° for the quadrant between the ascending node and the start of the path, and 180°, 180°, 360°, and 360°, respectively, for the four quadrants of the first path.

The closed forms of equations (20–6) and (20–7) are as follows:

$$x/R = \int_0^{\lambda'} [(H - S^2)/(1 + S^2)^{1/2}]d\lambda' - [S/(1 + S^2)^{1/2}] \ln \tan (\pi/4 + \phi'/2) \tag{20–6a}$$

$$y/R = (H + 1)\int_0^{\lambda'} [S/(1 + S^2)^{1/2}]d\lambda' + [1/(1 + S^2)^{1/2}] \ln \tan (\pi/4 + \phi'/2) \tag{20–7a}$$

Since these involve numerical integration for each point, the series forms, limiting numerical integration to once per satellite, are distinctly preferable. These are Fourier series, and equations (20–2) and (20–3) normally require integration from 0 to 2π, without the multiplier 4, but the symmetry of the circular orbit permits the simplification as shown for the nonzero coefficients.

For inverse formulas for the sphere, given R, i, P_2, P_1, λ_0, x, and y, with ϕ and λ required: Constants B, A_n, C_n, and λ_0 must be calculated from (20–1) through (20–3) and (20–11) just as they were for the forward equations.
Then,

$$\lambda = \arctan [(\cos i \sin \lambda' - \sin i \tan \phi')/\cos \lambda'] - (P_2/P_1)\lambda' + \lambda_0 \tag{20–13}$$

$$\phi = \arcsin (\cos i \sin \phi' + \sin i \cos \phi' \sin \lambda') \tag{20–14}$$

where the ATAN2 function of Fortran is useful for (20–13), except that it may be necessary to add or subtract 360° to place λ between long. 180° E. (+) and 180° W. (−), and

$$\lambda' = [x/R + Sy/R - A_2 \sin 2\lambda' - A_4 \sin 4\lambda' - S(C_1 \sin \lambda' + C_3 \sin 3\lambda')]/B \quad (20\text{–}15)$$

$$\ln \tan(\pi/4 + \phi'/2) = (1 + S^2)^{1/2}(y/R - C_1 \sin \lambda' - C_3 \sin 3\lambda') \quad (20\text{–}16)$$

Equation (20–15) is iterated by trying almost any λ' (preferably $x/(BR)$) in the right side, solving for λ' on the left and using the new λ' for the next trial, etc., until there is no significant change between successive trial values. Equation (20–16) uses the final λ' calculated from (20–15).

The closed form of equation (20–15) given below involves repeated numerical integration as well as iteration, making its use almost prohibitive:

$$(x + Sy)/R = \int_0^{\lambda'} [(H - S^2)/(1 + S^2)^{1/2}]d\lambda' + S(H+1)\int_0^{\lambda'} [S/(1 + S^2)^{1/2}]d\lambda' \quad (20\text{–}15a)$$

The following closed form of (20–16) requires the use of the last integral calculated from (20–15a):

$$\ln \tan(\pi/4 + \phi'/2) = (1 + S^2)^{1/2}\{(y/R) - (H+1)\int_0^{\lambda'} [S/(1 + S^2)^{1/2}]d\lambda'\} \quad (20\text{–}16a)$$

The original published forms of these equations include several other Fourier coefficient calculations which slightly save computer time when continuous mapping is involved. The resulting equations are more complicated, so they are omitted here for simplicity. The above equations are as accurate and only slightly less efficient.

The values of coefficients for Landsat ($P_2/P_1 = 18/251$; $i = 99.092°$) are listed here for convenience:

$A_2 = -0.0018820$
$A_4 = 0.0000007$
$B = 1.0075654142$ for λ' in radians
$= 0.0175853339$ for λ' in degrees
$C_1 = 0.1421597$
$C_3 = -0.0000296$

It is also of interest to determine values of ϕ, λ, or λ' along the groundtrack, given any one of the three (as well as P_2, P_1, i, and λ_0). Given ϕ,

$$\lambda' = \arcsin(\sin\phi/\sin i) \quad (20\text{–}17)$$

$$\lambda = \arctan[(\cos i \sin\lambda')/\cos\lambda'] - (P_2/P_1)\lambda' + \lambda_0 \quad (20\text{–}18)$$

If ϕ is given for a descending part of the orbit (daylight on Landsat), subtract λ' from the λ' of the nearest descending node (180°, 540°, . . .). If the orbit is ascending, add λ' to the λ' of the nearest ascending node

(0°, 360°, . . .). For a given path, only 180° and 360°, respectively, are involved.

Given λ,

$$\lambda' = \arctan(\tan \lambda_t / \cos i) \quad (20\text{–}19)$$

$$\lambda_t = \lambda - \lambda_0 + (P_2/P_1)\lambda' \quad (20\text{–}9)$$

$$\phi = \arcsin(\sin i \sin \lambda') \quad (20\text{–}20)$$

Given λ', equations (20–18) and (20–20) may be used for λ and ϕ, respectively. Equations (20–6) and (20–7), with $\phi' = 0$, convert these values to x and y. Equations (20–19) and (20–9) require joint iteration, using the same procedure as that for the pair of equations (20–8) and (20–9) given earlier. The λ calculated from equation (20–18) should have the same quadrant adjustment as that described for (20–13).

The formulas for scale factors h and k and maximum angular deformation ω are so lengthy that they are not given here. They are available in Snyder (1981). Table 30 lists these values as calculated for the spherical SOM using Landsat constants.

TABLE 30.—*Scale factors for the spherical Space Oblique Mercator projection using Landsat 1, 2, and 3 constants*

λ'	$\phi'=1°$				$\phi'=-1°$			
	h	k	$\omega°$	$m_{\phi'}$	h	k	$\omega°$	$m_{\phi'}$
0°	1.000154	1.000151	0.0006	1.000152	1.000154	1.000151	0.0006	1.000152
5	1.000153	1.000151	.0006	1.000151	1.000154	1.000151	.0006	1.000152
10	1.000153	1.000151	.0006	1.000151	1.000155	1.000151	.0006	1.000153
15	1.000153	1.000151	.0006	1.000150	1.000155	1.000151	.0006	1.000153
20	1.000152	1.000151	.0006	1.000150	1.000156	1.000151	.0006	1.000154
25	1.000152	1.000151	.0006	1.000150	1.000156	1.000151	.0006	1.000154
30	1.000152	1.000151	.0005	1.000149	1.000156	1.000151	.0005	1.000154
35	1.000152	1.000150	.0005	1.000149	1.000156	1.000151	.0005	1.000154
40	1.000152	1.000150	.0005	1.000150	1.000156	1.000151	.0005	1.000154
45	1.000152	1.000150	.0004	1.000150	1.000156	1.000151	.0005	1.000154
50	1.000152	1.000150	.0004	1.000150	1.000156	1.000151	.0004	1.000154
55	1.000152	1.000150	.0004	1.000150	1.000155	1.000151	.0004	1.000154
60	1.000153	1.000151	.0003	1.000151	1.000155	1.000151	.0003	1.000154
65	1.000153	1.000151	.0003	1.000151	1.000155	1.000151	.0003	1.000153
70	1.000153	1.000151	.0002	1.000152	1.000154	1.000151	.0002	1.000153
75	1.000153	1.000151	.0002	1.000152	1.000154	1.000151	.0002	1.000153
80	1.000153	1.000151	.0001	1.000152	1.000153	1.000152	.0001	1.000153
85	1.000153	1.000152	.0001	1.000152	1.000153	1.000152	.0001	1.000152
90	1.000152	1.000151	.0001	1.000152	1.000152	1.000152	.0000	1.000152

Notes: λ' = angular position along groundtrack, from ascending node.
ϕ' = angular distance away from groundtrack, positive in direction away from North Pole.
h = scale factor along meridian of longitude.
k = scale factor along parallel of latitude.
ω = maximum angular deformation.
$m_{\phi'}$ = scale factor along line of constant ϕ'.
$m_{\lambda'}$ = scale factor along line of constant λ'.
= sec ϕ', or 1.000152 at $\phi' = 1°$.
If $\phi' = 0°$, h, k, and $m_{\phi'} = 1.0$, while $\omega = 0$.

FORMULAS FOR THE ELLIPSOID

Since the SOM is intended to be used only for the mapping of relatively narrow strips, it is highly recommended that the ellipsoidal form be used to take advantage of the high accuracy of scale available, especially as the imagery is further developed and used for more precise measurement. In addition to the normal modifications to the above spherical formulas for ellipsoidal equivalents, an additional element is introduced by the fact that Landsat is designed to scan vertically, rather than in a geocentric direction. Therefore, "pseudotransformed" latitude ϕ'' and longitude λ'' have been introduced. They relate to a geocentric groundtrack for an orbit in a plane through the center of the Earth. The regular transformed coordinates ϕ' and λ' are related to the actual vertical groundtrack. The two groundtracks are only a maximum of 0.008° apart, although a lengthwise displacement of 0.028° for a given position may occur.

In addition, the following formulas accommodate a slight ellipticity in the satellite orbit. They provide a true-to-scale groundtrack for an orbit of any eccentricity, if the orbital motion follows Kepler's laws for two-bodied systems, but the areas scanned by the satellite as shown on the map are distorted beyond the accuracy desired if the eccentricity of the orbit exceeds about 0.05, well above the maximum reported eccentricity of Landsat orbits (about 0.002). For greater eccentricities, more complex formulas (Snyder, 1981) are recommended. If the orbital eccentricity is made zero, these formulas readily reduce to those for a circular orbit. If the eccentricity of the ellipsoid is made zero, the formulas further reduce to the spherical formulas above. These formulas vary slightly, but not significantly, from those previously published. In practice, the coordinates for each picture element (pixel) should not be calculated because of computer time required. Linear interpolation between occasional calculated points can be developed with adequate accuracy.

For the forward formulas, given a, e, i, P_2, P_1, λ_0, a', e', γ, ϕ, and λ, find x and y. As with the spherical formulas, the X axis passes through each ascending and descending node, x increasing in the direction of satellite motion, and the Y axis intersects perpendicularly at the ascending node for the time $t = 0$. Defining terms,

a, e = semimajor axis and eccentricity of ellipsoid, respectively (as for other ellipsoidal projections).

a', e' = semimajor axis and eccentricity of satellite orbit, respectively.

γ = longitude of the perigee relative to the ascending node (for a circular orbit, with $e' = 0$, γ is not involved).

i, P_2, P_1, λ_0, ϕ, λ are as defined for the spherical SOM formulas. For constants applying to the entire map:

$$B_1 = [1/(2\pi)]\int_0^{2\pi}[(HJ - S^2)/(J^2 + S^2)^{1/2}]d\lambda'' \quad (20\text{–}21)$$

$$B_2 = [1/(2\pi)]\int_0^{2\pi}[S(H+J)/(J^2+S^2)^{1/2}]d\lambda'' \quad (20\text{–}22)$$

$$A_n = [1/(\pi n)]\int_0^{2\pi}[(HJ - S^2)/(J^2 + S^2)^{1/2}] \cos n\lambda'' d\lambda'' \quad (20\text{–}23)$$

$$A_n' = [1/(\pi n)]\int_0^{2\pi}[(HJ - S^2)/(J^2 + S^2)^{1/2}] \sin n\lambda'' d\lambda'' \quad (20\text{–}24)$$

$$C_n = [1/(\pi n)]\int_0^{2\pi}[S(H+J)/(J^2 + S^2)^{1/2}] \cos n\lambda'' d\lambda'' \quad (20\text{–}25)$$

$$C_n' = [1/(\pi n)]\int_0^{2\pi}[S(H+J)/(J^2 + S^2)^{1/2}] \sin n\lambda'' d\lambda'' \quad (20\text{–}26)$$

$$J = (1 - e^2)^3 \quad (20\text{–}27)$$

$$W = [(1 - e^2 \cos^2 i)^2/(1 - e^2)^2] - 1 \quad (20\text{–}28)$$

$$Q = e^2 \sin^2 i/(1 - e^2) \quad (20\text{–}29)$$

$$T = e^2 \sin^2 i\,(2 - e^2)/(1 - e^2)^2 \quad (20\text{–}30)$$

$$H_1 = B_1/(B_1{}^2 + B_2{}^2)^{1/2} \quad (20\text{–}31)$$

$$S_1 = B_2/(B_1{}^2 + B_2{}^2)^{1/2} \quad (20\text{–}32)$$

$$j_n = (1/\pi)\int_0^{2\pi}\phi'' \sin n\lambda' d\lambda' \quad (20\text{–}33)$$

$$j_n' = (1/\pi)\int_0^{2\pi}\phi'' \cos n\lambda' d\lambda' \quad (20\text{–}34)$$

$$m_n = (1/\pi)\int_0^{2\pi}(\lambda'' - \lambda') \sin n\lambda' d\lambda' \quad (20\text{–}35)$$

$$m_n' = (1/\pi)\int_0^{2\pi}(\lambda'' - \lambda') \cos n\lambda' d\lambda' \quad (20\text{–}36)$$

where ϕ'' and λ'' are determined in these last four equations for the groundtrack as functions of λ', from equations (20–40a), (20–60), (20–59), (20–58), (20–61), and (20–45) through (20–49).

To calculate A_n, A_n', B_n, C_n, and C_n', the following functions, varying with λ'', are used:

$$S = (P_2/P_1)L' \sin i \cos \lambda'' \{(1 + T \sin^2 \lambda'')/[(1 + W \sin^2 \lambda'')(1 + Q \sin^2 \lambda'')]\}^{1/2} \quad (20\text{–}37)$$

$$H = \left[\frac{1 + Q \sin^2 \lambda''}{1 + W \sin^2 \lambda''}\right]^{1/2} \left[\frac{1 + W \sin^2 \lambda''}{(1 + Q \sin^2 \lambda'')^2} - (P_2/P_1)L' \cos i\right] \quad (20\text{–}38)$$

$$L' = (1 - e' \cos E')^2/(1 - e'^2)^{1/2} \quad (20\text{–}39)$$

$$E' = 2 \arctan \{\tan \tfrac{1}{2}(\lambda'' - \gamma)\,[(1 - e')/(1 + e')]^{1/2}\} \quad (20\text{–}40)$$

These constants may be determined from numerical integration, using Simpson's rule with 9° intervals. For noncircular orbits, integration must occur through the 360° or 2π cycle, as indicated. Many more terms are needed than for circular orbits.

For circular orbits, A_n', B_2, C_n', A_n if n is odd, C_n if n is even, S_1, j_n', m_n', j_n if n is even, and m_n if n is odd are all zero, while H_1 and L' are both 1.0. Numerical integration for the nonzero values of all the remaining coefficients for circular orbits may be carried out from 0 to $\pi/2$, multiplying the result by 4.

To find x and y from ϕ and λ:

$$x/a = x'H_1 + y'S_1 \quad (20\text{–}41)$$
$$y/a = y'H_1 - x'S_1 \quad (20\text{–}42)$$

where

$$x' = B_1\lambda'' + \sum_{n=1}^{n} A_n \sin n\lambda'' - \sum_{n=1}^{n} A'_n \cos n\lambda'' + \sum_{n=1}^{n} A'_n - [S/(J^2+S^2)^{1/2}] \ln\tan(\pi/4 + \phi''/2) \quad (20\text{–}43)$$

$$y' = B_2\lambda'' + \sum_{n=1}^{n} C_n \sin n\lambda'' - \sum_{n=1}^{n} C'_n \cos n\lambda'' + \sum_{n=1}^{n} C'_n + [J/(J^2+S^2)^{1/2}] \ln\tan(\pi/4 + \phi''/2) \quad (20\text{–}44)$$

$$\lambda'' = \arctan[\cos i \tan\lambda_t + (1-e^2)\sin i \tan\phi/\cos\lambda_t] \quad (20\text{–}45)$$
$$\lambda_t = \lambda - \lambda_0 + (P_2/P_1)(L+\gamma) \quad (20\text{–}46)$$
$$L = E' - e'\sin E' \quad (20\text{–}47)$$
$$E' = 2\arctan\{\tan \tfrac{1}{2}(\lambda''-\gamma)[(1-e')/(1+e')]^{1/2}\} \quad (20\text{–}48)$$
$$\phi'' = \arcsin\{[(1-e^2)\cos i \sin\phi - \sin i \cos\phi \sin\lambda_t]/(1-e^2\sin^2\phi)^{1/2}\} \quad (20\text{–}49)$$
$$\lambda_0 = 128.87° - (360°/251)p \text{ (Landsat 1, 2, 3 only)} \quad (20\text{–}11)$$

Function E' is called the "eccentric anomaly" along the orbit, and L is the "mean anomaly" or mean longitude of the satellite measured from perigee and directly proportional to time.

Equations (20–45) through (20–48) are solved by special iteration as described for equations (20–8) and (20–9) in the spherical formulas, except that λ'' replaces λ', and each trial λ'' is placed in (20–48), from which E' is calculated, then L from (20–47), λ_t from (20–46), and another trial λ'' from (20–45). This cycle is repeated until λ'' changes by less than the selected convergence. The last value of λ_t found is then used in (20–49) to find ϕ''.

For circular orbits, in calculating x and y from ϕ and λ, equations (20–41), (20–42), (20–47), and (20–48) may be eliminated, and equations (20–43) and (20–44) may be rewritten as follows:

$$x/a = B_1\lambda'' + A_2 \sin 2\lambda'' + A_4 \sin 4\lambda'' + \ldots - [S/(J^2+S^2)^{1/2}] \ln\tan(\pi/4 + \phi''/2) \quad (20\text{–}43a)$$

$$y/a = C_1 \sin\lambda'' + C_3 \sin 3\lambda'' + \ldots + [J/(J^2+S^2)^{1/2}] \ln\tan(\pi/4 + \phi''/2) \quad (20\text{–}44a)$$

Also, for circular orbits, $(L+\gamma)$ in (20–46) is replaced by λ'', and the two equations (20–45) and (20–46) are iterated together as were (20–8) and (20–9). Equation (20–49) is unchanged. For both circular and noncircular orbits, equation (20–37) is used to find S for the given λ'' in equations (20–43), (20–44), (20–43a), and (20–44a).

The closed forms of equations (20–43) and (20–44) are given below, but the repeated numerical integration necessitates replacement by the series forms.

$$x' = \int_0^{\lambda''} [(HJ - S^2)/(J^2 + S^2)^{1/2}]\, d\lambda'' - [S/(J^2 + S^2)^{1/2}] \ln \tan (\pi/4 + \phi''/2) \quad (20\text{–}43b)$$

$$y' = \int_0^{\lambda''} [S(H + J)/(J^2 + S^2)^{1/2}]\, d\lambda'' + [J/(J^2 + S^2)^{1/2}] \ln \tan (\pi/4 + \phi''/2) \quad (20\text{–}44b)$$

While the above equations are sufficient for plotting a graticule according to the SOM projection, it is also desirable to relate these points to the true vertical groundtrack. To find ϕ'' and λ'' in terms of ϕ' and λ', the shift between these two sets of coordinates is so small it is equivalent to an adjustment, requiring only small Fourier coefficients, and very lengthy calculations if they are not used. The use of Fourier series is therefore highly recommended, although the one-time calculation of coefficients is more difficult than the foregoing calculation of A_n, B_n, and C_n.

$$\phi'' = \phi' + \sum_{n=1}^{n} j_n \sin n\lambda' + \sum_{n=1}^{n} j_n' \cos n\lambda' - \sum_{n=1}^{n} j_n' \quad (20\text{–}50)$$

$$\lambda'' = \lambda' + \sum_{n=1}^{n} m_n \sin n\lambda' + \sum_{n=1}^{n} m_n' \cos n\lambda' - \sum_{n=1}^{n} m_n' \quad (20\text{–}51)$$

For the circular orbit, as outlined in discussing other Fourier constants,

$$\phi'' = \phi' + j_1 \sin \lambda' + j_3 \sin 3\lambda' + \ldots \quad (20\text{–}50a)$$

$$\lambda'' = \lambda' + m_2 \sin 2\lambda' + m_4 \sin 4\lambda' + \ldots \quad (20\text{–}51a)$$

For λ' in terms of time t from the initial ascending node,

$$\lambda' = \gamma + 2 \arctan \{(\tan \tfrac{1}{2} E')[(1 + e')/(1 - e')]^{1/2}\} \quad (20\text{–}52)$$

$$E' = e' \sin E' + L_0 + 2\pi t/P_2 \quad (20\text{–}53)$$

$$L_0 = E'_0 - e' \sin E'_0 \quad (20\text{–}54)$$

$$E'_0 = -2 \arctan \{\tan \tfrac{1}{2}\gamma\, [(1 - e')/(1 + e')]^{1/2}\} \quad (20\text{–}55)$$

Equation (20–53) requires iteration, converging rapidly by substituting an initial trial $E' = L_0 + 2\pi t/P_2$ in the right side, finding a new E' on the left, substituting it on the right, etc., until sufficient convergence occurs. For a circular orbit, λ' is merely $2\pi t/P_2$.

The equations for functions of the satellite groundtrack, both forward and inverse, are given here, since some are used in calculating j_n and m_n as well. In any case a, e, i, P_2, P_1, λ_0, a', e', and γ must be given. For λ' and λ, if ϕ is given:

$$\lambda' = \arcsin (\sin \phi_g / \sin i) \quad (20\text{–}56)$$

$$\phi_g = \phi - \arcsin \{a e^2 \sin \phi \cos \phi / [R_0 (1 - e^2 \sin^2 \phi)^{1/2}]\} \quad (20\text{–}57)$$

$$R_0 = a' (1 - e' \cos E') \quad (20\text{–}58)$$

$$E' = 2 \arctan \{\tan \tfrac{1}{2} (\lambda' - \gamma) [(1 - e')/(1 + e')]^{1/2}\} \quad (20\text{–}40a)$$

where ϕ_g is the geocentric latitude of the point geocentrically under the satellite, not the geocentric latitude corresponding to the vertical groundtrack latitude ϕ, and R_0 is the radius vector to the satellite from the center of the Earth.

These equations are solved as a group by iteration, inserting a trial $\lambda' = \arcsin(\sin\phi/\sin i)$ in (20–40a), solving (20–58), (20–57), and (20–56) for a new λ', etc. Each trial λ' must be adjusted for quadrant. If the satellite is traveling north, add 360° times the number of orbits completed at the nearest ascending node (0, 1, 2, etc.). If traveling south, subtract λ' from 360° times the number of orbits completed at the nearest descending node (1/2, 3/2, 5/2, etc.). For λ,

$$\lambda = \arctan[(\cos i \sin\lambda')/\cos\lambda'] - (P_2/P_1)(L+\gamma) + \lambda_0 \qquad (20\text{–}59)$$
$$L = E' - e' \sin E' \qquad (20\text{–}60)$$

using the λ' and E' finally found just above.

For a circular orbit, equations (20–58), (20–40a), and (20–60) are eliminated. R_0 becomes the radius of the orbit (7,294,690 m for Landsat), and $(L+\gamma)$ in (20–59) is replaced with λ'. Iteration is eliminated as well.

If λ of a point along the groundtrack is given, to find λ' and ϕ,

$$\lambda' = \arctan(\tan\lambda_t/\cos i) \qquad (20\text{–}19)$$
$$\lambda_t = \lambda - \lambda_0 + (P_2/P_1)(L+\gamma) \qquad (20\text{–}46)$$

and L is found from (20–60) and (20–40a) above. The four equations are iterated as a group, as above, but the first trial λ' and the quadrant adjustments should follow the procedures listed for equation (20–8).

$$\phi = \arcsin(\sin i \sin\lambda') + \arcsin\{ae^2 \sin\phi \cos\phi/ [R_0(1-e^2\sin^2\phi)^{1/2}]\} \qquad (20\text{–}61)$$

where R_0 is determined from (20–58) and (20–40a), using the λ' determined just above. Iteration is involved in (20–61), beginning with a trial ϕ of $\arcsin(\sin i \sin\lambda')$.

For a circular orbit, only equations (20–19), (20–46), and (20–61) are involved, using λ' in place of $(L+\gamma)$ in (20–46) and using the orbital radius for R_0. Iteration remains for calculations of both λ' and ϕ.

If λ' of a point along the groundtrack is given, ϕ is found from (20–61), (20–58), and (20–40a); while λ is found from (20–59), (20–60), and (20–40a). For the circular orbit, (20–61) is sufficient for ϕ, and (20–59) provides λ if λ' is substituted for $(L+\gamma)$. Only (20–61) requires iteration for these calculations, whether the orbit is circular or noncircular.

Inverse formulas for the ellipsoidal form of the SOM projection, with an orbit of 0.05 eccentricity or less, follow: Given a, e, i, P_2, P_1, λ_0, a', e', γ, x, and y, to find ϕ and λ, the general Fourier and other constants are first determined as described at the beginning of the forward equations. Then

$$\lambda = \lambda_t - (P_2/P_1)(L+\gamma) + \lambda_0 \quad (20\text{–}62)$$

where

$$\lambda_t = \arctan(V/\cos\lambda'') \quad (20\text{–}63)$$

$$V = \{[1-\sin^2\phi''/(1-e^2)]\cos i \sin\lambda'' - \sin i \sin\phi'' [(1+Q\sin^2\lambda'')(1-\sin^2\phi'') - U\sin^2\phi'']^{1/2}\}/[1-\sin^2\phi''(1+U)] \quad (20\text{–}64)$$

$$U = e^2\cos^2 i/(1-e^2) \quad (20\text{–}65)$$

while L is found from (20–60), E' from (20–48), and λ'' and ϕ'' from (20–68) and (20–69) below.

$$\phi = \arctan\{(\tan\lambda''\cos\lambda_t - \cos i \sin\lambda_t)/[(1-e^2)\sin i]\} \quad (20\text{–}66)$$

If $i=0$, equation (20–66) is indeterminate, but

$$\phi = \arcsin\{\sin\phi''/[(1-e^2)^2 + e^2\sin^2\phi'']^{1/2}\} \quad (20\text{–}67)$$

No iteration is involved in equations (20–62) through (20–67), and the ATAN2 function of Fortran should be used with (20–63), but not (20–66), adding or subtracting 360° to or from λ if necessary in (20–62) to place it between longs. 180° E. and W. For the circular orbit, (20–48) and (20–60) do not apply, and $(L+\gamma)$ in (20–62) is replaced with λ'' Other equations remain the same.

Iteration is required to find λ'' from x and y:

$$\lambda'' = \{x' + (S/J)y' - \sum_{n=1}^{n}[A_n + (S/J)C_n]\sin n\lambda'' + \sum_{n=1}^{n}[A'_n + (S/J)C'_n]\cos n\lambda'' - \sum_{n=1}^{n}[A'_n + (S/J)C'_n]\}/[B_1 + (S/J)B_2] \quad (20\text{–}68)$$

using equations (20–37), (20–70), (20–71), and various constants. Iteration involves substitution of a trial $\lambda'' = x'/B_1$ in the right side, finding a new λ'' on the left side, etc.

For ϕ'', the λ'' just calculated is used in the following equation:

$$\ln\tan(\pi/4 + \phi''/2) = (1 + S^2/J^2)^{1/2}(y' - B_2\lambda'' - \sum_{n=1}^{n} C_n \sin n\lambda'' + \sum_{n=1}^{n} C'_n \cos n\lambda'' - \sum_{n=1}^{n} C'_n) \quad (20\text{–}69)$$

where

$$x' = (x/a)H_1 - (y/a)S_1 \quad (20\text{–}70)$$

$$y' = (y/a)H_1 + (x/a)S_1 \quad (20\text{–}71)$$

For the circular orbit, equations (20–70) and (20–71) are eliminated, and (20–68) and (20–69) are rewritten thus:

$$\lambda'' = [x/a + (S/J)(y/a) - A_2\sin 2\lambda'' - A_4\sin 4\lambda'' - (S/J)(C_1\sin\lambda'' + C_3\sin 3\lambda'')]/B_1 \quad (20\text{–}68a)$$

$$\ln \tan (\pi/4 + \phi''/2) = (1 + S^2/J^2)^{1/2} (y/a - C_1 \sin \lambda'' - C_3 \sin 3 \lambda'') \qquad (20\text{–}69a)$$

The first is solved by iteration just as (20–68), using an initial $\lambda'' = x/aB_1$.

The closed forms of equations (20–68), (20–69), (20–68a), and (20–69a) involve both iteration and repeated numerical integration and are impractical:

$$x' + (S/J)y' = \int_0^{\lambda''} [(HJ - S^2)/(J^2 + S^2)^{1/2}] d\lambda'' + (S/J) \int_0^{\lambda''} [S(H+J)/(J^2+S^2)^{1/2}] d\lambda'' \qquad (20\text{–}68b)$$

$$\ln \tan (\pi/4 + \phi''/2) = [1 + (S/J)^2]^{1/2} \{y' - \int_0^{\lambda''} [S(H+J)/(J^2+S^2)^{1/2}] d\lambda''\} \qquad (20\text{–}69b)$$

(For the circular orbit, x' and y' are replaced by (x/a) and (y/a), respectively.)

For ϕ' and λ' in terms of ϕ'' and λ'', the same Fourier series developed for equations (20–50) and (20–51) may be used with reversal of signs, since the correction is so small. That is,

$$\phi' = \phi'' - \sum_{n=1}^{n} j_n \sin n\lambda'' - \sum_{n=1}^{n} j'_n \cos n\lambda'' + \sum_{n=1}^{n} j'_n \qquad (20\text{–}72)$$

$$\lambda' = \lambda'' - \sum_{n=1}^{n} m_n \sin n\lambda'' - \sum_{n=1}^{n} m'_n \cos n\lambda'' + \sum_{n=1}^{n} m'_n \qquad (20\text{–}73)$$

Equations (20–72) and (20–73) are, of course, not the exact inverses of (20–50) and (20–51), although the correct coefficients may be derived by an analogous numerical integration in terms of λ'', rather than λ'. The inverse values of ϕ' and λ' from (20–72) and (20–73) are within 0.000003° and 0.000009°, respectively, of the true inverses of (20–50) and (20–51) for the Landsat orbit.

For the circular orbit, as before, equations (20–72) and (20–73) simplify to the following:

$$\phi' = \phi'' - j_1 \sin \lambda'' - j_3 \sin 3\lambda'' - \ldots \qquad (20\text{–}72a)$$

$$\lambda' = \lambda'' - m_2 \sin 2\lambda'' - m_4 \sin 4\lambda'' - \ldots \qquad (20\text{–}73a)$$

The following values of Fourier coefficients for the ellipsoidal SOM are listed for Landsat orbits, using the Clarke 1866 ellipsoid (a = 6,378,206.4 m and e^2 = 0.00676866) and a circular orbit (R = 7,294,690 m, i = 99.092°, P_2/P_1 = 18/251):

B_1 = 1.005798138 for λ'' in radians
= 0.0175544891 for λ'' in degrees
A_2 = −0.0010979201
A_4 = −0.0000012928
A_6 = −0.0000000021
C_1 = 0.1434409899
C_3 = 0.0000285091
C_5 = −0.0000000011

$\dot{j}_1 = 0.00855567$ for ϕ'' and ϕ' in degrees
$\dot{j}_3 = 0.00081784$ "
$\dot{j}_5 = -0.00000263$ "
$m_2 = -0.02384005$ for λ'' and λ' in degrees
$m_4 = 0.00010606$ "
$m_6 = 0.00000019$ "

Additional Fourier constants have been developed in the published literature for other functions of circular orbits. They add to the complication of the equations, but not to the accuracy, and only slightly to continuous mapping efficiency. For noncircular orbits, too many terms are needed to justify their use on functions not otherwise requiring continuous integration. Therefore, they are omitted here. A further simplification from published formulas is the elimination of a function F, which nearly cancels out in the range involved in imaging.

As in the spherical form of the SOM, the formulas for scale factors h and k and maximum angular deformation ω are too lengthy to include here, although they are given by Snyder (1981). Table 31 presents these values for Landsat constants for the scanning range required.

TABLE 31.—*Scale factors for the ellipsoidal Space Oblique Mercator projection using Landsat 1, 2, and 3 constants*

λ''	ϕ''	h	k	$\omega°$	$\sin \frac{1}{2} \omega$
0°	1°	1.000154	1.000151	0.0006	0.000005
	0	1.000000	1.000000	.0000	.000000
	-1	1.000154	1.000151	.0006	.000005
15	1	1.000161	1.000151	.0022	.000019
	0	1.000000	1.000000	.0001	.000000
	-1	1.000147	1.000151	.0011	.000010
30	1	1.000167	1.000150	.0033	.000029
	0	1.000000	1.000000	.0001	.000001
	-1	1.000142	1.000150	.0025	.000021
45	1	1.000172	1.000150	.0036	.000031
	0	.999999	1.000000	.0001	.000001
	-1	1.000138	1.000150	.0031	.000027
60	1	1.000174	1.000150	.0031	.000027
	0	.999999	1.000000	.0002	.000001
	-1	1.000136	1.000150	.0028	.000025
75	1	1.000174	1.000152	.0019	.000016
	0	.999999	1.000000	.0001	.000000
	-1	1.000135	1.000150	.0019	.000016
90	1	1.000170	1.000156	.0008	.000007
	0	.999999	1.000000	.0000	.000000
	-1	1.000133	1.000151	.0010	.000009

Notes: λ'' = angular position along geocentric groundtrack, from ascending node.
ϕ'' = angular distance away from geocentric groundtrack, positive in direction away from North Pole.
h = scale factor along meridian of longitude.
k = scale factor along parallel of latitude.
ω = maximum angular deformation.
$\sin \frac{1}{2}\omega$ = maximum variation of scale factors from true conformal values.

MISCELLANEOUS MAP PROJECTIONS

Only two map projections described in this study cannot be satisfactorily placed in one of the four categories previously listed. If this study included many of the projections not used by the USGS, several additional categories would be shown, and those projections discussed below would be placed with similar projections and probably removed from the "miscellaneous" classification.

21. VAN DER GRINTEN PROJECTION

SUMMARY

- Neither equal-area nor conformal.
- Shows entire globe enclosed in a circle.
- Central meridian and Equator are straight lines.
- All other meridians and parallels are arcs of circles.
- A curved modification of the Mercator projection, with great distortion in the polar areas.
- Equator is true to scale.
- Used for world maps.
- Used only in the spherical form.
- Presented by van der Grinten in 1904.

HISTORY, FEATURES, AND USAGE

In a 1904 issue of a German geographical journal, Alphons J. van der Grinten (1852–?) of Chicago presented four projections showing the entire Earth. Aside from having a straight Equator and central meridian, three of the projections consist of arcs of circles for meridians and parallels; the other projection has straight-line parallels. The projections are neither conformal nor equal-area (van der Grinten, 1904; 1905). They were patented in the United States by van der Grinten in 1904.

The best-known Van der Grinten projection, his first (fig. 30), shows the world in a circle and was invented in 1898. It is designed for use in the spherical form only. There are no special features to preserve in an ellipsoidal form. It has been used by the National Geographic Society for their standard world map since 1943, printed at various scales and with the central meridian either through America or along the Greenwich meridian; this use has prompted others to employ the projection. The USGS has used one of the National Geographic maps as a base for a four-sheet set of maps of World Subsea Mineral Resources, 1970, one at a scale of 1:60,000,000 and three at 1:39,283,200 (a scale used by the National Geographic), and for three smaller maps in the *National Atlas*

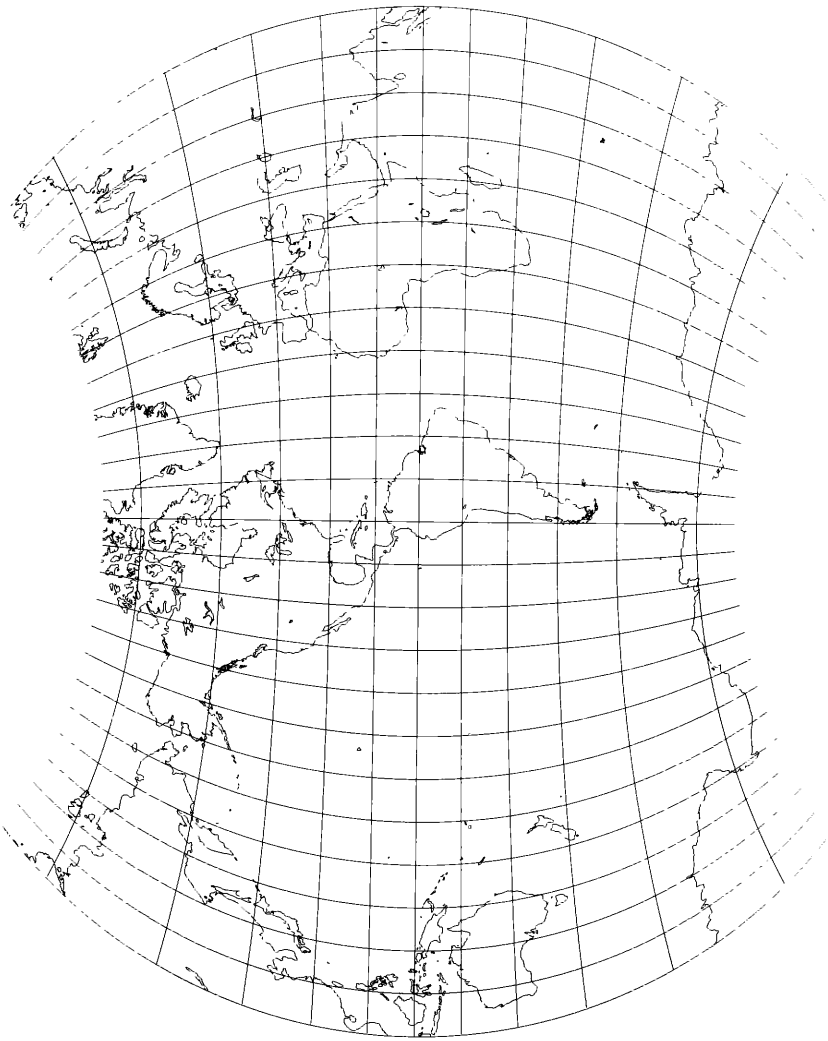


FIGURE 30.—Van der Grinten projection. A projection resembling the Mercator, but not conformal. Used by the USGS for special world maps, modifying a base map prepared by the National Geographic Society. This illustration is prepared by computer.

(USGS, 1970, p. 150–151, 332–335). All the USGS maps have a central meridian of long. 85° W., passing through the United States.

Van der Grinten emphasized that this projection blends the Mercator appearance with the curves of the Mollweide, an equal-area projection devised in 1805 and showing the world in an ellipse. He included a simple graphical construction and limited formulas showing the mathematical coordinates along the central meridian, the Equator, and the outer (180th) meridian. The meridians are equally spaced along the Equator, but the spacing between the parallels increases with latitude, so that the 75th parallels are shown about halfway between the Equator and the respective poles. Because of the polar exaggerations, most published maps using the Van der Grinten projection do not extend farther into the polar regions than the northern shores of Greenland and the outer rim of Antarctica.

The National Geographic Society prepared the base map graphically. General mathematical formulas have been published in recent years and are only useful with computers, since they are fairly complex for such a simply drawn projection (O'Keefe and Greenberg, 1977; Snyder, 1979b).

GEOMETRIC CONSTRUCTION

The meridians are circular arcs equally spaced on the Equator and joined at the poles. For parallels, referring to figure 31, semicircle *CDB* is drawn centered at *A*. Diagonal *CD* is drawn. Point *E* is marked so that the ratio of *EA* to *AD* is the same as the ratio of the latitude to 90°.

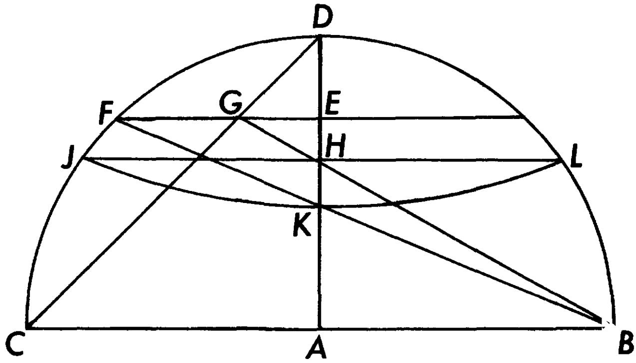


FIGURE 31.—Geometric construction of the Van der Grinten projection.

Line FE is drawn parallel to CB, and FB and GB are connected. At H, the intersection of GB and AD, JHL is drawn parallel to CB. A circular arc, representing the parallel of latitude, is then drawn through JKL.

FORMULAS FOR THE SPHERE

The general formulas published are in two forms. Both sets give identical results, but the 1979 formulas are somewhat shorter and are given here with some rearrangement and addition of new inverse equations. For the forward calculations, given R, λ_0, ϕ, and λ (giving true scale along the Equator), to find x and y:

$$x = \pm\pi R\{A(G-P^2)+[A^2(G-P^2)^2-(P^2+A^2)(G^2-P^2)]^{1/2}\}/(P^2+A^2) \quad (21\text{–}1)$$

taking the sign of $(\lambda-\lambda_0)$. Note that $(\lambda-\lambda_0)$ must fall between $+180°$ and $-180°$; if necessary, $360°$ must be added or subtracted. The X axis lies along the Equator, x increasing easterly, while the Y axis coincides with the central meridian λ_0.

$$y=\pm\pi R\{PQ-A[(A^2+1)(P^2+A^2)-Q^2]^{1/2}\}/(P^2+A^2) \quad (21\text{–}2)$$

taking the sign of ϕ,

where

$$A = \tfrac{1}{2}|\pi/(\lambda-\lambda_0)-(\lambda-\lambda_0)/\pi| \quad (21\text{–}3)$$

$$G = \cos\theta/(\sin\theta+\cos\theta-1) \quad (21\text{–}4)$$

$$P = G(2/\sin\theta-1) \quad (21\text{–}5)$$

$$\theta = \arcsin|2\phi/\pi| \quad (21\text{–}6)$$

$$Q = A^2+G \quad (21\text{–6a})$$

But if $\phi=0$ or $\lambda=\lambda_0$, these equations are indeterminant. In that case, if $\phi=0$,

$$x=R(\lambda-\lambda_0) \quad (21\text{–}7)$$

and

$$y=0$$

or if $\lambda=\lambda_0$,

$$x=0$$

and

$$y=\pm\pi R\tan(\theta/2) \quad (21\text{–}8)$$

taking the sign of ϕ. It may be noted that absolute values (symbol $|\ |$) are used in several cases. The origin is at the center ($\phi=0$, $\lambda=\lambda_0$).

For the inverse equations, given R, λ_0, x, and y, to find ϕ and λ: Because of the complications involved, the equations are given in the order of use. This is closely based upon a recent, non-iterative algorithm by Rubincam (1981):

$$X=x/(\pi R) \tag{21-9}$$

$$Y=y/(\pi R) \tag{21-10}$$

$$c_1=-|Y|(1+X^2+Y^2) \tag{21-11}$$

$$c_2=c_1-2Y^2+X^2 \tag{21-12}$$

$$c_3=-2c_1+1+2Y^2+(X^2+Y^2)^2 \tag{21-13}$$

$$d=Y^2/c_3+(2c_2^3/c_3^3-9c_1c_2/c_3^2)/27 \tag{21-14}$$

$$a_1=(c_1-c_2^2/3c_3)/c_3 \tag{21-15}$$

$$m_1=2(-a_1/3)^{1/2} \tag{21-16}$$

$$\theta_1=(1/3)\arccos(3d/a_1m_1) \tag{21-17}$$

$$\phi=\pm\pi[-m_1\cos(\theta_1+\pi/3)-c_2/3c_3] \tag{21-18}$$

taking the sign of y.

$$\lambda=\pi\{X^2+Y^2-1+[1+2(X^2-Y^2)+(X^2+Y^2)^2]^{1/2}\}/2X+\lambda_0 \tag{21-19}$$

but if $X=0$, equation (21–19) is indeterminate. Then

$$\lambda=\lambda_0 \tag{21-20}$$

The formulas for scale factors are quite lengthy and are not included here. Rectangular coordinates are given in table 32 for a map of the world with unit radius of the outer circle, or $R=1/\pi$. The longitude is measured from the central meridian. Only one quadrant of the map is given, but the map is symmetrical about both X and Y axes.

TABLE 32. – *Van der Grinten projection: Rectangular coordinates*

[*y*-coordinate in parentheses under *x*-coördinate]

Lat. \ Long.	0°	10°	20°	30°	40°
90°	0.00000 (1.00000)	0.00000 (1.00000)	0.00000 (1.00000)	0.00000 (1.00000)	0.00000 (1.00000)
80	.00000 (.60961)	.03491 (.61020)	.06982 (.61196)	.10473 (.61490)	.13963 (.61902)
70	.00000 (.47759)	.04289 (.47806)	.08581 (.47948)	.12878 (.48184)	.17184 (.48517)
60	.00000 (.38197)	.04746 (.38231)	.09495 (.38336)	.14252 (.38511)	.19020 (.38756)
50	.00000 (.30334)	.05045 (.30358)	.10094 (.30430)	.15149 (.30551)	.20215 (.30721)
40	.00000 (.23444)	.05251 (.23459)	.10504 (.23505)	.15764 (.23582)	.21031 (.23690)
30	.00000 (.17157)	.05392 (.17166)	.10787 (.17192)	.16185 (.17235)	.21588 (.17295)
20	.00000 (.11252)	.05485 (.11256)	.10972 (.11267)	.16460 (.11286)	.21951 (.11313)
10	.00000 (.05573)	.05538 (.05574)	.11077 (.05577)	.16616 (.05581)	.22156 (.05588)
0	.00000 (.00000)	.05556 (.00000)	.11111 (.00000)	.16667 (.00000)	.22222 (.00000)

TABLE 32. – *Van der Grinten projection: Rectangular coordinates* – Continued

Lat. \ Long.	50°	60°	70°	80°	90°
90°	0.00000 (1.00000)	0.00000 (1.00000)	0.00000 (1.00000)	0.00000 (1.00000)	0.00000 (1.00000)
80	.17450 (.62435)	.20932 (.63088)	.24403 (.63863)	.27859 (.64760)	.31293 (.65778)
70	.21498 (.48946)	.25821 (.49473)	.30152 (.50100)	.34488 (.50828)	.38827 (.51657)
60	.23800 (.39073)	.28594 (.39462)	.33403 (.39925)	.38225 (.40462)	.43059 (.41074)
50	.25293 (.30940)	.30385 (.31208)	.35492 (.31527)	.40614 (.31897)	.45750 (.32319)
40	.26308 (.23829)	.31596 (.24000)	.36897 (.24202)	.42210 (.24436)	.47535 (.24703)
30	.26998 (.17373)	.32415 (.17468)	.37841 (.17581)	.43275 (.17711)	.48718 (.17860)
20	.27445 (.11347)	.32944 (.11389)	.38446 (.11439)	.43953 (.11497)	.49464 (.11562)
10	.27697 (.05597)	.33239 (.05607)	.38782 (.05620)	.44327 (.05634)	.49872 (.05650)
0	.27778 (.00000)	.33333 (.00000)	.38889 (.00000)	.44444 (.00000)	.50000 (.00000)

TABLE 32. – *Van der Grinten projection: Rectangular coordinates* – Continued

Lat. \ Long.	100°	110°	120°	130°	140°
90°	0.00000	0.00000	0.00000	0.00000	0.00000
	(1.00000)	(1.00000)	(1.00000)	(1.00000)	(1.00000)
80	.34699	.38069	.41394	.44668	.47882
	(.66917)	(.68174)	(.69548)	(.71035)	(.72631)
70	.43163	.47493	.51810	.56110	.60385
	(.52588)	(.53621)	(.54756)	(.55992)	(.57328)
60	.47903	.52754	.57608	.62463	.67313
	(.41762)	(.42525)	(.43366)	(.44282)	(.45275)
50	.50899	.56059	.61228	.66404	.71585
	(.32792)	(.33317)	(.33894)	(.34524)	(.35207)
40	.52871	.58218	.63575	.68939	.74310
	(.25001)	(.25333)	(.25697)	(.26094)	(.26523)
30	.54168	.59626	.65091	.70562	.76038
	(.18026)	(.18209)	(.18411)	(.18631)	(.18869)
20	.54979	.60499	.66022	.71548	.77077
	(.11635)	(.11716)	(.11804)	(.11901)	(.12005)
10	.55419	.60967	.66516	.72066	.77617
	(.05668)	(.05688)	(.05710)	(.05734)	(.05760)
0	.55555	.61111	.66667	.72222	.77778
	(.00000)	(.00000)	(.00000)	(.00000)	(.00000)

TABLE 32. – *Van der Grinten projection: Rectangular coordinates* – Continued

Lat. \ Long.	150°	160°	170°	180°
90°	0.00000	0.00000	0.00000	0.00000
	(1.00000)	(1.00000)	(1.00000)	(1.00000)
80	.51028	.54101	.57093	.60000
	(.74331)	(.76130)	(.78021)	(.80000)
70	.64631	.68843	.73013	.77139
	(.58762)	(.60293)	(.61919)	(.63636)
60	.72156	.76988	.81804	.86603
	(.46344)	(.47488)	(.48707)	(.50000)
50	.76768	.81951	.87132	.92308
	(.35942)	(.36729)	(.37569)	(.38462)
40	.79686	.85066	.90448	.95831
	(.26986)	(.27482)	(.28010)	(.28571)
30	.81518	.87003	.92490	.97980
	(.19125)	(.19398)	(.19690)	(.20000)
20	.82609	.88143	.93678	.99216
	(.12117)	(.12237)	(.12365)	(.12500)
10	.83168	.88721	.94274	.99827
	(.05788)	(.05817)	(.05849)	(.05882)
0	.83333	.88889	.94444	1.00000
	(.00000)	(.00000)	(.00000)	(.00000)

Radius of map = 1.0. Radius of sphere = $1/\pi$.

Origin: $(x, y) = 0$ at (lat, long) = 0. Y axis increases north. One quadrant given. Other quadrants of world map are symmetrical.

22. SINUSOIDAL PROJECTION

SUMMARY

- Pseudocylindrical projection.
- Equal-area.
- Central meridian is a straight line; all other meridians are shown as sinusoidal curves.
- Parallels are equally spaced straight lines.
- Scale is true along central meridian and all parallels.
- Used for world maps with single central meridian or in interrupted form with several central meridians.
- Used for maps of South America and Africa.
- Used since the mid-16th century.

HISTORY

There is an almost endless number of possible projections with horizontal straight lines for parallels of latitude and curved lines for meridians. They are sometimes called pseudocylindrical because of their partial similarity to cylindrical projections. Scores of such projections have been presented, purporting various special advantages, although several are strikingly similar to other members of the group (Snyder, 1977). While there were rudimentary projections with straight parallels used as early as the 2nd century B.C. by Hipparchus, the first such projection still used for scientific mapping of the sphere is the Sinusoidal.

This projection (fig. 32), used for world maps as well as maps of continents and other regions, especially those bordering the Equator, has been given many names after various presumed originators, but it is most frequently called by the named used here. Among the first to show the Sinusoidal projection was Jean Cossin of Dieppe, who used it for a world map of 1570. In addition, it was used by Jodocus Hondius for maps of South America and Africa in some of his editions of Mercator's atlases of 1606–1609. This is probably the basis for one of the names of the projection: The Mercator Equal-Area. Nicolas Sanson (1600–67) of France used it in about 1650 for maps of continents, while John Flamsteed (1646–1719) of England later used it for star maps. Thus, the name "Sanson-Flamsteed" has often been applied to the Sinusoidal projection, even though they were not the originators (Keuning, 1955, p. 24; Deetz and Adams, 1934, p. 161).

While maps of North America are no longer drawn to the Sinusoidal, South America and Africa are still shown on this projection in recent Rand McNally atlases.

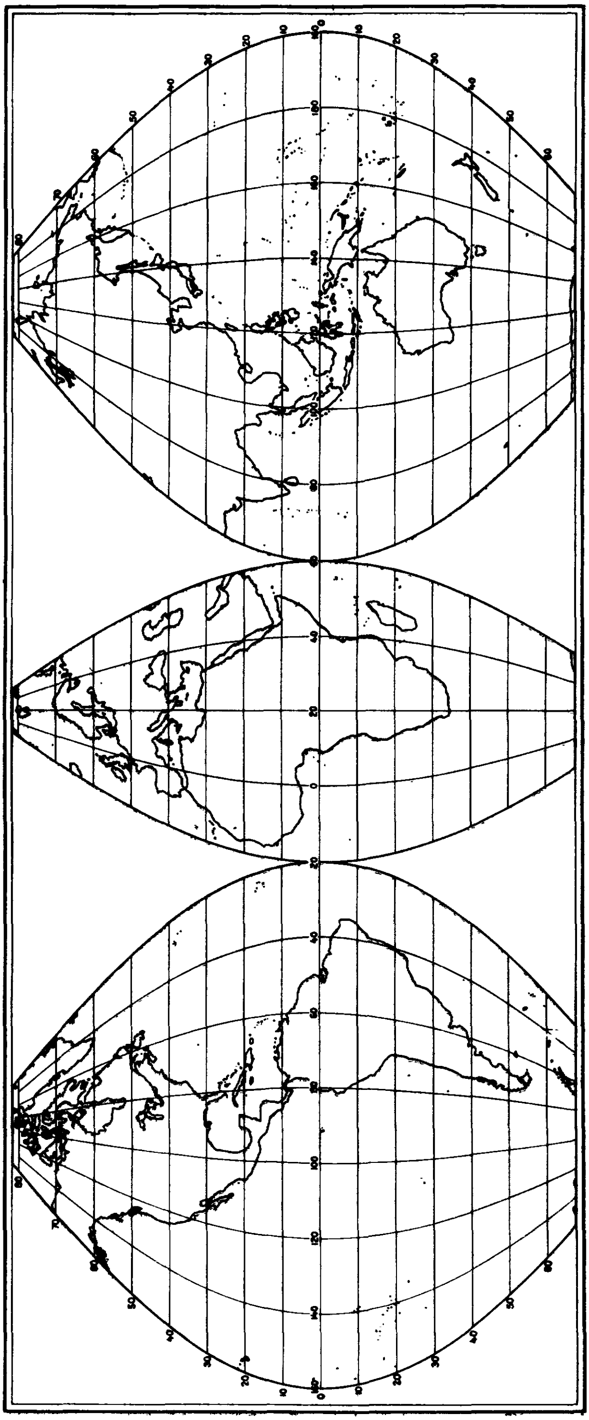


FIGURE 32. – Interrupted Sinusoidal projection as used by the USGS, although extended to the poles, which would be shown as points. (From Deetz and Adams, 1934.)

FEATURES AND USAGE

The simplicity of construction, either graphically or mathematically, combined with the useful features obtained, make the Sinusoidal projection not only popular to use, but a popular object of study for interruptions, transformations, and combination with other projections.

On the normal Sinusoidal projection, the parallels of latitude are equally spaced straight parallel lines, and the central meridian is a straight line crossing the parallels perpendicularly. The Equator is marked off from the central meridian equidistantly for meridians at the same scale as the latitude markings on the central meridian, so the Equator for a complete world map is twice as long as the central meridian. The other parallels of latitude are also marked off for meridians in proportion to the true distances from the central meridian. The meridians connect these markings from pole to pole. Since the spacings on the parallels are proportional to the cosine of the latitude, and since parallels are equally spaced, the meridians form curves which may be called cosine, sine, or sinusoidal curves; hence, the name.

Areas are shown correctly. There is no distortion along the Equator and central meridian, but distortion becomes pronounced near the outer meridians, especially in the polar regions.

Because of this distortion, J. Paul Goode (1862–1932) of The University of Chicago developed an interrupted form of the Sinusoidal with several meridians chosen as central meridians without distortion and a limited expanse east and west for each section. The central meridians may be different for Northern and Southern Hemispheres and may be selected to minimize distortion of continents or of oceans instead. Ultimately, Goode combined the portion of the interrupted Sinusoidal projection between about lats. 40° N. and S. with the portions of the Mollweide or Homolographic projection (mentioned earlier) not in this zone, to produce the Homolosine projection used in Rand McNally's *Goode's Atlas* (Goode, 1925).

In 1927, the Sinusoidal was shown interrupted in three symmetrical segments in the *Nordisk Världs Atlas*, Stockholm, serving as the base for the Sinusoidal as shown in Deetz and Adams (1934, p. 161). It is this interrupted form which served in turn as the base for a three-sheet set by the USGS in 1978 at a scale of 1:20,000,000, entitled Map of Prospective Hydrocarbon Provinces of the World. With interruptions occurring at longs. 160° W., 20° W., and 60° E., and the three central meridians equidistant from these limits, the sheets show (1) North and South America; (2) Europe, West Asia, and Africa; and (3) East Asia, Australia, and the Pacific; respectively. The maps extend pole to pole, but no data are shown for Antarctica. An inset of the Arctic region at the same scale is drawn to the polar Lambert Azimuthal Equal-Area projection. A similar map is being prepared by the USGS showing sedimentary basins of the world.

The Sinusoidal projection is normally used in the spherical form, adequate for the usual small-scale usage. The ellipsoidal form may be made by spacing parallels along the central meridian(s) true to scale for the ellipsoid (equation (3–21)) and meridians along each parallel also true to scale (equation (4–21)). The projection remains equal-area, while the parallels are not quite equally spaced, and the meridians are no longer perfect sinusoids.

FORMULAS FOR THE SPHERE

The formulas for the Sinusoidal projection are perhaps the simplest of those for any projection described in this bulletin, except for the Equidistant Cylindrical. For the forward case, given R, λ_0, ϕ, and λ, to find x and y:

$$x = R(\lambda - \lambda_0) \cos \phi \tag{22-1}$$
$$y = R\phi \tag{22-2}$$
$$h = [1 + (\lambda - \lambda_0)^2 \sin^2 \phi]^{1/2} \tag{22-3}$$
$$k = 1.0$$
$$\theta' = \arcsin (1/h) \tag{22-4}$$
$$\omega = 2 \arctan |\tfrac{1}{2}(\lambda - \lambda_0) \sin \phi| \tag{22-5}$$

where θ' is the angle of intersection of a given meridian and parallel (see equation (4–14)), and h, k, and ω are other distortion factors as previously used. The X axis coincides with the Equator, with x increasing easterly, while the Y axis follows the central meridian λ_0, y increasing northerly. It is necessary to adjust $(\lambda - \lambda_0)$, if it falls outside the range $\pm 180°$, by adding or subtracting $360°$. For the interrupted form, values of x are calculated for each section with respect to its own central meridian λ_0.

In equations (22–1) through (22–5), radians must be used, or ϕ and λ in degrees must be multiplied by $\pi/180°$.

For the inverse formulas, given R, λ_0, x, and y, to find ϕ and λ:

$$\phi = y/R \tag{22-6}$$
$$\lambda = \lambda_0 + x/R \cos \phi \tag{22-7}$$

but if $\phi = \pm \pi/2$, equation (22–7) is indeterminate, and λ may be given an arbitrary value such as λ_0.

APPENDIXES

APPENDIX A

NUMERICAL EXAMPLES

The numerical examples which follow should aid in the use of the many formulas in this study of map projections. Single examples are given for equations for forward and inverse functions of the projections, both spherical and ellipsoidal, when both are given. They are given in the order the projections are given. The order of equations used is based on the order of calculation, even though the equations may be originally listed in a somewhat different order. In some cases, the last digit may vary from check calculations, due to rounding off, or the lack of it.

AUXILIARY LATITUDES (SEE P. 16–22)

For all the examples under this heading, the Clarke 1866 ellipsoid is used: a is not needed here, $e^2 = 0.00676866$, or $e = 0.0822719$. Auxiliary latitudes will be calculated for geodetic latitude $\phi = 40°$:

Conformal latitude, using closed equation (3–1):

$$\chi = 2 \arctan \{\tan(45° + 40°/2) [(1 - 0.0822719 \sin 40°)/(1 + 0.0822719 \sin 40°)]^{0.0822719/2}\} - 90°$$
$$= 2 \arctan \{2.1445069 [0.8995456]^{0.0411360}\} - 90°$$
$$= 2 \arctan (2.1351882) - 90°$$
$$= 2 \times 64.9042961° - 90°$$
$$= 39.8085922° = 39°48'30.9''$$

Using series equation (3–2), obtaining χ first in radians:

$$\chi = 40° \times \pi/180° - (0.00676866/2 + 5 \times 0.00676866^2/24 + 3 \times 0.00676866^3/32) \times \sin(2 \times 40°) + (5 \times 0.00676866^2/48 + 7 \times 0.00676866^3/80) \times \sin(4 \times 40°) - (13 \times 0.00676866^3/480) \sin(6 \times 40°)$$
$$= 0.6981317 - (0.0033939) \times 0.9848078 + (0.0000048) \times 0.3420201 - (.0000000) \times (-0.8660254)$$
$$= 0.6947910 \text{ radian}$$
$$= 0.6947910 \times 180°/\pi = 39.8085923°$$

For inverse calculations, using closed equation (3–4) with iteration and given $\chi = 39.8085922°$, find ϕ:

First trial:

$$\phi = 2 \arctan \{\tan(45° + 39.8085922°/2) [(1 + 0.0822719 \sin 39.8085922°)/(1 - 0.0822719 \sin 39.8085922°)]^{0.0822719/2}\} - 90°$$
$$= 2 \arctan \{2.1351882 [1.1112023]^{0.0411360}\} - 90°$$
$$= 129.9992366° - 90°$$
$$= 39.9992366°$$

Second trial:

$\phi = 2 \arctan \{2.1351882 [(1 + 0.0822719 \sin 39.9992366°)/(1 - 0.0822719 \sin 39.9992366°)]^{0.0411360}\} - 90°$
$= 2 \arctan (2.1445068) - 90°$
$= 39.9999970°$

The third trial gives $\phi = 40.0000000°$, also given by the fourth trial.
Using series equation (3–5):

$\phi = 39.8085922° \times \pi/180° + (0.00676866/2 + 5 \times 0.00676866^2/24 + 0.00676866^3/12) \sin (2 \times 39.8085922°) + (7 \times 0.00676866^2/48 + 29 \times 0.00676866^3/240) \sin (4 \times 39.8085922°) + (7 \times 0.00676866^3/120) \sin (6 \times 39.8085922°)$
$= 0.6947910 + (0.0033939) \times 0.9836256 + (0.0000067) \times 0.3545461 + (0.0000000) \times (-0.8558300)$
$= 0.6981317$ radian
$= 0.6981317 \times 180°/\pi = 40.0000000°$

Isometric latitude, using equation (3–7):

$\psi = \ln \{\tan (45° + 40°/2) [(1 - 0.0822719 \sin 40°)/(1 + 0.0822719 \sin 40°)]^{0.0822719/2}\}$
$= \ln (2.1351882)$
$= 0.7585548$

Using equation (3–8) with the value of χ resulting from the above examples:

$\psi = \ln \tan (45° + 39.8085923°/2)$
$= \ln \tan 64.9042962°$
$= 0.7585548$

For inverse calculations, using equation (3–9) with $\psi = 0.7585548$:

$\chi = 2 \arctan e^{0.7585548} - 90°$
$= 2 \arctan (2.1351882) - 90°$
$= 39.8085922°$

From this value of χ, ϕ may be found from (3–4) or (3–5) as shown above.
Using iterative equation (3–10), with $\psi = 0.7585548$, to find ϕ:
First trial:

$\phi = 2 \arctan e^{0.7585548} - 90°$
$= 39.8085922°$, as just above.

Second trial:

$\phi = 2 \arctan \{e^{0.7585548} [(1 + 0.0822719 \sin 39.8085922°)/(1 - 0.0822719 \sin 39.8085922°)]^{0.0822719/2}\} - 90°$

$$= 2 \arctan (2.1351882 \times 1.0043469) - 90°$$
$$= 39.9992365°$$

Third trial:

$$\phi = 2 \arctan \{e^{0.7585548} [(1 + 0.0822719 \sin 39.9992365°)/(1 - 0.0822719 \sin 39.9992365°)]^{0.0822719/2}\} - 90°$$
$$= 39.9999970°$$

Fourth trial, substituting 39.9999970° in place of 39.9992365°:

$\phi = 40.0000000°$, also given by fifth trial.

Authalic latitude, using equations (3–11) and (3–12):

$$q = (1 - 0.00676866) \{\sin 40°/(1 - 0.00676866 \sin^2 40°) - [1/(2 \times 0.0822719)] \ln [(1 - 0.0822719 \sin 40°)/(1 + 0.0822719 \sin 40°)]\}$$
$$= 0.9932313 (0.6445903 - 6.0774117 \ln 0.8995456)$$
$$= 1.2792602$$
$$q_p = (1 - 0.00676866) \{\sin 90°/(1 - 0.00676866 \sin^2 90°) - [1/(2 \times 0.0822719)] \ln [(1 - 0.0822719 \sin 90°)/(1 + 0.0822719 \sin 90°)]\}$$
$$= 1.9954814$$
$$\beta = \arcsin (1.2792602/1.9954814)$$
$$= \arcsin 0.6410785$$
$$= 39.8722878° = 39°52'20.2''$$

Determining β from series equation (3–14) involves the same pattern as the example for equation (3–5) given above.

For inverse calculations, using equation (3–17) with iterative equation (3–16), given $\beta = 39.8722878°$, and $q_p = 1.9954814$ as determined above:

$$q = 1.9954814 \sin 39.8722878°$$
$$= 1.2792602$$

First trial:

$$\phi = \arcsin (1.2792602/2)$$
$$= 39.762435°$$

Second trial:

$$\phi = 39.7642435° + (180°/\pi) \{[(1 - 0.00676866 \sin^2 39.7642435°)^2/(2 \cos 39.7642435°)] [1.2792602/(1 - 0.00676866) - \sin 39.7642435°/(1 - 0.00676866 \sin^2 39.7642435°) + [1/(2 \times 0.0822719)] \ln [(1 - 0.0822719 \sin 39.7642435°)/(1 + 0.0822719 \sin 39.7642435°)]]\}$$
$$= 39.9996014°$$

Third trial, substituting 39.9996014° in place of 39.7642435°,

$\phi = 39.9999992°$

Fourth trial gives the same result.

Finding ϕ from β by series equation (3–18) involves the same pattern as the example for equation (3–5) given above.

Rectifying latitude, using equations (3–20) and (3–21):

$$M = a[(1 - 0.00676866/4 - 3\times 0.00676866^2/64 - 5\times 0.00676866^3/256) \times 40° \times \pi/180° - (3\times 0.00676866/8 + 3\times 0.00676866^2/32 + 45\times 0.00676866^3/1024) \sin (2\times 40°) + (15\times 0.00676866^2/256 + 45\times 0.00676866^3/1024) \sin (4\times 40°) - (35\times 0.00676866^3/3072) \sin (6\times 40°)]$$
$$= a[0.9983057\times 0.6981317 - 0.0025426 \sin 80° + 0.0000027 \sin 160° - 0.0000000 \sin 240°]$$
$$= 0.6944458a$$

$M_p = 1.5681349a$, using 90° in place of 40° in the above example.

$$\mu = 90°\times 0.6944458a/1.5681349a$$
$$= 39.8563451° = 39°51'22.8''$$

Calculation of μ from series (3–23), and the inverse ϕ from (3–26), is similar to the example for equation (3–2) except that e_1 is used rather than e. From equation (3–24),

$$e_1 = [1 - (1 - 0.00676866)^{1/2}]/[1 + (1 - 0.00676866)^{1/2}]$$
$$= 0.001697916$$

Geocentric latitude, using equation (3–28),

$$\phi_g = \arctan [(1 - 0.00676866) \tan 40°]$$
$$= 39.8085032° = 39°48'30.6''$$

Reduced latitude, using equation (3–31),

$$\eta = \arctan [(1 - 0.00676866)^{1/2} \tan 40°]$$
$$= 39.9042229° = 39°54'15.2''$$

Series examples for ϕ_g and η follow the pattern of (3–2) and (3–23).

DISTORTION FOR PROJECTIONS OF THE ELLIPSOID (SEE P. 28–31)

Radius of curvature and length of degrees, using the Clarke 1866 ellipsoid at lat. 40° N.:

From equation (4–18),

$$R' = 6378206.4\ (1 - 0.00676866)/(1 - 0.00676866 \sin^2 40°)^{3/2}$$
$$= 6{,}361{,}703.0 \text{ m}$$

From equation (4–19), using the figure just calculated,

$L_\phi = 6361703.0\ \pi/180° = 111{,}032.7$ m, the length of 1° of latitude at lat. 40° N.

From equation (4–20),

$$N = 6378206.4/(1 - 0.00676866 \sin^2 40°)^{1/2}$$
$$= 6{,}387{,}143.9 \text{ m}$$

From equation (4–21),

$$L_\lambda = [6378206.4 \cos 40°/(1 - 0.00676866 \sin^2 40°)^{1/2}]\ \pi/180°$$
$= 85{,}396.1$ m, the length of 1° of longitude at lat. 40° N.

MERCATOR PROJECTION (SPHERE)–FORWARD EQUATIONS (SEE P. 47, 50)

Given: Radius of sphere: $R = 1.0$ unit
Central meridian: $\lambda_0 = 180°$ W. long.
Point: $\phi = 35°$ N. lat.
$\lambda = 75°$ W. long.

Find: x, y, k.
Using equations (7–1) through (7–3),

$$x = \pi \times 1.0 \times [(-75°) - (-180°)]/180° = 1.8325957 \text{ units}$$
$$y = 1.0 \times \ln \tan (45° + 35°/2) = 1.0 \times \ln \tan (62.5°)$$
$$= \ln 1.9209821 = 0.6528366 \text{ unit}$$

or

$$y = 1.0 \times \text{arctanh} (\sin 35°) = \text{arctanh } 0.5735764$$
$$= 0.6528366 \text{ unit}$$
$$h = k = \sec 35° = 1/\cos 35° = 1/0.8191520 = 1.2207746$$

MERCATOR PROJECTION (SPHERE)–INVERSE EQUATIONS (SEE P. 50)

Inversing forward example:
Given: R, λ_0 for forward example
$x = 1.8325957$ units
$y = 0.6528366$ unit

Find: ϕ, λ

Using equations (7–4) and (7–5),

$$\phi = 90° - 2 \arctan (e^{-0.6528366/1.0})$$
$$= 90° - 2 \arctan (0.5205670) = 90° - 2 \times 27.5° = 35°$$
$= 35°$ N. lat, since the sign is "+."
$$\lambda = (1.8325957/1.0) \times 180°/\pi + (-180°)$$
$= 105° - 180° = -75° = 75°$ W. long., since the sign is "–."

The scale factor may then be determined as in equation (7–3) using the newly calculated ϕ.

MERCATOR PROJECTION (ELLIPSOID)–FORWARD EQUATIONS (SEE P. 50)

Given: Clarke 1866 ellipsoid: $a = 6378206.4$ m
$e^2 = 0.00676866$
or $e = 0.0822719$
Central meridian: $\lambda_0 = 180°$ W. long.
Point: $\phi = 35°$ N. lat.
$\lambda = 75°$ W. long.

Find: x, y, k

Using equations (7–6) through (7–8),

$$x = 6378206.4 \times [(-75°)-(-180°)] \times \pi/180° = 11688673.7 \text{ m}$$

$$y = 6378206.4 \ln \left[\tan(45° + 35°/2) \left(\frac{1-0.0822719 \sin 35°}{1+0.0822719 \sin 35°}\right)^{0.0822719/2}\right]$$

$$= 6378206.4 \ln [1.9209821 \times 0.9961223]$$

$$= 6378206.4 \ln 1.9135331 = 4{,}139{,}145.6 \text{ m}$$

$$k = (1-0.00676866 \sin^2 35°)^{1/2}/\cos 35°$$

$$= 1.2194146$$

MERCATOR PROJECTION (ELLIPSOID)–INVERSE EQUATIONS (SEE P. 50–51)

Inversing forward example:

Given: a, e; λ_0 for forward example
$x = 11688673.7$ m
$y = 4139145.6$ m

Find: ϕ, λ

Using equation (7–10),

$$t = e^{-4139145.6/6378206.4} = 0.5225935$$

From equation (7–11), the first trial $\phi = 90° - 2 \arctan 0.5225935 = 34.8174484°$. Using this value on the right side of equation (7–9),

$$\phi = 90° - 2 \arctan \{0.5225935[(1-0.0822719 \sin 34.8174484°)/(1 + 0.0822719 \sin 34.8174484°)]^{0.0822719/2}\}$$

$$= 34.9991687°$$

Replacing 34.8174484° with 34.9991687° for the second trial, recalculation using (7–9) gives $\phi = 34.9999969°$. The third trial gives $\phi = 35.0000006°$, which does not change (to 7 places) with recalculation. If it were not for rounding-off errors in the values of x and y, ϕ would be 35° N. lat.

For λ, using equation (7–12),

$$\lambda = (11688673.7/6378206.4) \times 180°/\pi + (-180°)$$

$$= -75.0000001° = 75.0000001° \text{ W. long.}$$

Using equations (7–13) and (3–5) instead, to find ϕ,

$\chi = 90° - 2 \arctan 0.5225935$
$= 90° - 55.1825516°$
$= 34.8174484°$

using t as calculated above from (7–10). Using (3–5), χ is inserted as in the example given above for inverse auxiliary latitude χ:

$\phi = 35.0000006°$

TRANSVERSE MERCATOR (SPHERE)–FORWARD EQUATIONS (SEE P. 64, 67)

Given: Radius of sphere: $R = 1.0$ unit
Origin: $\phi_0 = 0$
$\lambda_0 = 75°$ W. long.
Central scale factor: $k_0 = 1.0$
Point: $\phi = 40°30'$ N. lat.
$\lambda = 73°30'$ W. long.

Find: x, y, k

Using equation (8–5),

$B = \cos 40.5° \sin [(-73.5°) - (-75°)]$
$= \cos 40.5° \sin 1.5° = 0.0199051$

From equation (8–1),

$x = ½ \times 1.0 \times 1.0 \ln [(1 + 0.0199051)/(1 - 0.0199051)]$.
$= 0.0199077$ unit

From equation (8–3),

$y = 1.0 \times 1.0 \{\arctan [\tan 40.5°/\cos 1.5°] - 0\}$
$= 40.5096980° \ \pi/180° = 0.7070276$ unit

From equation (8–4),

$k = 1.0/(1 - 0.0199051^2)^{1/2} = 1.0001982$

TRANSVERSE MERCATOR (SPHERE)–INVERSE EQUATIONS (SEE P. 67)

Inversing forward example:

Given: R, ϕ_0, λ_0, k_0 for forward example
$x = 0.0199077$ unit
$y = 0.7070276$ unit

Find: ϕ, λ
Using equation (8–8),

$D = 0.7070276/(1.0 \times 1.0) + 0 = 0.7070276$ radian

For the hyperbolic functions of (x/Rk_0), the relationships

$$\sinh x = (e^x - e^{-x})/2$$

and

$$\cosh x = (e^x + e^{-x})/2$$

are recalled if the function is not directly available on a given computer or calculator. In this case,

$$\begin{aligned}\sinh (x/Rk_0) &= \sinh [0.0199077/(1.0 \times 1.0)] \\ &= (e^{0.0199077} - e^{-0.0199077})/2 \\ &= 0.0199090 \\ \cosh (x/Rk_0) &= (e^{0.0199077} + e^{-0.0199077})/2 \\ &= 1.0001982\end{aligned}$$

From equation (8–6), with D in radians, not degrees,

$$\begin{aligned}\phi &= \arcsin (\sin 0.7070276/1.0001982) = \arcsin (0.6495767/1.0001982) \\ &= 40.4999995^\circ \text{ N. lat.}\end{aligned}$$

From equation (8–7),

$$\begin{aligned}\lambda &= -75^\circ + \arctan [0.0199090/\cos 0.7070276] \\ &= -75^\circ + \arctan 0.0261859 = -75^\circ + 1.4999961 = -73.5000039^\circ \\ &= 73.5000039^\circ \text{ W. long.}\end{aligned}$$

If more decimals were supplied with the x and y calculated from the forward equations, the ϕ and λ here would agree more exactly with the original values.

TRANSVERSE MERCATOR (ELLIPSOID)–FORWARD EQUATIONS (SEE P. 67–68)

Given: Clarke 1866 ellipsoid:	$a = 6378206.4$ m
	$e^2 = 0.00676866$
Origin (UTM Zone 18):	$\phi_0 = 0$
	$\lambda_0 = 75^\circ$ W. long.
Central scale factor:	$k_0 = 0.9996$
Point:	$\phi = 40^\circ 30'$ N. lat.
	$\lambda = 73^\circ 30'$ W. long.

Find: x, y, k

Using equations (8–12) through (8–15) in order,

$$\begin{aligned}e'^2 &= 0.00676866/(1-0.00676866) = 0.0068148 \\ N &= 6378206.4/(1 - 0.00676866 \sin^2 40.5^\circ)^{1/2} = 6387330.5 \text{ m} \\ T &= \tan^2 40.5^\circ = 0.7294538 \\ C &= 0.0068148 \cos^2 40.5^\circ = 0.0039404 \\ A &= (\cos 40^\circ 30') \times [(-73.5^\circ) - (-75^\circ)]\ \pi/180^\circ = 0.0199074\end{aligned}$$

Instead of equation (3–21), we may use (3–22) for the Clarke 1866:

$$M = 111132.0894 \times (40.5^\circ) - 16216.94 \sin (2 \times 40.5^\circ) + 17.21 \sin (4 \times 40.5^\circ) - 0.02 \sin (6 \times 40.5^\circ)$$
$$= 4{,}484{,}837.67 \text{ m}$$
$$M_0 = 111132.0894 \times 0^\circ - 16216.94 \sin (2 \times 0^\circ) + 17.21 \sin (4 \times 0^\circ) - 0.02 \sin (6 \times 0^\circ)$$
$$= 0.00 \text{ m}$$

Equations (8–9) and (8–10) may now be used:

$$x = 0.9996 \times 6387330.5 \times [0.0199074 + (1 - 0.7294538 + 0.0039404) \times 0.0199074^3/6 + (5 - 18 \times 0.7294538 + 0.7294538^2 + 72 \times 0.0039404 - 58 \times 0.0068148) \times 0.0199074^5/120]$$
$$= 127{,}106.5 \text{ m}$$
$$y = 0.9996 \times \{4484837.7 - 0 + 6387330.5 \times 0.8540807 \times [0.0199074^2/2 + (5 - 0.7294538 + 9 \times 0.0039404 + 4 \times 0.0039404^2) \times 0.0199074^4/24 + (61 - 58 \times 0.7294538 + 0.7294538^2 + 600 \times 0.0039404 - 330 \times 0.0068148) \times 0.0199074^6/720]\}$$
$$= 4{,}484{,}124.4 \text{ m}$$

These values agree exactly with the UTM tabular values, except that 500,000.0 m must be added to x for "false eastings." To calculate k, using equation (8–11),

$$k = 0.9996 \times [1 + (1 + 0.0039404) \times 0.0199074^2/2 + (5 - 4 \times 0.7294538 + 42 \times 0.0039404 + 13 \times 0.0039404^2 - 28 \times 0.0068148) \times 0.0199074^4/24 + (61 - 148 \times 0.7294538 + 16 \times 0.7294538^2) \times 0.0199074^6/720]$$
$$= 0.9997989$$

Using equation (8–16) instead,

$$k = 0.9996 \times [1 + (1 + 0.0068148 \cos^2 40.5^\circ) \times 127106.5^2/(2 \times 0.9996^2 \times 6387330.5^2)]$$
$$= 0.9997989$$

TRANSVERSE MERCATOR (ELLIPSOID)–INVERSE EQUATIONS (SEE P. 68–69)

Inversing forward example:

Given: Clarke 1866 ellipsoid: $a = 6378206.4$ m
$e^2 = 0.00676866$
Origin (UTM Zone 18): $\phi_0 = 0$
$\lambda_0 = 75^\circ$ W. long.
Central scale factor: $k_0 = 0.9996$
Point: $x = 127106.5$ m
$y = 4484124.4$ m

Find: ϕ, λ

Calculating M_0 from equation (3–22),

$$M_0 = 111132.089\times 0° - 16216.9 \sin (2\times 0°) + 17.2 \sin (4\times 0°) - 0.02 \sin (6\times 0°)$$
$$= 0$$

From equation (8–12),

$$e'^2 = 0.00676866/(1-0.00676866) = 0.0068148$$

Using equation (8–20),

$$M = 0 + 4484124.4/0.9996 = 4485918.8 \text{ m}$$

From equation (3–24),

$$e_1 = [1-(1-0.00676866)^{1/2}]/[1+(1-0.00676866)^{1/2}]$$
$$= 0.001697916$$

From equation (8–19),

$$\mu = 4485918.8/[6378206.4\times(1-0.00676866/4-3\times 0.00676866^2/64 - 5\times 0.00676866^3/256)]$$
$$= 0.7045135 \text{ radian}$$

From equation (3–26), using μ in radians,

$$\phi_1 = 0.7045135 + (3\times 0.001697916/2 - 27\times 0.001697916^3/32) \sin (2\times 0.7045135) + (21\times 0.001697916^2/16 - 55\times 0.001697916^4/32) \sin (4\times 0.7045135) + (151\times 0.001697916^3/96) \sin (6\times 0.7045135)$$
$$= 0.7070283 \text{ radian}$$
$$= 0.7070283\times 180°/\pi$$
$$= 40.5097362°$$

Now equations (8–21) through (8–25) may be used:

$$C_1 = 0.0068148 \cos^2 40.5097362° = 0.0039393$$
$$T_1 = \tan^2 40.5097362° = 0.7299560$$
$$N_1 = 6378206.4/(1-0.00676866 \sin^2 40.5097362°)^{1/2}$$
$$= 6387334.2 \text{ m}$$
$$R_1 = 6378206.4\times(1-0.00676866)/(1-0.00676866 \sin^2 40.5097362°)^{3/2}$$
$$= 6{,}362{,}271.4 \text{ m}$$
$$D_1 = 127106.5/(6387334.2\times 0.9996) = 0.0199077$$

Returning to equation (8–17),

$$\phi = 40.5097362° - (6387334.2\times 0.8543746/6362271.4)\times[0.0199077^2/2 - (5\times 3\times 0.7299560 + 10\times 0.0039393 - 4\times 0.0039393^2 - 9\times 0.0068148)\times 0.0199077^4/24 + (61 + 90\times 0.7299560 + 298\times 0.0039393 + 45\times 0.7299560^2 - 252\times 0.0068148 - 3\times 0.0039393^2)\times 0.0199077^6/720]\times 180°/\pi$$
$$= 40.5000000° = 40°30' \text{ N. lat.}$$

From equation (8–18),

$$\lambda = -75° + \{[0.0199077 - (1 + 2\times 0.7299560 + 0.0039393)\times 0.01990\cdot/6 + (5 - 2\times 0.0039393 + 28\times 0.7299560 - 3\times 0.0039393^2 + 8 \times 0.0068148 + 24\times 0.7299560^2)\times 0.0199077^5/120]/\cos 40.5097362°\}\times 180°/\pi$$

$$= -75° + 1.5000000° = -73.5° = 73°30' \text{ W. long.}$$

OBLIQUE MERCATOR (SPHERE)–FORWARD EQUATIONS (SEE P. 76–78)

Given: Radius of sphere: $R = 1.0$ unit
Central scale factor: $k_0 = 1.0$
Central line through: $\phi_1 = 45°$ N. lat.
$\phi_2 = 0°$ lat.
$\lambda_1 = 0°$ long.
$\lambda_2 = 90°$ W. long.
Point: $\phi = 30°$ S. lat.
$\lambda = 120°$ E. long.

Find: x, y, k
Using equation (9–1),

$$\lambda_p = \arctan \{[\cos 45° \sin 0° \cos 0° - \sin 45° \cos 0° \cos(-90°)]/[\sin 45° \cos 0° \sin(-90°) - \cos 45° \sin 0° \sin 0°]\}$$
$$= \arctan \{[0 - 0]/[-0.7071068 - 0]\} = 0°$$

Since the denominator is negative, add or subtract 180° (the numerator has neither sign, but it doesn't matter). Thus,

$$\lambda_p = 0° + 180° = 180°$$

From equation (9–2),

$$\phi_p = \arctan[-\cos(180° - 0°)/\tan 45°]$$
$$= \arctan[+1/0.7071068] = 45°$$

The other pole is then at $\phi = -45°$, $\lambda = 0°$. From equation (9–6a),

$$\lambda_0 = 180° + 90° = 270°, \text{ equivalent to } 270° - 360° \text{ or } -90°.$$

From equation (9–6),

$$A = \sin 45° \sin(-30°) - \cos 45° \cos(-30°) \sin[120° - (-90°)]$$
$$= 0.7071068\times(-0.5) - 0.7071068\times 0.8660254\times(-0.5)$$
$$= -0.0473672$$

From equation (9–3),

$$x = -1.0\times 1.0 \arctan[\tan(-30°) \cos 45°/\cos(120° + 90°) + \sin 45° \tan(120° + 90°)]$$
$$= 0.7214592$$

Since cos (120° + 90°) is negative, subtract π, or $x = -2.4201335$ units
From equation (9–4),

$$y = \frac{1}{2} \times 1.0 \times 1.0 \ln [(1 - 0.0473672)/(1 + 0.0473672)]$$
$$= -0.0474026 \text{ unit}$$

From equation (9–5),

$$k = 1.0/[1 - (-0.0473672)^2]^{1/2} = 1.0011237$$

If the parameters are given in terms of a central point (for equations (9–7) and (9–8), we shall assume certain artificial parameters (calculated with different formulas) which give the same pole as above:

Given:	Radius of sphere:	$R = 1.0$ unit
	Central scale factor:	$k_0 = 1.0$
	Azimuth of central line:	$\beta = 48.8062990°$ east of north
	Center:	$\phi_c = 20°$ N. lat.
		$\lambda_c = 68.6557771°$ W. long.

Using equation (9–7),

$$\phi_p = \arcsin (\cos 20° \sin 48.8062990°)$$
$$= 45.0° \text{ N. lat.}$$

From equation (9–8),

$$\lambda_p = \arctan [-\cos 48.8062990°/(-\sin 20° \sin 48.8062990°)]$$
$$-68.6557771°$$
$$= 0°$$

Since the denominator of the argument of arctan is negative, add $-180°$ to λ_p, using "–" since the numerator is "–":

$$\lambda_p = 180° \text{ W. long.}$$

OBLIQUE MERCATOR (SPHERE)–INVERSE EQUATIONS (SEE P. 78)

Inversing forward example:

Given:	Radius of sphere:	$R = 1.0$ unit
	Central scale factor:	$k_0 = 1.0$
	Central line through:	$\phi_1 = 45°$ N. lat.
		$\phi_2 = 0°$ lat.
		$\lambda_1 = 0°$ long.
		$\lambda_2 = 90°$ W. long.
	Point:	$x = -2.4201335$ units
		$y = -0.0474026$ unit

Find: ϕ, λ

First, ϕ_p and λ_p are determined, exactly as for the forward example, so that λ_0 again is $-90°$, and $\phi_p = 45°$. Determining hyperbolic functions, if not readily available,

$$y/Rk_0 = -0.0747026/(1.0 \times 1.0) = -0.0474026$$
$$e^{-0.0474026} = 0.9537034$$
$$\sinh(y/Rk_0) = (0.9537034 - 1/0.9537034)/2 = -0.0474203$$
$$\cosh(y/Rk_0) = (0.9537034 + 1/0.9537034)/2 = 1.0011237$$
$$\tanh(y/Rk_0) = (0.9537034 - 1/0.9537034)/(0.9537034 + 1/0.9537034) = -0.0473671$$

From equation (9–9),

$$\phi = \arcsin\{\sin 45° \times (-0.0473671) + \cos 45° \sin [(-2.4201335/(1.0 \times 1.0)) \times 180°/\pi]/1.0011237$$
$$= \arcsin(-0.5000000)$$
$$= -30° = 30° \text{ S. lat.}$$

From equation (9–10),

$$\lambda = -90° + \arctan\{[\sin 45° \sin[-2.4201335 \times 180°/(\pi \times 1.0 \times 1.0)] - \cos 45° \times (-0.0474203)]/\cos[-2.4201335 \times 180°/(\pi \times 1.0 \times 1.0)]\}$$
$$= -90° + 30.0000041°$$
$$= -59.9999959°$$

but the main denominator is -0.7508428, which is negative, while the numerator is also negative. Therefore, add $(-180°)$ to λ, so $\lambda = -59.9999959° - 180° = -239.9999959° = 240°$ W. long. $= 120°$ E. long.

OBLIQUE MERCATOR (HOTINE ELLIPSOID)–FORWARD EQUATIONS (SEE P. 78–83)

For Alternate A:

Given: Clarke 1866 ellipsoid:	$a = 6378206.4$ m
	$e^2 = 0.00676866$
or	$e = 0.0822719$
Central scale factor:	$k_0 = 0.9996$
Center:	$\phi_0 = 40°$ N. lat.
Central line through:	$\phi_1 = 47°30'$ N. lat.
	$\lambda_1 = 122°18'$ W. long. (Seattle, Wash.)
	$\phi_2 = 25°42'$ N. lat.
	$\lambda_2 = 80°12'$ W. long. (Miami, Fla.)
False coordinates:	$x_0 = 4{,}000{,}000.0$ m

$y_0 = 500{,}000.0$ m
Point: $\phi = 40°48'$ N. lat.
$\lambda = 74°00'$ W. long. (New York City)

Find: x, y, k

Following equations (9–11) through (9–24) in order:

$B = [1 + 0.00676866 \cos^4 40°/(1 - 0.00676866)]^{1/2}$
$= 1.0011727$
$A = 6378206.4 \times 1.0011727 \times 0.9996 \times (1 - 0.00676866)^{1/2}/(1 - 0.00676866 \sin^2 40°)$
$= 6{,}379{,}333.2$ m
$t_0 = \tan(45° - 40°/2)/[(1 - 0.0822719 \sin 40°)/(1 + 0.0822719 \sin 40°)]^{0.0822719/2}$
$= 0.4683428$
$t_1 = \tan(45° - 47.5°/2)/[(1 - 0.0822719 \sin 47.5°)/(1 + 0.0822719 \sin 47.5°)]^{0.0822719/2}$
$= 0.3908266$
$t_2 = \tan(45° - 25.7°/2)/[1 - 0.0822719 \sin 25.7°)/(1 + 0.0822719 \sin 25.7°)]^{0.0822719/2}$
$= 0.6303639$
$D = 1.0011727 \times (1 - 0.00676866)^{1/2}/[\cos 40° \times (1 - 0.00676866 \sin^2 40°)^{1/2}]$
$= 1.3043327$
$E = [1.3043327 + (1.3043327^2 - 1)^{1/2}] \times 0.4683428^{1.0011727}$
$= 1.0021857$

using the "+" sign, since ϕ_0 is north or positive.

$H = 0.3908266^{1.0011727} = 0.3903963$
$L = 0.6303639^{1.0011727} = 0.6300229$
$F = 1.0021857/0.3903963 = 2.5670986$
$G = (2.5670986 - 1/2.5670986)/2 = 1.0887769$
$J = (1.0021857^2 - 0.6300229 \times 0.3903963)/(1.0021857^2 + 0.6300229 \times 0.3903963) = 0.6065716$
$P = (0.6300229 - 0.3903963)/(0.6300229 + 0.3903963)$
$= 0.2348315$
$\lambda_0 = ½[(-122.3°) + (-80.2°)] - \arctan\{0.6065716 \tan[1.0011727 \times (-122.3° + 80.2°)/2]/0.2348315\}/1.0011727$
$= -101.25° - \arctan(-0.9953887)/1.0011727$
$= -56.4349628°$
$\gamma_0 = \arctan\{\sin[1.0011727 \times (-122.3° + 56.4349628°)]/1.0887769\}$
$= -39.9858829°$
$\alpha_c = \arcsin[1.3043327 \sin(-39.9858829°)]$
$= -56.9466070°$

These are constants for the map. For the given ϕ and λ, following equations (9–25) through (9–34) in order:

$t = \tan(45° - 40.8°/2)/[(1 - 0.0822719 \sin 40.8°)/(1 + 0.0822719 \sin 40.8°)]^{0.0822719/2}$
$= 0.4598671$
$Q = 1.0021857/0.4598671^{1.0011727} = 2.1812805$
$S = (2.1812805 - 1/2.1812805)/2 = 0.8614171$
$T = (2.1812805 + 1/2.1812805)/2 = 1.3198634$
$V = \sin[1.0011727 \times (-74° + 56.4349628°)]$
$= -0.3021309$
$U = [0.3021309 \cos(-39.9858829°) + 0.8614171 \sin(-39.9858829°)]/1.3198634$
$= -0.2440041$
$v = 6379333.2 \ln[(1 + 0.2440041)/(1 - 0.2440041)]/(2 \times 1.0011727)$
$= 1{,}586{,}767.3$ m
$u = [[6379333.2 \arctan\{[0.8614171 \cos(-39.9858829°) + (-0.3021309) \sin(-39.9858829°)]/\cos[1.0011727 \times (-74° + 56.4349628°)]\}/1.0011727]] \times \pi/180°$
$= 4{,}655{,}443.7$ m

Note: Since $\cos[1.0011727 \times (-74° + 56.4349628°)] = 0.9532664$, which is positive, no correction is needed to the arctan in the equation for u. The ($\pi/180°$) is inserted, if arctan is calculated in degrees.

$k = 6379333.2 \cos[1.0011727 \times 4655443.7 \times 180°/(\pi \times 6379333.2)] \times (1 - 0.00676866 \sin^2 40.8°)^{1/2}/\{6378206.4 \cos 40.8° \cos[1.0011727 \times (-74° + 56.4349628°)]\}$
$= 1.0307554$
$x = 1586767.3 \cos(-56.9466070°) + 4655443.7 \sin(-56.9466070°) + 4000000$
$= 963{,}436.1$ m
$y = 4655443.7 \cos(-56.9466070°) - 1586767.3 \sin(-56.9466070°) + 500000$
$= 4{,}369{,}142.8$ m

For Alternate B (forward):

Given: Clarke 1866 ellipsoid:	$a = 6378206.4$ m
	$e^2 = 0.00676866$
or	$e = 0.0822719$
Central scale factor:	$k_0 = 1.0$
Center:	$\phi_0 = 36°$ N. lat.
	$\lambda_c = 77.7610558°$ W. long.
Azimuth of central line:	$\alpha_c = 14.3394883°$ east of north
Point:	$\phi = 38°48'33.166''$ N. lat.
	$= 38.8092128°$
	$\lambda = 76°52'14.863''$ W. long.
	$= -76.8707953°$

Find: u, v (example uses center of Zone 2, Path 16, Landsat mapping, with Hotine Oblique Mercator).

Using equations (9–11) through (9–39) in order,

$B=[1+0.00676866\cos^4 36°/(1-0.00676866)]^{1/2}$
$\quad=1.0014586$
$A=63780206.4\times 1.0014586\times 1.0\times(1-0.00676866)^{1/2}/(1-0.00676866\sin^2 36°)=6{,}380{,}777.0$ m
$t_0=\tan(45°-36°/2)/[(1-0.0822719\sin 36°)/(1+0.0822719\sin 36°)]^{0.0822719/2}$
$\quad=0.5115582$
$D=1.0014586\times(1-0.00676866)^{1/2}/[\cos 36°\times(1-0.00676866\sin^2 36°)^{1/2}]$
$\quad=1.2351194$
$F=1.2351194+(1.2351194^2-1)^{1/2}=1.9600471$

using the "+" sign since ϕ_0 is north or positive.

$E=1.9600471\times 0.5115582^{1.0014586}=1.0016984$
$G=(1.9600471-1/1.9600471)/2=0.7249276$
$\gamma_0=\arcsin[(\sin 14.3394883°)/1.2351194]$
$\quad=11.5673996°$
$\lambda_0=-77.7610558°-[\arcsin(0.7249276\tan 11.5673996°)]/1.0014586$
$\quad=-86.2814800°$
$u_{(36°,\,-77.76\ldots°)}=+(6380777.0/1.0014586)\arctan[(1.2351194^2-1)^{1/2}/\cos 14.3394883°]\times\pi/180°$
$\quad=4{,}092{,}868.9$ m

Note: The $\pi/180°$ is inserted, if arctan is calculated in degrees. These are constants for the map. The calculations of u, v, x, and y for (ϕ, λ) follow the same steps as the numerical example for equations (9–25) through (9–34) for alternate A. For $\phi=38.8092128°$ and $\lambda=-76.8707953°$, it is found that

$u=4{,}414{,}439.0$ m
$v=-2{,}356.3$ m

OBLIQUE MERCATOR (HOTINE ELLIPSOID)–INVERSE EQUATIONS
(SEE P. 83–84)

The above example for alternate A will be inverted, first using equations (9–11) through (9–24), then using equations (9–40) through (9–48). Since no new equations are involved for inverse alternate B, an example of the latter will be omitted. As stated with the inverse equations, the constants for the map are chosen as in the forward examples. Inversing forward example for alternate A:

Given: Clarke 1866 ellipsoid: $a=6{,}378{,}206.4$ m
$e^2=0.00676866$
or $e=0.0822719$
Central scale factor: $k_0=0.9996$
Center: $\phi_0=40°$ N. lat.

Center line through: $\phi_1 = 47°30'$ N. lat.
$\lambda_1 = 122°18'$ W. long.
$\phi_2 = 25°42'$ N. lat.
$\lambda_2 = 80°12'$ W. long.
False coordinates: $x_0 = 4{,}000{,}000.0$ m
$y_0 = 500{,}000.0$ m
Point: $x = 963{,}436.1$ m
$y = 4{,}369{,}142.8$ m

Find: ϕ, λ

Using equations (9–11) through (9–24) in order, again gives the following constants:

$B = 1.0011727$
$A = 6{,}379{,}333.2$ m
$E = 1.0021857$
$\lambda_0 = -56.4349628°$
$\gamma_0 = -39.9858829°$
$\alpha_c = -56.9466070°$

Following equations (9–40) through (9–48) in order:

$v = (963436.1 - 4000000.0) \cos(-56.9466070°) - (4369142.8 - 500000.0) \sin(-56.9466070°)$
$= 1{,}586{,}767.3$ m
$u = (4369142.8 - 500000.0) \cos(-56.9466070°) + (963436.1 - 4000000.0) \sin(-56.9466070°)$
$= 4{,}655{,}443.7$ m
$Q' = e^{-(1.0011727 \times 1586767.3/6379333.2)}$
$= e^{-0.2490273}$
$= 0.7795587$
$S' = (0.7795587 - 1/0.7795587)/2 = -0.2516092$
$T' = (0.7795587 + 1/0.7795587)/2 = 1.0311679$
$V' = \sin[(1.0011727 \times 4655443.7/6379333.2) \times 180°/\pi]$
$= \sin 41.8617535° = 0.6673356$
$U' = [0.6673356 \cos(-39.9858829°) - 0.2516092 \sin(-39.9858829°)]/1.0311679$
$= 0.6526562$
$t = \{1.0021857/[(1 + 0.6526562)/(1 - 0.6526562)]^{1/2}\}^{1/1.0011727}$
$= 0.4598671$

The first trial ϕ for equation (7–9) is

$\phi = 90° - 2 \arctan(0.4598671) = 40.6077096°$

Calculating a new trial ϕ:

$\phi = 90° - 2 \arctan\{0.4598671 \times [(1 - 0.0822719 \sin 40.6077096°)/(1 + 0.0822719 \sin 40.6077096°)]^{0.0822719/2}\}$
$= 40.7992509°$

Substituting 40.7992509° in place of 40.6077096° and recalculating, $\phi = 40.7999971°$. Using this ϕ for the third trial, $\phi = 40.8000000°$. The next trial gives the same value of ϕ. Thus,

$$\phi = 40.8° = 40°48' \text{ N. lat.}$$
$$\lambda = -56.4349628° - \arctan \{[-0.2516092 \cos(-39.9858829°) - 0.6673356 \sin(-39.9858829°)]/\cos [(1.0011727 \times 4655443.7/6379333.2) \times 180°/\pi]\} /1.0011727$$
$$= -74.0000000° = 74°00' \text{ W. long.}$$

Using series equation (3–5) with (7–13), to avoid iteration of (7–9), and beginning with equation (7–13),

$$\chi = 90° - 2 \arctan 0.4598671$$
$$= 40.6077096°$$

Since equation (3–5) is used in an example under *Auxiliary latitudes*, the calculation will not be shown here.

MILLER CYLINDRICAL (SPHERE)–FORWARD EQUATIONS (SEE P. 87–88)

Given: Radius of sphere: $R = 1.0$ unit
Central meridian: $\lambda_0 = 0°$ long.
Point: $\phi = 50°$ N. lat.
$\lambda = 75°$ W. long.

Find x, y, h, k

Using equations (10–1) through (10–5) in order,

$$x = 1.0 \times [-75° - 0°] \times \pi/180°$$
$$= -1.3089969 \text{ units}$$
$$y = 1.0 \times [\ln \tan (45° + 0.4 \times 50°)]/0.8$$
$$= (\ln \tan 65°)/0.8$$
$$= 0.9536371 \text{ unit}$$

or

$$y = 1.0 \times \{\text{arctanh} [\sin (0.8 \times 50°)]\}/0.8$$
$$= \text{arctanh } 0.6427876/0.8$$
$$= 0.9536371 \text{ unit}$$
$$h = \sec (0.8 \times 50°) = 1/\cos 40° = 1.3054073$$
$$k = \sec 50° = 1/\cos 50° = 1.5557238$$
$$\sin {}^1\!/_2\omega = (\cos 40° - \cos 50°)/(\cos 40° + \cos 50°)$$
$$= 0.0874887$$
$$\omega = 10.0382962°$$

MILLER CYLINDRICAL (SPHERE)–INVERSE EQUATIONS (SEE P. 88)

Inversing forward example:

Given: R, λ_0 for forward example

$x = -1.3089969$ units
$y = 0.9536371$ unit

Find: ϕ, λ

Using equations (10–6) and (10–7),

$$\begin{aligned}\phi &= 2.5 \arctan e^{(0.8\times 0.9536371/1.0)} - (5\pi/8)\times 180°/\pi \\ &= 2.5 \arctan e^{0.7629096} - 1.9634954\times 180°/\pi \\ &= 2.5 \arctan (2.1445069) - 1.9634954\times 180°/\pi \\ &= 2.5\times 65.0000006° - 112.5000000° \\ &= 50.0000015° = 50° \text{ N. lat.}\end{aligned}$$

or

$$\begin{aligned}\phi &= \arcsin [\tanh (0.8\times 0.9536371/1.0)]/0.8 \\ &= (\arcsin 0.6427876)/0.8 \\ &= 50.0000015° = 50° \text{ N. lat.} \\ \lambda &= 0° - (1.3089969/1.0)\times 180°/\pi \\ &= 0° - 74.9999978° = 75° \text{ W. long.}\end{aligned}$$

ALBERS CONICAL EQUAL-AREA (SPHERE)–FORWARD EQUATIONS
(SEE P. 95–96)

Given: Radius of sphere: $R = 1.0$ unit
Standard parallels: $\phi_1 = 29°30'$ N. lat.
$\phi_2 = 45°30'$ N. lat.
Origin: $\phi_0 = 23°$ N. lat.
$\lambda_0 = 96°$ W. long.
Point: $\phi = 35°$ N. lat.
$\lambda = 75°$ W. long.

Find: ρ, θ, x, y, k, h, ω

From equation (12–6),

$$\begin{aligned}n &= (\sin 29.5° + \sin 45.5°)/2 \\ &= 0.6028370\end{aligned}$$

From equation (12–5),

$$\begin{aligned}C &= \cos^2 29.5° + 2\times 0.6028370 \sin 29.5° \\ &= 1.3512213\end{aligned}$$

From equations (12–3) and (12–3a),

$$\begin{aligned}\rho &= 1.0\times (1.3512213 - 2\times 0.6028370 \sin 35°)^{1/2}/0.6028370 \\ &= 1.3473026 \text{ units} \\ \rho_0 &= 1.0\times (1.3512213 - 2\times 0.6028370 \sin 23°)^{1/2}/0.6028370 \\ &= 1.5562263 \text{ units}\end{aligned}$$

From equation (12–4),

$$\begin{aligned}\theta &= 0.6028370\times [(-75°) - (-96°)] \\ &= 12.6595771°\end{aligned}$$

From equation (12–1),

$$x = 1.3473026 \sin 12.6595771^\circ$$
$$= 0.2952720 \text{ unit}$$

From equation (12–2),

$$y = 1.5562263 - 1.3473026 \cos 12.6595771^\circ$$
$$= 0.2416774 \text{ unit}$$

From equation (12–7),

$$h = \cos 35^\circ/(1.3512213 - 2 \times 0.6028370 \sin 35^\circ)^{1/2}$$
$$= 1.0085547$$

and

$$k = 1/1.0085547 = 0.9915178$$

From equation (4–9),

$$\sin \tfrac{1}{2}\omega = |1.0085547 - 0.9915178|/(1.0085547 + 0.9915178)$$
$$\omega = 0.9761189^\circ$$

ALBERS CONICAL EQUAL-AREA (SPHERE)–INVERSE EQUATIONS (SEE P. 96)

Inversing forward example:

Given: R, ϕ_1, ϕ_2, ϕ_0, λ_0 for forward example
$x = 0.2952720$ unit
$y = 0.2416774$ unit

Find: ρ, θ, ϕ, λ

As in the forward example, from equation (12–6),

$$n = (\sin 29.5^\circ + \sin 45.5^\circ)/2$$
$$= 0.6028370$$

From equation (12–5),

$$C = \cos^2 29.5^\circ + 2 \times 0.6028370 \sin 29.5^\circ$$
$$= 1.3512213$$

From equation (12–3a),

$$\rho_0 = 1.0 \times (1.3512213 - 2 \times 0.6028370 \sin 23^\circ)^{1/2}/0.6028370$$
$$= 1.5562263 \text{ units}$$

From equation (12–10),

$$\rho = [0.2952720^2 + (1.5562263 - 0.2416774)^2]^{1/2}$$
$$= 1.3473026 \text{ units}$$

From equation (12–11),

$\theta = \arctan [0.2952720/(1.5562263 - 0.2416774)]$
$= 12.6595766°$. Since the denominator is positive, there is no adjustment to θ.

From equation (12–8),

$\phi = \arcsin \{[1.3512213 - (1.3473026 \times 0.6028370/1.0)^2]/(2 \times 0.6028370)\}$
$= \arcsin 0.5735764$
$= 35.0000007° = 35°$ N. lat.

From equation (12–9),

$\lambda = 12.6595766°/0.6028370 + (-96°)$
$= 20.9999992 - 96°$
$= -75.0000008° = 75°$ W. long.

ALBERS CONICAL EQUAL-AREA (ELLIPSOID)–FORWARD EQUATIONS
(SEE P. 96–97)

Given: Clarke 1866 ellipsoid: $a = 6378206.4$ m
$e^2 = 0.00676866$
or $e = 0.0822719$
Standard parallels: $\phi_1 = 29°30'$ N. lat.
$\phi_2 = 45°30'$ N. lat.
Origin: $\phi_0 = 23°$ N. lat.
$\lambda_0 = 96°$ W. long.
Point: $\phi = 35°$ N. lat.
$\lambda = 75°$ W. long.

Find: $\rho, \theta, x, y, k, h, \omega$

From equation (12–15),

$m_1 = \cos 29.5°/(1 - 0.00676866 \sin^2 29.5°)^{1/2}$
$= 0.8710708$
$m_2 = \cos 45.5°/(1 - 0.00676866 \sin^2 45.5°)^{1/2}$
$= 0.7021191$

From equation (3–12),

$q_1 = (1 - 0.00676866) \{\sin 29.5°/(1 - 0.00676866 \sin^2 29.5°) - [1/(2 \times 0.0822719)] \ln [(1 - 0.0822719 \sin 29.5°)/(1 + 0.0822719 \sin 29.5°)]\}$
$= 0.9792529$

Using the same formula for q_2 (with ϕ_2 instead of ϕ_1),

$q_2 = 1.4201080$

Using the same formula for q_0 (with ϕ_0 instead of ϕ_1),

$q_0 = 0.7767080$

From equation (12–14),

$$n = (0.8710708^2 - 0.7021191^2)/(1.4201080 - 0.9792529)$$
$$= 0.6029035$$

From equation (12–13),

$$C = 0.8710708^2 + 0.6029035 \times 0.9792529$$
$$= 1.3491594$$

From equation (12–12a),

$$\rho_0 = 6378206.4 \times (1.3491594 - 0.6029035 \times 0.7767080)^{1/2}/0.6029035$$
$$= 9{,}929{,}079.6 \text{ m}$$

These are the constants for the map. For $\phi = 35°$ N. lat. and $\lambda = 75°$ W. long.: Using equation (3–12), but with ϕ in place of ϕ_1,

$q = 1.1410831$

From equation (12–12),

$$\rho = 6378206.4 \times (1.3491594 - 0.6029035 \times 1.1410831)^{1/2}/0.6029035$$
$$= 8{,}602{,}328.2 \text{ m}$$

From equation (12–4),

$$\theta = 0.6029035 \times [-75° - (-96°)] = 12.6609735°$$

From equation (12–1),

$$x = 8602328.2 \sin 12.6609735° = 1{,}885{,}472.7 \text{ m}$$

From equation (12–2),

$$y = 9929079.6 - 8602328.2 \cos 12.6609735°$$
$$= 1{,}535{,}925.0 \text{ m}$$

From equation (12–15),

$$m = \cos 35°/(1 - 0.00676866 \sin^2 35°)^{1/2}$$
$$= 0.8200656$$

From equation (12–16),

$$k = 8602328.2 \times 0.6029035/(6378206.4 \times 0.8200656)$$
$$= 0.9915546$$

From equation (12–18),

$h = 1/0.9915546 = 1.0085173$

From equation (4–9),

$$\sin \tfrac{1}{2}\omega = |1.0085173 - 0.9915546|/(1.0085173 + 0.9915546)$$
$$\omega = 0.9718678°$$

ALBERS CONICAL EQUAL-AREA (ELLIPSOID)–INVERSE EQUATIONS
(SEE P. 97–98)

Inversing forward example:

Given: Clarke 1866 ellipsoid: $a = 6378206.4$ m
$e^2 = 0.00676866$
or $e = 0.0822719$
Standard parallel: $\phi_1 = 29°30'$ N. lat.
$\phi_2 = 45°30'$ N. lat.
Origin: $\phi_0 = 23°$ N. lat.
$\lambda_0 = 96°$ W. long.
Point: $x = 1{,}885{,}472.7$ m
$y = 1{,}535{,}925.0$ m

Find: ρ, θ, ϕ, λ

The same constants n, C, ρ_0 are calculated with the same equations as those used for the forward example. For the particular point:

From equation (12–10),

$$\rho = [1885472.7^2 + (9929079.6 - 1535925.0)^2]^{1/2}$$
$$= 8{,}602{,}328.3 \text{ m}$$

From equation (12–11),

$$\theta = \arctan\,[1885472.7/(9929079.6 - 1535925.0)]$$
$$= \arctan 0.2246441$$

$= 12.6609733°$. The denominator is positive; therefore θ is not adjusted. From equation (12–19),

$$q = [1.3491594 - (8602328.3 \times 0.6029035/6378206.4)^2]/0.6029035$$
$$= 1.1410831$$

Using for the first trial ϕ the arcsin of (1.1410831/2), or 34.7879983°, calculate a new ϕ from equation (12–19),

$$\phi = 34.7879983° + [(1 - 0.00676866 \sin^2 34.7879983°)^2/(2 \cos 34.7879983°)] \times \{1.1410831/(1 - 0.00676866) - \sin 34.7879983°/(1 - 0.00676866 \sin^2 34.7879983°) + [1/(2 \times 0.0822719)] \ln [(1 - 0.0822719 \sin 34.7879983°)/(1 + 0.0822719 \sin 34.7879983°)]\} \times 180°/\pi$$
$$= 34.9997335°$$

Note that $180°/\pi$ is included to convert to degrees. Replacing 34.7879983° by 34.9997335° for the second trial, the calculation using equation (12–19) now provides a third ϕ of 35.0000015°. A recalculation with this value results in no change to seven decimal places. (This does not give exactly 35° due to rounding-off errors in x and y.) Thus,

$$\phi = 35.0000015° \text{ N. lat.}$$

For the longitude use equation (12–9),

$$\begin{aligned}\lambda &= (-96°) + 12.6609733°/0.6029035 \\ &= -75.0000003° \text{ or } 75.0000003° \text{ W. long.}\end{aligned}$$

For scale factors, we revert to the forward example, since ϕ and λ are now known.

Series equation (3–18) may be used to avoid the iteration above. Beginning with equation (12–21),

$$\begin{aligned}\beta &= \arcsin [1.1410831/\{1-[(1-0.00676866)/(2\times 0.0822719)] \ln \\ &\quad [(1-0.0822719)/(1+0.0822719)]\}] \\ &= 34.8781793°\end{aligned}$$

An example is not shown for equation (3–18), since it is similar to the example for (3–5).

LAMBERT CONFORMAL CONIC (SPHERE)–FORWARD EQUATIONS (SEE P. 105)

Given: Radius of sphere: $R = 1.0$ unit
Standard parallels: $\phi_1 = 33°$ N. lat.
$\phi_2 = 45°$ N. lat.
Origin: $\phi_0 = 23°$ N. lat.
$\lambda_0 = 96°$ W. long.
Point: $\phi = 35°$ N. lat.
$\lambda = 75°$ W. long.

Find: ρ, θ, x, y, k

From equation (13–3),

$$\begin{aligned}n &= \ln (\cos 33°/\cos 45°)/\ln [\tan (45° + 45°/2)/\tan (45° + 33°/2)] \\ &= 0.6304777\end{aligned}$$

From equation (13–2),

$$\begin{aligned}F &= [\cos 33° \tan^{0.6304777} (45° + 33°/2)]/0.6304777 \\ &= 1.9550002 \text{ units}\end{aligned}$$

From equation (13–1a),

$$\begin{aligned}\rho_0 &= 1.0 \times 1.9550002/\tan^{0.6304777} (45° + 23°/2) \\ &= 1.5071429 \text{ units}\end{aligned}$$

The above constants apply to the map generally. For the specific ϕ and λ, using equation (13–1),

$$\rho = 1.0 \times 1.9550002/\tan^{0.6304777}(45° + 35°/2)$$
$$= 1.2953636 \text{ units}$$

From equation (12–4),

$$\theta = 0.6304777 \times [(-75°) - (-96°)]$$
$$= 13.2400316°$$

From equations (12–1) and (12–2),

$$x = 1.2953636 \sin 13.2400316°$$
$$= 0.2966785 \text{ unit}$$
$$y = 1.5071429 - 1.2953636 \cos 13.2400316°$$
$$= 0.2462112 \text{ unit}$$

From equation (13–4),

$$k = \cos 33° \tan^{0.6304777}(45° + 33°/2)/[\cos 35° \tan^{0.6304777}(45° + 35°/2)]$$
$$= 0.9970040$$

or from equation (4–5),

$$k = 0.6304777 \times 1.2953636/(1.0 \cos 35°)$$
$$= 0.9970040$$

LAMBERT CONFORMAL CONIC (SPHERE)–INVERSE EQUATIONS (SEE P. 105, 107)

Inversing forward example:

Given: R, ϕ_1, ϕ_2, ϕ_0, λ_0 for forward example
$x = 0.2966785$ unit
$y = 0.2462112$ unit

Find: ρ, θ, ϕ, λ

After calculating n, F, and ρ_0 as in the forward example, obtaining the same values, equation (12–10) is used:

$$\rho = [0.2966785^2 + (1.5071429 - 0.2462112)^2]^{1/2}$$
$$= 1.2953636 \text{ units}$$

From equation (12–11),

$$\theta = \arctan [0.2966785/(1.5071429 - 0.2462112)]$$

$= 13.2400329°$. Since the denominator is positive, θ is not adjusted.

From equation (12–9),

$$\lambda = 13.2400329°/0.6304777 + (-96°)$$
$$= -74.9999981° = 74.9999981° \text{ W. long.}$$

From equation (13–5),

$\phi = 2 \arctan (1.0 \times 1.9550002/1.2953636)^{1/0.6304777} - 90°$
$= 34.9999974°$ N. lat.

LAMBERT CONFORMAL CONIC (ELLIPSOID)–FORWARD EQUATIONS
(SEE P. 107–108)

Given: Clarke 1866 ellipsoid: $a = 6{,}378{,}206.4$ m
$e^2 = 0.00676866$
or $e = 0.0822719$
Standard parallels: $\phi_1 = 33°$ N. lat.
$\phi_2 = 45°$ N. lat.
Origin: $\phi_0 = 23°$ N. lat.
$\lambda_0 = 96°$ W. long.
Point: $\phi = 35°$ N. lat.
$\lambda = 75°$ W. long.

Find: ρ, θ, x, y, k

From equation (12–15),

$m_1 = \cos 33°/(1 - 0.00676866 \sin^2 33°)^{1/2}$
$= 0.8395138$
$m_2 = \cos 45°/(1 - 0.00676866 \sin^2 45°)^{1/2}$
$= 0.7083064$

From equation (13–9),

$t_1 = \tan (45° - 33°/2)/[(1 - 0.0822719 \sin 33°)/(1 + 0.0822719 \sin 33°)]^{0.0822719/2}$
$= 0.5449623$
$t_2 = 0.4162031$, using above with 45° in place of 33°.
$t_0 = 0.6636390$, using above with 23° in place of 33°.

From equation (13–8),

$n = \ln (0.8395138/0.7083064)/\ln (0.5449623/0.4162031)$
$= 0.6304965$

From equation (13–10),

$F = 0.8395138/(0.6304965 \times 0.5449623^{0.6304965})$
$= 1.9523837$

From equation (13–7a),

$\rho_0 = 6378206.4 \times 1.9523837 \times 0.6636390^{0.6304965}$
$= 9{,}615{,}955.2$ m

The above are constants for the map. For the specific ϕ, λ, using equation (13–9),

$t = 0.5225935$, using above calculation with 35° in place of 33°.

From equation (13–7),

$$\rho = 6378206.4 \times 1.9523837 \times 0.5225935^{0.6304965}$$
$$= 8{,}271{,}173.9 \text{ m}$$

From equation (12–4),

$$\theta = 0.6304965 \times [-75° - (-96°)] = 13.2404256°$$

From equations (12–1) and (12–2),

$$x = 8271173.9 \sin 13.2404256°$$
$$= 1{,}894{,}410.9 \text{ m}$$
$$y = 9615955.2 - 8271173.9 \cos 13.2404256°$$
$$= 1{,}564{,}649.5 \text{ m}$$

From equations (12–15) and (12–16),

$$m = \cos 35°/(1 - 0.00676866 \sin^2 35°)^{1/2}$$
$$= 0.8200656$$
$$k = 8271173.9 \times 0.6304965/(6378206.4 \times 0.8200656)$$
$$= 0.9970171$$

LAMBERT CONFORMAL CONIC (ELLIPSOID)–INVERSE EQUATIONS
(SEE P. 108–109)

Inversing forward example:

Given: Clarke 1866 ellipsoid: $a = 6{,}378{,}206.4$ m
$e^2 = 0.00676866$
or $e = 0.0822719$
Standard parallels: $\phi_1 = 33°$ N. lat.
$\phi_2 = 45°$ N. lat.
Origin: $\phi_0 = 23°$ N. lat.
$\lambda_0 = 96°$ W. long.
Point: $x = 1{,}894{,}410.9$ m
$y = 1{,}564{,}649.5$ m

The map constants n, F, and ρ_0 are calculated as in the forward example, obtaining the same values. Then, from equation (12–10),

$$\rho = [1894410.9^2 + (9615955.2 - 1564649.5)^2]^{1/2}$$
$$= 8{,}271{,}173.8 \text{ m}$$

From equation (12–11),

$$\theta = \arctan [1894410.9/(9615955.2 - 1564649.5)]$$
$= 13.2404257°$. The denominator is positive; therefore θ is not adjusted.

From equation (13–11),

$$t = [8271173.8/(6378206.4 \times 1.9523837)]^{1/0.6304965}$$
$$= 0.5225935$$

To use equation (7–9), an initial trial ϕ is found as follows:

$$\phi = 90° - 2 \arctan 0.5225935$$
$$= 34.8174484°$$

Inserting this into the right side of equation (7–9),

$$\phi = 90° - 2 \arctan \{0.5225935 \times [(1 - 0.0822719 \sin 34.8174484°)/ (1 + 0.0822719 \sin 34.8174484°)]^{0.0822719/2}$$
$$= 34.9991687°$$

Replacing 34.8174484° with 34.9991687° for the second trial, a ϕ of 34.9999969° is obtained. Recalculation with the new ϕ results in $\phi = 35.0000006°$, which does not change to 7 decimals with a fourth trial. (This is not exactly 35°, due to rounding-off errors.) Therefore,

$$\phi = 35.0000006° \text{ N. lat.}$$

From equation (12–9),

$$= 13.2404257°/0.6304965 + (-96°)$$
$$= -75.0000013° = 75.0000013° \text{ W. long.}$$

Examples using equations (3–5) and (7–13) are omitted here, since comparable examples for these equations have been given above.

BIPOLAR OBLIQUE CONIC CONFORMAL (SPHERE)–FORWARD EQUATIONS
(SEE P. 114–117)

This example will illustrate equations (14–11) through (14–23), assuming prior calculation of the constants from equations (14–1) through (14–13).

Given: Radius of sphere: $R = 6{,}370{,}997$ m
Point: $\phi = 40°$ N. lat.
$\lambda = 90°$ W. long.

Find: x, y, k

From equation (14–14),

$$z_B = \arccos \{\sin 45° \sin 40° + \cos 45° \cos 40° \cos [(-19°59'36'') - (-90°)]\}$$
$$= 50.22875°$$

From equation (14–15),

$$Az_B = \arctan \{\sin (-19°59'36'' + 90°)/[\cos 45° \tan 40° - \sin 45° \cos (-19°59'36'' + 90°)]\}$$
$$= 69.48856°$$

Since 69.48856° is less than 104.42834°, proceed to equation (14–16).

From equations (14–16) through (14–22),

$$\rho_B = 1.89725 \times 6370997 \tan^{0.63056}(\tfrac{1}{2} \times 50.22875°)$$
$$= 7{,}496{,}100 \text{ m}$$
$$k = 7{,}496{,}100 \times 0.63056/(6370997 \sin 50.22875°)$$
$$= 0.96527$$
$$\alpha = \arccos \{[\tan^{0.63056}(\tfrac{1}{2} \times 50.22875°) + \tan^{0.63056} \tfrac{1}{2}(104° - 50.22875°)]/1.27247\}$$
$$= 1.88279°$$
$$n(Az_{BA} - Az_B) = 0.63056 \times (104.42834° - 69.48856°) = 22.03163°$$

This is greater than α, so $\rho_B' = \rho_B$.

$$x' = 7{,}496{,}100 \sin [0.63056\,(104.42834° - 69.48855°)]$$
$$= 2{,}811{,}900 \text{ m}$$
$$y' = 7{,}496{,}100 \cos [0.63056\,(104.42834° - 69.48855°)] - 1.20709 \times 6{,}370{,}997$$
$$= -741{,}670 \text{ m}$$

From equations (14–32) and (14–33),

$$x = -2{,}811{,}900 \cos 45.81997° + 741670 \sin 45.81997°$$
$$= -1{,}427{,}800 \text{ m}$$
$$y = 741{,}670 \cos 45.81997° + 2811900 \sin 45.81997°$$
$$= 2{,}533{,}500 \text{ m}$$

BIPOLAR OBLIQUE CONIC CONFORMAL (SPHERE)–INVERSE EQUATIONS (SEE P. 117–118)

Inversing the forward example:

Given: Radius of sphere: $R = 6{,}370{,}997$ m
Point: $x = -1{,}427{,}800$ m
$y = 2{,}533{,}500$ m

Find: ϕ, λ
From equations (14–34) and (14–35),

$$x' = -(-1{,}427{,}800) \cos 45.81997° + 2{,}533{,}500 \sin 45.81997°$$
$$= 2{,}811{,}900 \text{ m}$$
$$y' = -(-1{,}427{,}800) \sin 45.81997° - 2{,}533{,}500 \cos 45.81997°$$
$$= -741{,}670 \text{ m}$$

Since x' is positive, go to equations (14–36) through (14–44) in order:

$$\rho_B' = [2{,}811{,}900^2 + (1.20709 \times 6{,}370{,}997 - 741{,}670)^2]^{1/2}$$
$$= 7{,}496{,}100 \text{ m}$$

$Az'_B = \arctan [2{,}811{,}900/(1.20709 \times 6{,}370{,}997 - 741{,}670)]$
$= 22.03150°$ (The denominator is positive, so there is no quadrant correction.)
$\rho_B = 7{,}496{,}100$ m
$z_B = 2 \arctan [7{,}496{,}100/(1.89725 \times 6{,}370{,}997)]^{1/0.63056}$
$= 50.22873°$
$\alpha = \arccos \{[\tan^{0.63056} (½ \times 50.22873°) + \tan^{0.63056} ½(104° - 50.22873°)]/1.27247\}$
$= 1.88279°$

Since Az'_B is greater than α, go to equation (14–42).

$Az_B = 104.42834° - 22.03150°/0.63056$
$= 69.48876°$
$\phi = \arcsin (\sin 45° \cos 50.22873° + \cos 45° \sin 50.22873° \cos 69.48876°)$
$= 39.99987°$ or 40° N. lat., if rounding off had not accumulated errors.
$\lambda = (-19°59'36'') - \arctan \{\sin 69.48876°/[\cos 45°/\tan 50.22873° - \sin 45° \cos 69.48876°]\}$
$= -89.99987°$ or 90° W. long., if rounding off had not accumulated errors.

POLYCONIC (SPHERE)–FORWARD EQUATIONS (SEE P. 128–129)

Given: Radius of sphere: $R = 1.0$ unit
Origin: $\phi_0 = 30°$ N. lat.
$\lambda_0 = 96°$ W. long.
Point: $\phi = 40°$ N. lat.
$\lambda = 75°$ W. long.

Find: x, y, h
From equations (15–2) through (15–4),

$E = (-75° + 96°) \sin 40°$
$= 13.4985398°$
$x = 1.0 \cot 40° \sin 13.4985398°$
$= 0.2781798$ unit
$y = 1.0 \times [40° \times \pi/180° - 30° \times \pi/180° + \cot 40° (1 - \cos 13.4985398°)]$
$= 0.2074541$ unit

From equations (15–6) and (15–5),

$D = \arctan [(13.4985398° \times \pi/180° - \sin 13.4985398°)/(\sec^2 40° - \cos 13.4985398°)]$
$= 0.17018327°$
$h = (1 - \cos^2 40° \cos 13.4985398°)/\sin^2 40° \cos 0.17018327°$
$= 1.0392385$

POLYCONIC (SPHERE)–INVERSE EQUATIONS (SEE P. 129)

Inversing the forward example:

Given: Radius of sphere: $R = 1.0$ unit
Origin: $\phi_0 = 30°$ N. lat.
$\lambda_0 = 96°$ W. long.
Point: $x = 0.2781798$ unit
$y = 0.2074541$ unit

Find: ϕ, λ
Since $y \neq -1.0 \times 30° \times \pi/180°$, use equations (15–7) and (15–8):

$A = 30° \times \pi/180° + 0.2074541/1.0$
$= 0.7310529$
$B = 0.2781798^2/1.0^2 + 0.7310529^2$
$= 0.6118223$

Assuming an initial $\phi_n = A = 0.7310529$ radians, it is simplest to work with equation (15–9) in *radians*:

$\phi_{n+1} = 0.7310529 - [\mathbf{0.7310529} \times (0.7310529 \tan 0.7310529 + 1) - 0.7310529 - \tfrac{1}{2}(0.7310529^2 + 0.6118223) \tan 0.7310529]/[(0.7310529 - \mathbf{0.7310529})/\tan 0.7310529 - 1]$
$= 0.6963533$ radian

Using 0.6963533 in place of 0.7310529 (except that the boldface retains the value of A) a new ϕ_{n+1} of 0.6981266 radian is obtained. Again substituting this value, 0.6981317 radian is obtained. The fourth iteration results in the same answer to seven decimal places. Therefore,

$\phi = 0.6981317 \times 180°/\pi = 40.0000004°$ or 40° N. lat.

From equation (15–10),

$$\lambda = [\arcsin(0.2781798 \tan 40°/1.0)]/\sin 40° + (-96°)$$
$$= -75.0000014° = 75° \text{ W. long.}$$

POLYCONIC (ELLIPSOID)–FORWARD EQUATIONS (SEE P. 129–130)

Given: Clarke 1866 ellipsoid: $a = 6{,}378{,}206.4$ m
$e^2 = 0.00676866$
Origin: $\phi_0 = 30°$ N. lat.
$\lambda_0 = 96°$ W. long.
Point: $\phi = 40°$ N. lat.
$\lambda = 75°$ W. long.

Find: x, y, h

From equation (3–21),

$$M = 6{,}378{,}206.4 \times [(1 - 0.00676866/4 - 3 \times 0.00676866^2/64 - 5 \times 0.00676866^3/256) \times 40° \times \pi 180° - (3 \times 0.00676866/8 + 3 \times 0.00676866^2/32 + 45 \times 0.00676866^3/1024) \sin(2 \times 40°) + (15 \times 0.00676866^2/256 + 45 \times 0.00676866^3/1024) \sin(4 \times 40°) - (35 \times 0.00676866^3/3072) \sin(6 \times 40°)]$$
$$= 4{,}429{,}318.9 \text{ m}$$

Using 30° in place of 40°,

$$M_0 = 3{,}319{,}933.3 \text{ m}$$

From equation (4–20),

$$N = 6{,}378{,}206.4/(1 - 0.00676866 \sin^2 40°)^{1/2}$$
$$= 6{,}387{,}143.9 \text{ m}$$

From equations (15–2), (15–12), and (15–13),

$$E = (-75° + 96°) \sin 40°$$
$$= 13.4985398°$$
$$x = 6{,}387{,}143.9 \cot 40° \sin 13.4985398°$$
$$= 1{,}776{,}774.5 \text{ m}$$
$$y = 4{,}429{,}318.9 - 3{,}319{,}933.3 + 6{,}387{,}143.9 \cot 40° (1 - \cos 13.4985398°)$$
$$= 1{,}319{,}657.8 \text{ m}$$

To calculate scale factor h, from equations (15–16) and (15–15),

$$D = \arctan\{(13.4985398° \times \pi/180° - \sin 13.4985398°)/[\sec^2 40° - \cos 13.4985398° - 0.00676866 \sin^2 40°/(1 - 0.00676866 \sin^2 40°)]\}$$
$$= 0.1708380522°$$
$$h = [1 - 0.00676866 + 2(1 - 0.00676866 \sin^2 40°) \sin^2 \tfrac{1}{2}(13.4985398°)/\tan^2 40°]/(1 - 0.00676866) \cos 0.1708380522°$$
$$= 1.0393954$$

POLYCONIC (ELLIPSOID)–INVERSE EQUATIONS (SEE P. 130–131)

Inversing the forward example:

Given: Clarke 1866 ellipsoid: $a = 6{,}378{,}206.4$ m
$e^2 = 0.00676866$
Origin: $\phi_0 = 30°$ N. lat.
$\lambda_0 = 96°$ W. long.
Point: $x = 1{,}776{,}774.5$ m
$y = 1{,}319{,}657.8$ m

Find: ϕ, λ

First calculating M_0 from equation (3–21), as in the forward example,

$M_0 = 3{,}319{,}933.3$ m

Since $y \neq M_0$, from equations (15–18) and (15–19),

$$A = (3{,}319{,}933.3 + 1{,}319{,}657.8)/6{,}378{,}206.4$$
$$= 0.7274131$$
$$B = 1{,}776{,}774.5^2/6{,}378{,}206.4^2 + 0.7274131^2$$
$$= 0.6067309$$

Assuming an initial value of $\phi_n = 0.7274131$ radian, the following calculations are made in radians from equations (15–20), (3–21), (15–17), and (15–21):

$$C = (1 - 0.00676866 \sin^2 0.7274131)^{1/2} \tan 0.7274131$$
$$= 0.8889365$$
$$M_n = 4{,}615{,}626.1 \text{ m}$$
$$M'_n = 1 - 0.00676866/4 - 3 \times 0.00676866^2/64 - 5 \times 0.00676866^3/256 - 2 \times (3 \times 0.00676866/8 + 3 \times 0.00676866^2/32 + 45 \times 0.00676866^3/1024) \cos (2 \times 0.7274131) + 4 \times (15 \times 0.00676866^2/256 + 45 \times 0.00676866^3/1024) \cos (4 \times 0.7274131) - 6 \times (35 \times 0.00676866^3/3072) \cos (6 \times 0.7274131)$$
$$= 0.9977068$$
$$M_a = 4{,}615{,}626.1/6{,}378{,}206.4 = 0.7236558$$
$$\phi_{n+1} = 0.7274131 - [\mathbf{0.7274131} \times (0.8889365 \times 0.7236558 + 1) - 0.7236558 - \tfrac{1}{2}(0.7236558^2 + 0.6067309) \times 0.8889365]/ [0.00676866 \sin (2 \times 0.7274131) \times (0.7236558^2 + 0.6067309 - 2 \times \mathbf{0.7274131} \times 0.7236558)/(4 \times 0.8889365) + (\mathbf{0.7274131} - 0.7236558) \times (0.8889365 \times 0.9977068 - 2/\sin (2 \times 0.7274131)) - 0.9977068]$$
$$= 0.6967280 \text{ radian}$$

Substitution of 0.6967280 in place of 0.7274131 in equations (15–20), (3–21), (15–17), and (15–21), except for boldface values, which are A, not ϕ_n, a new ϕ_{n+1} of 0.6981286 is obtained. Using this in place of the previous value results in a third ϕ_{n+1} of 0.6981317, which is unchanged by recalculation to seven decimals. Thus,

$$\phi = 0.6981317 \times 180°/\pi = 40.0000005° = 40° \text{ N. lat.}$$

From equation (15–22), using the finally calculated C of 0.8379255,

$$\lambda = [\arcsin(1{,}776{,}774.5 \times 0.8379255/6{,}378{,}206.4)]/\sin 40° + (-96°)$$
$$= -75° = 75° \text{ W. long.}$$

ORTHOGRAPHIC (SPHERE)–FORWARD EQUATIONS (SEE P. 146–147)

Given: Radius of sphere: $R = 1.0$ unit
Center: $\phi_1 = 40°$ N. lat.
$\lambda_0 = 100°$ W. long.
Point: $\phi = 30°$ N. lat.
$\lambda = 110°$ W. long.

Find: x, y

In general calculations, to determine whether this point is beyond viewing, using equation (5–3),

$$\cos c = \sin 40° \sin 30° + \cos 40° \cos 30° \cos(-110° + 100°)$$
$$= 0.9747290$$

Since this is positive, the point is within view.

Using equations (16–3) and (16–4),

$$x = 1.0 \cos 30° \sin(-110° + 100°)$$
$$= -0.1503837$$
$$y = 1.0 [\cos 40° \sin 30° - \sin 40° \cos 30° \cos(-110° + 100°)]$$
$$= -0.1651911$$

Examples of other forward equations are omitted, since the oblique case applies generally.

ORTHOGRAPHIC (SPHERE)–INVERSE EQUATIONS (SEE P. 147)

Inversing forward example:

Given: Radius of sphere: $R = 1.0$ unit
Center: $\phi_1 = 40°$ N. lat.
$\lambda_0 = 100°$ W. long.
Point: $x = -0.1503837$ unit
$y = -0.1651911$ unit

Find: ϕ, λ

Using equations (16–18) and (16–19),

$$\rho = [(-0.1503837)^2 + (-0.1651911)^2]^{1/2}$$
$$= 0.2233906$$
$$c = \arcsin(0.2233906/1.0)$$
$$= 12.9082572°$$

Using equations (16–14) and (16–15),

$\phi = \arcsin [\cos 12.9082572° \sin 40° + (-0.1651911 \sin 12.9082572° \cos 40°/0.2233906)]$
$= 30.0000007°$, or 30° N. lat. if rounding off did not occur.
$\lambda = -100° + \arctan [-0.1503837 \sin 12.9082572°/(0.2233906 \cos 40° \cos 12.9082572° + 0.1651911 \sin 40° \sin 12.9082572°)]$
$= -100° + \arctan [-0.0335943/0.1905228]$
$= -100° + (-9.9999964°)$
$= -109.9999964°$, or 110° W. long. if rounding off did not occur.

Since the denominator of the argument of arctan is positive, no adjustment for quadrant is necessary.

STEREOGRAPHIC (SPHERE)–FORWARD EQUATIONS (SEE P. 158–159)

Given: Radius of sphere: $R = 1.0$ unit
Center: $\phi_1 = 40°$ N. lat.
$\lambda_0 = 100°$ W. long.
Central scale factor: $k_0 = 1.0$
Point: $\phi = 30°$ N. lat.
$\lambda = 75°$ W. long.

Find: x, y, k
Using equations (17–4), (17–2), and (17–3) in order,

$k = 2 \times 1.0/[1 + \sin 40° \sin 30° + \cos 40° \cos 30° \cos (-75° + 100°)]$
$= 1.0402304$
$x = 1.0 \times 1.0402304 \cos 30° \sin (-75° + 100°)$
$= 0.3807224$ unit
$y = 1.0 \times 1.0402304 [\cos 40° \sin 30° - \sin 40° \cos 30° \cos (-75° + 100°)]$
$= -0.1263802$ unit

Examples of other forward equations are omitted, since the above equations are general.

STEREOGRAPHIC (SPHERE)–INVERSE EQUATIONS (SEE P. 159)

Inversing forward example:

Given: Radius of sphere: $R = 1.0$ unit
Center: $\phi_1 = 40°$ N. lat.
$\lambda_0 = 100°$ W. long.
Central scale factor: $k_0 = 1.0$
Point: $x = 0.3807224$ unit
$y = -0.1263802$ unit

Find: ϕ, λ
Using equations (16–18) and (17–15),

$$\rho = [0.3807224^2 + (-0.1263802)^2]^{1/2} = 0.4011502 \text{ units}$$
$$c = 2 \arctan [0.4011502/(2 \times 1.0 \times 1.0)]$$
$$= 22.6832261°$$

Using equations (16–14) and (16–15),

$$\phi = \arcsin [\cos 22.6832261° \sin 40° + (-0.1263802) \sin 22.6832261° \cos 40°/0.4011502]$$
$$= \arcsin 0.5000000 = 30° = 30° \text{ N. lat.}$$
$$\lambda = -100° + \arctan [0.3807224 \sin 22.6832261°/(0.4011502 \cos 40° \cos 22.6832261° + 0.1263802 \sin 40° \sin 22.6832261°)]$$
$$= -100° + \arctan (0.1468202/0.3148570)$$
$$= -100° + 25.0000013°$$
$$= -74.9999987° = 75° \text{ W. long.}$$

except for effect of rounding-off input data. Since the denominator of the argument of arctan is positive, no quadrant adjustment is necessary. If it were negative, 180° should be added.

STEREOGRAPHIC (ELLIPSOID)–FORWARD EQUATIONS (SEE P. 160, 162–163)

Oblique aspect:

Given: Clarke 1866 ellipsoid:	$a = 6{,}378{,}206.4$ m
	$e^2 = 0.00676866$
or	$e = 0.0822719$
Center:	$\phi_1 = 40°$ N. lat.
	$\lambda_0 = 100°$ W. long.
Central scale factor:	$k_0 = 0.9999$
Point:	$\phi = 30°$ N. lat.
	$\lambda = 90°$ W. long.

Find: x, y, k
From equation (3–1),

$$\chi_1 = 2 \arctan \{\tan (45° + 40°/2) [(1 - 0.0822719 \sin 40°)/(1 + 0.0822719 \sin 40°)]^{0.0822719/2}\} - 90°$$
$$= 2 \arctan 2.1351882 - 90°$$
$$= 39.8085922°$$
$$\chi = 2 \arctan \{\tan (45° + 30°/2)[(1 - 0.0822719 \sin 30°)/(1 + 0.0822719 \sin 30°)]^{0.0822719/2}\} - 90°$$
$$= 2 \arctan 1.7261956 - 90°$$
$$= 29.8318339°$$

From equation (12–15),

$$m_1 = \cos 40°/(1 - 0.00676866 \sin^2 40°)^{1/2}$$
$$= 0.7671179$$
$$m = \cos 30°/(1 - 0.00676866 \sin^2 30°)^{1/2}$$
$$= 0.8667591$$

From equation (17–27),

$$A = 2 \times 6{,}378{,}206.4 \times 0.9999 \times 0.7671179/\{\cos 39.8085922° [1 \times \sin 39.8085922° \sin 29.8318339° + \cos 39.8085922° \cos 29.8318339° \cos(-90° + 100°)]\}$$
$$= 6{,}450{,}107.7 \text{ m}$$

From equations (17–24), (17–25), and (17–26),

$$x = 6{,}450{,}107.7 \cos 29.8318339° \sin(-90° + 100°)$$
$$= 971{,}630.8 \text{ m}$$
$$y = 6{,}450{,}107.7 [\cos 39.8085922° \sin 29.8318339° - \sin 39.8085922° \cos 29.8318339° \cos(-90° + 100°)]$$
$$= -1{,}063{,}049.3 \text{ m}$$
$$k = 6{,}450{,}107.7 \cos 29.8318339°/[6{,}378{,}206.4 \times 0.8667591]$$
$$= 1.0121248$$

Polar aspect with known k_0:

Given: International ellipsoid:	$a = 6{,}378{,}388.0$ m
	$e^2 = 0.00672267$
or	$e = 0.0819919$
Center: South Pole	$\phi_1 = 90°$ S. lat.
	$\lambda_0 = 100°$ W. long. (meridian along pos. Y axis)
Central scale factor:	$k_0 = 0.994$
Point:	$\phi = 75°$ S. lat.
	$\lambda = 150°$ E. long.

Find: x, y, k

Since this is the south polar aspect, for calculations change signs of x, y, ϕ, λ, and λ_0 (ϕ_c is not used): $\lambda_0 = 100°$ E. long., $\phi = 75°$ N. lat., $\lambda = 150°$ W. long. Using equations (13–9) and (17–33),

$$t = \tan(45° - 75°/2)/[(1 - 0.0819919 \sin 75°)/(1 + 0.0819919 \sin 75°)]^{0.0819919/2}$$
$$= 0.1325120$$
$$\rho = 2 \times 6{,}378{,}388.0 \times 0.994 \times 0.1325120/[(1 + 0.0819919)^{(1+0.0819919)} \times (1 - 0.0819919)^{(1-0.0819919)}]^{1/2}$$
$$= 1{,}674{,}638.5 \text{ m}$$

Using equations (17–30) and (17–31), changing signs of x and y for the south polar aspect,

$$x = -1{,}674{,}638.5 \sin(-150° - 100°)$$
$$= -1{,}573{,}645.4 \text{ m}$$
$$y = +1{,}674{,}638.5 \cos(-150° - 100°)$$
$$= -572{,}760.1 \text{ m}$$

From equation (12–15),

$$m = \cos 75°/(1 - 0.00672267 \sin^2 75°)^{1/2}$$
$$= 0.2596346$$

From equation (17–32),

$$k = 1{,}674{,}638.5/(6{,}378{,}388 \times 0.2596346)$$
$$= 1.0112245$$

Polar aspect with known ϕ_c not at the pole:

Given: International ellipsoid: $a = 6{,}378{,}388.0$ m
$e^2 = 0.00672267$
or $e = 0.0819919$
Standard parallel: $\phi_c = 71°$ S. lat.
$\lambda_0 = 100$ W. long. (meridian along pos. Y axis)
Point: $\phi = 75°$ S. lat.
$\lambda = 150°$ E. long.

Find: x, y, k

Since ϕ_c is southern, for calculations change signs of x, y, ϕ_c, ϕ, λ, and λ_0: $\phi_c = 71°$ N. lat., $\phi = 75°$ N. lat., $\lambda = 150°$ W. long., $\lambda_0 = 100°$ E. long. Using equation (13–9), t for 75° has been calculated in the preceding example, or

$$t = 0.1325120$$

For t_c, substitute 71° in place of 75° in (13–9), and

$$t_c = 0.1684118$$

From equation (12–15),

$$m_c = \cos 71°/(1 - 0.00672267 \sin^2 71°)^{1/2}$$
$$= 0.3265509$$

From equation (17–34),

$$\rho = 6{,}378{,}388.0 \times 0.3265509 \times 0.1325120/0.1684118$$
$$= 1{,}638{,}869.6 \text{ m}$$

Equations (17–30), (17–31), and (17–32) are used as in the preceding south polar example, changing signs of x and y.

$$x = -1{,}638{,}869.6 \sin(-150° - 100°)$$
$$= -1{,}540{,}033.6 \text{ m}$$
$$y = +1{,}638{,}869.6 \cos(-150° - 100°)$$
$$= -560{,}526.4 \text{ m}$$
$$k = 1{,}638{,}869.6/(6{,}378{,}388.0 \times 0.2596346)$$
$$= 0.9896255$$

where m is calculated in the preceding example.

STEREOGRAPHIC (ELLIPSOID)–INVERSE EQUATIONS (SEE P. 163–164)

Oblique aspect (inversing forward example):

Given: Clarke 1866 ellipsoid:	$a = 6{,}378{,}206.4$ m
	$e^2 = 0.00676866$
or	$e = 0.0822719$
Center:	$\phi_1 = 40°$ N. lat.
	$\lambda_0 = 100°$ W. long.
Center scale factor:	$k_0 = 0.9999$
Point:	$x = 971{,}630.8$ m
	$y = -1{,}063{,}049.3$ m

Find: ϕ, λ
From equation (12–15),

$$m_1 = \cos 40°/(1 - 0.00676866 \sin^2 40°)^{1/2}$$
$$= 0.7671179$$

From equation (3–1), as in the forward oblique example,

$$\chi_1 = 39.8085922°$$

From equations (16–18) and (17–38),

$$\rho = [971{,}630.8^2 + (-1{,}063{,}049.3)^2]^{1/2}$$
$$= 1{,}440{,}187.6 \text{ m}$$
$$c_e = 2 \arctan [1{,}440{,}187.6 \cos 39.8085922°/(2 \times 6{,}378{,}206.4 \times 0.9999 \times 0.7671179)]$$
$$= 12.9018251°$$

From equation (17–37),

$$\chi = \arcsin [\cos 12.9018251° \sin 39.8085922° + (-1{,}063{,}049.3 \sin 12.9018251° \cos 39.8085922°/1{,}440{,}187.6)]$$
$$= 29.8318337°$$

Using χ as the first trial ϕ in equation (3–4),

$$\phi = 2 \arctan \{\tan (45° + 29.8318337°/2) \times [(1 + 0.0822719 \sin 29.8318337°)/(1 - 0.0822719 \sin 29.8318337°)]^{0.0822719/2}\} - 90°$$
$$= 29.9991438°$$

Using this new trial value in the same equation for ϕ, not for χ,

$$\phi = 2 \arctan \{\tan (45° + 29.8318337°/2) \times [(1 + 0.0822719 \sin 29.9991438°)/(1 - 0.0822719 \sin 29.9991438°)]^{0.0822719/2}\} - 90°$$
$$= 29.9999953°$$

Repeating with 29.9999953° in place of 29.9991438°, the next trial ϕ is

$$\phi = 29.9999997°$$

The next trial calculation produces the same ϕ to seven decimals. Therefore, this is ϕ.

Using equation (17–36),

$$\lambda = -100° + \arctan [971{,}630.8 \sin 12.9018251°/(1{,}440{,}187.6 \cos 39.8085922° \cos 12.9018251° + 1{,}063{,}049.3 \sin 39.8085922° \sin 12.9018251°)]$$
$$= -100° + \arctan (216{,}946.9/1{,}230{,}366.8)$$
$$= -100° + 10.0000000°$$
$$= -90.0000000° = 90° \text{ W. long.}$$

Since the denominator of the arctan argument is positive, no quadrant adjustment is necessary. If it were negative, it would be necessary to add or subtract 180°, whichever would place the final λ between +180° and −180°.

Instead of the iterative equation (3–4), series equation (3–5) may be used:

$$\phi = 29.8318337° \times \pi/180° + (0.00676866/2 + 5 \times 0.00676866^2/24 + 0.00676866^3/12) \sin (2 \times 29.8318337°) + (7 \times 0.00676866^2/48 + 29 \times 0.00676866^3/240) \sin (4 \times 29.8318337°) + (7 \times 0.00676866^3/120) \sin (6 \times 29.8318337°)$$
$$= 0.5235988 \text{ radian}$$
$$= 29.9999997°$$

Polar aspect with known k_0 (inversing forward example):

Given: International ellipsoid: $a = 6{,}378{,}388.0$ m
$e^2 = 0.00672267$
or $e = 0.0819919$
Center: South Pole $\phi_1 = 90°$ S. lat.
$\lambda_0 = 100°$ W. long. (meridian along pos. Y axis)

Central scale factor: $k_0 = 0.994$
Point: $x = -1{,}573{,}645.4$ m
$y = -572{,}760.1$ m

Find: ϕ, λ

Since this is the south polar aspect, change signs as stated in text: For calculation, use $\phi_c = 90°$, $\lambda_0 = 100°$ E. long., $x = 1{,}573{,}645.4$ m, $y = 572{,}760.1$ m. From equations (16–18) and (17–39),

$$\rho = (1{,}573{,}645.4^2 + 572{,}760.1^2)^{1/2}$$
$$= 1{,}674{,}638.5 \text{ m}$$
$$t = 1{,}674{,}638.5 \times [(1 + 0.0819919)^{(1+0.0819919)}$$
$$(1 - 0.0819919)^{(1-0.0819919)}]^{1/2}/(2 \times 6{,}378{,}388.0 \times 0.994)$$
$$= 0.1325120$$

To iterate with equation (7–9), use as the first trial ϕ,

$$\phi = 90° - 2 \arctan 0.1325120$$
$$= 74.9031975°$$

Substituting in (7–9),

$$\phi = 90° - 2 \arctan \{0.1325120 \times [(1 - 0.0819919 \sin 74.9031975°)/$$
$$(1 + 0.0819919 \sin 74.9031975°)]^{0.0819919/2}\}$$
$$= 74.9999546°$$

Using this second trial ϕ in the same equation instead of 74.9031975°,

$$\phi = 74.9999986°.$$

The third trial gives the same value to seven places, so, since the sign of ϕ must be reversed for the south polar aspect,

$$\phi = -74.9999986°, = 75° \text{ S. lat. (disregarding effects of rounding off).}$$

If the series equation (3–5) is used instead of (7–9), χ is first found from (7–13):

$$\chi = 90° - 2 \arctan 0.1325120$$
$$= 74.9031975°$$

Substituting this into (3–5), after converting χ to radians for the first term, ϕ is found in radians and is converted to degrees, then given a reversal of sign for the south polar aspect, giving the same result as the iteration.

From equation (16–16),

$$\lambda = +100° + \arctan [1{,}573{,}645.4/(-572{,}760.1)]$$
$$= 100° + (-69.9999995°)$$
$$= 30.0000005°$$

However, since the denominator of the argument of arctan is negative, 180° must be added to λ (added, not substracted, since the numerator is positive), *then* the sign of λ must be changed for the south polar aspect:

$$\lambda = -(30.0000005° + 180°)$$
$$= -210.0000005°$$

To place this between +180° and −180°, add 360°, so

$\lambda = +149.9999995°$ or 150° E. long., disregarding effects of rounding off.

Polar aspect with known ϕ_c not at the pole (inversing forward example):

Given: International ellipsoid:	$a = 6{,}378{,}388.0$ m
	$e^2 = 0.00672267$
or	$e = 0.0819919$
Standard parallel:	$\phi_c = 71°$ S. lat.
	$\lambda_0 = 100°$ W. long. (meridian along pos. Y axis)
Point:	$x = -1{,}540{,}033.6$ m
	$y = -560{,}526.4$ m

Find: ϕ, λ

Since this is south polar, change signs as stated in text: For calculation, $\phi_c = 71°$ N. lat., $\lambda_0 = 100°$ E. long., $x = 1{,}540{,}033.6$, $y = 560{,}526.4$. From equations (13–9) and (12–15), as calculated in the corresponding forward example,

$$t_c = \tan(45° - 71°/2)/[(1 - 0.0819919 \sin 71°)/(1 + 0.0819919 \sin 71°)]^{0.0819919/2}$$
$$= 0.1684118$$
$$m_c = \cos 71°/(1 - 0.00672267 \sin^2 71°)^{1/2}$$
$$= 0.3265509$$

From equations (16–18) and (17–40),

$$\rho = (1{,}540{,}033.6^2 + 560{,}526.4^2)^{1/2}$$
$$= 1{,}638{,}869.5 \text{ m}$$
$$t = 1{,}638{,}869.5 \times 0.1684118/(6{,}378{,}388.0 \times 0.3265509)$$
$$= 0.1325120$$

For the first trial ϕ in equation (7–9),

$$\phi = 90° - 2 \arctan 0.1325120$$
$$= 74.903197°$$

Substituting in (7–9),

$$\phi = 90° - 2 \arctan \{0.1325120 [(1 - 0.0819919 \sin 74.903197°)/(1 + 0.0819919 \sin 74.903197°)]^{0.0819919/2}\}$$
$$= 74.9999586°$$

Replacing 74.903197° with 74.9999586°, the next trial ϕ is

$$\phi = 75.0000026°$$

The next iteration results in the same ϕ to seven places, so changing signs,

$\phi = -75.0000026° = 75°$ S. lat., disregarding effects of rounding off.

The use of series equation (3–5) with (7–13) to avoid iteration follows the same procedure as the preceding example. For λ, equation (16–16) is used, calculating with reversed signs:

$$\lambda = +100° + \arctan [1{,}540{,}033.6/(-560{,}526.4)]$$
$$= 100° + (-69.9999997°)$$
$$= 30.0000003°$$

Since the denominator in the argument for arctan is negative, add 180°:

$$\lambda = 210.0000003°$$

Now subtract 360° to place λ between +180° and −180°:

$$\lambda = -149.9999997°$$

Finally, reverse the sign to account for the south polar aspect:

$\lambda = +149.9999997° = 150°$ E. long., disregarding rounding off in the input.

LAMBERT AZIMUTHAL EQUAL-AREA (SPHERE)–FORWARD EQUATIONS (SEE P. 170, 172–173)

Given: Radius of sphere: $R = 3.0$ units
Center: $\phi_1 = 40°$ N. lat.
$\lambda_0 = 100°$ W. long.
Point: $\phi = 20°$ S. lat.
$\lambda = 100°$ E. long.

Find: x, y
Using equation (18–4),

$$k' = \{2/[1 + \sin 40° \sin(-20°) + \cos 40° \cos(-20°) \cos(100° + 100°)]\}^{1/2}$$
$$= 4.3912175$$

Using equations (18–2) and (18–3),

$$x = 3.0 \times 4.3912175 \cos(-20°) \sin(100° + 100°)$$
$$= -4.2339303 \text{ units}$$
$$y = 3.0 \times 4.3912175 [\cos 40° \sin(-20°) - \sin 40° \cos(-20°) \cos(100° + 100°)]$$
$$= 4.0257775 \text{ units}$$

Examples for the polar and equatorial reductions, equations (18–5) through (18–16), are omitted, since the above general equations give the same results.

LAMBERT AZIMUTHAL EQUAL-AREA (SPHERE)–INVERSE EQUATIONS (SEE P. 173)

Inversing forward example:

Given: Radius of sphere: $R = 3.0$ units
Center: $\phi_1 = 40°$ N. lat.
$\lambda_0 = 100°$ W. long.
Point: $x = -4.2339303$ units
$y = 4.0257775$ units

Find: ϕ, λ
Using equations (16–18) and (18–18),

$$\rho = [(-4.2339303)^2 + 4.0257775^2]^{1/2}$$
$$= 5.8423497 \text{ units}$$
$$c = 2 \arcsin[5.8423497/(2 \times 3.0)]$$
$$= 153.6733917°$$

From equation (16–14),

$$\phi = \arcsin[\cos 153.6733917° \sin 40° + 4.0257775 \sin 153.6733917° \cos 40°/5.8423497]$$
$$= -19.9999993° = 20° \text{ S. lat., disregarding rounding-off effects.}$$

From equation (16–15),

$$\lambda = -100° + \arctan[-4.2339303 \sin 153.6733917°/(5.8423497 \cos 40° \cos 153.6733917° - 4.0257775 \sin 40° \sin 153.6733917°)]$$
$$= -100° + \arctan[-1.8776951/(-5.1589246)]$$
$$= -100° + 20.0000005°$$
$$= -79.9999995°$$

Since the denominator of the argument of arctan is negative, add 180°:

$\lambda = 100.0000005° = 100°$ E. long., disregarding rounding-off effects.

In polar spherical cases, the calculation of λ from equations (16–16) or (16–17) is simpler than the above, but the quadrant adjustment follows the same rules.

LAMBERT AZIMUTHAL EQUAL-AREA (ELLIPSOID)–FORWARD EQUATIONS
(SEE P. 173–175)

Oblique aspect:

Given: Clarke 1866 ellipsoid: $a = 6{,}378{,}206.4$ m
$e^2 = 0.00676866$
or $e = 0.0822719$
Center: $\phi_1 = 40°$ N. lat.
$\lambda_0 = 100°$ W. long.
Point: $\phi = 30°$ N. lat.
$\lambda = 110°$ W. long.

Find: x, y
Using equation (3–12),

$$q = (1 - 0.00676866) \{\sin 30°/(1 - 0.00676866 \sin^2 30°) - [1/(2 \times 0.0822719)] \ln [(1 - 0.0822719 \sin 30°)/(1 + 0.0822719 \sin 30°)]\}$$
$$= 0.9943535$$

Inserting $\phi_1 = 40°$ in place of 30° in the same equation,

$$q_1 = 1.2792602$$

Inserting 90° in place of 30°,

$$q_p = 1.9954814$$

Using equation (3–11),

$$\beta = \arcsin (0.9943535/1.9954814)$$
$$= 29.8877622°$$
$$\beta_1 = \arcsin (1.2792602/1.9954814)$$
$$= 39.8722878°$$

Using equation (3–13),

$$R_q = 6{,}378{,}206.4 \times (1.9954814/2)^{1/2}$$
$$= 6{,}370{,}997.2 \text{ m}$$

Using equation (12–15),

$$m_1 = \cos 40°/(1 - 0.00676866 \sin^2 40°)^{1/2}$$
$$= 0.7671179$$

Using equations (18–21) and (18–22),

$$B = 6{,}370{,}997.2 \times \{2/[1 + \sin 39.8722878° \sin 29.8877622° + \cos 39.8722878° \cos 29.8877622° \cos(-110° + 100°)]\}^{1/2}$$
$$= 6{,}411{,}606.1 \text{ m}$$
$$D = 6{,}378{,}206.4 \times 0.7671179/(6{,}370{,}997.2 \cos 39.8722878°)$$
$$= 1.0006653$$

Using equations (18–19) and (18–20),

$$x = 6{,}411{,}606.1 \times 1.0006653 \cos 29.8877622° \sin(-110° + 100°)$$
$$= -965{,}932.1 \text{ m}$$
$$y = (6{,}411{,}606.1/1.0006653)[\cos 39.8722878° \sin 29.8877622° - \sin 39.8722878° \cos 29.8877622° \cos(-110° + 100°)]$$
$$= -1{,}056{,}814.9 \text{ m}$$

Polar aspect:

Given: International ellipsoid:	$a = 6{,}378{,}388.0$ m
	$e^2 = 0.00672267$
or	$e = 0.0819919$
Center: North Pole	$\phi_1 = 90°$ N. lat.
	$\lambda_0 = 100°$ W. long. (meridian along neg. Y axis)
Point:	$\phi = 80°$ N. lat.
	$\lambda = 5°$ E. long.

Find: ϕ, λ, h, k

From equation (3–12),

$$q = (1 - 0.00672267)\{\sin 80°/(1 - 0.00672267 \sin^2 80°) - [1/(2 \times 0.0819919)] \ln[(1 - 0.0819919 \sin 80°)/(1 + 0.0819919 \sin 80°)]\}$$
$$= 1.9649283$$

Using the same equation with 90° in place of 80°,

$$q_p = 1.9955122$$

From equation (12–15),

$$m = \cos 80°/(1 - 0.00672267 \sin^2 80°)^{1/2}$$
$$= 0.1742171$$

Using equations (18–25), (17–30), (17–31), and (17–32),

$$\rho = 6{,}378{,}388.0 \times (1.9955122 - 1.9649283)^{1/2}$$
$$= 1{,}115{,}468.3 \text{ m}$$
$$x = 1{,}115{,}468.3 \sin(5° + 100°)$$
$$= 1{,}077{,}459.7 \text{ m}$$

$y = -1{,}115{,}468.3 \cos (5° + 100°)$
$= 288{,}704.5$ m
$k = 1{,}115{,}468.3/(6{,}378{,}388.0 \times 0.1742171)$
$= 1.0038193$
$h = 1/1.0038193 = 0.9961952$

LAMBERT AZIMUTHAL EQUAL-AREA (ELLIPSOID)–INVERSE EQUATIONS
(SEE P. 175–177)

Oblique aspect (inversing forward example):

Given: Clarke 1866 ellipsoid: $a = 6{,}378{,}206.4$ m
$e^2 = 0.00676866$
or $e = 0.0822719$
Center: $\phi_1 = 40°$ N. lat.
$\lambda_0 = 100°$ W. long.
Point: $x = -965{,}932.1$ m
$y = -1{,}056{,}814.9$ m

Find: ϕ, λ
Since these are the same map parameters as those used in the forward example, calculations of map constants not affected by ϕ and λ are not repeated here.

$q_p = 1.9954814$
$\beta_1 = 39.8722878°$
$R_q = 6{,}370{,}997.2$ m
$D = 1.0006653$

Using equations (18–30), (18–31), and (18–29),

$\rho = \{[-965{,}932.1/1.0006653]^2 + [1.0006653 \times (-1{,}056{,}814.9)]^2\}^{1/2}$
$= 1{,}431{,}827.1$ m
$c_e = 2 \arcsin [1{,}431{,}827.1/(2 \times 6{,}370{,}997.2)]$
$= 12.9039908°$
$q = 1.9954814 [\cos 12.9039908° \sin 39.8722878°$
$+ 1.0006653 \times (-1{,}056{,}814.9) \sin 12.9039908°$
$\cos 39.8722878°/1{,}431{,}827.1]$
$= 0.9943535$

For the first trial ϕ in equation (3–16),

$\phi = \arcsin (0.9943535/2)$
$= 29.8133914°$

Substituting into equation (3–16),

$$\begin{aligned}\phi &= 29.8133914° + [(1-0.00676866 \sin^2 29.8133914°)^2/\\ &\quad (2 \cos 29.8133914°)] \times \{0.9943535/(1-0.00676866)\\ &\quad - \sin 29.8133914°/(1-0.00676866 \sin^2 29.8133914)\\ &\quad + [1/(2\times 0.0822719)] \ln [(1-0.0822719\\ &\quad \sin 29.8133914°)/(1+0.0822719 \sin 29.8133914°)]\} \times 180°/\pi\\ &= 29.9998293°\end{aligned}$$

Substituting 29.9998293° in place of 29.8133914° in the same equation, the new trial ϕ is found to be

$$\phi = 30.0000002°$$

The next iteration results in no change to seven decimal places; therefore,

$$\phi = 30° \text{ N. lat.}$$

Using equation (18–28),

$$\begin{aligned}\lambda &= -100° + \arctan \{-965{,}932.1 \sin 12.9039908°/[1.0006653\\ &\quad \times 1{,}431{,}827.1 \cos 39.8722878° \cos 12.9039908°\\ &\quad - 1.0006653^2 (-1{,}056{,}814.9) \sin 39.8722878°\\ &\quad \sin 12.9039908°]\}\\ &= -100° + \arctan (-215{,}710.0/1{,}223{,}352.4)\\ &= -100° - 9.9999999°\\ &= -109.9999999° = 110° \text{ W. long.}\end{aligned}$$

Since the denominator of the argument for arctan is positive, no quadrant adjustment is necessary.

Polar aspect (inversing forward example):

Given: International ellipsoid:	$a = 6{,}378{,}388.0$ m
	$e^2 = 0.00672267$
or	$e = 0.0819919$
Center: North Pole	$\phi_1 = 90°$ N. lat.
	$\lambda_0 = 100°$ W. long. (meridian along neg. Y axis)
Point:	$x = 1{,}077{,}459.7$ m
	$y = 288{,}704.5$ m

Find: ϕ, λ

First q_p is found to be 1.9955122 from equation (3–12), as in the corresponding forward example for the polar aspect. From equations (16–18) and (16–33),

$$\begin{aligned}\rho &= (1{,}077{,}459.7^2 + 288{,}704.5^2)^{1/2}\\ &= 1{,}115{,}468.4 \text{ m}\end{aligned}$$

$$q = +[1.9955122 - (1{,}115{,}468.4/6{,}378{,}388.0)^2]$$
$$= 1.9649283$$

Iterative equation (3–16) may be used to find ϕ. The first trial ϕ is

$$\phi = \arcsin\,(1.9649283/2)$$
$$= 79.2542275°$$

When this is used in equation (3–16) as in the oblique inverse example, the next trial ϕ is found to be

$$\phi = 79.9744304°$$

Using this value instead, the next trial is

$$\phi = 79.9999713°$$

and the next,

$$\phi = 80.0000036°$$

The next value is the same, so

$$\phi = 80° \text{ N. lat.}$$

From equation (16–16),

$$\lambda = -100° + \arctan\,[1{,}077{,}459.7/(-288{,}704.5)]$$
$$= -174.9999978°$$

Since the denominator of the argument for arctan is negative, add 180°, or

$$\lambda = 5.0000022° = 5° \text{ E. long.}$$

AZIMUTHAL EQUIDISTANT (SPHERE)–FORWARD EQUATIONS
(SEE P. 184–185)

Given: Radius of sphere: $R = 3.0$ units
Center: $\phi_1 = 40°$ N. lat.
$\lambda_0 = 100°$ W. long.
Point: $\phi = 20°$ S. lat.
$\lambda = 100°$ E. long.

Find: x, y
Using equations (5–3) and (19–2),

$$\cos c = \sin 40° \sin(-20°) + \cos 40° \cos(-20°) \cos(100° + 100°)$$
$$= -0.8962806$$
$$c = 153.6733925°$$
$$k' = (153.6733925° \times \pi/180°)/\sin 153.6733925°$$
$$= 6.0477621$$

Using equations (18–2) and (18–3),

$$x = 3.0 \times 6.0477621 \cos(-20°) \sin(100° + 100°)$$
$$= -5.8311398 \text{ units}$$
$$y = 3.0 \times 6.0477621 [\cos 40° \sin(-20°) - \sin 40° \cos(-20°) \cos(100° + 100°)]$$
$$= 5.5444634 \text{ units}$$

Since the above equations are general, examples of other forward formulas are not given.

AZIMUTHAL EQUIDISTANT (SPHERE)–INVERSE EQUATIONS (SEE P. 185)

Inversing forward example:

Given: Radius of sphere: $R = 3.0$ units
Center: $\phi_1 = 40°$ N. lat.
$\lambda_0 = 100°$ W. long.
Point: $x = -5.8311398$ units
$y = 5.5444634$ units

Find: ϕ, λ

Using equations (16–18) and (19–15),

$$\rho = [(-5.8311398)^2 + 5.5444634^2]^{1/2}$$
$$= 8.0463200 \text{ units}$$
$$c = 8.0463200/3.0$$
$$= 2.6821067 \text{ radians}$$
$$= 2.6821067 \times 180°/\pi = 153.6733925°$$

Using equation (16–14),

$$\phi = \arcsin(\cos 153.6733925° \sin 40° + 5.5444634 \sin 153.6733925° \cos 40°/8.0463200)$$
$$= -19.9999999°$$
$$= 20° \text{ S. lat., disregarding effects of rounding off.}$$

Using equation (16–15),

$$\lambda = -100° + \arctan[(-5.8311398) \sin 153.6733925°/(8.0463200 \cos 40° \cos 153.6733925° - 5.5444634 \sin 40° \sin 153.6733925°)]$$
$$= -100° + \arctan[(-2.5860374)/(-7.1050794)]$$
$$= -100° - \arctan 0.3639702$$
$$= -80.0000001°$$

but since the denominator of the argument of arctan is negative, add or subtract 180°, whichever places the final result between +180° and −180°:

$\lambda = -80.0000001° + 180°$
$= 99.9999999°$
$= 100°$ E. long., disregarding effects of rounding off.

AZIMUTHAL EQUIDISTANT (ELLIPSOID)–FORWARD EQUATIONS
(SEE P. 185–189)

Polar aspect:

Given: International ellipsoid: $a = 6{,}378{,}388.0$ m
$e^2 = 0.00672267$
Center: North Pole $\phi_1 = 90°$ N. lat.
$\lambda_0 = 100°$ W. long. (meridian along neg. Y axis)
Point: $\phi = 80°$ N. lat.
$\lambda = 5°$ E. long.

Find: x, y, k
Using equation (3–21),

$$M = 6{,}378{,}388.0 \times [(1 - 0.00672267/4 - 3 \times 0.00672267^2/64 - 5 \times 0.00672267^3/256) \times 80° \times \pi/180° - (3 \times 0.00672267/8 + 3 \times 0.00672267^2/32 + 45 \times 0.00672267^3/1024) \sin(2 \times 80°) + (15 \times 0.00672267^2/256 + 45 \times 0.00672267^3/1024) \sin(4 \times 80°) - (35 \times 0.00672267^3/3072) \sin(6 \times 80°)]$$
$$= 8{,}885{,}403.1 \text{ m}$$

Using the same equation (3–21), but with 90° in place of 80°,

$$M_p = 10{,}002{,}288.3 \text{ m}$$

Using equation (12–15),

$$m = \cos 80°/(1 - 0.00672267 \sin^2 80°)^{1/2}$$
$$= 0.1742171$$

Using equations (19–16), (17–30), (17–31), and (17–32),

$$\rho = 10{,}002{,}288.3 - 8{,}885{,}403.1$$
$$= 1{,}116{,}885.2 \text{ m}$$
$$x = 1{,}116{,}885.2 \sin(5° + 100°)$$
$$= 1{,}078{,}828.3 \text{ m}$$
$$y = -1{,}116{,}885.2 \cos(5° + 100°)$$
$$= 289{,}071.2 \text{ m}$$
$$k = 1{,}116{,}885.2/(6{,}378{,}388.0 \times 0.1742171)$$
$$= 1.0050946$$

Oblique aspect (Guam projection):

Given: Clarke 1866 ellipsoid: $a = 6{,}378{,}206.4$ m
$e^2 = 0.00676866$
Center: $\phi_1 = 13°28'20.87887''$ N. lat.
$\lambda_0 = 144°44'55.50254''$ E. long.
False origin: $x_0 = 50{,}000$ m
$y_0 = 50{,}000$ m
Point: $\phi = 13°20'20.53846''$ N. lat.
$\lambda = 144°38'07.19265''$ E. long.

Find: x, y

Using equation (19–18), after converting angles to degrees and decimals: ($\phi_1 = 13.472466353°$, $\lambda_0 = 144.748750706°$, $\phi = 13.339038461°$, $\lambda = 144.635331292°$),

$$x = [6{,}378{,}206.4 \times (144.635331292° - 144.748750706°) \cos 13.339038461° / (1 - 0.00676866 \sin^2 13.339038461°)^{1/2}] \times \pi/180°$$
$$= -12{,}287.52 \text{ m}$$

Since 50,000 m is added to the origin for the Guam projection,

$$x = -12{,}287.52 + 50{,}000.0$$
$$= 37{,}712.48 \text{ m}$$

From equation (3–21),

$$M = 6{,}378{,}206.4 \times [(1 - 0.00676866/4 - 3 \times 0.00676866^2/64 - 5 \times 0.00676866^3/256) \times 13.339038461° \times \pi/180° - (3 \times 0.00676866/8 + 3 \times 0.00676866^2/32 + 45 \times 0.00676866^3/1024) \sin(2 \times 13.339038461°) + (15 \times 0.00676866^2/256 + 45 \times 0.00676866^3/1024) \sin(4 \times 13.339038461°) - (35 \times 0.00676866^3/3072) \sin(6 \times 13.339038461°)]$$
$$= 1{,}475{,}127.96 \text{ m}$$

Substituting $\phi_1 = 13.472466353°$ in place of 13.339038461° in the same equation,

$$M_1 = 1{,}489{,}888.76 \text{ m}$$

Using equation (19–19), and using the x without false origin,

$$y = 1{,}475{,}127.96 - 1{,}489{,}888.76 + (-12{,}287.52)^2 \tan 13.339038461° \times (1 - 0.00676866 \sin^2 13.339038461°)^{1/2} / (2 \times 6{,}378{,}206.4)$$
$$= -14{,}758.00 \text{ m}$$

Adding 50,000 meters for the false origin,
$$y = 35{,}242.00 \text{ m}$$

Oblique aspect (Micronesia form):

Given: Clarke 1866 ellipsoid: $a = 6{,}378{,}206.4$ m
$e^2 = 0.00676866$
Center: Saipan Island $\phi_1 = 15°11'05.6830''$ N. lat.
$\lambda_0 = 145°44'29.9720''$ E. long.
False origin: $x_0 = 28{,}657.52$ m
$y_0 = 67{,}199.99$ m
Point: Station Petosukara $\phi = 15°14'47.4930''$ N. lat.
$\lambda = 145°47'34.9080''$ E. long.

Find: x, y

First convert angles to degrees and decimals:

$\phi_1 = 15.18491194°$
$\lambda_0 = 145.7416589°$
$\phi = 15.24652583°$
$\lambda = 145.7930300°$

From equations (4–20a), (4–20), (19–20), and (19–21) in order,

$$N_1 = 6{,}378{,}206.4/(1 - 0.00676866 \times \sin^2 15.18491194°)^{1/2}$$
$$= 6{,}379{,}687.9 \text{ m}$$
$$N = 6{,}378{,}206.4/(1 - 0.00676866 \times \sin^2 15.24652583°)^{1/2}$$
$$= 6{,}379{,}699.7 \text{ m}$$
$$\psi = \arctan [(1 - 0.00676866) \tan 15.24652583° + 0.00676866 \times 6379687.9 \sin 15.18491194°/(6{,}379{,}699.7 \times \cos 15.24652583°)]$$
$$= 15.2461374°$$

$$Az = \arctan \{\sin (145.79303° - 145.7416589°)/[\cos 15.18491194° \times \tan 15.2461374° - \sin 15.18491194° \times \cos (145.79303° - 145.7416589°)]\}$$
$$= 38.9881345°$$

Since $\sin Az \neq 0$, from equation (19–22a),

$$s = \arcsin [\sin (145.79303° - 145.7416589°) \times \cos 15.2461374°/\sin 38.9881345°]$$
$= 0.001374913$ radians, since s is used only in radians.

From equations (19–23) through (19–27) in order,

$$G = 0.00676866^{1/2} \sin 15.18491194°/(1 - 0.00676866)^{1/2}$$
$$= 0.02162319$$
$$H = 0.00676866^{1/2} \cos 15.18491194° \cos 38.9881345°/(1 - 0.00676866)^{1/2}$$
$$= 0.06192519$$

$$c = 6{,}379{,}687.9 \times 0.001374913 \times \{1 - 0.001374913^2 \times 0.06192519^2 \times (1 - 0.06192519^2)/6 + (0.001374913^3/8) \times 0.02162319 \times 0.06192519 \times (1 - 2 \times 0.06192519^2) + (0.001374913^4/120) \times [0.06192519^2 \times (4 - 7 \times 0.06192519^2) - 3 \times 0.02162319^2 \times (1 - 7 \times 0.06192519^2)] - (0.001374913^5/48) \times 0.02162319 \times 0.06192519\}$$
$$= 8{,}771.52 \text{ m}$$
$$x = 8{,}771.52 \times \sin 38.9881345° + 28{,}657.52$$
$$= 34{,}176.20 \text{ m}$$
$$y = 8{,}771.52 \times \cos 38.9881345° + 67{,}199.99$$
$$= 74{,}017.88 \text{ m}$$

AZIMUTHAL EQUIDISTANT (ELLIPSOID)–INVERSE EQUATIONS (SEE P. 189–192)

Polar aspect (inversing forward example):

Given: International ellipsoid: $a = 6{,}378{,}388.0$ m
$e^2 = 0.00672267$
Center: North Pole $\phi_1 = 90°$ N. lat.
$\lambda_0 = 100°$ W. long. (meridian along neg. Y axis)
Point: $x = 1{,}078{,}828.3$ m
$y = 289{,}071.2$ m

Find: ϕ, λ
Using equation (3–21), as in the corresponding forward example,

$$M_p = 10{,}002{,}288.3 \text{ m}$$

Using equations (16–18), (19–28), and (8–19),

$$\rho = (1{,}078{,}828.3^2 + 289{,}071.2^2)^{1/2}$$
$$= 1{,}116{,}885.2 \text{ m}$$
$$M = 10{,}002{,}288.3 - 1{,}116{,}885.2$$
$$= 8{,}885{,}403.1 \text{ m}$$
$$\mu = 8{,}885{,}403.1/[6{,}378{,}388.0 \times (1 - 0.00672267/4 - 3 \times 0.00672267^2/64 - 5 \times 0.00672267^3/256)]$$
$$= 1.3953965 \text{ radians}$$
$$= 1.3953965 \times 180°/\pi = 79.9503324°$$

Using equations (3–24) and (3–26),

$$e_1 = [1 - (1 - 0.00672267)^{1/2}]/[1 + (1 - 0.00672267)^{1/2}]$$
$$= 0.0016863$$
$$\phi = 1.3953965 \text{ radians} + (3 \times 0.0016863/2 - 27 \times 0.0016863^3/32) \sin (2 \times 79.9503324°) + (21 \times 0.0016863^2/16 - 55 \times 0.0016863^4/32) \sin (4 \times 79.9503324°) + (151 \times 0.0016863^3/96) \sin (6 \times 79.9503324°)$$

$= 1.3962634$ radians
$= 1.3962634 \times 180°/\pi = 79.9999999°$
$= 80°$ N. lat., rounding off.

Using equation (16–16),

$$\lambda = -100° + \arctan [1{,}078{,}828.3/(-289{,}071.2)]$$
$$= -100° - 74.9999986° + 180°$$
$$= 5.0000014°$$
$$= 5° \text{ E. long., rounding off.}$$

The 180° is added because the denominator in the argument for arctan is negative.

Oblique aspect (Guam projection, inversing forward example):

Given: Clarke 1866 ellipsoid:	$a = 6{,}378{,}206.4$ m
	$e^2 = 0.00676866$
Center:	$\phi_1 = 13.472466353°$ N. lat.
	$\lambda_0 = 144.748750706°$ E. long.
False origin:	$x_0 = 50{,}000$ m
	$y_0 = 50{,}000$ m
Point:	$x = 37{,}712.48$ m
	$y = 35{,}242.00$ m

Find: ϕ, λ
First subtract 50,000 m from x and y to relate them to actual projection origin: $x = -12{,}287.52$ m, $y = -14{,}758.00$ m. Calculation of M_1 from equation (3–21) is exactly the same as in the forward example, or

$$M_1 = 1{,}489{,}888.76 \text{ m}$$

From equation (19–30), the first trial M is found from an assumed $\phi = \phi_1$:

$$M = 1{,}489{,}888.76 + (-14{,}758.00) - (-12{,}287.52)^2 \tan 13.472466353° \times (1 - 0.00676866 \sin^2 13.472466353°)^{1/2}/(2 \times 6{,}378{,}206.4)$$
$$= 1{,}475{,}127.92 \text{ m}$$

Using equation (8–19) and the above trial M,

$$\mu = 1{,}475{,}127.92/[6{,}378{,}206.4 \times (1 - 0.00676866/4 - 3 \times 0.00676866^2/64 - 5 \times 0.00676866^3/256)]$$
$$= 0.2316688 \text{ radian}$$

Using equation (3–24),

$$e_1 = [1 - (1 - 0.00676866)^{1/2}]/[1 + (1 - 0.00676866)^{1/2}]$$
$$= 0.0016979$$

Using equation (3–26) in *radians*, although it could be converted to degrees,

$$\phi = 0.2316688 + (3\times 0.0016979/2 - 27\times 0.0016979^3/32) \sin(2\times 0.2316688) + (21\times 0.0016979^2/16 - 55 \times 0.0016979^4/32) \sin(4\times 0.2316688) + (151 \times 0.0016979^3/96) \sin(6\times 0.2316688)$$
$$= 0.2328101 \text{ radian}$$
$$= 0.2328101\times 180°/\pi = 13.3390381°$$

If this new trial value of ϕ is used in place of ϕ_1 in equation (19–30), a new value of M is found:

$$M = 1{,}475{,}127.95 \text{ m}$$

This in turn, used in (8–19), gives

$$\mu = 0.2316688 \text{ radian}$$

and from (3–26)

$$\phi = 13.3390384°$$

The third trial, through the above equations and starting with this value of ϕ, produces no change to seven decimal places. Thus, this is the final value of ϕ. Converting to degrees, minutes, and seconds,

$$\phi = 13°20'20.538'' \text{ N. lat.}$$

Using equation (19–31) for longitude,

$$\lambda = 144.748750706° + [(-12{,}287.52)\times(1 - 0.00676866 \sin^2 13.3390384°)^{1/2}/(6{,}378{,}206.4 \cos 13.3390384°)]\times 180°/\pi$$
$$= 144.6353313°$$
$$= 144°38'07.193'' \text{ E. long.}$$

Oblique aspect (Micronesia form, inversing forward example):

Given: Clarke 1866 ellipsoid:	$a = 6{,}378{,}206.4$ m
	$e^2 = 0.00676866$
Center: Saipan Island	$\phi_1 = 15.18491194°$ N. lat.
	$\lambda_0 = 145.7416589°$ E. long.
False origin:	$x_0 = 28{,}657.52$ m
	$y_0 = 67{,}199.99$ m
Point:	$x = 34{,}176.20$ m
	$y = 74{,}017.88$ m

Find: ϕ, λ

From equations (19–32) through (19–41) in order,

$$c = [(34{,}176.20 - 28{,}657.52)^2 + (74{,}017.88 - 67{,}199.99)^2]^{1/2}$$
$$= 8{,}771.51 \text{ m}$$

$Az = \arctan\,[(34{,}176.20 - 28{,}657.52)/(74{,}017.88 - 67{,}199.99)]$
$= 38.9881292°$
$N_1 = 6{,}378{,}206.4/(1 - 0.00676866 \sin^2 15.18491194°)^{1/2}$
$= 6{,}379{,}687.9$ m
$A = -0.00676866 \cos^2 15.18491194° \cos^2 38.9881292°/ (1 - 0.00676866)$
$= -0.003834730$
$B = 3 \times 0.00676866 \times (1 + 0.003834730) \sin 15.18491194° \cos 15.18491194° \times \cos 38.9881292°/(1 - 0.00676866)$
$= 0.004032465$
$D = 8{,}771.51/6{,}379{,}687.9$
$= 0.001374913$
$E = 0.001374913 + 0.003834730 \times (1 - 0.003834730) \times 0.001374913^3/6 - 0.004032465 \times (1 - 3 \times 0.003834730) \times 0.001374913^4/24$
$= 0.001374913$. This is in radians for use in equation (19–38).

For use as degrees in equations (19–39) and (19–40),

$E = 0.001374913 \times 180°/\pi = 0.07877669°$
$F = 1 + 0.003834730 \times 0.001374913^2/2 - 0.004032465 \times 0.001374913^3/6$
$= 1.000000004$
$\psi = \arcsin\,(\sin 15.18491194° \cos 0.07877669° + \cos 15.18491194° \times \sin 0.07877669° \cos 38.9881292°)$
$= 15.2461374°$
$\lambda = 145.7416589° + \arcsin\,(\sin 38.9881292° \sin 0.07877669°/ \cos 15.2461374°)$
$= 145.7416589° + 0.0513711°$
$= 145.7930300°$
$= 145°47'34.908''$ E. long.
$\phi = \arctan\,[(1 - 0.00676866 \times 1.000000004 \sin 15.18491194°/ \sin 15.2461374°) \times \tan 15.2461374°/(1 - 0.00676866)]$
$= 15.2465258°$
$= 15°14'47.493''$ N. lat.

SPACE OBLIQUE MERCATOR (SPHERE)–FORWARD EQUATIONS
(SEE P. 198–200)

Given: Radius of sphere: $R = 6{,}370{,}997.0$ m
Landsat orbit: $i = 99.092°$
$P_2/P_1 = 18/251$
Path $= 15$
Point: $\phi = 40°$ N. lat.
$\lambda = 73°$ W. long.

Find: x, y for point taken during daylight northern (first) quadrant of orbit.

Assuming that this is only one of several points to be located, the Fourier constants should first be calculated. Simpson's rule may be written as follows, using λ' as the main variable:

If

$$F=\int_a^b f(\lambda')d\lambda'$$

a close approximation of the integral is

$$F=(\Delta\lambda'/3)[f(\lambda'_a)+4f(\lambda'_a+\Delta\lambda')+2f(\lambda'_a+2\Delta\lambda')+4f(\lambda'_a+3\Delta\lambda') + 2f(\lambda'_a+4\Delta\lambda')+\ldots+4f(\lambda'_b-\Delta\lambda')+f(\lambda'_b)]$$

where $f(\lambda')$ is calculated for λ' equal to a, and for λ' at each equal interval $\Delta\lambda'$ until $\lambda'=b$. The values $f(\lambda')$ are alternately multiplied by 4 and 2 as the formula indicates, except for the two end values, and all the resulting values are added and multiplied by one-third of the interval. The interval $\Delta\lambda'$ must be chosen so there is an even number of intervals.

Applying this rule to equation (20–1) with the suggested 9° interval in λ', the function $f(\lambda')=(H-S^2)/(1+S^2)^{1/2}$ is calculated for a λ' of 0°, 9°, 18°, 27°, 36°, . . ., 81°, and 90°, with ten 9° intervals. The calculation for $\lambda'=9°$ is as follows, using equations (20–4) and (20–5):

$$H=1-(18/251)\cos 99.092° = 1.0113321$$
$$S=(18/251)\sin 99.092°\cos 9° = 0.0699403$$
$$f(\lambda')=(1.0113321-0.0699403^2)/(1+0.0699403^2)^{1/2} = 1.0039879$$

To calculate B, the following table may be figuratively prepared, although a computer or calculator program would normally be used instead (H is a constant):

λ'	S	$f(\lambda')$	Multiplier	Summation
0°	0.0708121	1.0038042	×1=	1.0038042
9	.0699403	1.0039879	×4=	4.0159516
18	.0673463	1.0045212	×2=	2.0090423
27	.0630941	1.0053522	×4=	4.0214087
36	.0572882	1.0064001	×2=	2.0128001
45	.0500717	1.0075627	×4=	4.0302507
54	.0416223	1.0087263	×2=	2.0174526
63	.0321480	1.0097770	×4=	4.0391079
72	.0218822	1.0106114	×2=	2.0212227
81	.0110775	1.0111474	×4=	4.0445895
90	.0000000	1.0113321	×1=	1.0113321
			Total =	30.2269624

To convert to B, again referring to equation (20–1) and remaining in degrees for the final multipliers, since they cancel,

$$B = (2/180°) \times (9°/3) \times 30.2269624$$
$$= 1.0075654$$

This is the Fourier coefficient B for equation (20–6) with λ' in radians. To use λ' in degrees, multiply B by $\pi/180°$:

$$B = 1.0075654 \times \pi/180$$
$$= 0.017585334$$

Calculations of A_n and C_n are similar, except that the calculations of the function involve an additional trigonometric term at each step. For example, to calculate C_3 for $\lambda' = 9°$, using equation (20–3) and the S found above from equation (20–5),

$$f(\lambda') = [S/(1+S^2)^{1/2}] \cos 3\lambda'$$
$$= [0.0699403/(1+0.0699403^2)^{1/2}] \cos (3 \times 9°)$$
$$= 0.06216542$$

The sums for A_n corresponding to 30.2269624 for B are as follows:

for A_2: -0.0564594
for A_4: 0.000041208

To convert to the desired constants,

$$A_2 = [4/(180° \times 2)] \times (9°/3) \times (-0.0564594)$$
$$= -0.00188198$$
$$A_4 = [4/(180° \times 4)] \times (9°/3) \times (0.000041208)$$
$$= 0.0000006868$$

The sums for C_n:

for C_1: 1.0601909
for C_3: -0.0006626541

To convert,

$$C_1 = [4 \times (1.0113321 + 1)/(180° \times 1)] \times (9°/3) \times (1.0601909)$$
$$= 0.1421597$$
$$C_3 = [4 \times (1.0113321 + 1)/(180° \times 3)] \times (9°/3) \times (-0.0006626541)$$
$$= -0.0000296182$$

These constants, rounded to seven decimal places except for B, will be used below:

Using equation (20–11),

$$\lambda_0 = 128.87° - (360°/251) \times 15$$
$$= 107.36°$$

To solve equations (20–8) and (20–9) by iteration, determine λ'_p from equation (20–12) and the discussion following the equation, with $N=0$:

$$\lambda'_p = 90° \times (4\times 0 + 2 - 1)$$
$$= 90°$$

Then $\lambda_{t_p} = -73° - 107.36° + (18/251)\times 90°$
$= -173.9058167°$
$\cos \lambda_{t_p} = -0.9943487$

Using λ'_p as the first trial value of λ' in equation (20–9), using extra decimal places for illustration:

$$\lambda_t = -73° - 107.36° + (18/251)\times 90°$$
$= -173.9058167°$, as before.

Using equation (20–8),

$$\lambda' = \arctan [\cos 99.092° \tan (-173.9058167°) + \sin 99.092° \tan 40°/\cos (-173.9058167°)]$$
$$= -40.36910525°$$

For quadrant correction, from the discussion following equation (20–12), using the sign of $\cos \lambda_{t_p}$ as calculated above,

$$\lambda' = -40.36910525° + 90° - 90° \sin 90° \times (-1)$$
$$= -40.36910525° + 180°$$
$$= 139.6308947°$$

This is the next trial λ'. Using equation (20–9),

$$\lambda_t = -73° - 107.36° + (18/251)\times 139.6308947°$$
$$= -170.3466291°$$

Substituting this value of λ_t in place of $-173.9058167°$ in equation (20–8),

$$\lambda' = -40.9362858°$$

The same quadrant adjustment applies:

$$\lambda' = -40.9362858° + 180°$$
$$= 139.0637142°$$

Substituting this in equation (20–9),

$$\lambda_t = -170.3873034°$$

and from equation (20–8),

$$\lambda' = 139.0707998°$$

From the 4th iteration,

$$\lambda_t = -170.3867952^\circ$$
$$\lambda' = 139.0707113^\circ$$

From the 5th iteration,

$$\lambda_t = -170.3868016^\circ$$
$$\lambda' = 139.0707124^\circ$$

From the 6th iteration,

$$\lambda_t = -170.3868015^\circ$$
$$\lambda' = 139.0707124^\circ$$

Since λ' has not changed to seven decimal places, the last iteration is taken as the final value. Using equation (20–10), with the final value of λ_t,

$$\begin{aligned}\phi' &= \arcsin\,[\cos 99.092^\circ \sin 40^\circ - \sin 99.092^\circ \cos 40^\circ \sin\\ &\quad (-170.3868015^\circ)]\\ &= 1.4179606^\circ\end{aligned}$$

From equation (20–5),

$$\begin{aligned}S &= (18/251) \sin 99.092^\circ \cos 139.0707124^\circ\\ &= -0.0534999\end{aligned}$$

From equations (20–6) and (20–7),

$$\begin{aligned}x &= 6{,}370{,}997 \times \{0.017585334 \times 139.0707124^\circ + (-0.001882[illegible])\\ &\quad \sin (2 \times 139.0707124^\circ) + 0.0000007 \sin (4 \times 139.0707124^\circ)\\ &\quad - [-0.0534999/(1 + (-0.0534999)^2)^{1/2}] \ln \tan\\ &\quad (45^\circ + 1.4179606^\circ/2)\}\\ &= 15{,}601{,}233.74 \text{ m}\\ y &= 6{,}370{,}997 \times \{0.1421597 \sin 139.0707124^\circ + (-0.000029[illegible])\\ &\quad \sin (3 \times 139.0707124^\circ) + [1/(1 + (-0.0534999)^2)^{1/2}]\\ &\quad \ln \tan (45^\circ + 1.4179606^\circ/2)\}\\ &= 750{,}650.37 \text{ m}\end{aligned}$$

SPACE OBLIQUE MERCATOR (SPHERE)–INVERSE EQUATIONS (SEE P. 200–202)

Inversing forward example:

Given: Radius of sphere: $R = 6{,}370{,}997.0$ m
Landsat orbit: $i = 99.092^\circ$
$P_2/P_1 = 18/251$
Path = 15

Point: $x = 15,601,233.74$ m
$y = 750,650.37$ m

Find: ϕ, λ

Constants A_2, A_4, B, C_1, C_3, and λ_0 are calculated exactly and have the same values as in the forward example above. To solve equation (20–15) by iteration, the first trial λ' is x/BR, using the value of B for λ' in degrees in this example:

$$\lambda' = 15,601,233.74/(0.017585334 \times 6370997.0) \\ = 139.2518341°$$

Using equation (20–5) to find S for this trial λ',

$$S = (18/251) \sin 99.092° \cos 139.2518341° \\ = -0.0536463$$

Inserting these values in the right side of equation (20–15),

$$\lambda' = \{15,601,233.74/6,370,997.0 + (-0.0536463) \\ \times 750,650.37/6,370,997.0 - (-0.0018820) \sin (2 \times 139.2518341°) \\ - 0.0000007 \sin (4 \times 139.2518341°) - (-0.0536463) \\ \times [0.1421597 \sin 139.2518341° + (-0.0000296) \\ \sin (3 \times 139.2518341°)]\}/0.017585334 \\ = 139.0695675°$$

Substituting this new trial value of λ' in (20–5) for a new S, then both in (20–15) for a new λ', the next trial value is

$$\lambda' = 139.0707197°$$

The fourth value is

$$\lambda' = 139.0707124°$$

and the fifth does not change to seven decimal places. Therefore, this λ' is the final value. The corresponding S last calculated from (20–5) is

$$S = (18/251) \sin 99.092° \cos 139.0707124° \\ = -0.0534999$$

Using equation (20–16),

$$\ln \tan (45° + \phi'/2) = [1 + (-0.0534999)^2]^{1/2} \times [750650.37/ \\ 6370997.0 - 0.1421597 \sin 139.0707124° \\ -(-0.0000296) \sin (3 \times 139.0707124°)] \\ = 0.02475061$$

$$\tan (45° + \phi'/2) = e^{0.02475061} \\ = 1.0250594$$

$$45° + \phi'/2 = \arctan 1.0250594 \\ = 45.7089803°$$

$$\phi' = 2 \times (45.7089803° - 45°) \\ = 1.4179606°$$

Using equation (20–13),

$$\lambda = \arctan[(\cos 99.092° \sin 139.0707124° - \sin 99.092° \tan 1.4179606°)/\cos 139.0707124°] - (18/251)\, 139.0707124° + 107.36°$$
$$= \arctan[-0.1279654/(-0.7555187)] + 97.3868015°$$
$$= 9.6131985° + 97.3868015°$$
$$= 107.0000000°$$

Since the denominator of the argument of arctan is negative, and the numerator is negative, 180° must be subtracted from λ, or

$$\lambda = 107.0000000° - 180° = -73.0000000°$$
$$= 73° \text{ W. long.}$$

Using equation (20–14),

$$\phi = \arcsin(\cos 99.092° \sin 1.4179606° + \sin 99.092° \cos 1.4179606° \sin 139.0707124°)$$
$$= 40.0000000°$$
$$= 40° \text{ N. lat.}$$

For groundtrack calculations, equations (20–17) through (20–20) are used, given the same Landsat parameters as above for R, i, P_2/P_1, and path 15, with $\lambda_0 = 107.36°$, and $\phi = 40°$ S. lat. on the daylight (descending) part of the orbit. Using equation (20–17),

$$\lambda' = \arcsin[\sin(-40°)/\sin 99.092°]$$
$$= -40.6145062°$$

To adjust for quadrant, subtract from 180°, which is the λ' of the descending node:

$$\lambda' = 180° - (-40.6145062°)$$
$$= 220.6145062°$$

Using equation (20–18),

$$\lambda = \arctan[(\cos 99.092° \sin 220.6145062°)/\cos 220.6145062°] - (18/251) \times 220.6145062° + 107.36°$$
$$= \arctan[0.1028658/(-0.7591065)] + 91.5390394°$$
$$= 83.8219462°$$

Since the denominator of the argument for arctan is negative, add 180°, but 360° must be then subtracted to place λ between +180° and −180°:

$$\lambda = 83.8219462 + 180° - 360°$$
$$= -96.1780538°$$
$$= 96°10'40.99'' \text{ W. long.}$$

If λ is given instead, with the above λ used for the example, equations (20–19) and (20–9) are iterated together using the same type of initial trial λ' as that used in the forward example for equations (20–8) and (20–9). In this case, as described following equation (20–12), λ'_p is 270°, but this is only known from the final results. If $\lambda'_p=90°$ is chosen, the same answer will be obtained, since there is considerable overlap in actual regions for which two adjacent λ'_p's may be used. If $\lambda'_p=450°$ is chosen, the λ' calculated will be about 487.9°, or the position on the next orbit for this λ. Using $\lambda'_p=270°$ and the equation for λ_{tp} following equation (20–12),

$$\lambda_{tp} = -96.1780538° - 107.36° + (18/251)\times 270°$$
$$= -184.1755040°$$

for which the cosine is negative. From equation (20–9), the first trial λ_t is the same as λ_{tp}. From equation (20–19),

$$\lambda' = \arctan\,[\tan\,(-184.1755040°)/\cos\,99.092°]$$
$$= 24.7970120°$$

For quadrant adjustment, using the procedure following (20–12),

$$\lambda' = 24.7970120 + 270° - 90°\,\sin\,270° \times (-1)$$
$$= 204.7970120°$$

where the (-1) takes the sign of $\cos\,\lambda_{tp}$.

Substituting this as the trial λ' in (20–9),

$$\lambda_t = -96.1780538° - 107.36° + (18/251)\times 204.7970120°$$
$$= -188.8514155°$$

Substituting this in place of $-184.1755040°$ in (20–19),

$$\lambda' = 44.5812628°$$

but with the same quadrant adjustment as before,

$$\lambda' = 224.5812628°$$

Repeating the iteration, successive values of λ' are

$$\lambda' = 219.5419815°, \text{ then}$$
$$= 220.8989682°, \text{ then}$$
$$= 220.5386678°, \text{ then}$$
$$= 220.6346973°, \text{ then}$$
$$= 220.6091287°, \text{ then}$$
$$= 220.6159384°, \text{ etc.}$$

After a total of about 16 iterations, a value which does not change to seven decimal places is obtained:

$$\lambda' = 220.6145063°$$

Using equation (20–20),

$$\phi = \arcsin(\sin 99.092° \sin 220.6145063°)$$
$$= -40.0000000°$$
$$= 40° \text{ S. lat.}$$

SPACE OBLIQUE MERCATOR (ELLIPSOID)–FORWARD EQUATIONS
(SEE P. 203–207)

While equations are given for orbits of small eccentricity, the calculations are so lengthy that examples will only be given for the circular Landsat orbit, thus eliminating or simplifying several of the equations given in the text.

Given: Clarke 1866 ellipsoid: $a = 6{,}378{,}206.4$ m
$e^2 = 0.00676866$
Landsat orbit: $i = 99.092°$
$P_2/P_1 = 18/251$
$R_0 = 7{,}294{,}690.0$ m
Path = 15
Point: $\phi = 40°$ N. lat.
$\lambda = 73°$ W. long.

Find: x, y for point taken during daylight northern (first) quadrant of orbit.

The calculation of Fourier constants for the map follows the same basic procedure as that given for the forward example for the spherical form, except for greater complications in computing each step for the Simpson's numerical integration. The formula for Simpson's rule (see above) is not repeated here, but an example of calculation of a function $f(\lambda'')$ for constant A_2 at $\lambda'' = 18°$ is given below, as represented in equation (20–23).

$$f(\lambda'') = [(HJ - S^2)/(J^2 + S^2)^{1/2}] \cos 2\lambda''$$

Using equations (20–27) through (20–30) in order,

$$J = (1 - 0.00676866)^3$$
$$= 0.9798312$$
$$W = [(1 - 0.00676866 \cos^2 99.092°)^2/(1 - 0.00676866)^2] - 1$$
$$= 0.0133334$$
$$Q = 0.00676866 \sin^2 99.092°/(1 - 0.00676866)$$
$$= 0.0066446$$
$$T = 0.00676866 \sin^2 99.092° \times (2 - 0.00676866)/(1 - 0.00676866)^2$$
$$= 0.0133345$$

Using equations (20–37) and (20–38), remembering that $L'=1.0$ for the circular orbit (as can be readily determined from (20–39) with $e'=0$),

$$S=(18/251)\times 1.0 \sin 99.092° \cos 18°\times[(1+0.0133345 \sin^2 18°)/(1+0.0133334 \sin^2 18°)(1+0.0066446 \sin^2 18°)]^{1/2}$$
$$=0.0673250$$
$$H=[(1+0.0066446 \sin^2 18°)/(1+0.0133334 \sin^2 18°)]^{1/2} \times [(1+0.0133334 \sin^2 18°)/(1+0.0066446 \sin^2 18°)^2 -(18/251)\times 1.0 \cos 99.092°]$$
$$=1.0110133$$

Calculating the function $f(\lambda'')$ as given above,

$$f(\lambda'')=[(1.0110133\times 0.9798312-0.0673250^2)/(0.9798312^2+0.0673250^2)^{1/2}] \cos(2\times 18°)$$
$$=0.8122693$$

In tabular form, using 9° intervals in λ'', the calculation of A_2 proceeds as follows, integrating only to 90° for the circular orbit:

λ''	H	S	$f(\lambda'')$	Multiplier	Summation
0°	1.0113321	0.0708121	1.0035971	×1=	1.0035971
9	1.0112504	0.0699346	0.9545807	×4=	3.8183229
18	1.0110133	0.0673250	0.8122693	×2=	1.6245386
27	1.0106439	0.0630509	0.5904356	×4=	2.3617425
36	1.0101782	0.0572226	0.3106003	×2=	0.6212007
45	1.0096617	0.0499888	0.0000000	×4=	0.0000000
54	1.0091450	0.0415321	−0.3110197	×2=	−0.6220394
63	1.0086787	0.0320636	−0.5919529	×4=	−2.3678116
72	1.0083085	0.0218167	−0.8151437	×2=	−1.6302874
81	1.0080708	0.0110417	−0.9585531	×4=	−3.8342122
90	1.0079888	0.0000000	−1.0079888	×1=	−1.0079888
				Total=	−0.0329376

To convert to A_2, referring to equation (20–23), but multiplying by 4 because of the single-quadrant integration,

$$A_2=[4/(180°\times 2)]\times(9°/3)\times(-0.0329376)$$
$$=-0.0010979$$

Similar calculations of A_4, B_1, C_1, and C_3 lead to the values given in the text following equation (20–73a):

$B_1=0.0175544891$ for λ'' in degrees
$A_4=-0.0000013$
$C_1=0.1434410$
$C_3=0.0000285$

Since the calculations of j_n and m_n are not necessary for calculation of x and y from ϕ and λ, or the inverse, and are also lengthy, they will be omitted in these examples. The examples given will, however, assist in the understanding of the text concerning their calculations. The other general constant needed is λ_0, determined from (20–11), as in the forward spherical formulas and example:

$$\lambda_0 = 128.87° - (360°/251) \times 15$$
$$= 107.36°$$

For coordinates of the specific point, equations (20–45) and (20–46) are iterated together, replacing $(L+\gamma)$ with λ'' in (20–46) for the circular orbit. Except for the additional factor of $(1-e^2)$ in (20–45), the procedure is identical to the forward spherical example for solving (20–8) and (20–9). The calculations of λ'_p and the first trial λ_t are identical with that example since ϕ and λ have been made the same. The sign of $\cos \lambda_{tp}$ is also negative.

$$\lambda'_p = 90°$$
$$\lambda_t = -173.9058167°$$

Using equation (20–45),

$$\lambda'' = \arctan [\cos 99.092° \tan(-173.9058167°) + (1-0.00676866) \sin 99.092° \tan 40°/\cos(-173.9058167°)]$$
$$= -40.1810005°$$

For quadrant correction,

$$\lambda'' = -40.1810005° + 90° - 90° \sin 90° \times (-1)$$
$$= 139.8189995°$$

Successive iterations give

(2) $\lambda_t = -170.3331395°$
$\lambda'' = 139.2478915°$
(3) $\lambda_t = -170.3740954°$
$\lambda'' = 139.2550483°$
(4) $\lambda_t = -170.3735822°$
$\lambda'' = 139.2549587°$
(5) $\lambda_t = -170.3735886°$
$\lambda'' = 139.2549598°$
(6) $\lambda_t = -170.3735885°$
$\lambda'' = 139.2549598°$

These last values do not change within seven decimal places in subsequent iterations.

Using equation (20–49) with the final value of λ_t,

$$\phi'' = \arcsin \{[(1-0.00676866) \cos 99.092° \sin 40° - \sin 99.092° \cos 40° \sin(-170.3735885°)]/(1-0.00676866 \sin^2 40°)^{1/2}\}$$
$$= 1.4692784°$$

From equation (20–37), using 139.2549598° in place of 18° in the example for calculation of Fourier constants,

$S = -0.0535730$

From equations (20–43a) and (20–44a),

$$x = 6{,}378{,}206.4 \times \{0.0175544891 \times 139.2549598° + (-0.0010979) \sin (2 \times 139.2549598°) + (-0.0000013) \sin (4 \times 139.2549598°) - [-0.0535730/(0.9798312^2 + (-0.0535730)^2)^{1/2}] \ln \tan (45° + 1.4692784°/2)\}$$
$$= 15{,}607{,}700.94 \text{ m}$$
$$y = 6{,}378{,}206.4 \times \{0.1434410 \sin 139.2549598° + 0.0000285 \sin (3 \times 139.2549598°) + [0.9798312/(0.9798312^2 + (-0.0535730)^2)^{1/2}] \ln \tan (45° + 1.4692784°/2)\}$$
$$= 760{,}636.33 \text{ m}$$

For calculation of positions along the groundtrack for a circular orbit, these examples use the same basic Landsat parameters as those in the preceding example, except that $\phi = 40°$ S. lat. on the daylight (descending) part of the orbit. To find λ', ϕ_g is first calculated from equation (20–57):

$$\phi_g = (-40°) - \arcsin \{6{,}378{,}206.4 \times 0.00676866 \sin (-40°) \cos (-40°)/[7{,}294{,}690.0 \times (1 - 0.00676866 \sin^2 (-40°))^{1/2}]$$
$$= -40° - (-0.1672042°)$$
$$= -39.8327958°$$

From equation (20–56),

$$\lambda' = \arcsin [\sin (-39.8327958°)/\sin 99.092°]$$
$$= -40.4436361°$$

To adjust for quadrant, since the satellite is traveling south, subtract from ½ × 360°:

$$\lambda' = 180° - (-40.4436361°)$$
$$= 220.4436361°$$

Using equation (20–59), replacing $(L + \gamma)$ with λ' for the circular orbit,

$$\lambda = \arctan [(\cos 99.092° \sin 220.4436361°)/\cos 220.4436361°] - (18/251) \times 220.4436361° + 107.36°$$
$$= \arctan [0.1025077/(-0.7610445)] + 91.5512930°$$
$$= 83.8800995°$$

Since the denominator of the argument for arctan is negative, add 180°, but 360° must also be subtracted to place λ between +180° and −180°:

$$\lambda = 83.8800995° + 180° - 360°$$
$$= -96.1199005°$$
$$= 96°07'11.64'' \text{ W. long.}$$

If λ is given instead, with the above λ used in the example, equations (20–19) and (20–46) are iterated together, with λ' in place of $(L+\gamma)$ in the latter for the circular orbit. The technique is the same as that used previously for solving (20–8) and (20–9) in the forward spherical example. See also the discussion for the corresponding spherical groundtrack example, using equations (20–19) and (20–9), near the end of the inverse example. Since the formulas for the circular orbit are the same for ellipsoid or sphere for this particular calculation, the various iterations are not shown here. With $\lambda = -96.1199005°$, λ' is found to be 220.4436361°. To find the corresponding ϕ from equation (20–61), a trial $\phi = \arcsin(\sin 99.092° \sin 220.4436361°) = -39.8327958°$ is inserted:

$$\begin{aligned}\phi &= \arcsin(\sin 99.092° \sin 220.4436361°) \\ &\quad + \arcsin\{6{,}378{,}206.4 \times 0.00676866 \sin(-39.8327958°) \\ &\quad \cos(-39.8327958°)/[7{,}294{,}690.0 \times (1-0.00676866 \\ &\quad \sin^2(-39.8327958°))^{1/2}]\} \\ &= -39.9998234°\end{aligned}$$

Substituting $-39.9998234°$ in place of $-39.8327958°$ in the same equation, a new value of ϕ is obtained:

$$\phi = -39.9999998°$$

With the next iteration,

$$\phi = -40.0000000°$$

which does not change to seven decimal places. Thus,

$$\phi = 40° \text{ S. lat.}$$

SPACE OBLIQUE MERCATOR (ELLIPSOID)–INVERSE EQUATIONS

(SEE P. 207–210)

This example is limited to the circular Landsat orbit, using the parameters of the forward example.

Inversing forward example:

Given: Clarke 1866 ellipsoid: $a = 6{,}378{,}206.4$ m
$e^2 = 0.00676866$
Landsat orbit: $i = 99.092°$
$P_2/P_1 = 18/251$
$R_0 = 7{,}294{,}690.0$ m
Path = 15 (thus $\lambda_0 = 107.36°$ as in forward example)
Point: $x = 15{,}607{,}700.94$ m
$y = 760{,}636.33$ m

Find: ϕ, λ

All constants J, W, Q, T, A_n, B_1, and C_n, as calculated in the forward example, must be calculated or otherwise provided for use for inverse calculations.

To find λ'' from equation (20–68a) by iteration, the procedure is identical to that given for (20–15) in the inverse spherical example, except for the use of different constants. For the initial $\lambda'' = x/aB_1$,

$$\lambda'' = 15{,}607{,}700.94/(6{,}378{,}206.4 \times 0.0175544891)$$
$$= 139.3965968°$$

Using equation (20–37) to find S for this value of λ'',

$$S = (18/251) \times 1.0 \sin 99.092° \cos 139.3965968° \times [(1 + 0.0133345 \sin^2 139.3965968°)/(1 + 0.0133334 \sin^2 139.3965968°)(1 + 0.0066446 \sin^2 139.3965968°)]^{1/2}$$
$$= -0.0536874$$

Inserting these values into (20–68a),

$$\lambda'' = \{15{,}607{,}700.94/6{,}378{,}206.4 + (-0.0536874/0.9798312) \times (760{,}636.33/6{,}378{,}206.4) - (-0.0010979) \sin (2 \times 139.3965968°) - (-0.0000013) \sin (4 \times 139.3965968°) - (-0.0536874/0.9798312) \times [0.1434410 \sin 139.3965968° + 0.0000285 \sin (3 \times 139.3965968°)]\}/0.0175544891$$
$$= 139.2539963°$$

Substituting this new trial value of λ'' into (20–37) for a new S, then both into (20–68a), the next trial value is

$$\lambda'' = 139.2549663°$$

and the fourth trial value is

$$\lambda'' = 139.2549597°$$

The fifth trial value is

$$\lambda'' = 139.2549598°$$

which does not change with another iteration to seven decimal places. Therefore, this is the final value of λ''. The corresponding S last calculated from (20–37) using this value of λ'' is -0.0535730. Using equation (20–69a),

$$\ln \tan (45° + \phi''/2) = [1 + (-0.0535730)^2/0.9798312^2]^{1/2} \times [760{,}636.33/6{,}378{,}206.4 - 0.1434410 \sin 139.2549598° - 0.0000285 \sin (3 \times 139.2549598°)]$$
$$= 0.0256466$$

$$\tan(45° + \phi''/2) = e^{0.0256466}$$
$$= 1.0259783$$
$$45° + \phi''/2 = \arctan 1.0259783$$
$$= 45.7346392°$$
$$\phi'' = 2 \times (45.7346392° - 45°)$$
$$= 1.4692784°$$

Using equations (20–65), (20–64), and (20–63) in order,

$$U = 0.00676866 \cos^2 99.092°/(1 - 0.00676866)$$
$$= 0.0001702$$
$$V = \{[1 - \sin^2 1.4692784°/(1 - 0.00676866)] \cos 99.092° \sin 139.2549598° - \sin 99.092° \sin 1.4692784° \times [(1 + 0.0066446 \sin^2 139.2549598°) \times (1 - \sin^2 1.4692784°) - 0.0001702 \sin^2 1.4692784°]^{1/2}\}/ [1 - \sin^2 1.4692784° (1 + 0.0001702)]$$
$$= -0.1285013$$
$$\lambda_t = \arctan(-0.1285013/\cos 139.2549598°)$$
$$= \arctan[-0.1285013/(-0.7576215)]$$
$$= 9.6264115°$$

Since the denominator of the argument for arctan is negative, and the numerator is negative, subtract 180°:

$$\lambda_t = 9.6264115° - 180°$$
$$= -170.3735885°$$

Using equation (20–62), with λ'' in place of $(L+\gamma)$ for the circular orbit,

$$\lambda = -170.3735885° - (18/251) \times 139.2549598° + 107.36°$$
$$= -73.0000000°$$
$$= 73° \text{ W. long.}$$

Using equation (20–66),

$$\phi = \arctan\{[\tan 139.2549598° \cos(-170.3735885°) - \cos 99.092° \sin(-170.3735885°)]/[(1 - 0.00676866) \sin 99.092°]\}$$
$$= 40.0000000°$$
$$= 40° \text{ W. lat.}$$

VAN DER GRINTEN (SPHERE)–FORWARD EQUATIONS (SEE P. 214)

Given: Radius of sphere: $R = 1.0$ unit
Central meridian: $\lambda_0 = 85°$ W. long.
Point: $\phi = 50°$ S. lat.
$\lambda = 160°$ W. long.

Find: x, y

From equations (21–6), (21–3), (21–4), (21–5), and (21–6a) in order,

$$\theta = \arcsin |2\times(-50°)/180°|$$
$$= \arcsin 0.5555556$$
$$= 33.7489886°$$
$$A = ½|180°/[(-160°)-(-85°)]-[(-160°)-(-85°)]/180°|$$
$$= ½|-2.4000000-(-0.4166667)|$$
$$= 0.9916667$$
$$G = \cos 33.7489886°/(\sin 33.7489886° + \cos 33.7489886° - 1)$$
$$= 2.1483315$$
$$P = 2.1483315\times(2/\sin 33.7489886° - 1)$$
$$= 5.5856618$$
$$Q = 0.9916667^2 + 2.1483315 = 3.1317342$$

From equation (21–1),

$$x = -\pi\times 1.0\times\{0.9916667\times(2.1483315-5.5856618^2) + [0.9916667^2\times(2.1483315-5.5856618^2)^2 - (5.5856618^2+0.9916667^2)\times(2.1483315^2-5.5856618^2)]^{1/2}\}/(5.5856618^2+0.9916667^2)$$
$$= -1.1954154 \text{ units}$$

taking the initial "−" sign because $(\lambda-\lambda_0)$ is negative. Note that π is not converted to 180° here, since there is no angle in degrees to offset it.

From equation (21–2),

$$y = -\pi\times 1.0\times\{5.5856618\times 3.1317342 - 0.9916667 \times[(0.9916667^2+1)\times(5.5856618^2+0.9916667^2) - 3.1317342^2]^{1/2}\}/(5.5856618^2+0.9916667^2)$$

$= -0.9960733$ units, taking the initial "−" sign because ϕ is negative.

VAN DER GRINTEN (SPHERE)–INVERSE EQUATIONS (SEE P. 214–216)

Inversing forward example:

Given: Radius of sphere:	$R = 1.0$ unit
Central meridian:	$\lambda_0 = 85°$ W. long.
Point:	$x = -1.1954154$ units
	$y = -0.9960733$ unit

Find: ϕ, λ

Using equations (21–9) through (21–19) in order,

$$X = -1.1954154/(\pi\times 1.0)$$
$$= -0.3805125$$
$$Y = -0.9960733/(\pi\times 1.0)$$
$$= -0.3170600$$
$$c_1 = -0.3170600\times[1+(-0.3805125)^2+(-0.3170600)^2]$$
$$= -0.3948401$$

$c_2 = -0.3948401 - 2\times(-0.3170600)^2 + (-0.3805125)^2$
$= -0.4511044$
$c_3 = -2\times(-0.3948401) + 1 + 2\times(-0.3170600)^2$
$+[(-0.3805125)^2 + (-0.3170600)^2]^2$
$= 2.0509147$
$d = (-0.3170600)^2/2.0509147 + [2\times(-0.4511044)^3/2.0509147^3$
$-9\times(-0.3948401)\times(-0.4511044)/2.0509147^2]/27$
$= 0.0341124$
$a_1 = [-0.3948401 - (-0.4511044)^2/(3\times2.0509147)]/2.0509147$
$= -0.2086455$
$m_1 = 2\times(0.2086455/3)^{1/2}$
$= 0.5274409$
$\theta_1 = (1/3)\arccos[3\times0.0341124/(-0.2086455\times0.5274409)]$
$= (1/3)\arccos(-0.9299322)$
$= 52.8080831°$
$\phi = -180°\times[-0.5274409\times\cos(52.8080831°+60°)$
$-(-0.4511044)/(3\times2.0509147)]$
$= -50° = 50°$ S. lat., taking the initial "−" sign because y is negative.
$\lambda = 180°\times\{(-0.3805125)^2 + (-0.3170600)^2 - 1 +$
$[1 + 2\times((-0.3805125)^2 - (-0.3170600)^2)$
$+((-0.3805125)^2 + (-0.3170600)^2)^2]^{1/2}\}/$
$[2\times(-0.3805125)] + (-85°)$
$= -160° = 160°$ W. long.

SINUSOIDAL (SPHERE)–FORWARD EQUATIONS (SFE P. 222)

Given: Radius of sphere: $R = 1.0$ unit
Central meridian: $\lambda_0 = 90°$ W. long.
Point: $\phi = 50°$ S. lat.
$\lambda = 75°$ W. long.

Find: x, y, h, k, θ', ω
From equations (22–1) through (22–5) in order,

$x = 1.0\times[-75° - (-90°)]\times\cos(-50°)\times\pi/180°$
$= 0.1682814$ unit
$y = 1.0\times(-50°)\times\pi/180°$
$= -0.8726646$ unit
$h = \{1 + [-75° - (-90°)]^2\times(\pi/180°)^2\times\sin^2(-50°)\}^{1/2}$
$= 1.0199119$
$k = 1.0$
$\theta' = \arcsin(1/1.0199119)$
$= 78.6597719°$
$\omega = 2\arctan|½[-75° - (-90°)]\times(\pi/180°)\times\sin(-50°)|$
$= 11.4523842°$

SINUSOIDAL (SPHERE)–INVERSE EQUATIONS (SEE P. 222)

Inversing forward example:

Given: Radius of sphere: $R = 1.0$ unit
Central meridian: $\lambda_0 = 90°$ W. long.
Point: $x = 0.1682814$ unit
$y = -0.8726646$ unit

Find: ϕ, λ
From equations (22–6) and (22–7),

$\phi = (-0.8726646/1.0) \times 180°/\pi$
$= -49.9999985°$
$= 50°$ S. lat. rounding off.

$\lambda = -90° + [0.1682814/(1.0 \times \cos(-49.9999985°))] \times 180°/\pi$
$= -75.0000007°$
$= 75°$ W. long.

NOTES FOR NUMERICAL EXAMPLES

APPENDIX B

USE OF MAP PROJECTIONS BY U.S. GEOLOGICAL SURVEY—SUMMARY

Note: This list is not exhaustive. For further details, see text.

Class, Projection	*Maps*
Cylindrical	
Mercator	Northeast Equatorial Pacific Indonesia (Tectonic) Other planets and satellites
Transverse Mercator	7½′ and 15′ quadrangles for 22 States North America
Universal Transverse Mercator	1° lat. × 2° long. quadrangles of U.S. metric quadrangles and County maps.
"Modified Transverse Mercator"	Alaska
Oblique Mercator	Grids in southeast Alaska Landsat Satellite Imagery
Miller Cylindrical	World
Equidistant Cylindrical	United States and State Index Maps
Conic	
Albers Equal-Area Conic	United States and sections
Lambert Conformal Conic	7½′ and 15′ quadrangles for 32 States Quadrangles for Puerto Rico, Virgin Islands, and Samoa State Base Maps Quadrangles for International Map of the World Other planets and satellites
Bipolar Oblique Conic Conformal	North America (Geologic)
Polyconic	Quadrangles for all States
Modified Polyconic	Quadrangles for International Map of the World

Azimuthal

Orthographic (oblique)	Pictorial views of Earth or portions
Stereographic (oblique)	Other planets and satellites
(polar)	Antarctica Arctic regions Other planets and satellites
Lambert Azimuthal Equal-Area	
(oblique)	Pacific Ocean
(polar)	Arctic regions (Hydrocarbon Provinces) North and South Polar regions (polar expeditions)
Azimuthal Equidistant (oblique)	World Quadrangles for Guam and Micronesia

Space

Space Oblique Mercator	Satellite image mapping

Miscellaneous

Van der Grinten	World (Subsea Mineral Resources, misc.)
Sinusoidal (interrupted)	World (Hydrocarbon Provinces)

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