Mtg 2: Tue, 5 Jan 10

2 ways to construct f_n:

1) Taylor series
2) Int. poly. ← Interp.

f_n as Taylor series of f:

Example: $I = \int_0^1 \frac{e^x - 1}{x} dx$

$f(x) = \frac{e^x - 1}{x}$ has a removable singularity at $x = 0$

HW: Find $\lim f(x)$ as $x \to 0$.
Plot $f(x)$, $x \in [0, 1]$

Taylor series

$$e^x = \sum_{j=0}^{\infty} \frac{x^j}{j!} = 1 + \sum_{j=1}^{\infty} \frac{x^j}{j!}$$

$\frac{x^0}{0!} = \frac{1}{1}$ $(j = 0)$

$$e^x - 1 = \sum_{j=1}^{\infty} \frac{x^j}{j!}$$

$$\frac{e^x - 1}{x} = \sum_{j=1}^{\infty} \frac{x^{j-1}}{j!} \quad (1)$$

Thm on Taylor Series (A. p. 4, Thm 1.4)

$f(\cdot)$ s.t. $f^{(n+1)}$ exists and cont.

$$f^{(n+1)}(x) := \frac{d^{n+1}}{dx^{n+1}} f(x) \quad (2)$$

$$f(x) = \underbrace{p_n(x)}_{\text{poly. of order } n} + R_{n+1}(x) \quad (3)$$

$p_n(x) \equiv f_n(x)$

(ident. or equiv. to)

$$(4) \quad p_n(x) := f(x_0) + \frac{(x - x_0)}{1!} f^{(1)}(x_0) + \cdots + \frac{(x - x_0)^n}{n!} f^{(n)}(x_0)$$

$$(5) \quad R_{n+1}(x) := \frac{1}{n!} \int_{x_0}^{x} \underbrace{(x-t)^n}_{w(t)} \underbrace{f^{(n+1)}(t)}_{g(t)} dt$$

(1) $$R_{n+1}(x) = \frac{(x-x_0)^{n+1}}{(n+1)!} f^{(n+1)}(\xi)$$

for $\xi \in [x_0, x]$.

HW: Find $p_n(x)$ and $R_{n+1}(x)$ of e^x.

Note: (5) p. 2-2 = (1) p. 2-3

int. mean value thm ≡ IMVT

Int. Mean Value Thm (IMVT)

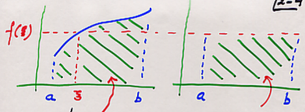
$$\int_a^b W(x) f(x) dx = f(\xi) \int_a^b W(x) dx$$

with $W(x) \geq 0 \quad \forall x \in [a, b]$.

$W(\cdot)$ is non-neg.

Special case: $W(x) = 1 \quad \forall x \in [a, b]$

$$\int_a^b f(x) dx = f(\xi) \underbrace{(b-a)}_{\int_a^b 1\, dx}$$



$$\int_a^b f(x)\,dx = f(\xi)(b-a)$$

$f(\cdot)$ cont. on $[a,b]$

$$m \leq f(x) \leq M \quad \forall x \in [a,b]$$

$$m = \min_x f(x) \qquad M = \max_x f(x)$$

$$\Rightarrow \underbrace{\int_a^b m\,dx}_{m(b-a)} \leq \int_a^b f(x)\,dx \leq \underbrace{\int_a^b M\,dx}_{M(b-a)}$$

$\exists\, \xi \in [a,b]$ st $m(b-a) \leq f(\xi)(b-a) \leq M(b-a)$

↑ "there exists"