An Estimate for the Difference between the Consecutive Primes

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In this paper we will estimate the difference between the consecutive primes. This result is a new result on the distribution of the prime number. This paper is a continuation of [6] and [7].

1. Introduction

By the theorem 2 of the paper [6], there exist the optimum points $\overline{\lambda_0} = \left(\lambda_1^0,\, \lambda_2^0, \cdots, \lambda_m^0\right) \in R^m$ in m-dimensional real space R^m of the function

$$H\left(\overline{\lambda}\right) = H\left(\lambda_1,\, \lambda_2, \cdots, \lambda_m\right) = \frac{\exp\left(\exp\left(e^{-\gamma} \cdot F\left(\overline{\lambda}\right)\right)\right)}{p_1^{\lambda_1} \cdot p_2^{\lambda_2} \cdots p_m^{\lambda_m}}, \tag{1}$$

where

$$F\left(\overline{\lambda}\right) = F\left(\lambda_1,\, \lambda_2, \cdots, \lambda_m\right) = \prod_{i=1}^{m} \frac{1 - p_i^{-\lambda_i - 1}}{1 - p_i^{-1}}, \tag{2}$$

and $\gamma = 0.577\cdots$ is Euler's constant ([1,2,5]).

And here we assume that $\overline{\lambda} = (\lambda_1, \lambda_2, \cdots, \lambda_m)$ are real numbers and $\lambda_1 \geq \lambda_1 \geq \cdots \geq \lambda_m \geq 1$. Let $p_1 = 2,\ p_2 = 3,\ p_3 = 5, \cdots, p_m, \cdots$ be the consecutive primes. We will choose $p_m \geq 5$ arbitrarily and fix it. Then the optimum points $\overline{\lambda}_0 = (\lambda_1^0,\ \lambda_2^0, \cdots \lambda_m^0) \in R^m$ are estimated as follows. There is a number $k\ (1 < k < m)$ such that for any $i\ (1 \leq i \leq k)$ it holds that

$$\lambda_i^0 = \left(\frac{\log p_m}{\log p_i} + \frac{\log\log p_m}{\log p_i} - 1 \right) + \mathrm{O}\left(\frac{1}{\log p_i \cdot \log p_m} \right) \tag{3}$$

and for any $i\ (k+1 \leq i \leq m)$ it holds that $\lambda_i^0 = 1$. So we have

$$\lambda_1^0 > \lambda_2^0 > \cdots > \lambda_k^0 > \lambda_{k+1}^0 = \cdots = \lambda_m^0 = 1. \tag{4}$$

And for any $i\ (1 \leq i \leq k)$ it holds that

$$p_1^{\lambda_1^0+1} = p_2^{\lambda_2^0+1} = \cdots = p_k^{\lambda_k^0+1} = \left(e^{-\gamma} F(\overline{\lambda}_0)\right) \cdot \exp\left(e^{-\gamma} F(\overline{\lambda}_0)\right) + 1. \tag{5}$$

On the other hand, for the number k these hold that

$$\lambda_k^0 = 1 + \mathrm{O}\left(\frac{1}{\log p_m} \right),\ p_k = \sqrt{p_m \cdot \log p_m} \cdot \left(1 + \mathrm{O}\left(\frac{1}{\log p_m} \right) \right), \tag{6}$$

$$k = 2\sqrt{m} \cdot \left(1 + \mathrm{O}\left(\frac{\log\log p_m}{\log p_m} \right) \right) = 2\sqrt{\frac{p_m}{\log p_m}} \cdot \left(1 + \mathrm{O}\left(\frac{\log\log p_m}{\log p_m} \right) \right). \tag{7}$$

This shows that the function value $H(\lambda_1^0,\ \lambda_2^0, \cdots, \lambda_m^0)$ is dependent only on p_m. So we can put

$$C_m = H(\lambda_1^0,\ \lambda_2^0, \cdots, \lambda_m^0) = \frac{\exp\left(\exp\left(e^{-\gamma} \cdot F(\overline{\lambda}_0)\right)\right)}{p_1^{\lambda_1^0} \cdot p_2^{\lambda_2^0} \cdots p_m^{\lambda_m^0}}. \tag{8}$$

In this connection, we will put

$$\begin{cases} n_0 = p_1^{\lambda_1^0} p_2^{\lambda_2^0} \cdots p_k^{\lambda_k^0} \cdot p_{k+1}^1 \cdots p_m^1, \quad n_0' = n_0 \cdot p_m^{-1}, \\ \overline{\lambda}_0' = \left(\lambda_1^0,\, \lambda_2^0, \cdots, \lambda_{m-1}^0\right) \in R^{m-1}, \\ C_{m-1}' = H\left(\overline{\lambda}_0'\right) = H\left(\lambda_1^0,\, \lambda_2^0, \cdots, \lambda_{m-1}^0\right) \end{cases} \tag{9}$$

and

$$C_{m-1} = \max_{(\lambda_1,\, \lambda_2, \cdots, \lambda_{m-1}) \in R^m} H\left(\lambda_1,\, \lambda_2, \cdots, \lambda_{m-1}\right). \tag{10}$$

Then it is clear that $C_{m-1}' \le C_{m-1}$.

Let $\overline{\lambda}' = \left(\lambda_1',\, \lambda_2',\, \cdots, \lambda_{m-1}'\right)$ be the optimum points of the function $H\left(\lambda_1,\, \lambda_2, \cdots, \lambda_{m-1}\right)$ with $(m-1)$-variable in the space R^{m-1}. In general, then we have

$$\lambda_1' > \lambda_2' > \cdots > \lambda_{k-1}' > \lambda_k' = \cdots = \lambda_{m-1}' = 1. \tag{11}$$

Rarely, the last bigger number than 1 in $\left\{\lambda_1',\, \lambda_2',\, \cdots, \lambda_{m-1}'\right\}$ could be k. But it is not essential. It is important that for any $i\ (1 \le i \le k-1)$

$$p_1^{\lambda_1'+1} = p_2^{\lambda_2'++1} = \cdots = p_{k-1}^{\lambda_{k-1}'+1} = \left(e^{-\gamma} \cdot F\left(\overline{\lambda}'\right)\right) \cdot \exp\left(e^{-\gamma} \cdot F\left(\overline{\lambda}'\right)\right) + 1 \tag{12}$$

holds. We note that it doesn't exceed one in $\left\{\lambda_1^0,\, \lambda_2^0, \cdots, \lambda_m^0\right\}$.

We also put

$$\begin{cases} n' = p_1^{\lambda_1'}\ p_2^{\lambda_2'}\ \cdots p_{k-1}^{\lambda_k'} \cdot p_k^1 \cdots p_{m-1}^1, \\ n_+' = p_1^{\lambda_1'}\ p_2^{\lambda_2'}\ \cdots p_{k-1}^{\lambda_k'} \cdot p_k^1 \cdots p_{m-1}^1 \cdot p_m^1 = n' \cdot p_m^1, \\ \overline{\lambda}_+' = \left(\lambda_1',\, \lambda_2',\, \cdots, \lambda_{m-1}',\, 1\right),\ C_m' = H\left(\overline{\lambda}_+'\right). \end{cases} \tag{13}$$

On the other hand, it is well known that

$$\sum_{p \le p_m} \frac{1}{p} = \log\log p_m + b_0 + E_0\left(p_m\right), \tag{14}$$

where

$$b_0 = \gamma + \sum_p \left(\log\left(1 - \frac{1}{p}\right) + \frac{1}{p}\right) = 0.241\cdots \tag{15}$$

([1,2,5]). And there exists a constant $a>0$ such that

$$E_0\left(p_m\right)=\mathrm{O}\left(\exp\left(-a\sqrt{\log p_m}\right)\right). \tag{16}$$

2. The estimate of $F\left(\bar{\lambda}_0\right)$

We have

Theorem 1. For the optimum points $\bar{\lambda}_0=\left(\lambda_1^0,\lambda_2^0,\cdots\lambda_m^0\right)\in R^m$ of the function $H\left(\lambda_1,\lambda_2,\cdots,\lambda_m\right)$ we have

$$F\left(\bar{\lambda}_0\right)=e^{\gamma}\cdot\log p_m\cdot\left(1+E_0\left(p_m\right)-\frac{4}{\sqrt{p_m}\cdot\log^{3/2}p_m}+\varepsilon\left(p_m\right)\right), \tag{17}$$

where $\varepsilon\left(p_m\right)=\mathrm{O}\left(E_0^2\left(p_m\right)\right)$. Hence we also have

$$\begin{aligned}&\left(e^{-\gamma}\cdot F\left(\bar{\lambda}_0\right)\right)\cdot\exp\left(e^{-\gamma}\cdot F\left(\bar{\lambda}_0\right)\right)=\\&=p_m\cdot\log p_m\cdot\left(1+\left(\log p_m+1\right)\cdot E_0\left(p_m\right)-\frac{4\cdot\left(\log p_m+1\right)}{\sqrt{p_m}\cdot\log^{3/2}p_m}+\tilde{\varepsilon}\left(p_m\right)\right),\end{aligned} \tag{18}$$

where $\tilde{\varepsilon}\left(p_m\right)=\mathrm{O}\left(\log^2 p_m\cdot E_0^2\left(p_m\right)\right)$.

Proof. From (2) and (5), it is clear that

$$\begin{aligned}\log F\left(\bar{\lambda}_0\right)&=\log\left(\prod_{i=1}^{m}\frac{1-p_i^{-\lambda_i^0-1}}{1-p_i^{-1}}\right)=\\&=\sum_{i=1}^{k}\log\left(1-\frac{1}{p_i^{\lambda_i^0+1}}\right)+\sum_{i=k+1}^{m}\log\left(1-\frac{1}{p_i^2}\right)+\sum_{i=1}^{m}\log\left(1-\frac{1}{p_i}\right)^{-1}=\\&=A_1+A_2+A_3,\end{aligned} \tag{19}$$

where

$$A_1=\sum_{i=1}^{k}\log\left(1-\frac{1}{p_i^{\lambda_i^0+1}}\right),\ A_2=\sum_{i=k+1}^{m}\log\left(1-\frac{1}{p_i^2}\right),\ A_3=\sum_{i=1}^{m}\log\left(1-\frac{1}{p_i}\right)^{-1}. \tag{20}$$

First let's see A_1. By Mertens' theorem ([1,2]), preliminarily, we have

$$F\left(\overline{\lambda}_0\right)=\prod_{i=1}^{m}\frac{1-p_i^{-\lambda_i^0-1}}{1-p_i^{-1}}=e^{\gamma}\cdot\log p_m\cdot\left(1+O\left(\frac{1}{\log^2 p_m}\right)\right). \tag{21}$$

So we have

$$\left(e^{-\gamma}\cdot F\left(\overline{\lambda}_0\right)\right)\cdot\exp\left(e^{-\gamma}\cdot F\left(\overline{\lambda}_0\right)\right)+1=p_m\cdot\log p_m\cdot\left(1+O\left(\frac{1}{\log p_m}\right)\right). \tag{22}$$

Hence From (5) and (7) for any $i\ (1\le i\le k)$ we have

$$\begin{aligned}A_1&=\sum_{i=1}^{k}\log\left(1-\frac{1}{p_i^{\lambda_i^0+1}}\right)=k\cdot\log\left(1-\frac{1}{p_i^{\lambda_i^0+1}}\right)=\\&=-\frac{k}{p_i^{\lambda_i^0+1}}+O\left(\frac{k}{p_i^{2\cdot\left(\lambda_i^0+1\right)}}\right)=\\&=-\frac{2}{\sqrt{p_m}\cdot\left(\log p_m\right)^{3/2}}\cdot\left(1+O\left(\frac{\log\log p_m}{\log p_m}\right)\right).\end{aligned} \tag{23}$$

Next let's see A_2. Now we put

$$T\left(x\right)=\sum_{p\le x}\frac{1}{p}=\log\log x+b_0+E_0(x). \tag{24}$$

Then we have $dT\left(x\right)=\dfrac{dx}{x\cdot\log x}+dE_0(x)$. So we have

$$\begin{aligned}\sum_{i=k+1}^{m}\frac{1}{p_i^2}&=\int_{p_k}^{p_m}\frac{dT\left(t\right)}{t}=\int_{p_k}^{p_m}\frac{1}{t}\cdot\left(\frac{dt}{t\cdot\log t}+dE_0\left(t\right)\right)=\\&=\frac{1}{p_k\cdot\log p_k}-\frac{1}{p_m\cdot\log p_m}+\int_{p_k}^{p_m}\frac{dE_0\left(t\right)}{t}=\\&=\frac{2}{\sqrt{p_m}\cdot\left(\log p_m\right)^{3/2}}\cdot\left(1+O\left(\frac{\log\log p_m}{\log p_m}\right)\right)\end{aligned} \tag{25}$$

and

$$A_2=\sum_{i=k+1}^{m}\log\left(1-\frac{1}{p_i^2}\right)=-\sum_{i=k+1}^{m}\frac{1}{p_i^2}+\mathrm{O}\left(\sum_{i=k+1}^{m}\frac{1}{p_i^4}\right)=$$
$$=\frac{-2}{\sqrt{p_m}\cdot\left(\log p_m\right)^{3/2}}\cdot\left(1+\mathrm{O}\left(\frac{\log\log p_m}{\log p_m}\right)\right). \tag{26}$$

Next let’s see A_3. By (15) we have

$$\sum_{p\le p_m}\left(\log\left(1-\frac{1}{p}\right)+\frac{1}{p}\right)=b_0-\gamma-\sum_{p>p_m}\left(\log\left(1-\frac{1}{p}\right)+\frac{1}{p}\right). \tag{27}$$

And it is clear that

$$\left|\log\left(1-\frac{1}{p}\right)+\frac{1}{p}\right|=\left|-\sum_{j=1}^{\infty}\frac{1}{j\cdot p^j}+\frac{1}{p}\right|=$$
$$=\left|\frac{1}{2p^2}+\frac{1}{3p^3}+\cdots+\frac{1}{j\cdot p^j}+\cdots\right|\le \tag{28}$$
$$\le\left|\frac{1}{p^2}+\frac{1}{p^3}+\cdots+\frac{1}{p^j}+\cdots\right|=\frac{1}{p\cdot\left(p-1\right)}.$$

So we have

$$-\sum_{p>p_m}\left(\log\left(1-\frac{1}{p}\right)+\frac{1}{p}\right)\le\sum_{p>p_m}\frac{1}{p\cdot\left(p-1\right)}\le$$
$$\le\sum_{n>p_m}\frac{1}{n\cdot\left(n-1\right)}=\sum_{n>p_m}\left(\frac{1}{n-1}-\frac{1}{n}\right)=\frac{1}{n}=\mathrm{O}\left(\frac{1}{p_m}\right). \tag{29}$$

By (14) we have

$$A_3=\sum_{i=1}^{m}\frac{1}{p_i}-\sum_{i=1}^{m}\left(\log\left(1-\frac{1}{p_i}\right)+\frac{1}{p_i}\right)=$$
$$=\log\log p_m+b_0+E_0\left(p_m\right)-\left(b_0-\gamma-\sum_{p>p_m}\left(\log\left(1-\frac{1}{p}\right)+\frac{1}{p}\right)\right)= \tag{30}$$
$$=\log\log p_m+\gamma+E_0\left(p_m\right)+\mathrm{O}\left(\frac{1}{p_m}\right).$$

From (23), (26) and (30) we have

$$\log F\left(\bar{\lambda}_0\right) = \log\log p_m + \gamma + E_0\left(p_m\right) + \frac{-4}{\sqrt{p_m}\cdot\log^{3/2} p_m}\left(1+\mathrm{O}\left(\frac{\log\log p_m}{\log p_m}\right)\right). \tag{31}$$

and hence we have

$$\left(e^{-\gamma}\cdot F\left(\bar{\lambda}_0\right)\right) = \log p_m \cdot \left(1+E_0\left(p_m\right)-\frac{4}{\sqrt{p_m}\log^{3/2} p_m}+\varepsilon\left(p_m\right)\right),$$

where $\varepsilon\left(p_m\right)=\mathrm{O}\left(E_0^2\left(p_m\right)\right)$. Therefore we have

$$\left(e^{-\gamma}\cdot F\left(\bar{\lambda}_0\right)\right)\cdot\exp\left(e^{-\gamma}\cdot F\left(\bar{\lambda}_0\right)\right) =$$
$$= p_m\cdot\log p_m\cdot\left(1+\left(\log p_m+1\right)\cdot E_0\left(p_m\right)-\frac{4\cdot\left(\log p_m+1\right)}{\sqrt{p_m}\log^{3/2} p_m}+\tilde{\varepsilon}\left(p_m\right)\right),$$

where $\tilde{\varepsilon}\left(p_m\right)=\mathrm{O}\left(\log^2 p_m\cdot E_0^2\left(p_m\right)\right)$.

This completes the proof of the theorem 1. □

3. The estimate of $\left(\log C_{m-1}-\log C'_{m-1}\right)$

We get

Theorem 2. There exists a number m_0 such that for any $m \ge m_0$ we have

$$\log C_{m-1}-\log C'_{m-1} = \frac{p_m-p_{m-1}}{\sqrt{p_{m-1}\cdot\log p_{m-1}}}\cdot\left(\frac{p_m-p_{m-1}}{p_{m-1}}\right)\cdot\left(1+\tilde{E}\left(p_m\right)\right), \tag{32}$$

where $\tilde{E}\left(p_m\right)=\mathrm{O}\left(\dfrac{1}{\log p_m}\right).$ (33)

Proof. From (9) and (10), we have

$$\begin{aligned}
&\log C_{m-1}-\log C'_{m-1} = \\
&=\left(\exp\left(e^{-\gamma}\cdot F\left(\bar{\lambda}'\right)\right)-\log n'\right)-\left(\exp\left(e^{-\gamma}\cdot F\left(\bar{\lambda}'_0\right)\right)-\log n'_0\right) = \\
&=\left(\exp\left(e^{-\gamma}\cdot F\left(\bar{\lambda}'\right)\right)-\exp\left(e^{-\gamma}\cdot F\left(\bar{\lambda}'_0\right)\right)\right)-\left(\log n'-\log n'_0\right) = \\
&= R_1-R_2,
\end{aligned} \tag{34}$$

where

$$R_1=\left(\exp\left(e^{-\gamma}\cdot F\left(\overline{\lambda}'\right)\right)-\exp\left(e^{-\gamma}\cdot F\left(\overline{\lambda}_0'\right)\right)\right),\quad R_2=\left(\log n'-\log n_0'\right). \tag{35}$$

Let's see R_1. We can write as

$$\begin{aligned}\left(-R_1\right)&=\left(\exp\left(e^{-\gamma}\cdot F\left(\overline{\lambda}_0'\right)\right)-\exp\left(e^{-\gamma}\cdot F\left(\overline{\lambda}'\right)\right)\right)=\\&=\exp\left(e^{-\gamma}\cdot F\left(\overline{\lambda}'\right)\right)\cdot\left(\frac{\exp\left(e^{-\gamma}\cdot F\left(\overline{\lambda}_0'\right)\right)}{\exp\left(e^{-\gamma}\cdot F\left(\overline{\lambda}'\right)\right)}-1\right)\end{aligned} \tag{36}$$

and here we have

$$\begin{aligned}\log\left(\frac{\exp\left(e^{-\gamma}\cdot F\left(\overline{\lambda}_0'\right)\right)}{\exp\left(e^{-\gamma}\cdot F\left(\overline{\lambda}'\right)\right)}\right)&=\left(e^{-\gamma}\cdot F\left(\overline{\lambda}_0'\right)\right)-\left(e^{-\gamma}\cdot F\left(\overline{\lambda}'\right)\right)=\\&=\left(e^{-\gamma}\cdot F\left(\overline{\lambda}'\right)\right)\cdot\left(\frac{F\left(\overline{\lambda}_0'\right)}{F\left(\overline{\lambda}'\right)}-1\right).\end{aligned} \tag{37}$$

By the Taylor's expansion of the function $\log(1+x)$ $(0<x<1)$, (5) and (12), for any $i\,(1\le i\le k-1)$ we have

$$\begin{aligned}\log\left(\frac{F\left(\overline{\lambda}_0'\right)}{F\left(\overline{\lambda}'\right)}\right)&=\log F\left(\overline{\lambda}_0'\right)-\log F\left(\overline{\lambda}'\right)=\\&=\log\left(\prod_{i=1}^{m-1}\frac{1-p_i^{-\lambda_i^0-1}}{1-p_i^{-1}}\right)-\log\left(\prod_{i=1}^{m-1}\frac{1-p_i^{-\lambda_i'-1}}{1-p_i^{-1}}\right)=\\&=(k)\cdot\log\left(\frac{1-p_i^{-\lambda_i^0-1}}{1-p_i^{-1}}\right)-(k-1)\cdot\log\left(\frac{1-p_i^{-\lambda_i'-1}}{1-p_i^{-1}}\right)-\log\left(1+\frac{1}{p_k}\right)=\\&=(k)\cdot\log\left(\frac{1-p_i^{-\lambda_i^0-1}}{1-p_i^{-\lambda_i'-1}}\right)+\log\left(\frac{1-p_i^{-\lambda_i'-1}}{1-p_k^{-2}}\right)=\\&=(k)\cdot\log\left(1+\frac{p_i^{\lambda_i^0+1}-p_i^{\lambda_i'+1}}{p_i^{\lambda_i^0+1}\cdot\left(p_i^{\lambda_i'+1}-1\right)}\right)+\log\left(1+\frac{p_i^{\lambda_i'+1}-p_k^2}{p_i^{\lambda_i'+1}\cdot\left(p_k^2-1\right)}\right)=\\&=(k)\cdot\left(\frac{p_i^{\lambda_i^0+1}-p_i^{\lambda_i'+1}}{p_i^{\lambda_i^0+1}\cdot\left(p_i^{\lambda_i'+1}-1\right)}\right)-\frac{(k)}{2}\cdot\left(\frac{p_i^{\lambda_i^0+1}-p_i^{\lambda_i'+1}}{p_i^{\lambda_i^0+1}\cdot\left(p_i^{\lambda_i'+1}-1\right)}\right)^2+\end{aligned} \tag{38}$$

$$+\left(\frac{p_i^{\lambda_i'+1}-p_k^2}{p_i^{\lambda_i'+1}\cdot\left(p_k^2-1\right)}\right)-\frac{1}{2}\cdot\left(\frac{p_i^{\lambda_i'+1}-p_k^2}{p_i^{\lambda_i'+1}\cdot\left(p_k^2-1\right)}\right)^2+\beta_1\left(p_m\right)$$

where $\beta_1\left(p_m\right)=\mathrm{O}\left(k\cdot\left(\frac{p_i^{\lambda_i^0+1}-p_i^{\lambda_i'+1}}{p_i^{\lambda_i^0+1}\cdot\left(p_i^{\lambda_i'+1}-1\right)}\right)^3\right)$. (39)

Hence we have

$$\left(\frac{F\left(\bar{\lambda}_0'\right)}{F\left(\bar{\lambda}'\right)}\right)=1+(k)\cdot\left(\frac{p_i^{\lambda_i^0+1}-p_i^{\lambda_i'+1}}{p_i^{\lambda_i^0+1}\cdot\left(p_i^{\lambda_i'+1}-1\right)}\right)+\frac{k^2}{2}\left(\frac{p_i^{\lambda_i^0+1}-p_i^{\lambda_i'+1}}{p_i^{\lambda_i^0+1}\cdot\left(p_i^{\lambda_i'+1}-1\right)}\right)^2-$$

$$-\frac{(k)}{2}\cdot\left(\frac{p_i^{\lambda_i^0+1}-p_i^{\lambda_i'+1}}{p_i^{\lambda_i^0+1}\cdot\left(p_i^{\lambda_i'+1}-1\right)}\right)^2+\left(\frac{p_i^{\lambda_i'+1}-p_k^2}{p_i^{\lambda_i'+1}\cdot\left(p_k^2-1\right)}\right)+\frac{1}{2}\cdot\left(\frac{p_i^{\lambda_i'+1}-p_k^2}{p_i^{\lambda_i'+1}\cdot\left(p_k^2-1\right)}\right)^2-$$

$$-\frac{1}{2}\cdot\left(\frac{p_i^{\lambda_i'+1}-p_k^2}{p_i^{\lambda_i'+1}\cdot\left(p_k^2-1\right)}\right)^2+\beta_1'\left(p_m\right)= \qquad (40)$$

$$=1+(k)\cdot\left(\frac{p_i^{\lambda_i^0+1}-p_i^{\lambda_i'+1}}{p_i^{\lambda_i^0+1}\cdot\left(p_i^{\lambda_i'+1}-1\right)}\right)+\frac{(k)\cdot(k-1)}{2}\cdot\left(\frac{p_i^{\lambda_i^0+1}-p_i^{\lambda_i'+1}}{p_i^{\lambda_i^0+1}\cdot\left(p_i^{\lambda_i'+1}-1\right)}\right)^2+$$

$$+\left(\frac{p_i^{\lambda_i'+1}-p_k^2}{p_i^{\lambda_i'+1}\cdot\left(p_k^2-1\right)}\right)+\beta_1'\left(p_m\right),$$

where $\beta_1'\left(p_m\right)=\mathrm{O}\left(k^3\cdot\left(\frac{p_i^{\lambda_i^0+1}-p_i^{\lambda_i'+1}}{p_i^{\lambda_i^0+1}\cdot\left(p_i^{\lambda_i'+1}-1\right)}\right)^3\right)$. (41)

From (40), the expression (37) is

$$\log\left(\frac{\exp\left(e^{-\gamma}\cdot F\left(\bar{\lambda}_0'\right)\right)}{\exp\left(e^{-\gamma}\cdot F\left(\bar{\lambda}'\right)\right)}\right)=\left(e^{-\gamma}\cdot F\left(\bar{\lambda}'\right)\right)\cdot\left(\frac{F\left(\bar{\lambda}_0'\right)}{F\left(\bar{\lambda}'\right)}-1\right)=$$

$$=\left(e^{-\gamma}\cdot F\left(\bar{\lambda}'\right)\right)\cdot\left((k)\cdot\left(\frac{p_i^{\lambda_i^0+1}-p_i^{\lambda_i'+1}}{p_i^{\lambda_i^0+1}\cdot\left(p_i^{\lambda_i'+1}-1\right)}\right)+\right. \qquad (42)$$

$$\left.+\frac{(k)\cdot(k-1)}{2}\cdot\left(\frac{p_i^{\lambda_i^0+1}-p_i^{\lambda_i'+1}}{p_i^{\lambda_i^0+1}\cdot\left(p_i^{\lambda_i'+1}-1\right)}\right)^2+\left(\frac{p_i^{\lambda_i'+1}-p_k^2}{p_i^{\lambda_i'+1}\cdot\left(p_k^2-1\right)}\right)+\beta_1'\left(p_m\right)\right)$$

and so we have

$$\left(\frac{\exp\left(e^{-\gamma}\cdot F\left(\bar{\lambda}_0'\right)\right)}{\exp\left(e^{-\gamma}\cdot F\left(\bar{\lambda}'\right)\right)}\right)=1+(k)\cdot\left(e^{-\gamma}\cdot F\left(\bar{\lambda}'\right)\right)\cdot\left(\frac{p_i^{\lambda_i^0+1}-p_i^{\lambda_i'+1}}{p_i^{\lambda_i^0+1}\cdot\left(p_i^{\lambda_i'+1}-1\right)}\right)+$$

$$+\frac{k^2}{2}\cdot\left(e^{-\gamma}\cdot F\left(\bar{\lambda}'\right)\right)^2\cdot\left(\frac{p_i^{\lambda_i^0+1}-p_i^{\lambda_i'+1}}{p_i^{\lambda_i^0+1}\cdot\left(p_i^{\lambda_i'+1}-1\right)}\right)^2+$$

$$+\left(e^{-\gamma}\cdot F\left(\bar{\lambda}'\right)\right)\cdot\left(\frac{p_i^{\lambda_i'+1}-p_k^2}{p_i^{\lambda_i'+1}\cdot\left(p_k^2-1\right)}\right)+\frac{\left(e^{-\gamma}\cdot F\left(\bar{\lambda}'\right)\right)^2}{2}\cdot\left(\frac{p_i^{\lambda_i'+1}-p_k^2}{p_i^{\lambda_i'+1}\cdot\left(p_k^2-1\right)}\right)^2+ \tag{43}$$

$$+\frac{(k)\cdot(k-1)}{2}\cdot\left(e^{-\gamma}\cdot F\left(\bar{\lambda}'\right)\right)\cdot\left(\frac{p_i^{\lambda_i^0+1}-p_i^{\lambda_i'+1}}{p_i^{\lambda_i^0+1}\cdot\left(p_i^{\lambda_i'+1}-1\right)}\right)^2+\beta_1''(p_m),$$

where $\beta_1''(p_m)=\mathrm{O}\left(k^3\cdot\left(e^{-\gamma}\cdot F\left(\bar{\lambda}'\right)\right)^3\cdot\left(\frac{p_i^{\lambda_i^0+1}-p_i^{\lambda_i'+1}}{p_i^{\lambda_i^0+1}\cdot\left(p_i^{\lambda_i'+1}-1\right)}\right)^3\right)$. (44)

Hence we have

$$(-R_1)=\exp\left(e^{-\gamma}\cdot F\left(\bar{\lambda}'\right)\right)\cdot\left(\frac{\exp\left(e^{-\gamma}\cdot F\left(\bar{\lambda}_0'\right)\right)}{\exp\left(e^{-\gamma}\cdot F\left(\bar{\lambda}'\right)\right)}-1\right)=$$

$$=\exp\left(e^{-\gamma}\cdot F\left(\bar{\lambda}'\right)\right)\cdot\left((k)\cdot\left(e^{-\gamma}\cdot F\left(\bar{\lambda}'\right)\right)\cdot\left(\frac{p_i^{\lambda_i^0+1}-p_i^{\lambda_i'+1}}{p_i^{\lambda_i^0+1}\cdot\left(p_i^{\lambda_i'+1}-1\right)}\right)+\right.$$

$$+\frac{k^2}{2}\cdot\left(e^{-\gamma}\cdot F\left(\bar{\lambda}'\right)\right)^2\cdot\left(\frac{p_i^{\lambda_i^0+1}-p_i^{\lambda_i'+1}}{p_i^{\lambda_i^0+1}\cdot\left(p_i^{\lambda_i'+1}-1\right)}\right)^2+\left(e^{-\gamma}\cdot F\left(\bar{\lambda}'\right)\right)\cdot\left(\frac{p_i^{\lambda_i'+1}-p_k^2}{p_i^{\lambda_i'+1}\cdot\left(p_k^2-1\right)}\right)+$$

$$+\frac{1}{2}\cdot\left(e^{-\gamma}\cdot F\left(\bar{\lambda}'\right)\right)^2\cdot\left(\frac{p_i^{\lambda_i'+1}-p_k^2}{p_i^{\lambda_i'+1}\cdot\left(p_k^2-1\right)}\right)^2+$$

$$\left.+\frac{(k)\cdot(k-1)}{2}\cdot\left(e^{-\gamma}\cdot F\left(\bar{\lambda}'\right)\right)\cdot\left(\frac{p_i^{\lambda_i^0+1}-p_i^{\lambda_i'+1}}{p_i^{\lambda_i^0+1}\cdot\left(p_i^{\lambda_i'+1}-1\right)}\right)^2+\beta_1''(p_m)\right). \tag{45}$$

From (12), since $\left(e^{-\gamma}F\left(\overline{\lambda}'\right)\right)\cdot\exp\left(e^{-\gamma}F\left(\overline{\lambda}'\right)\right)=p_i^{\lambda_i'+1}-1\quad\left(1\le i\le k-1\right)$,

we have

$$\begin{aligned}\left(-R_1\right)&=\left(k\right)\cdot\left(\frac{p_i^{\lambda_i^0+1}-p_i^{\lambda_i'+1}}{p_i^{\lambda_i^0+1}}\right)+\left(\frac{p_i^{\lambda_i'+1}-1}{p_i^{\lambda_i'+1}}\right)\cdot\left(\frac{p_i^{\lambda_i'+1}-p_k^2}{p_k^2-1}\right)+\\&+\frac{k^2}{2}\cdot\frac{\left(e^{-\gamma}\cdot F\left(\overline{\lambda}'\right)\right)}{\left(p_i^{\lambda_i'+1}-1\right)}\cdot\left(\frac{p_i^{\lambda_i^0+1}-p_i^{\lambda_i'+1}}{p_i^{\lambda_i^0+1}}\right)^2+\\&+\frac{1}{2}\cdot\frac{\left(p_i^{\lambda_i'+1}-1\right)}{p_i^{\lambda_i'+1}}\cdot\frac{\left(e^{-\gamma}\cdot F\left(\overline{\lambda}'\right)\right)}{p_i^{\lambda_i'+1}}\cdot\left(\frac{p_i^{\lambda_i'+1}-p_k^2}{p_k^2-1}\right)^2+\\&+\frac{1}{2}\cdot\frac{\left(k\right)\cdot\left(k-1\right)}{\left(p_i^{\lambda_i'+1}-1\right)}\cdot\left(\frac{p_i^{\lambda_i^0+1}-p_i^{\lambda_i'+1}}{p_i^{\lambda_i^0+1}}\right)^2+\beta_1'''\left(p_m\right),\end{aligned}\tag{46}$$

where

$$\begin{aligned}\beta_1'''\left(p_m\right)&=\exp\left(e^{-\gamma}\cdot F\left(\overline{\lambda}'\right)\right)\cdot\beta_1''\left(p_m\right)=\\&=\mathrm{O}\left(k^3\cdot\left(\frac{e^{-\gamma}\cdot F\left(\overline{\lambda}'\right)}{p_i^{\lambda_i'+1}-1}\right)^2\cdot\left(\frac{p_i^{\lambda_i^0+1}-p_i^{\lambda_i'+1}}{p_i^{\lambda_i^0+1}}\right)^3\right).\end{aligned}\tag{47}$$

Next let's see R_2. It is not difficult to see that for any $i\left(1\le i\le k-1\right)$

$$\begin{aligned}\left(-R_2\right)&=\left(\log n_0'-\log n'\right)=\log\left(\frac{p_1^{\lambda_1^0}\cdot p_2^{\lambda_2^0}\cdots p_k^{\lambda_k^0}\,p_{k+1}^1\cdots p_{m-1}^1}{p_1^{\lambda_1'}\,p_2^{\lambda_2'}\cdots p_{k-1}^{\lambda_k'}\cdot p_k^1\cdots p_{m-1}^1}\right)=\\&=\left(k\right)\cdot\log\left(\frac{p_i^{\lambda_i^0+1}}{p_i^{\lambda_i'+1}}\right)-\log\left(\frac{p_i^{\lambda_i^0+1}}{p_i^{\lambda_i'+1}}\right)+\log\left(\frac{p_k^{\lambda_k^0+1}}{p_k^2}\right)=\\&=\left(k\right)\cdot\log\left(1+\frac{p_i^{\lambda_i^0+1}-p_i^{\lambda_i'+1}}{p_i^{\lambda_i'+1}}\right)+\log\left(1+\frac{p_i^{\lambda_i'+1}-p_k^2}{p_k^2}\right)=\\&=\left(k\right)\cdot\left(\frac{p_i^{\lambda_i^0+1}-p_i^{\lambda_i'+1}}{p_i^{\lambda_i'+1}}\right)-\left(\frac{k}{2}\right)\cdot\left(\frac{p_i^{\lambda_i^0+1}-p_i^{\lambda_i'+1}}{p_i^{\lambda_i'+1}}\right)^2+\\&+\left(\frac{p_i^{\lambda_i'+1}-p_k^2}{p_k^2}\right)-\frac{1}{2}\cdot\left(\frac{p_i^{\lambda_i'+1}-p_k^2}{p_k^2}\right)^2+\beta_2\left(p_m\right),\end{aligned}\tag{48}$$

where $\beta_2\left(p_m\right) = \mathrm{O}\left(k \cdot \left(\frac{p_i^{\lambda_i^0+1} - p_i^{\lambda_i'+1}}{p_i^{\lambda_i'+1}}\right)^3\right)$. (49)

Therefore we have

$$\left(-\right)\cdot\left(\log C_{m-1} - \log C'_{m-1}\right) = -\left(R_1 - R_2\right) =$$
$$= \left(k\right)\cdot\left(\frac{p_i^{\lambda_i^0+1} - p_i^{\lambda_i'+1}}{p_i^{\lambda_i^0+1}}\right) + \left(\frac{p_i^{\lambda_i'+1} - 1}{p_i^{\lambda_i'+1}}\right)\cdot\left(\frac{p_i^{\lambda_i'+1} - p_k^2}{p_k^2 - 1}\right) +$$
$$+ \left(\frac{k^2}{2}\right)\cdot\frac{\left(e^{-\gamma}\cdot F\left(\overline{\lambda}'\right)\right)}{\left(p_i^{\lambda_i'+1} - 1\right)}\cdot\left(\frac{p_i^{\lambda_i^0+1} - p_i^{\lambda_i'+1}}{p_i^{\lambda_i^0+1}}\right)^2 +$$
$$+ \frac{1}{2}\cdot\frac{\left(p_i^{\lambda_i'+1} - 1\right)}{p_i^{\lambda_i'+1}}\cdot\frac{\left(e^{-\gamma}\cdot F\left(\overline{\lambda}'\right)\right)}{p_i^{\lambda_i'+1}}\cdot\left(\frac{p_i^{\lambda_i'+1} - p_k^2}{p_k^2 - 1}\right)^2 +$$
$$+ \frac{1}{2}\cdot\frac{\left(k\right)\cdot\left(k-1\right)}{\left(p_i^{\lambda_i'+1} - 1\right)}\cdot\left(\frac{p_i^{\lambda_i^0+1} - p_i^{\lambda_i'+1}}{p_i^{\lambda_i^0+1}}\right)^2 + \beta_1'''\left(p_m\right) -$$
$$- \left(k\right)\cdot\left(\frac{p_i^{\lambda_i^0+1} - p_{k-1}^{\lambda_i'+1}}{p_i^{\lambda_i'+1}}\right) + \left(\frac{k}{2}\right)\cdot\left(\frac{p_i^{\lambda_i^0+1} - p_i^{\lambda_i'+1}}{p_i^{\lambda_i'+1}}\right)^2 -$$
$$- \left(\frac{p_i^{\lambda_i'+1} - p_k^2}{p_k^2}\right) + \frac{1}{2}\cdot\left(\frac{p_i^{\lambda_i'+1} - p_k^2}{p_k^2}\right)^2 - \beta_2\left(p_m\right). \quad (50)$$

Here the term without k in its coefficients is

$$\frac{1}{2}\cdot\frac{\left(p_i^{\lambda_i'+1} - 1\right)}{p_i^{\lambda_i'+1}}\cdot\frac{\left(e^{-\gamma}\cdot F\left(\overline{\lambda}'\right)\right)}{p_i^{\lambda_i'+1}}\cdot\left(\frac{p_i^{\lambda_i'+1} - p_k^2}{p_k^2 - 1}\right)^2 +$$
$$+ \frac{\left(p_i^{\lambda_i'+1} - 1\right)}{p_i^{\lambda_i'+1}}\cdot\frac{p_i^{\lambda_i'+1} - p_k^2}{\left(p_k^2 - 1\right)} - \left(\frac{p_i^{\lambda_i'+1} - p_k^2}{p_k^2}\right) + \frac{1}{2}\cdot\left(\frac{p_i^{\lambda_i'+1} - p_k^2}{p_k^2}\right)^2 =$$
$$= \frac{1}{2}\cdot\left(\frac{p_i^{\lambda_i'+1} - p_k^2}{p_k^2}\right)^2\cdot\left(\frac{\left(p_i^{\lambda_i'+1} - 1\right)}{p_i^{\lambda_i'+1}}\cdot\frac{\left(e^{-\gamma}\cdot F\left(\overline{\lambda}'\right)\right)}{p_i^{\lambda_i'+1}}\cdot\left(\frac{p_k^2}{p_k^2 - 1}\right)^2 + 1\right) + \quad (51)$$
$$+ \left(\frac{p_i^{\lambda_i'+1} - p_k^2}{p_k^2}\right)\cdot\left(\frac{\left(p_i^{\lambda_i'+1} - 1\right)}{p_i^{\lambda_i'+1}}\cdot\frac{p_k^2}{\left(p_k^2 - 1\right)} - 1\right) =$$

$$=\frac{1}{2}\cdot\left(\frac{p_i^{\lambda_i'+1}-p_k^2}{p_k^2}\right)^2\cdot\left(1+\frac{\left(p_i^{\lambda_i'+1}-1\right)}{p_i^{\lambda_i'+1}}\cdot\frac{\left(e^{-\gamma}\cdot F\left(\overline{\lambda}'\right)\right)}{p_i^{\lambda_i'+1}}\cdot\left(\frac{p_k^2}{p_k^2-1}\right)^2\right)+$$

$$+\left(\frac{p_i^{\lambda_i'+1}-p_k^2}{p_k^2}\right)\cdot\left(\frac{\left(p_i^{\lambda_i'+1}-1\right)\cdot p_k^2-p_i^{\lambda_i'+1}\cdot\left(p_k^2-1\right)}{p_i^{\lambda_i'+1}\cdot\left(p_k^2-1\right)}\right)=$$

$$=\frac{1}{2}\cdot\left(\frac{p_i^{\lambda_i'+1}-p_k^2}{p_k^2}\right)^2\cdot\left(1+\frac{\left(p_i^{\lambda_i'+1}-1\right)}{p_i^{\lambda_i'+1}}\cdot\frac{\left(e^{-\gamma}\cdot F\left(\overline{\lambda}'\right)\right)}{p_i^{\lambda_i'+1}}\cdot\left(\frac{p_k^2}{p_k^2-1}\right)^2\right)+$$

$$+\left(\frac{p_i^{\lambda_i'+1}-p_k^2}{p_k^2}\right)^2\cdot\left(\frac{p_k^2}{p_i^{\lambda_i'+1}\cdot\left(p_k^2-1\right)}\right)=$$

$$=\frac{k}{2}\cdot\left(\frac{p_i^{\lambda_i^0+1}-p_i^{\lambda_i'+1}}{p_i^{\lambda_i'+1}}\right)^2\cdot\alpha_1\left(p_m\right),$$

where

$$\alpha_1\left(p_m\right)=\frac{1}{k}\cdot\left(\frac{p_i^{\lambda_i'+1}-p_k^2}{p_k^2}\right)^2\cdot\left(\frac{p_i^{\lambda_i^0+1}-p_i^{\lambda_i'+1}}{p_i^{\lambda_i'+1}}\right)^{-2}\times$$

$$\times\left(1+\frac{\left(p_i^{\lambda_i'+1}-1\right)}{p_i^{\lambda_i'+1}}\cdot\frac{\left(e^{-\gamma}\cdot F\left(\overline{\lambda}'\right)\right)}{p_i^{\lambda_i'+1}}\cdot\left(\frac{p_k^2}{p_k^2-1}\right)^2+\frac{2\cdot p_k^2}{p_i^{\lambda_i'+1}\cdot\left(p_k^2-1\right)}\right)= \qquad (52)$$

$$=\mathrm{O}\left(\sqrt{\frac{\log p_m}{p_m}}\right).$$

and the term with k in its coefficients is

$$k\cdot\left(\frac{p_i^{\lambda_i^0+1}-p_i^{\lambda_i'+1}}{p_i^{\lambda_i^0+1}}\right)-k\cdot\left(\frac{p_i^{\lambda_i^0+1}-p_i^{\lambda_i'+1}}{p_i^{\lambda_i'+1}}\right)+\frac{k}{2}\cdot\left(\frac{p_i^{\lambda_i^0+1}-p_i^{\lambda_i'+1}}{p_i^{\lambda_i'+1}}\right)^2=$$

$$=k\cdot\left(\frac{p_i^{\lambda_i^0+1}-p_i^{\lambda_i'+1}}{p_i^{\lambda_i^0+1}}-\frac{p_i^{\lambda_i^0+1}-p_i^{\lambda_i'+1}}{p_i^{\lambda_i'+1}}\right)+\frac{k}{2}\cdot\left(\frac{p_i^{\lambda_i^0+1}-p_i^{\lambda_i'+1}}{p_i^{\lambda_i'+1}}\right)^2=$$

$$=-k\cdot\frac{\left(p_i^{\lambda_i^0+1}-p_i^{\lambda_i'+1}\right)^2}{p_i^{\lambda_i^0+1}\cdot p_i^{\lambda_i'+1}}+\frac{k}{2}\cdot\left(\frac{p_i^{\lambda_i^0+1}-p_i^{\lambda_i'+1}}{p_i^{\lambda_i'+1}}\right)^2= \qquad (53)$$

$$=-\frac{k}{2}\cdot\left(\frac{p_i^{\lambda_i^0+1}-p_i^{\lambda_i'+1}}{p_i^{\lambda_i'+1}}\right)^2\cdot\left(2\cdot\frac{p_i^{\lambda_i'+1}}{p_i^{\lambda_i^0+1}}-1\right)=$$

$$=-\frac{k}{2}\cdot\left(\frac{p_i^{\lambda_i^0+1}-p_i^{\lambda_i'+1}}{p_i^{\lambda_i'+1}}\right)^2\cdot\left(1-2\cdot\frac{p_i^{\lambda_i^0+1}-p_i^{\lambda_i'+1}}{p_i^{\lambda_i^0+1}}\right)=$$

$$=-\frac{k}{2}\cdot\left(\frac{p_i^{\lambda_i^0+1}-p_i^{\lambda_i'+1}}{p_i^{\lambda_i'+1}}\right)^2\cdot\left(1+\alpha_2\left(p_m\right)\right),$$

where $\alpha_2\left(p_m\right)=-2\cdot\frac{p_i^{\lambda_i^0+1}-p_i^{\lambda_i'+1}}{p_i^{\lambda_i^0+1}}=\mathrm{O}\left(\frac{1}{p_m^{\theta}}\right)\ \left(0<\theta<1/2\right).$ (54)

And the term with k^2 in its coefficients is

$$\frac{k^2}{2}\cdot\frac{\left(e^{-\gamma}\cdot F\left(\overline{\lambda}'\right)\right)}{\left(p_i^{\lambda_i'+1}-1\right)}\cdot\left(\frac{p_i^{\lambda_i^0+1}-p_i^{\lambda_i'+1}}{p_i^{\lambda_i^0+1}}\right)^2+\frac{1}{2}\cdot\frac{(k)\cdot(k-1)}{\left(p_i^{\lambda_i'+1}-1\right)}\cdot\left(\frac{p_i^{\lambda_i^0+1}-p_i^{\lambda_i'+1}}{p_i^{\lambda_i^0+1}}\right)^2=$$

$$=\frac{k^2}{2}\cdot\left(\frac{p_i^{\lambda_i^0+1}-p_i^{\lambda_i'+1}}{p_i^{\lambda_i^0+1}}\right)^2\cdot\left(\frac{\left(e^{-\gamma}\cdot F\left(\overline{\lambda}'\right)\right)}{\left(p_i^{\lambda_i'+1}-1\right)}+\frac{k-1}{k\cdot\left(p_i^{\lambda_i'+1}-1\right)}\right)=$$

$$=\frac{k^2}{2}\cdot\left(\frac{p_i^{\lambda_i^0+1}-p_i^{\lambda_i'+1}}{p_i^{\lambda_i'+1}}\right)^2\cdot\left(\frac{p_i^{\lambda_i'+1}}{p_i^{\lambda_i^0+1}}\right)\cdot\left(\frac{\left(e^{-\gamma}\cdot F\left(\overline{\lambda}'\right)\right)}{\left(p_i^{\lambda_i'+1}-1\right)}+\frac{k-1}{k\cdot\left(p_i^{\lambda_i'+1}-1\right)}\right)=$$

$$=\frac{k}{2}\cdot\left(\frac{p_i^{\lambda_i^0+1}-p_i^{\lambda_i'+1}}{p_i^{\lambda_i'+1}}\right)^2\cdot\alpha_3\left(p_m\right),$$

(55)

where

$$\alpha_3\left(p_m\right)=k\cdot\left(\frac{p_i^{\lambda_i'+1}}{p_i^{\lambda_i^0+1}}\right)\cdot\left(\frac{\left(e^{-\gamma}\cdot F\left(\overline{\lambda}'\right)\right)}{\left(p_i^{\lambda_i'+1}-1\right)}+\frac{k-1}{k\cdot\left(p_i^{\lambda_i'+1}-1\right)}\right)=$$
$$=\mathrm{O}\left(\sqrt{\frac{\log p_m}{p_m}}\right).$$ (56)

And we have

$$\beta_1'''\left(p_m\right)-\beta_2\left(p_m\right)=\left(\frac{k}{2}\right)\cdot\left(\frac{p_i^{\lambda_i^0+1}-p_i^{\lambda_i'+1}}{p_i^{\lambda_i'+1}}\right)^2\cdot\beta_0\left(p_m\right),$$ (57)

where $\beta_0\left(p_m\right)=\mathrm{O}\left(\frac{1}{p_m^{\theta}}\right)\left(0<\theta<1/2\right)$. Hence we have

$$\log\left(\frac{C'_{m-1}}{C_{m-1}}\right) = -\frac{k}{2}\cdot\left(\frac{p_i^{\lambda_i^0+1} - p_i^{\lambda_i'+1}}{p_i^{\lambda_i'+1}}\right)^2\cdot\left(1+\alpha_0\left(p_m\right)\right) \tag{58}$$

where

$$\begin{gathered}\alpha_0\left(p_m\right) = \alpha_2\left(p_m\right) - \alpha_1\left(p_m\right) - \alpha_3\left(p_m\right) + \\ + \beta_1'''\left(p_m\right) - \beta_2\left(p_m\right) = \mathrm{O}\left(\frac{1}{p_m^{\theta}}\right).\end{gathered} \tag{59}$$

On the other hand, by (11) and (12) we have

$$p_i^{\lambda_i^0+1} = p_m\cdot\log p_m\cdot\left(1+\varepsilon_1\left(p_m\right)\right),\ \ \varepsilon_1\left(p_m\right) = \mathrm{O}\left(\frac{1}{\log p_m}\right) \tag{60}$$

and

$$p_i^{\lambda_i'+1} = p_{m-1}\cdot\log p_{m-1}\cdot\left(1+\varepsilon_2\left(p_{m-1}\right)\right),\ \ \varepsilon_2\left(p_{m-1}\right) = \mathrm{O}\left(\frac{1}{\log p_m}\right). \tag{61}$$

and

$$k = 2\cdot\sqrt{\frac{p_{m-1}}{\log p_{m-1}}}\left(1+\varepsilon_3\left(p_m\right)\right),\ \ \varepsilon_3\left(p_m\right) = \mathrm{O}\left(\frac{\log\log p_m}{\log p_m}\right).$$

Hence

$$\frac{p_i^{\lambda_i^0+1} - p_i^{\lambda_i'+1}}{p_i^{\lambda_i'+1}} = \frac{p_m - p_{m-1}}{p_{m-1}}\cdot\left(1+\varepsilon_4\left(p_m\right)\right), \tag{62}$$

where $\varepsilon_4\left(p_m\right) = \mathrm{O}\left(\frac{1}{\log p_m}\right)$. From this we have

$$\begin{gathered}\log C_{m-1} - \log C'_{m-1} = \frac{k}{2}\cdot\left(\frac{p_i^{\lambda_i^0+1} - p_i^{\lambda_i'+1}}{p_i^{\lambda_i'+1}}\right)^2\cdot\left(1+\alpha_0\left(p_m\right)\right) = \\ = \frac{\left(p_m - p_{m-1}\right)}{\sqrt{p_{m-1}\cdot\log p_{m-1}}}\cdot\left(\frac{p_m - p_{m-1}}{p_{m-1}}\right)\cdot\left(1+\beta_0\left(p_m\right)\right).\end{gathered} \tag{63}$$

where

$$1+\beta_0(p_m)=(1+\varepsilon_1(p_m))\cdot(1+\varepsilon_2(p_m))^2\cdot(1+\alpha_0(p_m))= \\ =1+\mathrm{O}\left(\frac{1}{\log p_m}\right). \tag{64}$$

This is the proof of the theorem 2. □

4. The estimate of $(p_m - p_{m-1})$

We have

Theorem 3. There exist a number m_0 such that for any $m \geq m_0$ we have

$$(p_m - p_{m-1}) = \mathrm{O}\left(\sqrt{p_{m-1}}\cdot\log^{5/2} p_{m-1}\right). \tag{65}$$

Proof. It is easy to see that

$$\log C_m - \log C'_m = \left(\exp\left(e^{-\gamma}\cdot F\left(\bar{\lambda}_0\right)\right) - \log n_0\right) - \left(\exp\left(e^{-\gamma}\cdot F\left(\bar{\lambda}'_+\right)\right) - \log n'_+\right) = \\ = \left(\exp\left(e^{-\gamma}\cdot F\left(\bar{\lambda}'_0\right)\right) - \exp\left(e^{-\gamma}\cdot F\left(\bar{\lambda}'\right)\right)\right) - \left(\log n'_0 - \log n'\right) + \\ + \left(\exp\left(e^{-\gamma}\cdot F\left(\bar{\lambda}_0\right)\right) - \exp\left(e^{-\gamma}\cdot F\left(\bar{\lambda}'_0\right)\right)\right) - \\ - \left(\exp\left(e^{-\gamma}\cdot F\left(\bar{\lambda}'_+\right)\right) - \exp\left(e^{-\gamma}\cdot F\left(\bar{\lambda}'\right)\right)\right) = \\ = \left(\log C'_{m-1} - \log C_{m-1}\right) + R_0 , \tag{66}$$

where

$$R_0 = \left(\exp\left(e^{-\gamma}\cdot F\left(\bar{\lambda}_0\right)\right) - \exp\left(e^{-\gamma}\cdot F\left(\bar{\lambda}'_0\right)\right)\right) - \\ - \left(\exp\left(e^{-\gamma}\cdot F\left(\bar{\lambda}'_+\right)\right) - \exp\left(e^{-\gamma}\cdot F\left(\bar{\lambda}'\right)\right)\right). \tag{67}$$

On other hand, by (18) in the theorem 1 we have

$$\left(\exp\left(e^{-\gamma}\cdot F\left(\bar{\lambda}_0\right)\right) - \exp\left(e^{-\gamma}\cdot F\left(\bar{\lambda}'_0\right)\right)\right) = \\ = \exp\left(e^{-\gamma}\cdot F\left(\bar{\lambda}'_0\right)\right)\cdot\left(\exp\left(e^{-\gamma}\cdot F\left(\bar{\lambda}'_0\right)\cdot\frac{1}{p_m}\right) - 1\right) = \tag{68}$$

$$=\frac{1}{p_m}\cdot\left(e^{-\gamma}\cdot F\left(\overline{\lambda}_0'\right)\right)\cdot\exp\left(e^{-\gamma}\cdot F\left(\overline{\lambda}_0'\right)\right)\cdot\left(1+\mathrm{O}\left(\frac{\log p_m}{p_m}\right)\right)=$$

$$=\frac{1}{p_m}\cdot\left(\left(e^{-\gamma}\cdot F\left(\overline{\lambda}_0\right)\right)\cdot\exp\left(e^{-\gamma}\cdot F\left(\overline{\lambda}_0\right)\right)\right)\times$$

$$\times\left(\exp\left(-e^{-\gamma}\cdot F\left(\overline{\lambda}_0'\right)\cdot\frac{1}{p_m}\right)\right)\cdot\left(1-\frac{1}{p_m+1}\right)\cdot\left(1+\mathrm{O}\left(\frac{\log p_m}{p_m}\right)\right)=$$

$$=\frac{1}{p_m}\cdot\left(\left(e^{-\gamma}\cdot F\left(\overline{\lambda}_0\right)\right)\cdot\exp\left(e^{-\gamma}\cdot F\left(\overline{\lambda}_0\right)\right)\right)\cdot\left(1+\mathrm{O}\left(\frac{\log p_m}{p_m}\right)\right)=$$

$$=\log p_m\cdot\left(1+\left(\log p_m+1\right)\cdot E_0\left(p_m\right)-\frac{4\cdot\left(\log p_m+1\right)}{\sqrt{p_m}\log^{3/2}p_m}+\Theta_1\left(p_m\right)\right),$$

where $\Theta_1\left(p_m\right)=\mathrm{O}\left(\log^2 p_m\cdot E_0^2\left(p_m\right)\right)$.

And we have

$$\exp\left(e^{-\gamma}\cdot F\left(\overline{\lambda}'\right)\right)\cdot\left(\exp\left(e^{-\gamma}\cdot F\left(\overline{\lambda}'\right)\cdot\frac{1}{p_m}\right)-1\right)=$$

$$=\frac{p_{m-1}}{p_m}\cdot\log p_{m-1}\cdot\times\left(1+\left(\log p_{m-1}+1\right)\cdot E_0\left(p_{m-1}\right)-\right.\tag{69}$$

$$\left.-\frac{4\cdot\left(\log p_{m-1}+1\right)}{\sqrt{p_{m-1}}\log^{3/2}p_{m-1}}+\Theta_2\left(p_{m-1}\right)\right),$$

where $\Theta_2\left(p_{m-1}\right)=\mathrm{O}\left(\log^2 p_{m-1}\cdot E_0^2\left(p_{m-1}\right)\right)$.

From this we obtain

$$R_0=\log p_m\cdot\left(1+\left(\log p_m+1\right)\cdot E_0\left(p_m\right)-\frac{4\cdot\left(\log p_m+1\right)}{\sqrt{p_m}\log^{3/2}p_m}+\Theta_1\left(p_m\right)\right)-$$

$$-\frac{p_{m-1}\cdot\log p_{m-1}}{p_m}\cdot\left(1+\left(\log p_{m-1}+1\right)\cdot E_0\left(p_{m-1}\right)-\right.$$

$$\left.-\frac{4\cdot\left(\log p_{m-1}+1\right)}{\sqrt{p_{m-1}}\log^{3/2}p_{m-1}}+\Theta_2\left(p_{m-1}\right)\right)=\tag{70}$$

$$=\left(\log p_m-\frac{p_{m-1}}{p_m}\cdot\log p_{m-1}\right)+\log p_m\cdot\left(\log p_m+1\right)\cdot E_0\left(p_m\right)-$$

$$-\frac{p_{m-1}\cdot\log p_{m-1}}{p_m}\cdot\left(\log p_{m-1}+1\right)\cdot E_0\left(p_{m-1}\right)+$$

$$+\frac{4\cdot\log p_m\cdot\left(\log p_m+1\right)}{\sqrt{p_m}\log^{3/2}p_m}-\frac{p_{m-1}\cdot\log p_{m-1}}{p_m}\cdot\frac{4\cdot\left(\log p_{m-1}+1\right)}{\sqrt{p_{m-1}}\log^{3/2}p_{m-1}}+$$

$$+\log p_m\cdot\Theta_1\left(p_m\right)-\frac{p_{m-1}\cdot\log p_{m-1}}{p_m}\cdot\Theta_2\left(p_{m-1}\right).$$

Here we have

$$\begin{aligned}
&\log p_m-\frac{p_{m-1}}{p_m}\cdot\log p_{m-1}=\log p_m-\left(1-\frac{p_m-p_{m-1}}{p_m}\right)\cdot\log p_{m-1}=\\
&=\left(\log p_m-\log p_{m-1}\right)+\left(\frac{p_m-p_{m-1}}{p_m}\right)\cdot\log p_{m-1}=\\
&=\log\left(1+\frac{p_m-p_{m-1}}{p_{m-1}}\right)+\left(\frac{p_m-p_{m-1}}{p_m}\right)\cdot\log p_{m-1}= \qquad (71)\\
&=\frac{p_m-p_{m-1}}{p_{m-1}}-\frac{1}{2}\left(\frac{p_m-p_{m-1}}{p_{m-1}}\right)^2+\left(\frac{p_m-p_{m-1}}{p_m}\right)\cdot\log p_{m-1}=\\
&=\frac{p_m-p_{m-1}}{p_{m-1}}\cdot\log p_{m-1}\cdot r_1\left(p_m\right),
\end{aligned}$$

where $r_1\left(p_m\right)=1+O\left(\frac{1}{\log p_m}\right)=O(1)$.

By (14) it is true that

$$\begin{aligned}
&E_0\left(p_m\right)=\sum_{i=1}^{m}\frac{1}{p_i}-\log\log p_m-b_0=\\
&=\left(\sum_{i=1}^{m-1}\frac{1}{p_i}-\log\log p_{m-1}-b_0\right)-\log\log p_m+\log\log p_{m-1}+\frac{1}{p_m}= \qquad (72)\\
&=E_0\left(p_{m-1}\right)-\log\left(\frac{\log p_m}{\log p_{m-1}}\right)+\frac{1}{p_m}.
\end{aligned}$$

Hence we have

$$
\begin{aligned}
&\log p_m \cdot (\log p_m + 1)\cdot E_0(p_m) - \frac{p_{m-1}\cdot \log p_{m-1}}{p_m}\cdot(\log p_{m-1}+1)\cdot E_0(p_{m-1}) = \\
&= \log p_m \cdot (\log p_m + 1)\cdot E_0(p_m) - \log p_{m-1}\cdot(\log p_{m-1}+1)\cdot E_0(p_{m-1}) + \\
&+ \left(\frac{p_m - p_{m-1}}{p_m}\right)\log p_{m-1}\cdot(\log p_{m-1}+1)\cdot E_0(p_{m-1}) = \\
&= \log p_m \cdot (\log p_m + 1)\cdot\left(E_0(p_m) - E_0(p_{m-1})\right) + \\
&+ \left(\log p_m \cdot(\log p_m+1) - \log p_{m-1}\cdot(\log p_{m-1}+1)\right)\cdot E_0(p_{m-1}) + \\
&+ \left(\frac{p_m - p_{m-1}}{p_m}\right)\log p_{m-1}\cdot(\log p_{m-1}+1)\cdot E_0(p_{m-1}) = \\
&= \log p_m \cdot (\log p_m + 1)\cdot\left(-\log\left(\frac{\log p_m}{\log p_{m-1}}\right) + \frac{1}{p_m}\right) + \\
&+ \left(\log p_m \cdot(\log p_m+1) - \log p_{m-1}\cdot(\log p_{m-1}+1)\right)\cdot E_0(p_{m-1}) + \\
&+ \left(\frac{p_m - p_{m-1}}{p_m}\right)\log p_{m-1}\cdot(\log p_{m-1}+1)\cdot E_0(p_{m-1}) = \\
&= \left(\frac{p_m - p_{m-1}}{p_{m-1}}\right)\log^2 p_{m-1}\cdot r_2(p_m),
\end{aligned}
\tag{73}
$$

where $r_2(p_m) = \mathrm{O}\left(\frac{1}{p_m - p_{m-1}} + \frac{1}{\log p_m}\right) = \mathrm{O}(1)$.

Next, it is clear that

$$
\begin{aligned}
&\frac{4\cdot\log p_m\cdot(\log p_m+1)}{\sqrt{p_m}\log^{3/2} p_m} - \frac{p_{m-1}\cdot\log p_{m-1}}{p_m}\cdot\frac{4\cdot(\log p_{m-1}+1)}{\sqrt{p_{m-1}}\log^{3/2} p_{m-1}} = \\
&= \frac{p_m - p_{m-1}}{p_{m-1}}\cdot\log p_{m-1}\cdot r_3(p_m),
\end{aligned}
\tag{74}
$$

and

$$
\begin{aligned}
&\log p_m\cdot\Theta_1(p_m) - \frac{p_{m-1}\cdot\log p_{m-1}}{p_m}\cdot\Theta_2(p_{m-1}) = \\
&= \frac{p_m - p_{m-1}}{p_{m-1}}\cdot\log p_{m-1}\cdot r_4(p_m)
\end{aligned}
\tag{75}
$$

where $r_3(p_m) = \mathrm{O}\left(\frac{\log p_m}{\sqrt{p_m}}\right)$ and $r_4(p_m) = \mathrm{O}\left(\frac{1}{\log p_m}\right)$.

Therefore we have

$$R_0 = \frac{p_m - p_{m-1}}{p_{m-1}} \cdot \log^2 p_{m-1} \cdot \delta_0(p_m), \tag{76}$$

where $\delta_0(p_m) = r_2(p_m) + \frac{1}{\log p_m} \cdot \left(r_1(p_m) + r_3(p_m) r_4(p_m)\right) = \mathrm{O}(1)$.

On the other hand, since $C'_m \le C_m$ we have

$$0 < \left(\log C_{m-1} - \log C'_{m-1}\right) < R_0. \tag{77}$$

Thus from the theorem 2, we have

$$\begin{aligned} &\frac{(p_m - p_{m-1})}{\sqrt{p_{m-1} \cdot \log p_{m-1}}} \cdot \left(\frac{p_m - p_{m-1}}{p_{m-1}}\right) \cdot \left(1 + \beta_0(p_m)\right) \le \\ &\le \left(\frac{p_m - p_{m-1}}{p_{m-1}}\right) \cdot \log^2 p_{m-1} \cdot \delta_0(p_m). \end{aligned} \tag{78}$$

Therefore we have

$$(p_m - p_{m-1}) = \mathrm{O}\left(\sqrt{p_{m-1}} \cdot \log^{5/2} p_{m-1}\right). \tag{79}$$

This is the proof of the theorem 3. □

5. The estimate of $E_0(p_m)$

We get

Theorem 4. There exists a number m_0 such that for any $m \ge m_0$ we have

$$E_0(p_m) = \mathrm{O}\left(\frac{\log^2 p_m}{\sqrt{p_m}}\right). \tag{80}$$

Proof. Since

$$F(\bar{\lambda}_0) = F(\bar{\lambda}'_0) \cdot \left(1 + \frac{1}{p_m}\right), \tag{81}$$

we have

$$\log p_m \cdot \left(1 + E_0\left(p_m\right) - \frac{4}{\sqrt{p_m}\log^{3/2} p_m} + \varepsilon\left(p_m\right) \right) =$$
$$= \log p_{m-1} \cdot \left(1 + E_0\left(p_{m-1}\right) - \frac{4}{\sqrt{p_m}\log^{3/2} p_m} + \varepsilon\left(p_{m-1}\right) \right) \cdot \left(1 + \frac{1}{p_m} \right). \tag{82}$$

From this we have

$$\log p_m \cdot E_0\left(p_m\right) - \log p_{m-1} \cdot \left(1 + \frac{1}{p_m} \right) \cdot E_0\left(p_{m-1}\right) =$$
$$= -\left(\log p_m - \log p_{m-1} - \frac{\log p_{m-1}}{p_m} \right) + \tag{83}$$
$$+ \frac{4}{\sqrt{p_m}\log^{3/2} p_m} \cdot \left(\log p_m - \log p_{m-1} - \frac{\log p_{m-1}}{p_m} \right) +$$
$$+ \log p_m \cdot \varepsilon\left(p_m\right) - \log p_{m-1} \cdot \left(1 + \frac{1}{p_m} \right) \cdot \varepsilon\left(p_{m-1}\right).$$

From (73) the left hand side of (83) is

$$\log p_m \cdot E_0\left(p_m\right) - \log p_{m-1} \cdot \left(1 + \frac{1}{p_m} \right) \cdot E_0\left(p_{m-1}\right) =$$
$$= \log p_m \cdot \left(E_0\left(p_{m-1}\right) - \log\left(\frac{\log p_m}{\log p_{m-1}} \right) + \frac{1}{p_m} \right) -$$
$$- \log p_{m-1} \cdot \left(1 + \frac{1}{p_m} \right) \cdot E_0\left(p_{m-1}\right) = \tag{84}$$
$$= \left(\log p_m - \log p_{m-1} - \frac{\log p_{m-1}}{p_m} \right) \cdot E_0\left(p_{m-1}\right) -$$
$$- \log p_m \cdot \left(\log\left(\frac{\log p_m}{\log p_{m-1}} \right) - \frac{1}{p_m} \right).$$

On the other hand, here we have

$$-\log p_m \cdot \left(\log\left(\frac{\log p_m}{\log p_{m-1}} \right) - \frac{1}{p_m} \right) = -\left(\log p_m - \log p_{m-1} - \frac{\log p_{m-1}}{p_m} \right) +$$

$$+\left(\log p_m - \log p_{m-1} - \frac{\log p_{m-1}}{p_m} \right) \cdot \left(\frac{\log p_m - \log p_{m-1}}{\log p_{m-1}} \right) + \tag{85}$$

$$+\log p_m \cdot \left(\frac{1}{2} \cdot \left(\frac{\log p_m - \log p_{m-1}}{\log p_{m-1}} \right)^2 + \Delta\left(p_m\right) \right),$$

where $\Delta\left(p_m\right) = \mathrm{O}\left(\left(\frac{\log p_m - \log p_{m-1}}{\log p_{m-1}} \right)^3 \right).$ (86)

And we have

$$\log p_m \cdot \varepsilon\left(p_m\right) - \log p_{m-1} \cdot \left(1 + \frac{1}{p_m} \right) \cdot \varepsilon\left(p_{m-1}\right) =$$

$$= \mathrm{O}\left(\left(\frac{\log p_m - \log p_{m-1}}{\log p_{m-1}} \right)^2 \right). \tag{87}$$

So from (83) we have

$$\left(\log p_m - \log p_{m-1} - \frac{\log p_{m-1}}{p_m} \right) \cdot E_0\left(p_{m-1}\right) -$$

$$-\left(\log p_m - \log p_{m-1} - \frac{\log p_{m-1}}{p_m} \right) \cdot \left(1 + \frac{\log p_m - \log p_{m-1}}{\log p_{m-1}} \right)$$

$$-\log p_m \cdot \left(\frac{1}{2} \cdot \left(\frac{\log p_m - \log p_{m-1}}{\log p_{m-1}} \right)^2 + \Delta\left(p_m\right) \right) = \tag{88}$$

$$= -\left(\log p_m - \log p_{m-1} - \frac{\log p_{m-1}}{p_m} \right) + \frac{4}{\sqrt{p_m} \log^{3/2} p_m} \times$$

$$\times \left(\log p_m - \log p_{m-1} - \frac{\log p_{m-1}}{p_m} \right) + \mathrm{O}\left(\left(\frac{\log p_m - \log p_{m-1}}{\log p_{m-1}} \right)^2 \right).$$

Now if we would exactly write not only the term with $E_0\left(p_m\right)$, but also the term with $E_0^2\left(p_m\right)$ and more $E_0^n\left(p_m\right)$, and would repeat above process, then we would have $\left(\log p_m - \log p_{m-1} - \dfrac{\log p_{m-1}}{p_m}\right)$ in the every term.

So we will eliminate it from both hand sides of (88). Then since

$$\begin{aligned}&\left(\log p_m - \log p_{m-1}\right)\cdot\left(\log p_m - \log p_{m-1} - \frac{\log p_{m-1}}{p_m}\right)^{-1} = \\ &= \left(1 - \frac{\log p_{m-1}}{p_m \cdot \left(\log p_m - \log p_{m-1}\right)}\right)^{-1} = \mathrm{O}(1)\end{aligned} \tag{89}$$

and

$$\begin{aligned}&\frac{\log p_m - \log p_{m-1}}{\log p_{m-1}} = \frac{1}{\log p_{m-1}}\cdot\log\left(1 + \frac{p_m - p_{m-1}}{p_{m-1}}\right) = \\ &= \frac{1}{\log p_{m-1}}\cdot\frac{p_m - p_{m-1}}{p_{m-1}}\cdot\left(1 + \mathrm{O}\left(\frac{p_m - p_{m-1}}{p_{m-1}}\right)\right),\end{aligned} \tag{90}$$

by the theorem 3 we have

$$E_0\left(p_m\right) = \mathrm{O}\left(\frac{\log^2 p_m}{\sqrt{p_m}}\right). \tag{91}$$

This is the proof of the theorem 4. □

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