Canard cycles in generic slow-fast systems on the two-torus

How many ducks can dance on the torus?

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Saint Petersburg, Russia

Slow-fast systems: definitions

Definition

- The slow-fast system is a system of the following form:

$$\begin{cases} \dot{x} = f(x, y, \varepsilon), \\ \dot{y} = \varepsilon g(x, y, \varepsilon), \end{cases} \qquad \varepsilon \in (\mathbb{R}, 0). \tag{1}$$

- Variables: x is a fast variable, and y is a slow one, ε is a small parameter.
- Slow curve is a set $M := \{(x, y) \mid f(x, y, 0) = 0\}$.

Remark

Outside of any fixed neighborhood of the slow curve M, for ε small enough, the fast variable x changes much more rapidly than the slow variable y.

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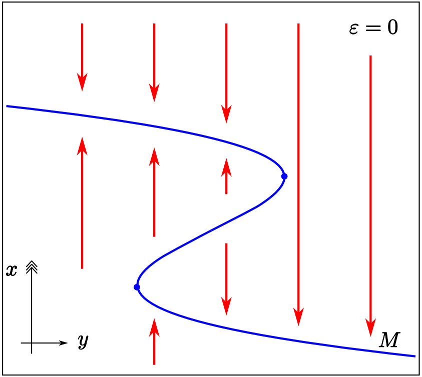
Fast system

- Fast dynamics for $\varepsilon = 0$: slow variable y is a constant.
- Attracting part of the slow curve M consist of stable fixed points.
- Repelling part of the slow curve M consist of unstable fixed points.
- Folds are neutral fixed points.



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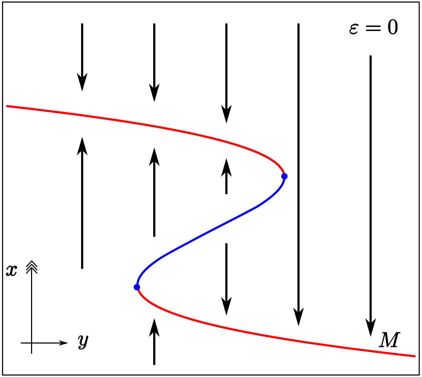

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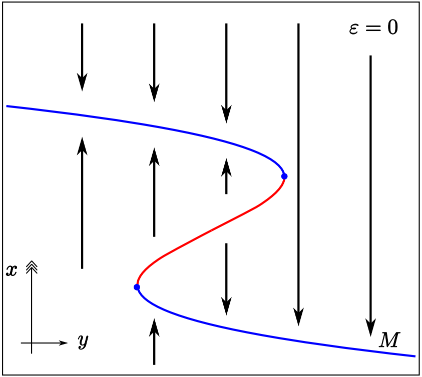

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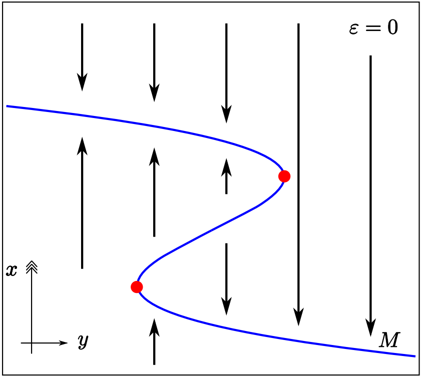

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Slow-fast dynamics: generic planar case

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- it quickly falls on *attracting* segment of M
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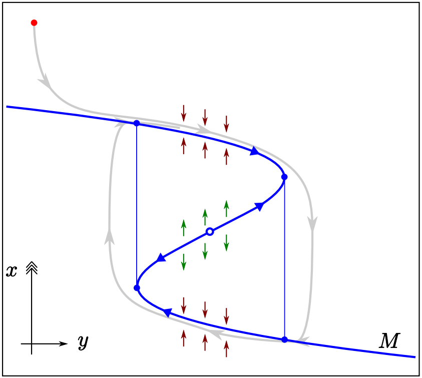

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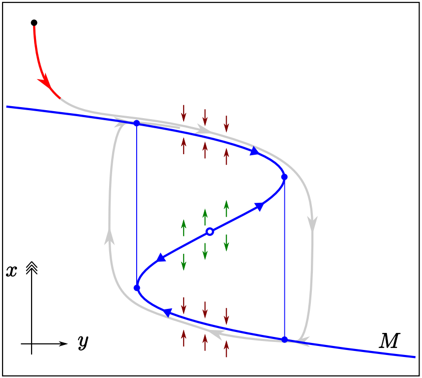

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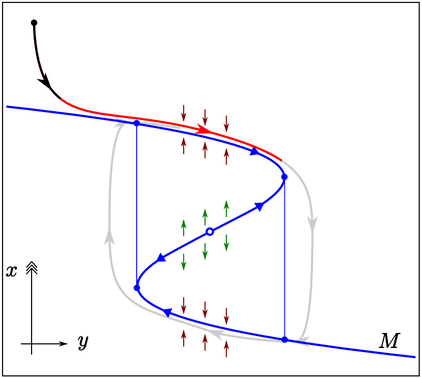

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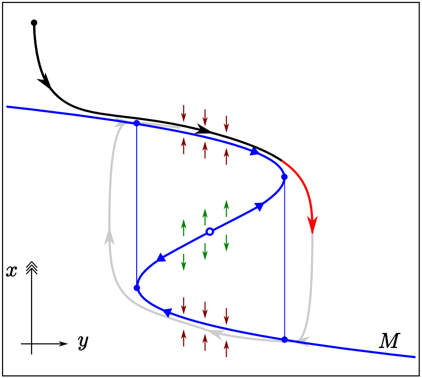

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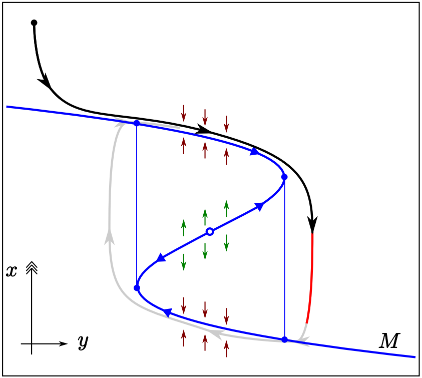

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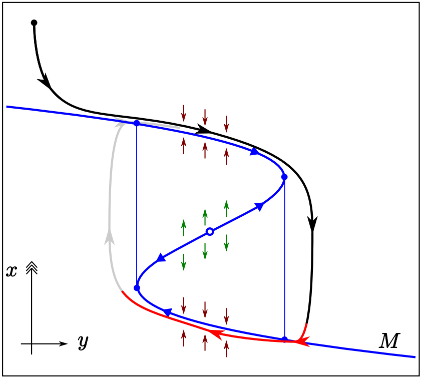

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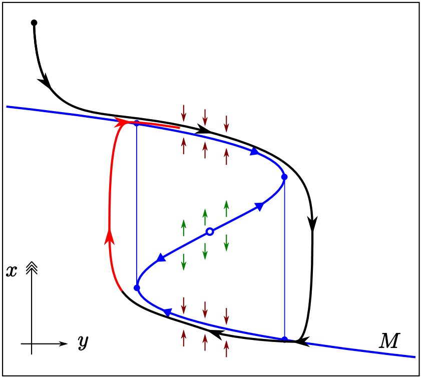

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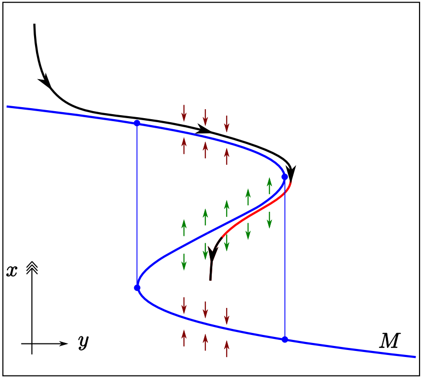
Canard solutions

Definition 1

Duck (or *canard*) solutions are solutions, whose phase curves contain an arc of length bounded away from 0 uniformly in ε, that keeps close to the *unstable* part of the slow curve

Definition 2

Canard cycle is a limit cycle which is a canard.



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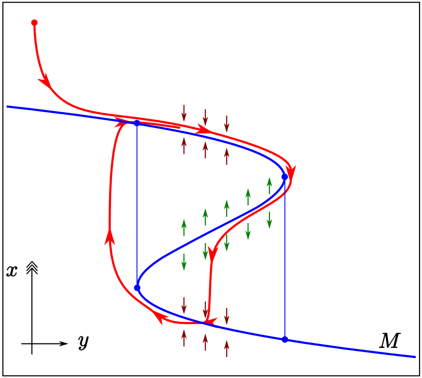

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Remark

There's no attracting canard cycles in generic planar systems.

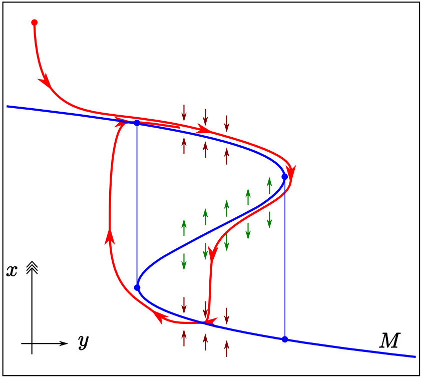
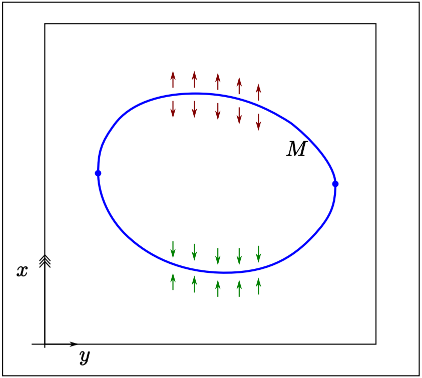


Figure: Canards

Ducks on the torus: introduction

- Consider slow-fast system on the two-torus
- Pick a point far from M
- Consider its trajectory in forward time
- Reverse the time
- When ε decreases, L moves down, and R moves up.
- For some ε, we've got canard cycle.



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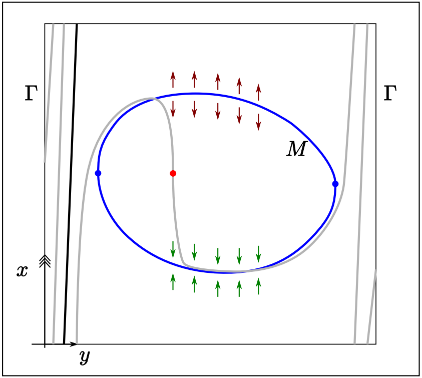

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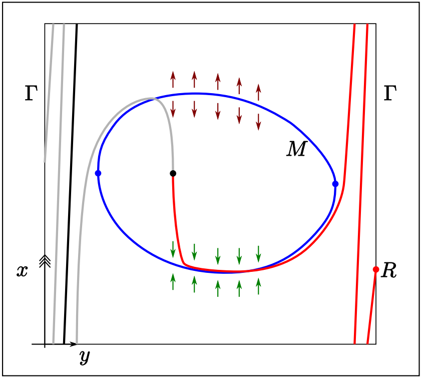

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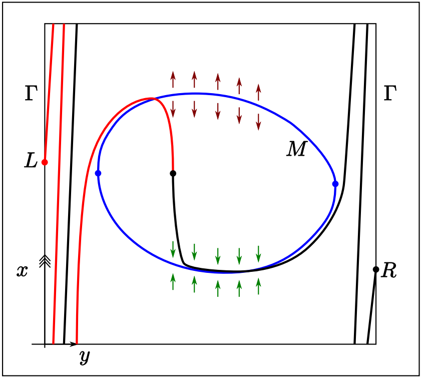

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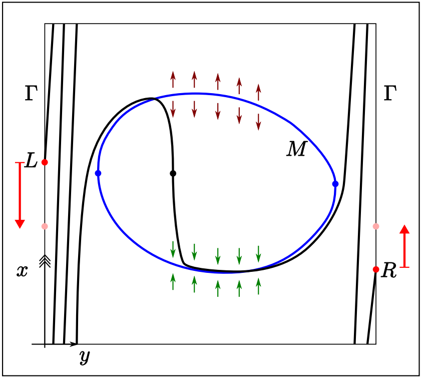

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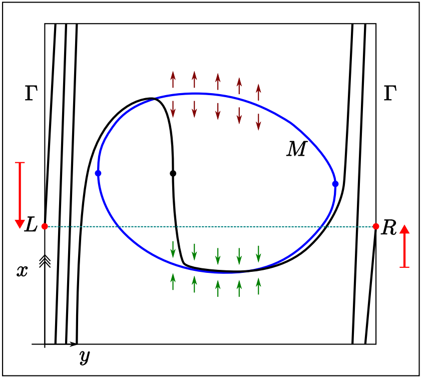

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Main results: the structure of ducky area

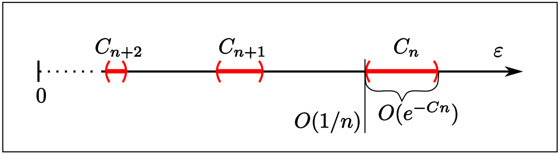

Figure: Intervals C_n: the ducks live here

- There exists a sequence of intervals $\{C_n\}_{n=1}^{\infty}$ such that for every $\varepsilon \in C_n$ the system has *attracting* canard cycles.
- Intervals C_n are exponentially small.
- They accumulate to 0.
- Their density is 0 near $\varepsilon = 0$.

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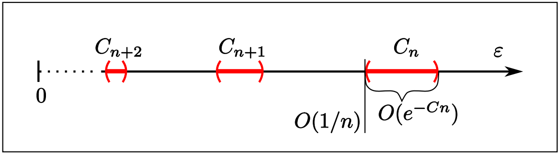

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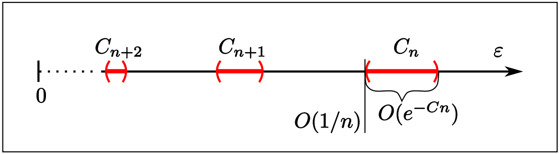

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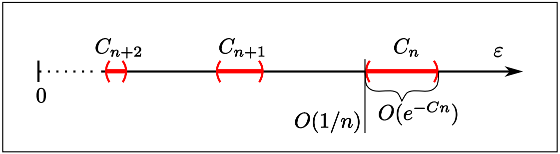

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How many ducks can dance on the torus?

Theorem 1 (Upper estimate for the number of canards)

Consider slow-fast system on the two-torus, i.e. $(x, y) \in \mathbb{T}^2$, *and the speed of the slow motion is bounded away from zero* ($g > 0$). *Assume* M *is connected nondegenerate curve with* $2N$ *fold points,* $N < \infty$, *and some additional nondegenericity assumptions hold. Then there exists number* $0 < K \leq N$, *such that the following assertions hold:*

- *There exists a sequence* $\{C_n\}_{n=1}^{\infty}$ *of intervals on the ray* $\{\varepsilon > 0\}$, *accumulating to 0, such that for every* $\varepsilon \in C_n$, *the system has exactly* $2K$ *canard cycles (*K *attracting and* K *repelling).*
- *For any* $\varepsilon > 0$ *small enough, the number of limit cycles that make one rotation along y-axis is bounded by* $2K$.
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Illustrations for main result: $N = K = 2$

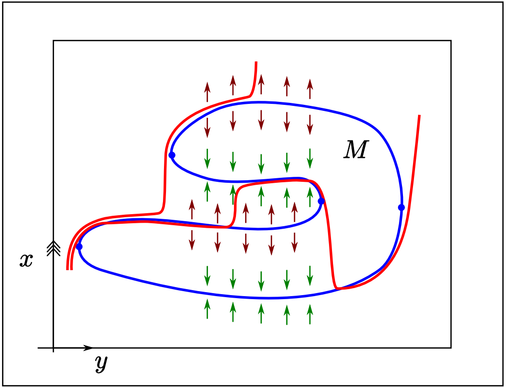

Figure: Simple example: two pair of folds, two attracting ducks (two repelling ducks are not shown)

How many ducks can dance on the torus? (2)

Remark

It follows from theorem 1, that for convex slow curve, there exists exactly one pair of canard cycles

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The number K of canard cycles can be effectively computed without intergration of the system.

Theorem 2 (Sharp estimate for K)

For every $N > 0$ there exists an open set in the space of slow-fast systems on the two-torus for which the number of canard cycles reaches its maximum: $K = N$.

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The duck farm

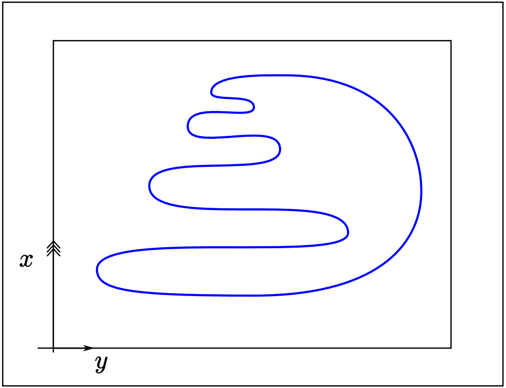

Figure: The construction of open set of slow-fast systems with maximal number of canard cycles. E.g. $K = N = 4$ on the figure.