Research Plan for the Riemann Hypothesis

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Abstract

The Riemann hypothesis is well known (RH). The RH is closely related with many problems of the analytic number theory. In particular, the RH shows the distribution of prime numbers in natural numbers. And the RH is also very important in the computer science, too. In this article we will show the research plan to discuss the RH. This plan is based on a new idea with respect to the relation between the sum of divisors function and the Hardy-Ramanujan's number.

1. Introduction

The function $\zeta(s)$ defined by an absolute convergent Dirichlet's series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

in complex half-plane $\operatorname{Re} s > 1$ is called the Riemann's zeta function ([4]). The Riemann's zeta function has a simple pole with the residue 1 at $s = 1$ and except the point $s = 1$ the function $\zeta(s)$ is analytically continued to whole complex plane ([4]). And $\zeta(s)$ is expressed for $\operatorname{Re} s > 1$ as

$$\zeta(s) = \prod_p \left(1 - p^{-s}\right)^{-1},$$

where infinite product runs over all the prime numbers. Also for $\operatorname{Re} s > 1$ the function $\zeta(s)$ satisfies the functional equation

$$\zeta(s) = 2 \cdot (2\pi)^{s-1} \cdot \Gamma(1-s) \cdot \sin\left(\frac{\pi s}{2}\right) \cdot \zeta(1-s),$$

where $\Gamma(s)$ is the gamma function ([4])

$$\Gamma(s) = \int_0^{+\infty} e^{-s} x^{s-1} dx .$$

From the infinite product of $\zeta(s)$ the Riemann's zeta function has no zeros in $\operatorname{Re} s > 1$ and from the functional equation of $\zeta(s)$ it has trivial zeros $-2, -4, -6, \cdots$ in $\operatorname{Re} s < 0$. The zeros of $\zeta(s)$ in $0 \le \operatorname{Re} s \le 1$ are called the nontrivial zeros of $\zeta(s)$([4]). In 1859 G. Riemann conjectured that all the nontrivial zeros of $\zeta(s)$ would lie on the line $\operatorname{Re} s = 1/2$. This is just the Riemenn's hypothesis ([4]).

There have been published many research results on the RH ([1~10]). But the RH is unsolved until now ([1]). The below research plan gives us a certain possibility to discuss the RH. This plan is based on a new concept that any natural number has such the properties as the following three-dimensional structure.

Let N be the set of the natural numbers. Suppose that $n = q_1^{\lambda_1} \cdot q_2^{\lambda_2} \cdots q_m^{\lambda_m}$ is a prime factorization of $n \in N$, where $q_1, q_2, \cdots q_m$ are distinct primes and $\lambda_1, \lambda_2, \cdots, \lambda_m$ are non-negative integers.

We assume that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \geq 1$. And we put $\overline{\lambda}(n) = (\lambda_1, \lambda_2, \cdots, \lambda_m)$, $\overline{q}(n) = (q_1, q_2, \cdots, q_m)$ and $\omega(n) = m$, which would be called an exponent pattern, a prime factor pattern, an exponent length of n, respectively. Here $\omega(n) = \sum_{p|n} 1$ ([4]) is the number of the prime factors of a given n. Then we could write any natural number n and the set N as

$$n = n(\overline{q}(n),\ \overline{\lambda}(n),\ \omega(n))$$

and

$$N = \bigcup_{\omega(n)} \bigcup_{\overline{\lambda}(n)} \bigcup_{\overline{q}(n)} n(\overline{q}(n), \overline{\lambda}(n), \omega(n))$$

respectively. Hence we can say that any natural number n has the three-dimensional structure. Of course, both $\overline{\lambda}(n) = (\lambda_1, \lambda_2, \cdots, \lambda_m)$ and $\overline{q}(n) = (q_1, q_2, \cdots, q_m)$ are dependent on $\omega(n) = m$. But, if we have such the standpoint at the study of the arithmetical functions, then we would obtain better results. In fact, for example, let's see below the Robin's inequality (RI) ([1~6])

$$\sigma(n) \leq e^{\gamma} \cdot n \cdot \log\log n,$$

where $\gamma = 0.577\cdots$ is Euler's constant ([4]).

Let $p_1 = 2,\ p_2 = 3,\ p_3 = 5, \cdots, p_n, \cdots$ be the first primes ([1]). Here then p_n is n-th prime number. If $\overline{\lambda}(n)$ and $\omega(n)$ are fixed in a given number $n = q_1^{\lambda_1} \cdot q_2^{\lambda_2} \cdots q_m^{\lambda_m}$, then we put $r_0(n) = p_1^{\lambda_1} \cdot p_2^{\lambda_2} \cdots p_m^{\lambda_m}$ to $n = q_1^{\lambda_1} \cdot q_2^{\lambda_2} \cdots q_m^{\lambda_m}$.

If $n = r_0(n)$ then the number n is called a Hardy-Ramanujan number (HRN) ([3]). In other words, the HRN is just a natural number of such forms as $p_1^{\lambda_1} \cdot p_2^{\lambda_2} \cdots p_m^{\lambda_m}$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \geq 1$. The HRN has special properties. In particular, the HRN plays very important role in the RI. We put ([9,10])

$$S(\bar{\lambda}, m) = \{\, n \in N \mid \bar{\lambda} = \bar{\lambda}(n) = (\lambda_1, \lambda_2, \cdots, \lambda_m),\ \omega(n) = m \},$$

and

$$HR(m) = \{\, n \in N \mid n = r_0(n), \omega(n) = m \}.$$

Then $S(\bar{\lambda}, m)$ consists of the natural numbers with $\bar{\lambda}(n) = (\lambda_1, \lambda_2, \cdots, \lambda_m)$ and $\omega(n) = m$. And the set $HR(m)$ consists of the HRN with $\omega(n) = m$.

The function $\sigma(n) = \sum_{d|n} d$ is called the sum of divisors function of n ([2,4]).

Then the function $\sigma(n)$ is multiplicative on the co-prime numbers ([4]). And for $n = q_1^{\lambda_1} \cdot q_2^{\lambda_2} \cdots q_m^{\lambda_m}$ it holds that

$$\frac{\sigma(n)}{n} = \prod_{i=1}^{m} \frac{1 - q_i^{-\lambda_i - 1}}{1 - q_i^{-1}}.$$

We can rewrite the RI as

$$\frac{\exp\left(\exp\left(e^{-\gamma} \cdot \sigma(n)/n\right)\right)}{n} < 1.$$

We could easily have ([9]) that

$$r_0(n) = \min_{(q_1, q_2, \cdots q_m)} S(\bar{\lambda}, m),\quad \frac{\sigma(r_0(n))}{r_0(n)} = \max_{n \in S(\bar{\lambda}, m)} \left\{ \frac{\sigma(n)}{n} \right\}.$$

Therefore for any $n \in S(\bar{\lambda}, m)$ we have

$$\frac{\exp\left(\exp\left(e^{-\gamma} \cdot \sigma(n)/n\right)\right)}{n} \leq \frac{\exp\left(\exp\left(e^{-\gamma} \cdot \sigma(r_0(n))/r_0(n)\right)\right)}{r_0(n)}.$$

Hence to consider the RI is reduced to one of the inequality

$$\frac{\exp\left(\exp\left(e^{-\gamma}\cdot\sigma\left(r_0\left(n\right)\right)/r_0\left(n\right)\right)\right)}{r_0\left(n\right)}<1 \quad \text{for } n\geq 5041.$$

But this inequality has two parameters with $\bar{\lambda}$ and m. In other words, for the RI with three-dimensional structure it is sufficient to consider the above inequality with two-dimensional structure on the set $\bigcup_m HR\left(m\right)$.

So we put ([10])

$$H\left(\lambda_1,\lambda_2,\cdots,\lambda_m\right)=\frac{\exp\left(\exp\left(e^{-\gamma}\cdot\sigma\left(r_0\left(n\right)\right)/r_0\left(n\right)\right)\right)}{r_0\left(n\right)}.$$

Then we can consider $H\left(\lambda_1,\lambda_2,\cdots,\lambda_m\right)$ as the function with m-variable, and we could also obtained that there exist a point $\bar{\lambda}_0=\left(\lambda_1^0,\lambda_2^0,\cdots,\lambda_m^0\right)$ in the real m-dimensional space R^m such that

$$H\left(\lambda_1,\lambda_2,\cdots,\lambda_m\right)\leq H\left(\bar{\lambda}_0\right).$$

Here $\bar{\lambda}_0=\left(\lambda_1^0,\lambda_2^0,\cdots,\lambda_m^0\right)$ would be estimated ([10]) as

$$\lambda_i^0=\frac{\log p_m}{\log p_i}+\frac{\log\log p_m}{\log p_i}+\mathrm{O}\left(\frac{1}{\log p_i\cdot\log p_m}\right)\quad\left(1\leq i\leq m\right).$$

We put $n_0=p_1^{\lambda_1^0}\cdot p_2^{\lambda_2^0}\cdots p_m^{\lambda_m^0}$, which would be called a special HRN ([9]).

In this connection, the number $\in HR(m)$ would call general HRN ([9]).

Then the function value $H\left(\lambda_1^0,\lambda_2^0,\cdots,\lambda_m^0\right)$ is determined only on p_m.

So we put $C_m=H\left(\lambda_1^0,\lambda_2^0,\cdots,\lambda_m^0\right)$, which has one-dimensional structure.

Our work to consider the RI would have the end at $\sup\limits_{\substack{\omega(n)=m\\ n\geq 5041}} C_m<1$.

The RI is very beautiful, but it has one troublesome with for any $n\geq 5041$. By paper [5], we could easily have that the RI holds if and only if there exists a constant $c_0\geq 1$ (or $c_0'\geq 1$) such that for any $n\geq 2$

$$\sigma(n) \le e^{\gamma} \cdot n \cdot \log\log(c_0 \cdot n)$$

and, more,

$$\sigma(n) \le e^{\gamma} \cdot n \cdot \log\log\left(c_0' \cdot n \cdot \exp\left(\sqrt{\log n} \cdot \exp\left(\sqrt{\log\log(n+1)}\right)\right)\right)$$

holds ([9,10]). Hence the RI is

$$\sup_{n \ge 5041} \frac{\exp\left(\exp\left(e^{-\gamma} \cdot \sigma(n)/n\right)\right)}{n} < 1$$

and it is equivalent to

$$c_0 = \sup_{n \ge 2} \frac{\exp\left(\exp\left(e^{-\gamma} \cdot \sigma(n)/n\right)\right)}{n} < +\infty$$

or

$$c_0' = \sup_{n \ge 2} \frac{\exp\left(\exp\left(e^{-\gamma} \cdot \sigma(n)/n\right)\right)}{n \cdot \exp\left(\sqrt{\log n} \cdot \exp\left(\sqrt{\log\log(n+1)}\right)\right)} < +\infty .$$

Clearly, there is no any limited condition here, and it is easy to consider.

Thus we have that $\sup\limits_{\substack{\omega(n)=m \\ n \ge 5041}} C_m < 1$ is equivalent to $c_0 = \sup\limits_{m \ge 1} C_m < +\infty$.

We could have ([10]) $c_0 = C_1 = \exp\left(\exp\left(e^{-\gamma} \cdot \sigma(2)/2\right)\right)/2 = 5.0951\cdots$ and

$$c_0' = \frac{\exp\left(\exp\left(e^{-\gamma} \cdot \sigma(2)/2\right)\right)}{2 \cdot \exp\left(\sqrt{\log 2} \cdot \exp\left(\sqrt{\log\log 3}\right)\right)} = 1.643\cdots .$$

Therefore, we could say that the RH would be determined by the prime number $p = 2$.

We put

$$H(n) = \frac{\exp\left(\exp\left(e^{-\gamma} \cdot \sigma(n)/n\right)\right)}{n} .$$

This function $H(n)$ we would like to call an σ-index of the natural number $n \in N$ ([9]). In the future, we will consider the RH with such standpoint as in above description.

2. Research Plan

Title: An σ- Index of Natural Number and its Applications to the Study of Riemann Hypothesis

1) <u>Background</u>

The Riemann Hypothesis (RH) is well known [4]. There have been published many research results on the RH, in particular, on the function $\pi(x)$ and the Chebyshev's function $\vartheta(x)$ [4,8]. The sum of divisors function $\sigma(n)$ is one of the most important arithmetical functions, but its properties are not well known in the RH. In the past, the study of the function $\sigma(n)$ had been mostly limited to the relation with the Euler's function $\varphi(n)$ and to the study of the perfect numbers, but for the RH it has been studied after 1980 ([1~7, 9,10]).

Recently, we have studied the function $\sigma(n)$ for the RH. We obtained a theorem that the RH is true if and only if the function $\sigma(n)$ satisfies a condition, which would be called an equivalence theorem (ET) and an equivalence condition (EC) respectively ([9,10]). By the ET, we have more obtained some conditions equivalent to the RH. These theorems are different in the forms each other. The EC is generalized rather than the Robin's criterion or the Lagarias's one ([5,6]).

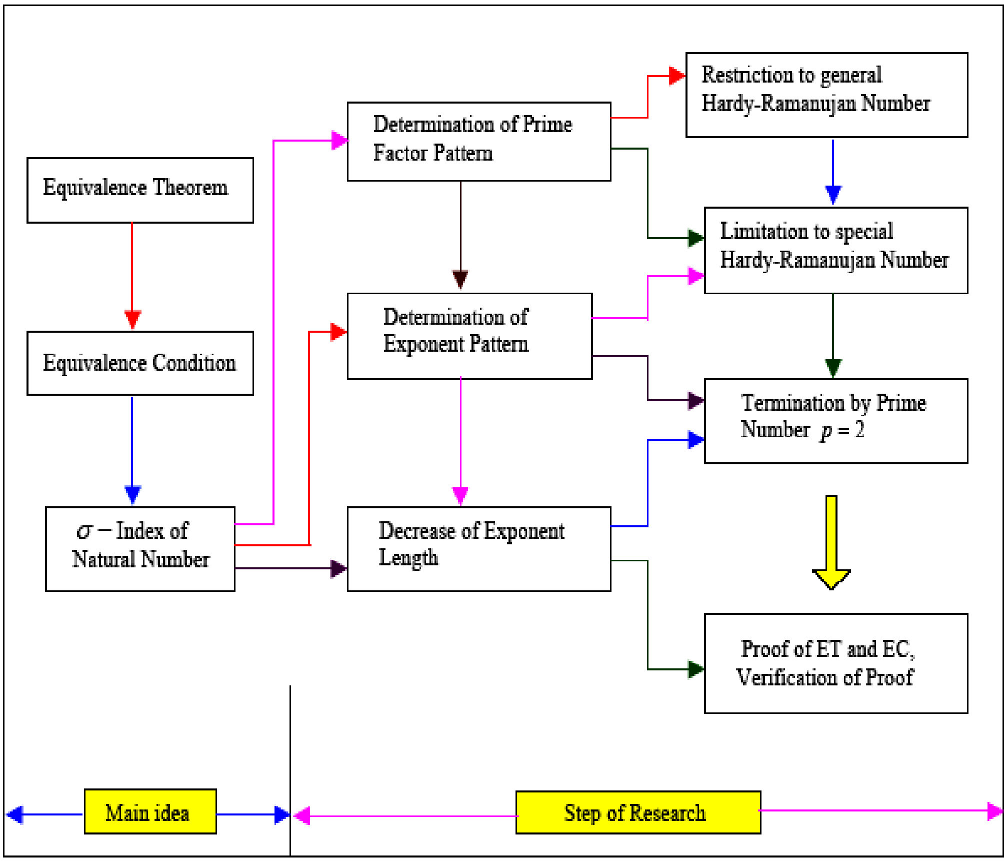
We have a new idea for the proof that the EC holds unconditionally. The idea is to introduce a notion, which would be called an $\sigma-$index of the natural number, and to estimate up it. On the basis of the idea, we plan to progress following process; first, the $\sigma-$index of any natural number is determined by the general Hardy-Ramanujan number (HRN). Second, the $\sigma-$index of the HRN is estimated by a special HRN. Finally, the $\sigma-$index of the special HRN is terminated by the prime number $p = 2$. This process would be accompanied by such courses as the restriction of the prime factor pattern, the limitation of the exponent pattern, the decrease of the exponent length of the natural number. By this process, the EC would be holden unconditionally. To complete the research for the RH we need the objective verification on the results obtained from above process.

2) <u>Detailed Research Plan</u>

1. A new proposition equivalent to the RH
   - −* Necessary condition for the RH by the function $\sigma(n)$
   - −* Sufficient condition for the RH by the function $\sigma(n)$
   - −* Necessary and Sufficient condition for the RH by the function $\sigma(n)$
2. Relations between the $\sigma -$ index and characteristic index of the natural number
   - −* The $\sigma -$ index and the prime factor pattern
   - −* The $\sigma -$ index and the exponent pattern
   - −* The $\sigma -$ index and the exponent length
3. Relations of the $\sigma -$ index and the Hardy-Ramanujan number
   - −* Restriction to the general HRN
   - −* Limitation to the special HRN
   - −* Termination by the prime number $p = 2$
4. Complete conclusion
   - −* The proof of the ET and the EC
   - − The verification of the proof

(−*: This part has been accomplished in the main)

3. The schematic description of the Research Plan



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