



# **Special relativity and steps towards general relativity: SR**

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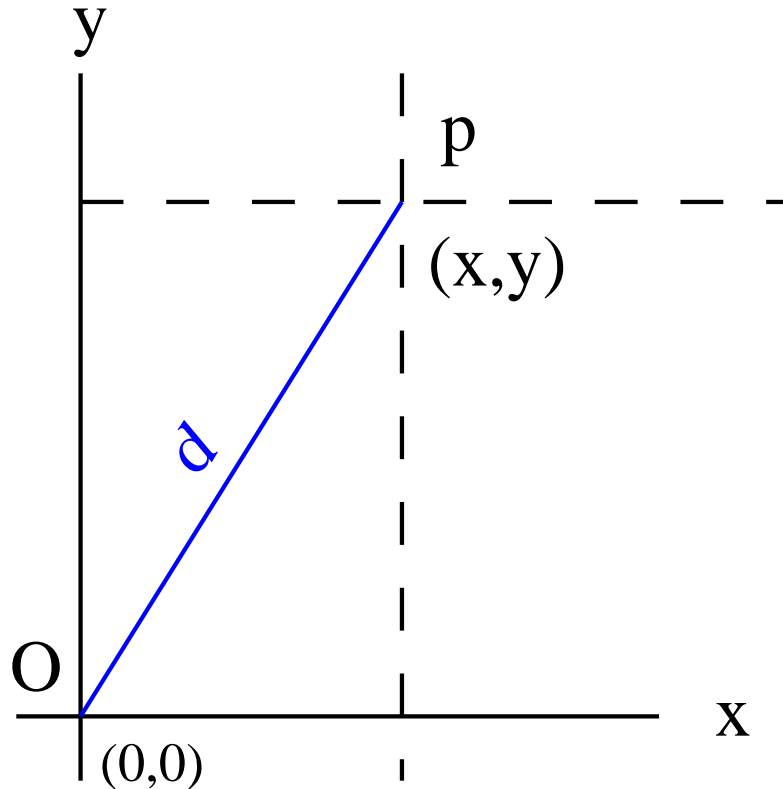


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- GR: spacetime = a solution of the w:Einstein field equations



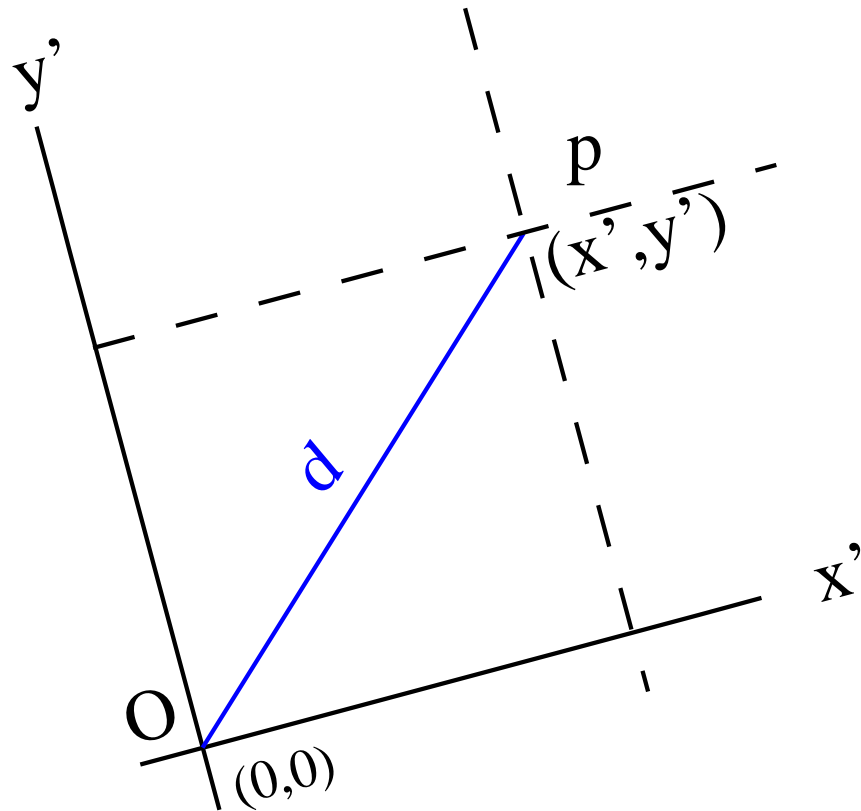
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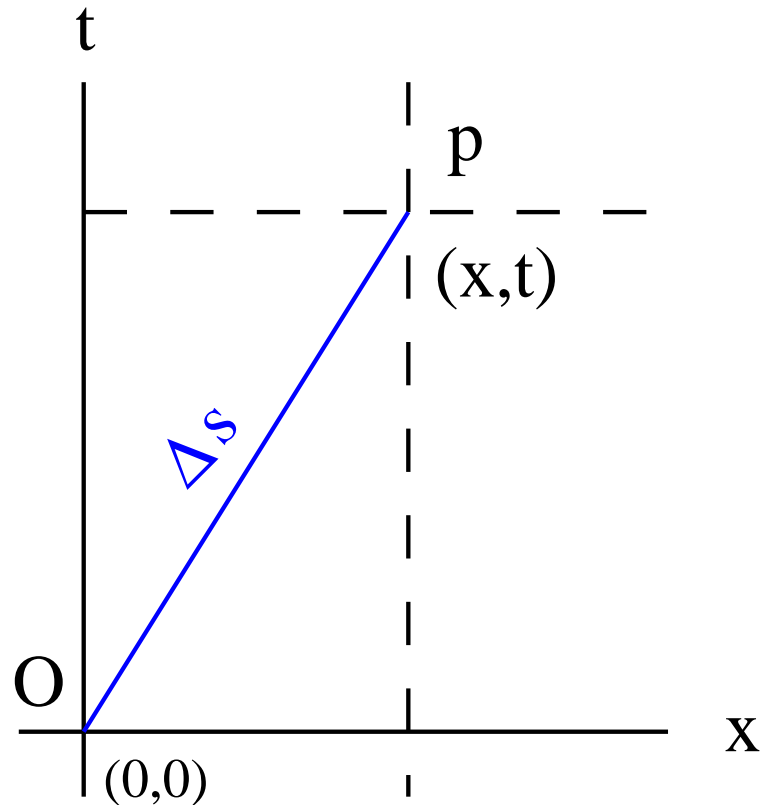
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$$\begin{pmatrix} p_{x'} \\ p_{y'} \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} p_x \\ p_y \end{pmatrix}$$

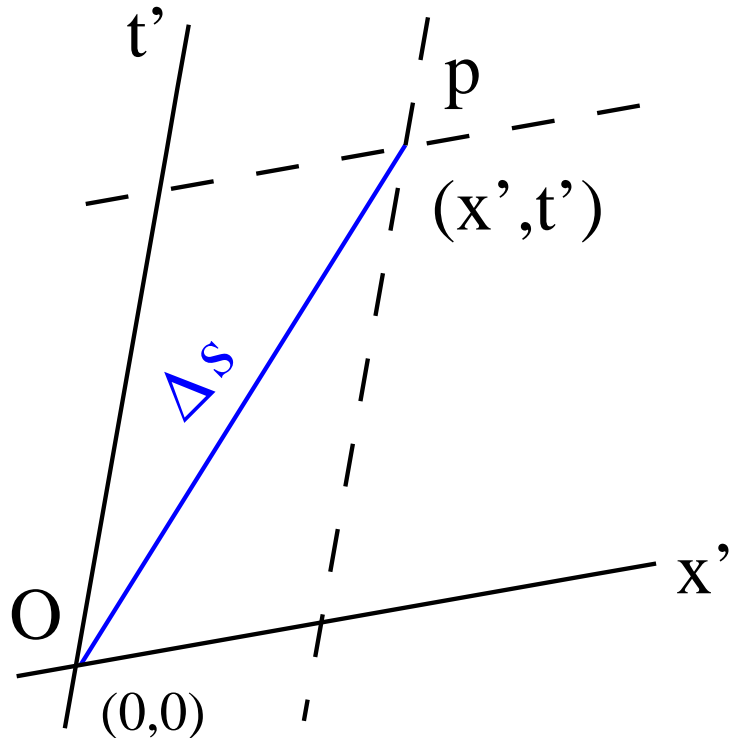
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$p$  at  $(x, t)$ , w:invariant interval from observer at  $O$  is  $\Delta s$   
 where  $(\Delta s)^2 = -(\Delta t)^2 + (\Delta x)^2$

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$$\begin{pmatrix} p_{x'} \\ p_{t'} \end{pmatrix} = \begin{pmatrix} \cosh \phi & -\sinh \phi \\ -\sinh \phi & \cosh \phi \end{pmatrix} \begin{pmatrix} p_x \\ p_t \end{pmatrix}$$

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# SR: Lorentz transformation

$$\Lambda(\phi) := \begin{pmatrix} \cosh \phi & -\sinh \phi \\ -\sinh \phi & \cosh \phi \end{pmatrix}$$

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  $= \frac{1 \text{ s}}{1 \text{ s}} = 1$  (dimensionless)



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where velocity  $\beta := v/c \equiv v = \tanh \phi$



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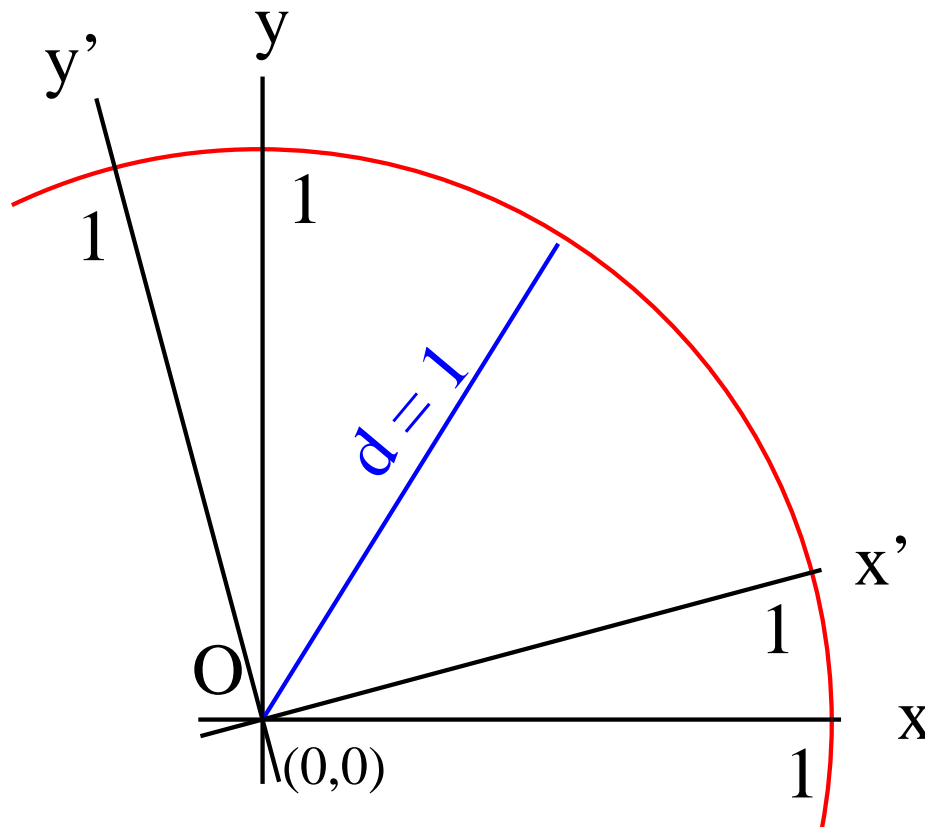
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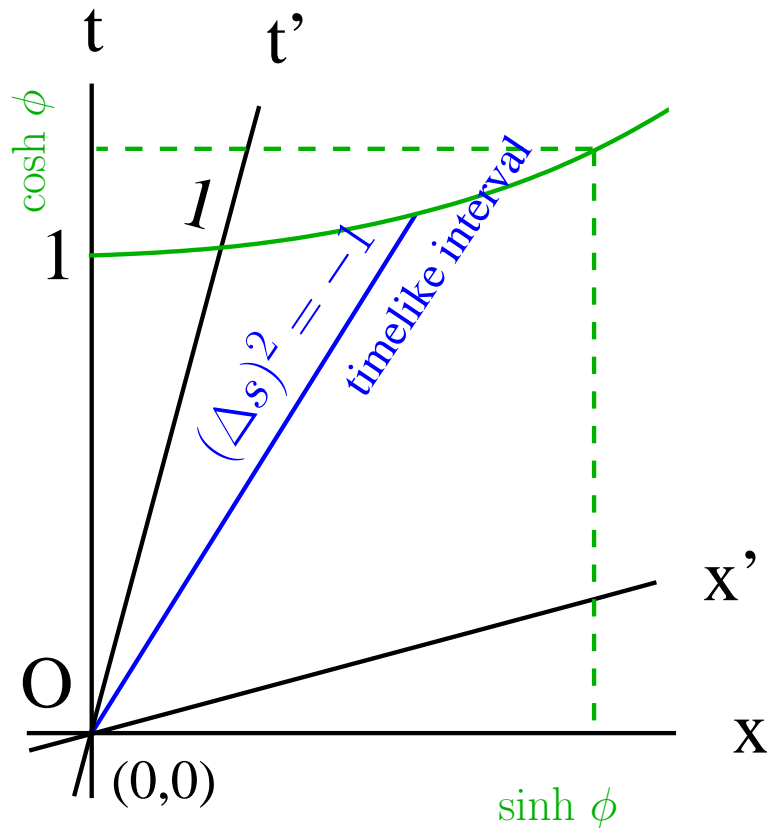


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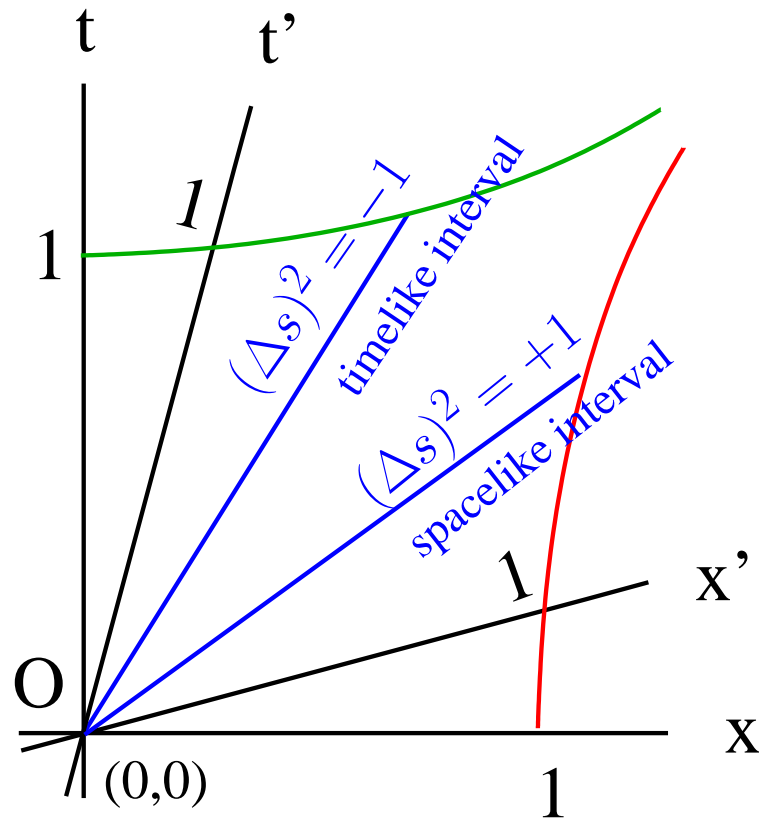
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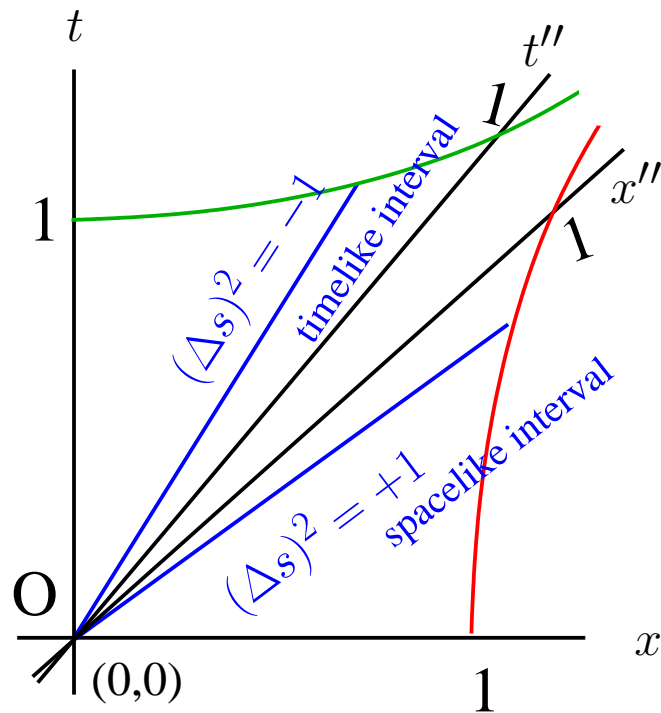


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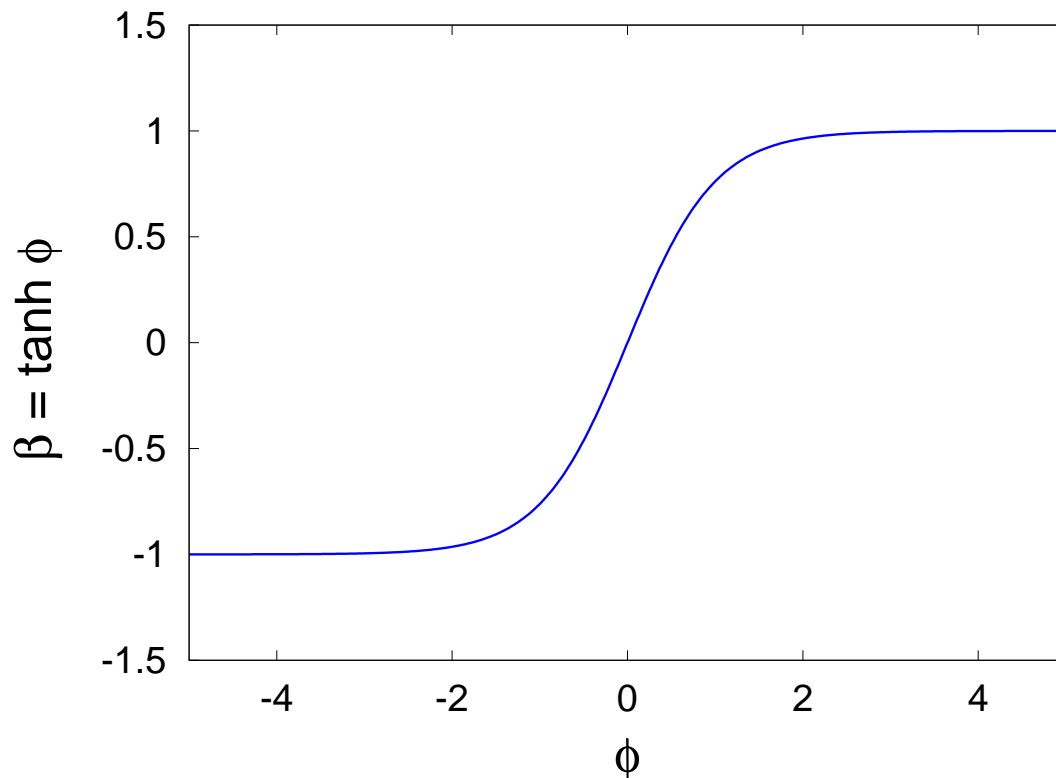


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- w:Michelson-Morley experiment (1887)

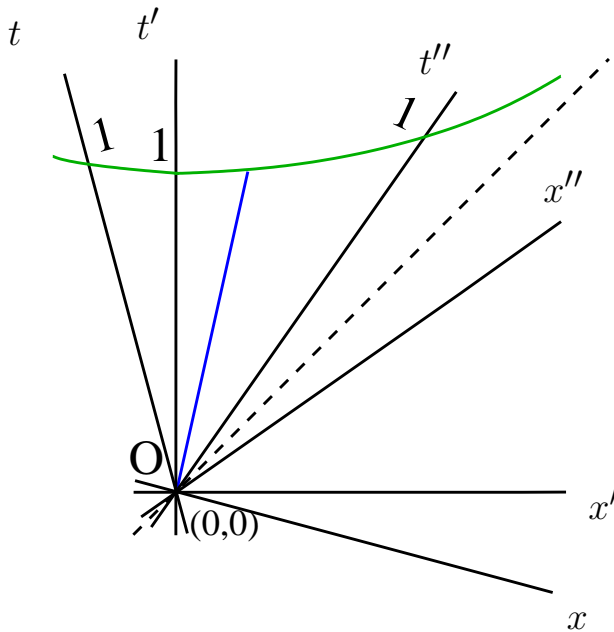


# SR: adding velocities

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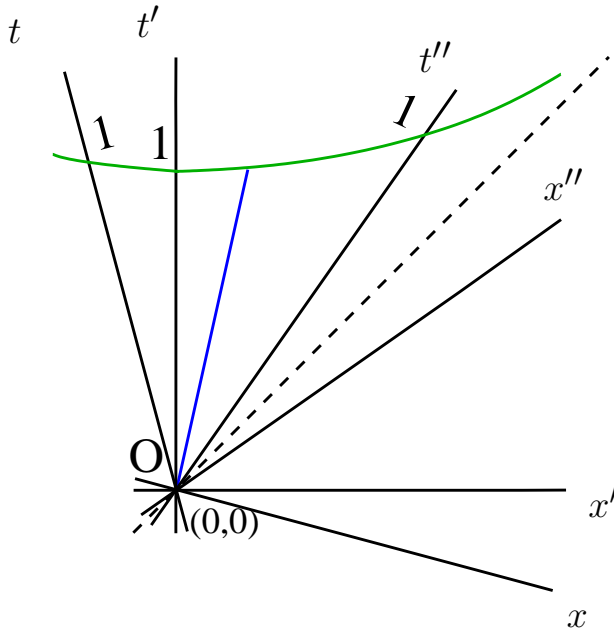


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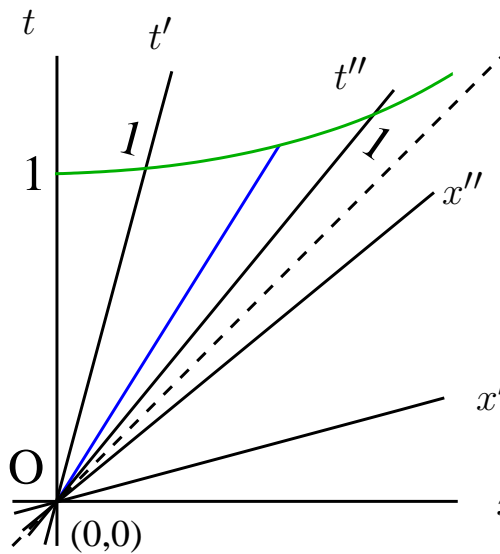


$$\begin{pmatrix} x'' \\ t'' \end{pmatrix} = \begin{pmatrix} \cosh \phi_2 & -\sinh \phi_2 \\ -\sinh \phi_2 & \cosh \phi_2 \end{pmatrix} \begin{pmatrix} x' \\ t' \end{pmatrix}$$

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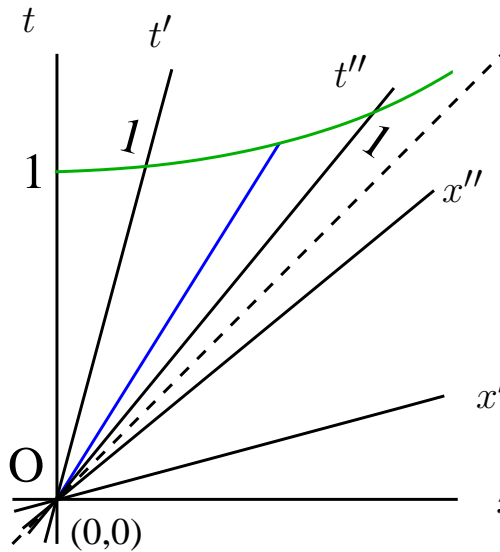


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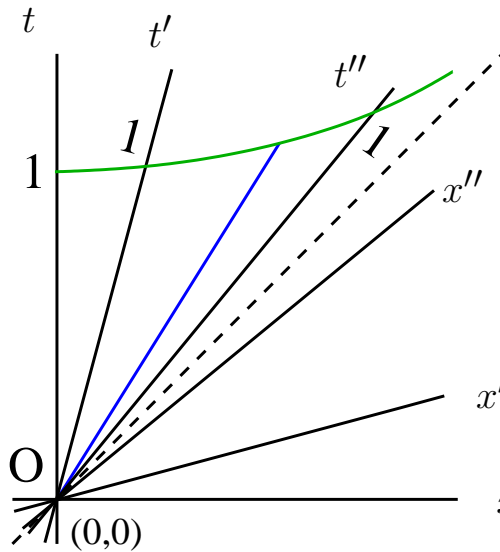
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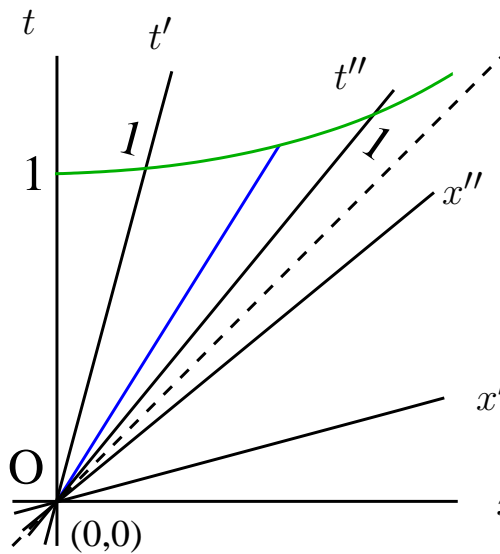


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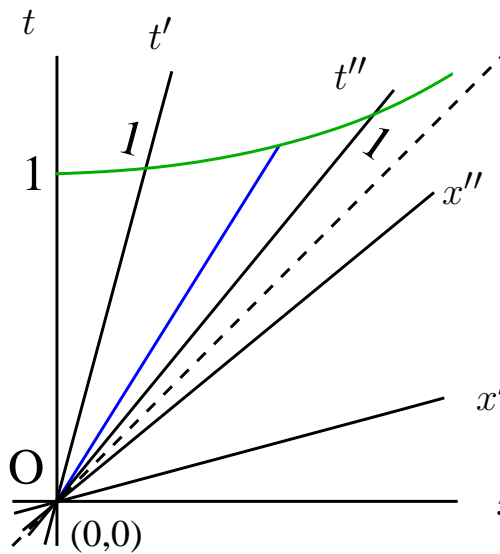
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cf. rotation  $\theta_1$  "plus" rotation  $\theta_2 =$  rotation  $\theta_1 + \theta_2$

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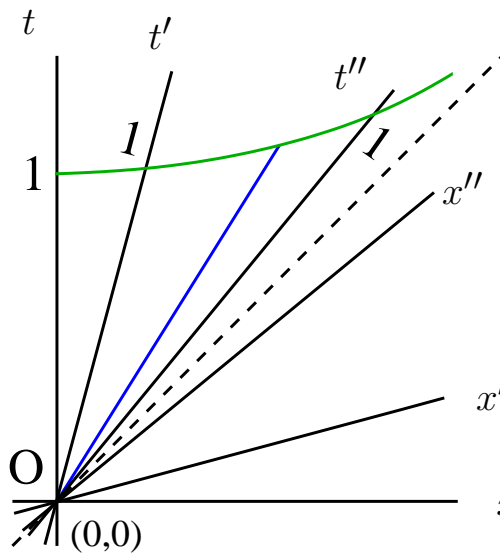
but  $\Lambda(\phi_2)\Lambda(\phi_1) = \Lambda(\phi_1 + \phi_2)$

so  $\beta_3 = \tanh(\phi_1 + \phi_2)$



# SR: adding velocities

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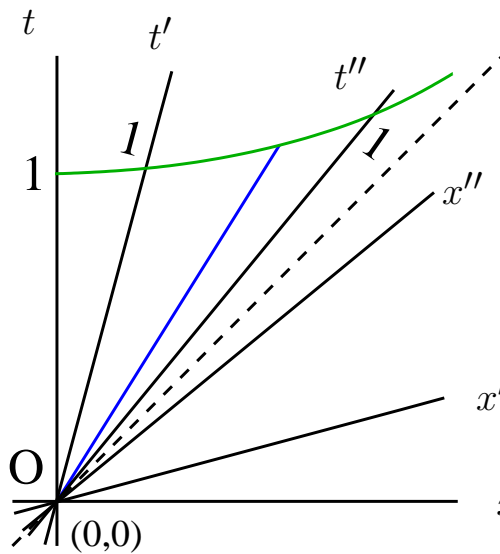
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so  $\beta_3 = \tanh(\phi_1 + \phi_2) = \frac{\tanh \phi_1 + \tanh \phi_2}{1 + \tanh \phi_1 \tanh \phi_2}$

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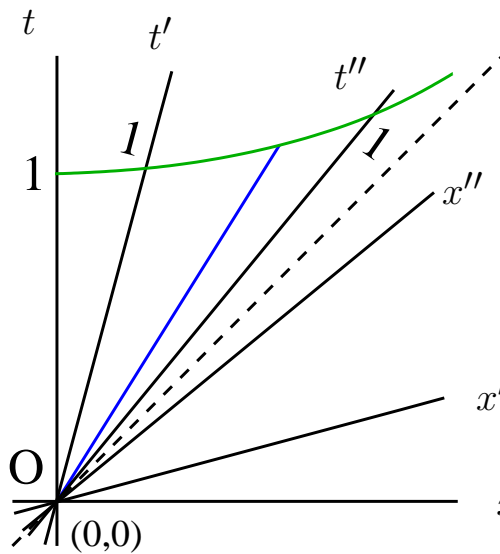
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$$\text{so } \beta_3 = \tanh(\phi_1 + \phi_2) = \frac{\beta_1 + \beta_2}{1 + \beta_1\beta_2} = \frac{0.1 + 0.5}{1 + 0.1 \times 0.5} \approx 0.57$$

# SR: Lorentz factor



$\Lambda$ : alternative to hyperbolic trig functions



# SR: Lorentz factor



$\Lambda$ : alternative to hyperbolic trig functions

$$\Lambda(\phi) := \begin{pmatrix} \cosh \phi & -\sinh \phi \\ -\sinh \phi & \cosh \phi \end{pmatrix}$$

$$\cosh \phi := \frac{e^\phi + e^{-\phi}}{2}$$

$$\sinh \phi := \frac{e^\phi - e^{-\phi}}{2}$$

w:hyperbolic function



# SR: Lorentz factor



$\Lambda$ : alternative to hyperbolic trig functions

$$\Lambda(\beta) := \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix}$$

$$\beta = \tanh \phi$$

$$\gamma := (1 - \beta^2)^{-1/2} =$$

**Lorentz factor**

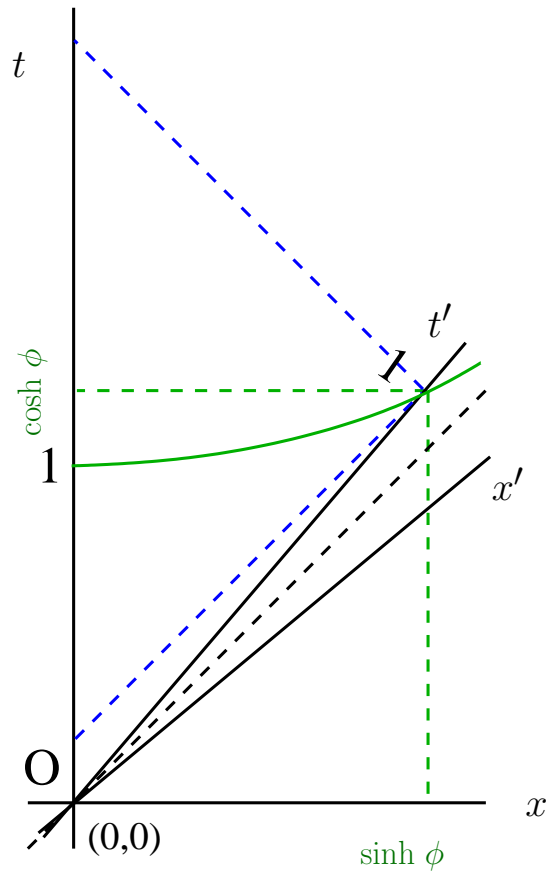
$$\gamma = \cosh \phi$$

$$\beta\gamma = \sinh \phi$$

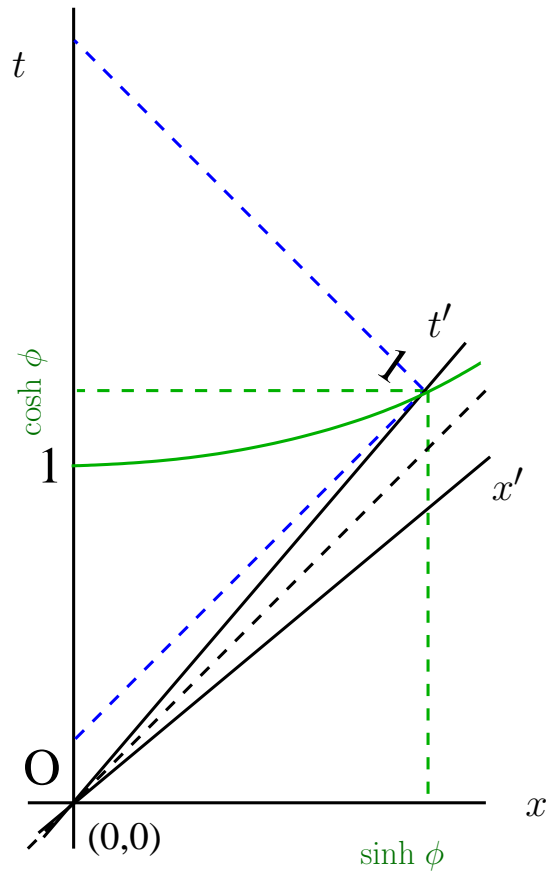




# SR: worldline time dilation



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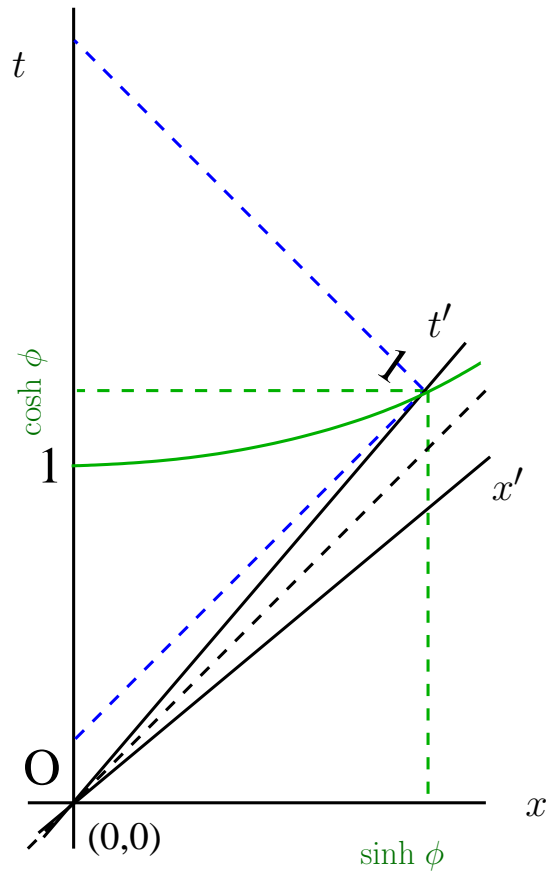


$$\cosh \phi \equiv \gamma$$





# SR: worldline time dilation

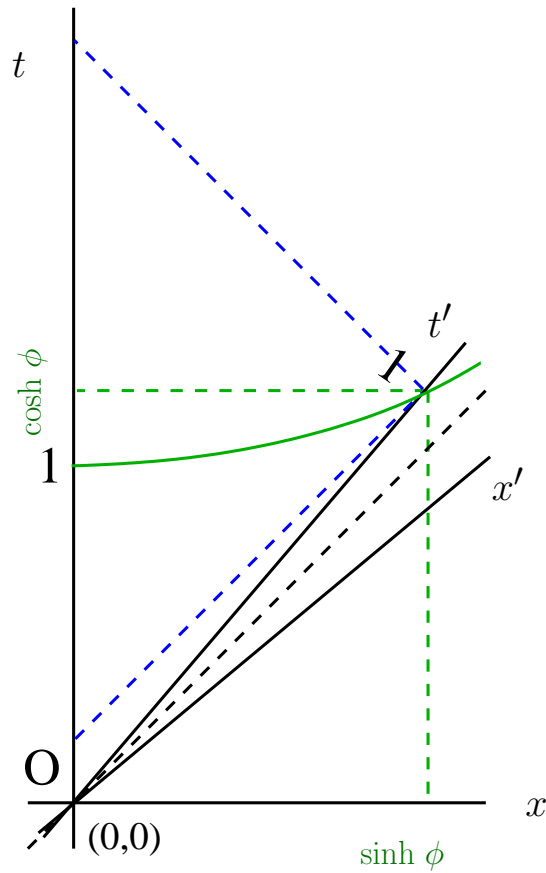


$$\cosh \phi \equiv \gamma \equiv \frac{1}{\sqrt{1-\beta^2}}$$





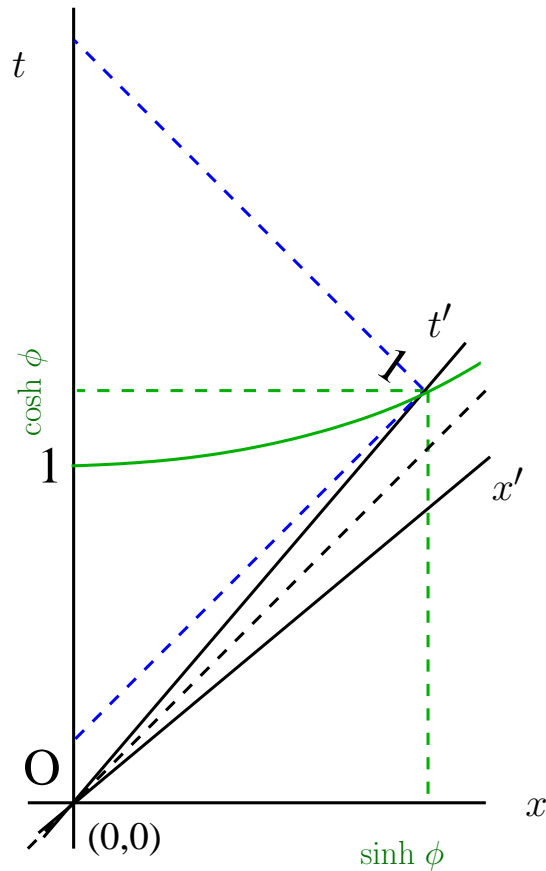
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$$\cosh \phi \equiv \gamma \equiv \frac{1}{\sqrt{1-\beta^2}} > 1$$



# SR: worldline time dilation

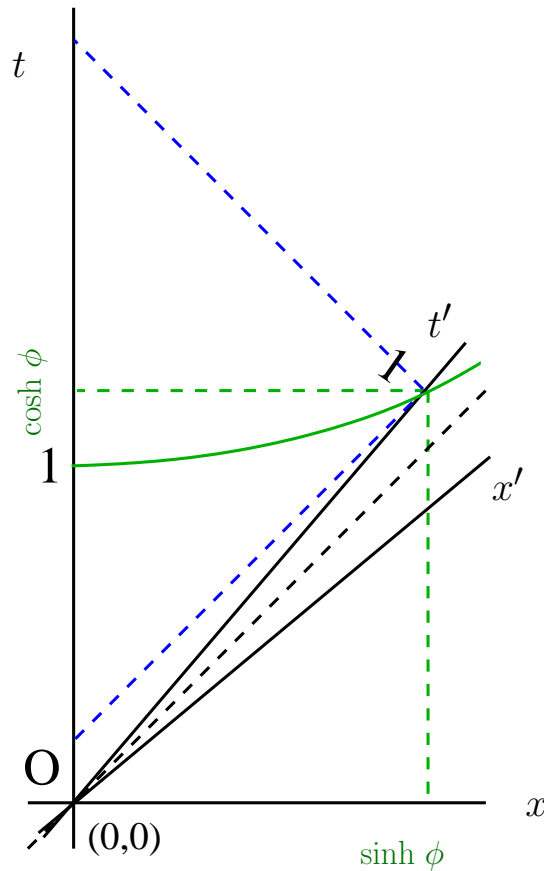


worldline “time dilation”

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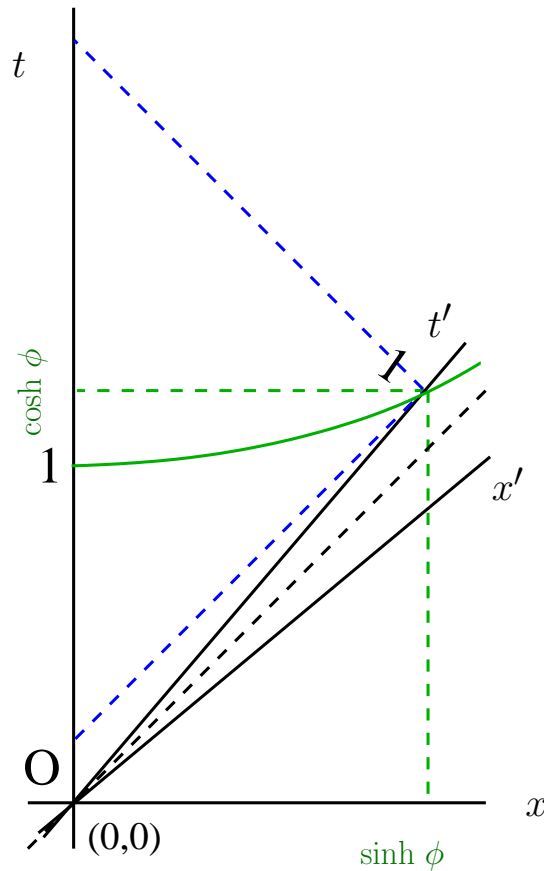
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muons: mean lifetime  
2197 ns  $\ll$  15 km

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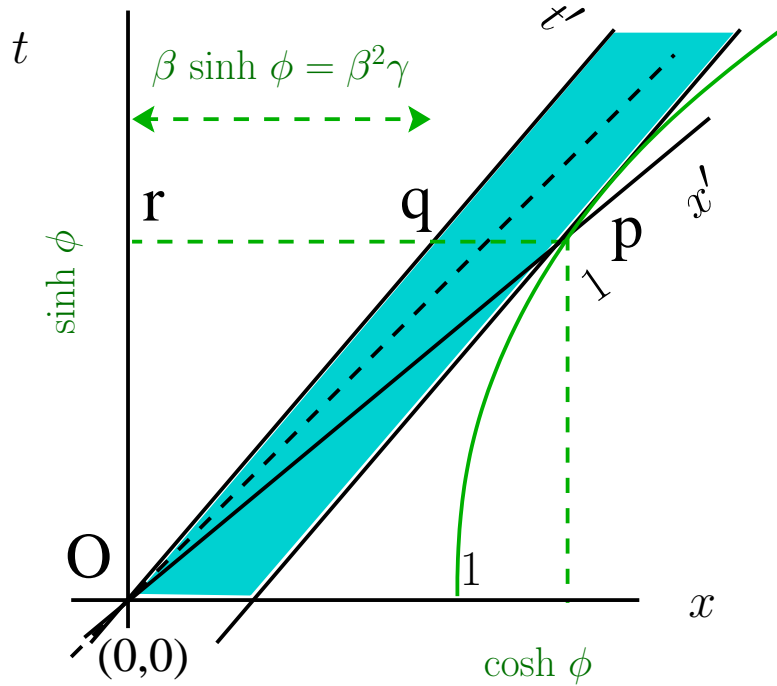
worldline “time dilation”

**muons:** mean lifetime  
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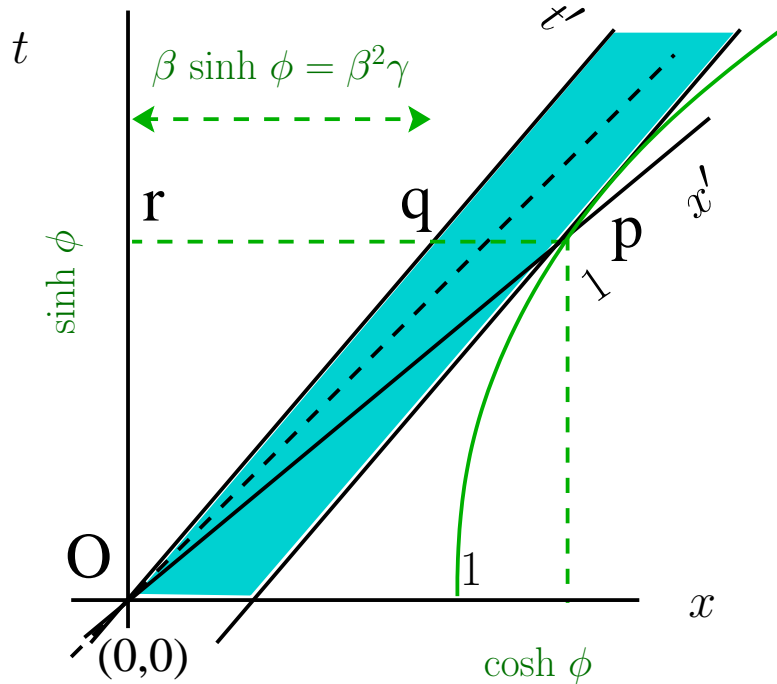
time dilation  $\Rightarrow$  muons  
can hit the ground

$$\cosh \phi \equiv \gamma \equiv \frac{1}{\sqrt{1-\beta^2}} > 1$$

# SR: worldsheet space contraction

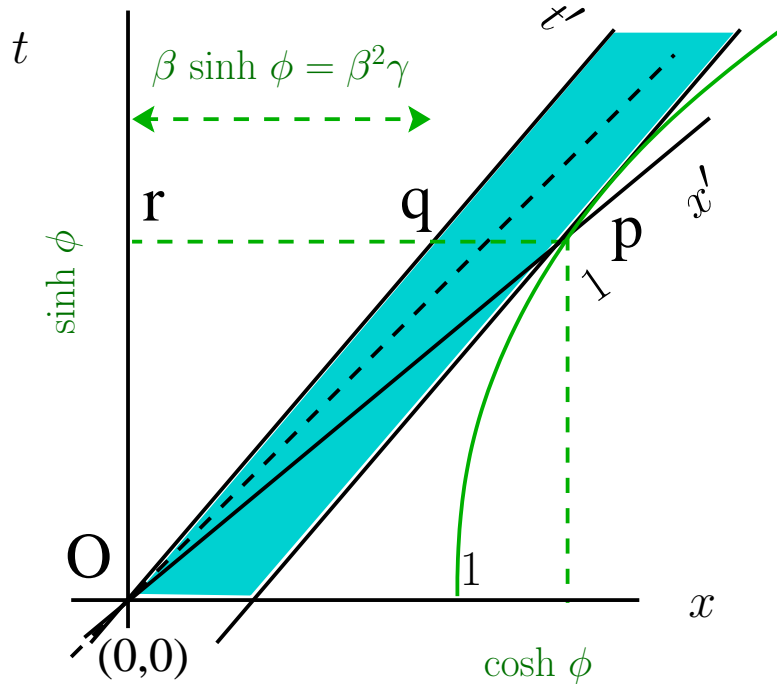


# SR: worldsheet space contraction



$$\sqrt{\Delta s^2(q, p)} =$$

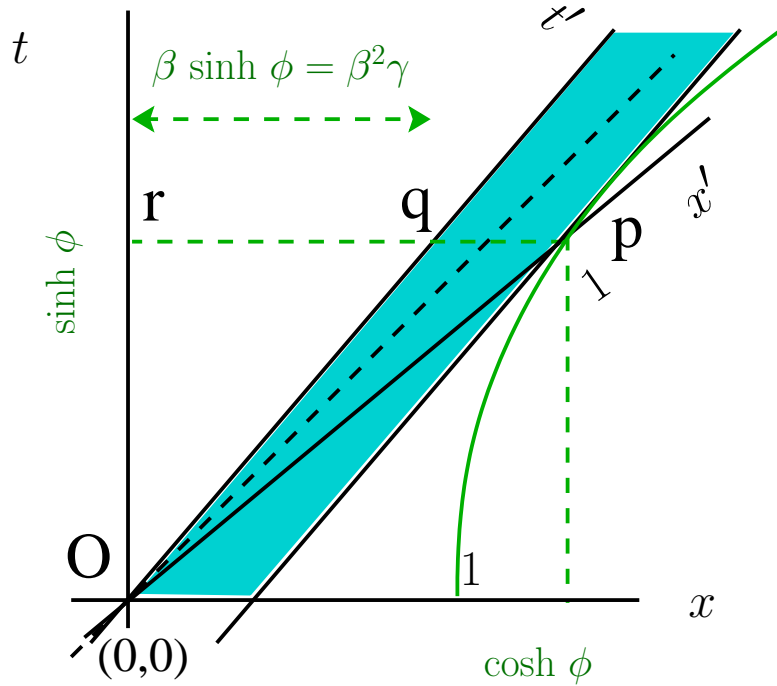
# SR: worldsheet space contraction



$$\sqrt{\Delta s^2(q, p)} = \cosh \phi - \beta \sinh \phi$$

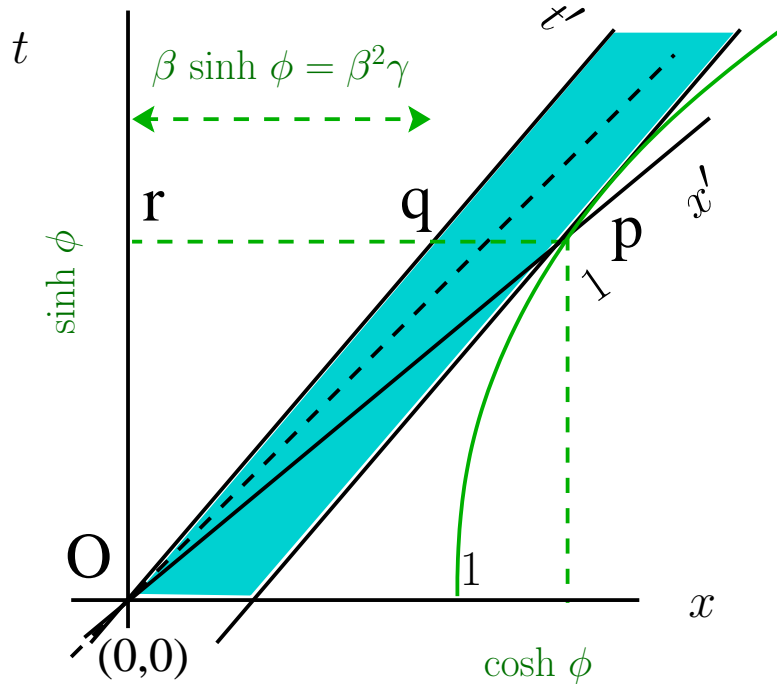


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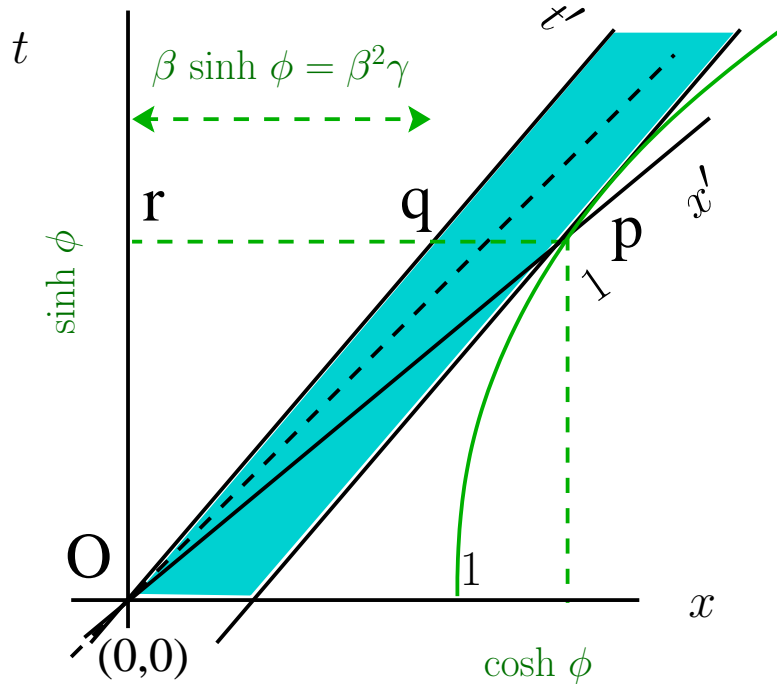
$$\sqrt{\Delta s^2(q, p)} = \gamma - \beta\beta\gamma$$

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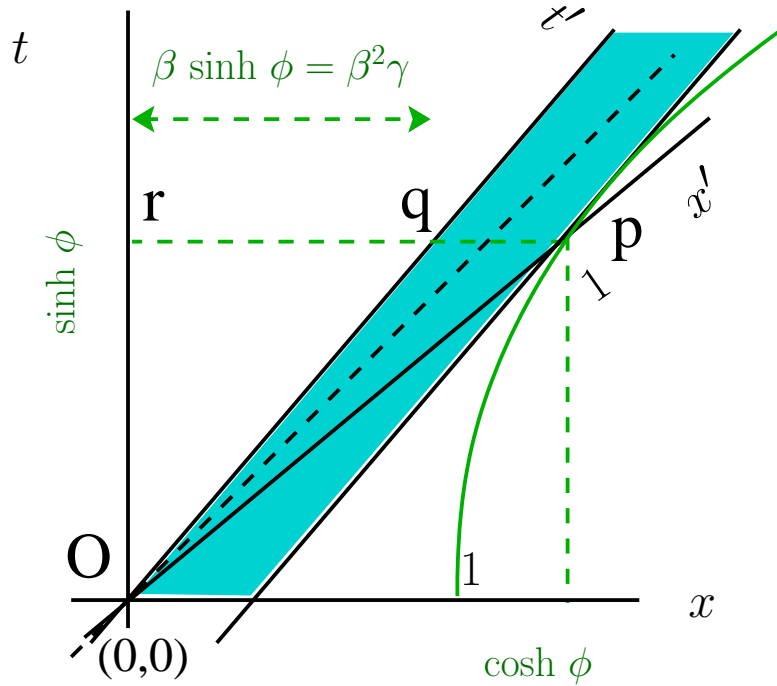
$$\sqrt{\Delta s^2(q, p)} = (1 - \beta^2)\gamma$$

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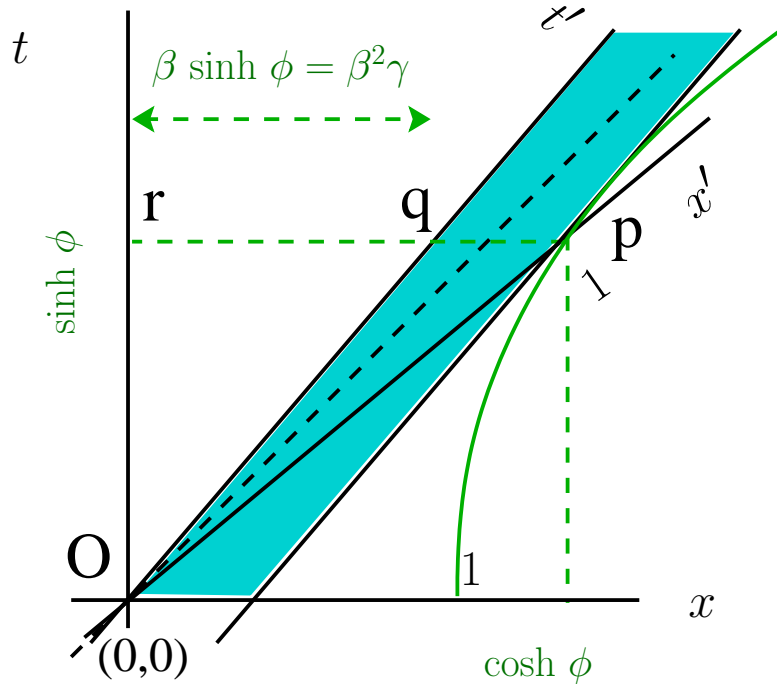
$$\sqrt{\Delta s^2(q, p)} = (1 - \beta^2)^{1+(-1/2)}$$

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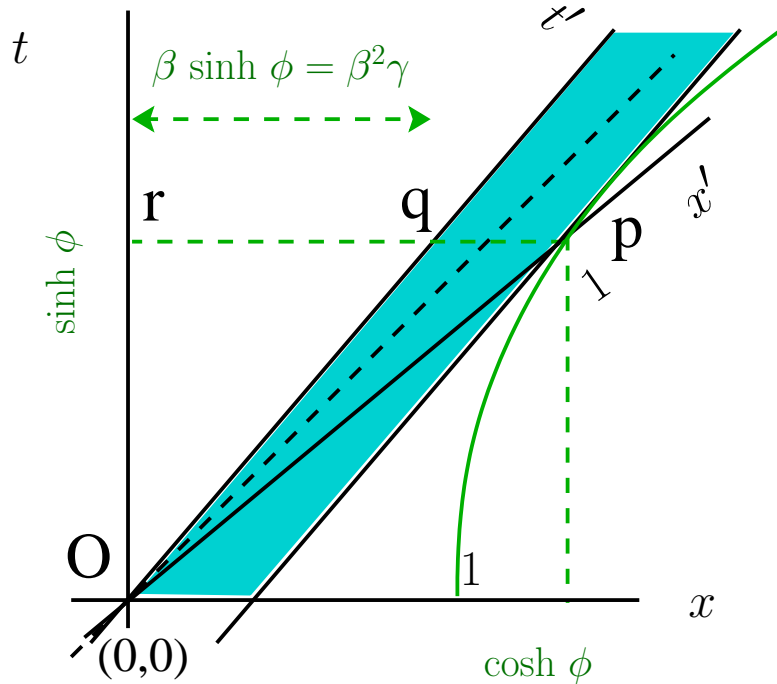
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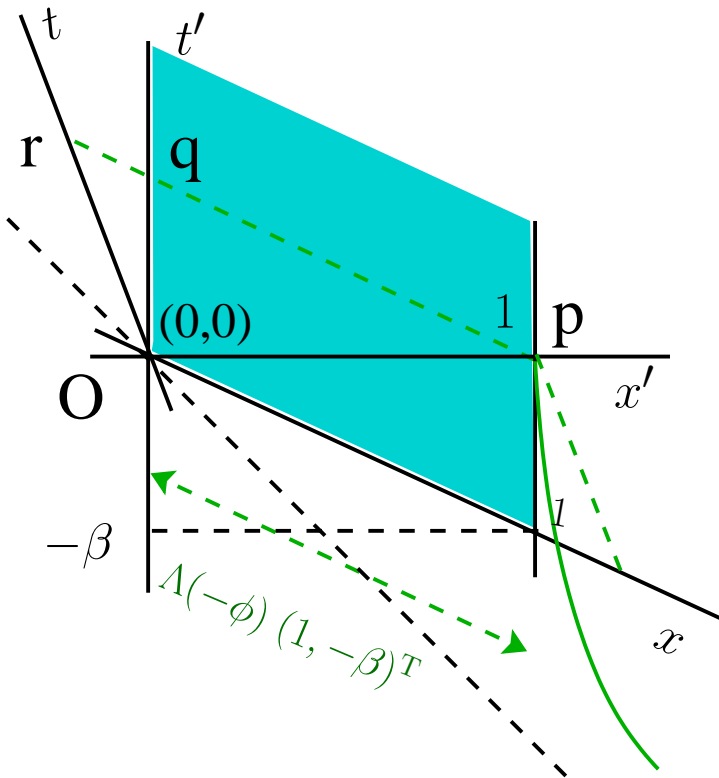
$$\sqrt{\Delta s^2(q, p)} = \gamma^{-1} < 1$$

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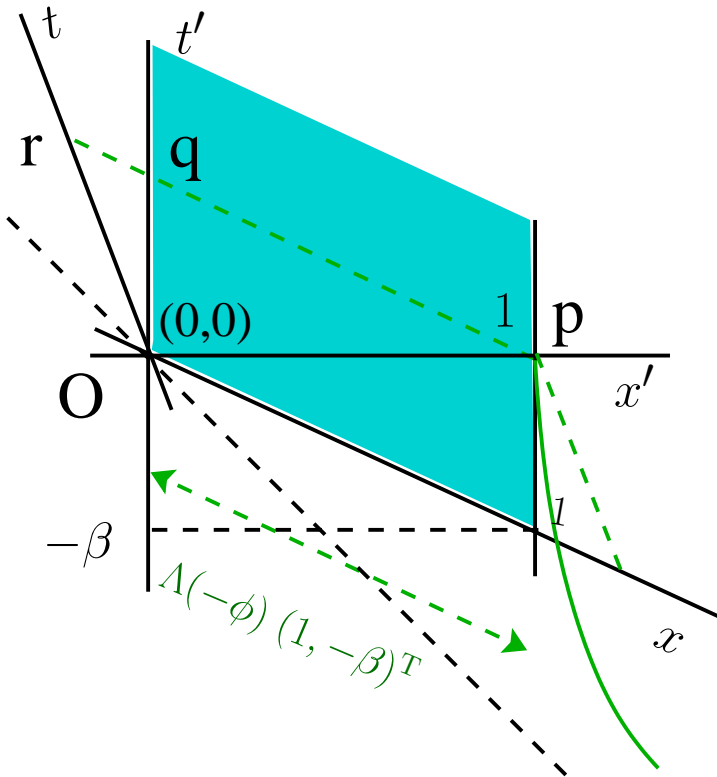


$$\sqrt{\Delta s^2(q, p)} = \gamma^{-1} < 1 \quad \text{worldsheet "space contraction"}$$

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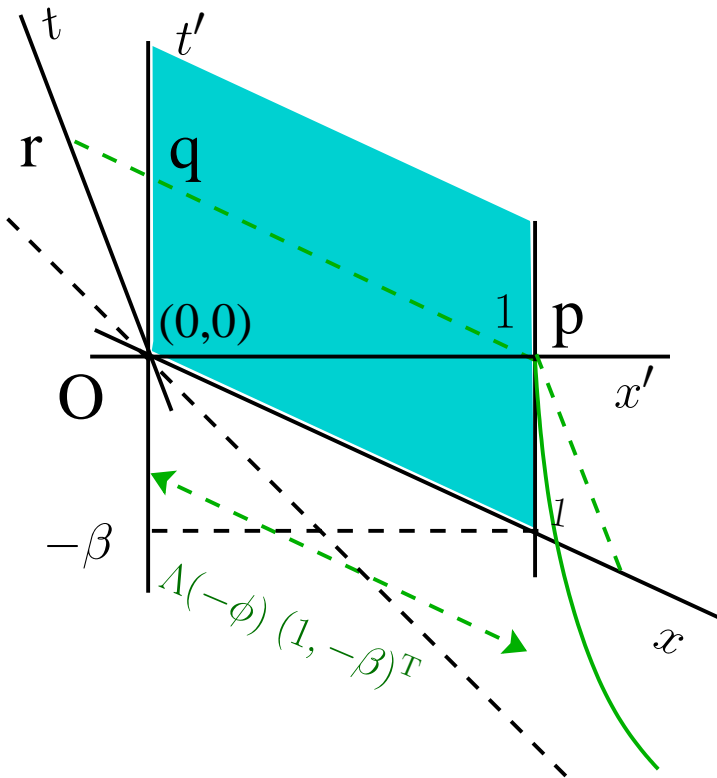
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$$\Lambda^{-1} \begin{pmatrix} 1 \\ -\beta \end{pmatrix} = \begin{pmatrix} \cosh \phi - \beta \sinh \phi \\ \sinh \phi - \beta \cosh \phi \end{pmatrix}$$

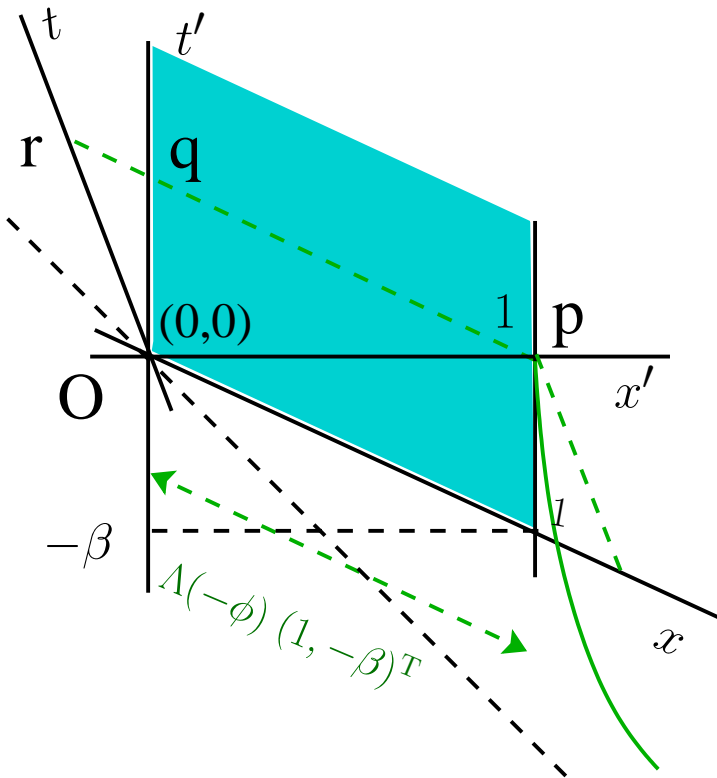


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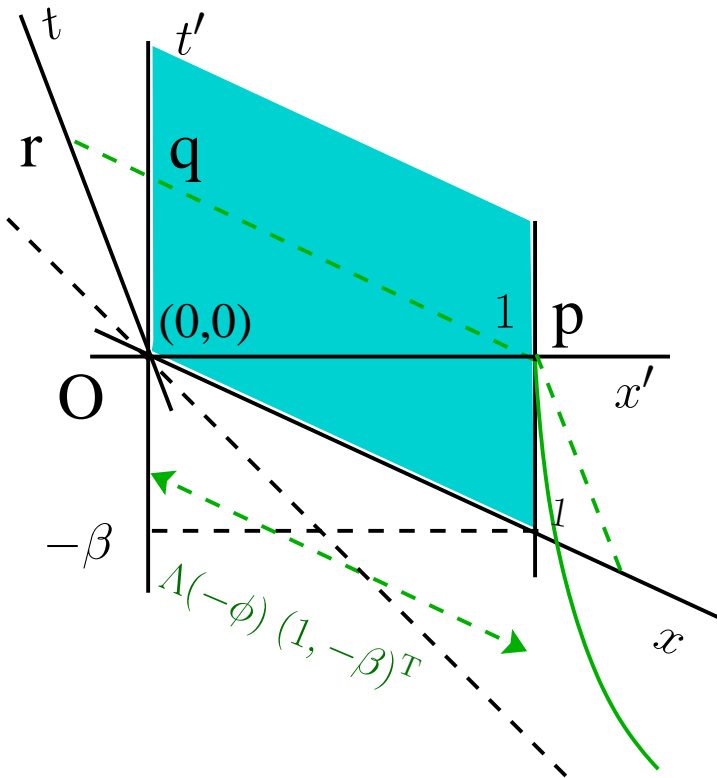
$$\Lambda^{-1} \begin{pmatrix} 1 \\ -\beta \end{pmatrix} = \begin{pmatrix} \cosh \phi (1 - \beta^2) \\ \cosh \phi (\tanh \phi - \tanh \phi) \end{pmatrix}$$

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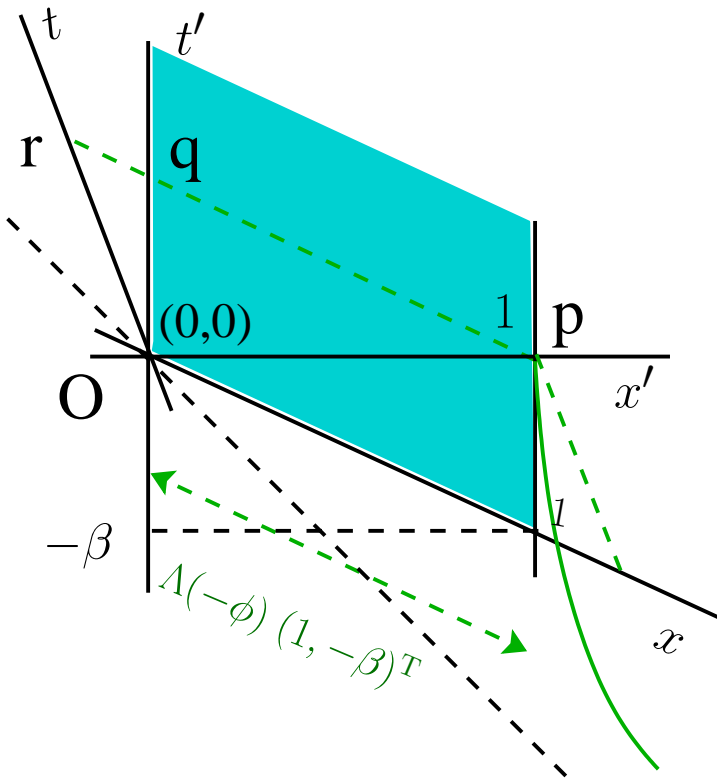
$$\Lambda^{-1} \begin{pmatrix} 1 \\ -\beta \end{pmatrix} = \begin{pmatrix} \gamma(1 - \beta^2) \\ 0 \end{pmatrix}$$

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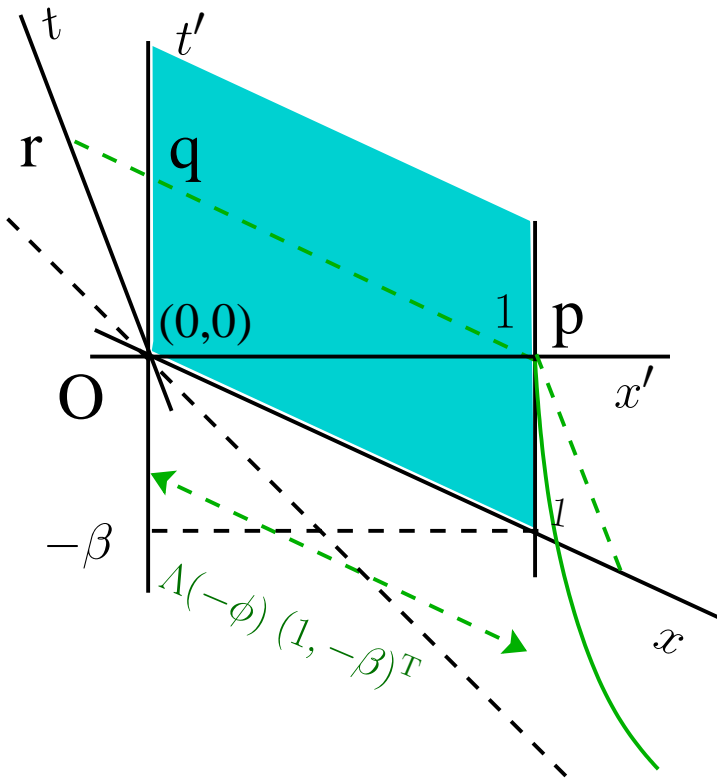
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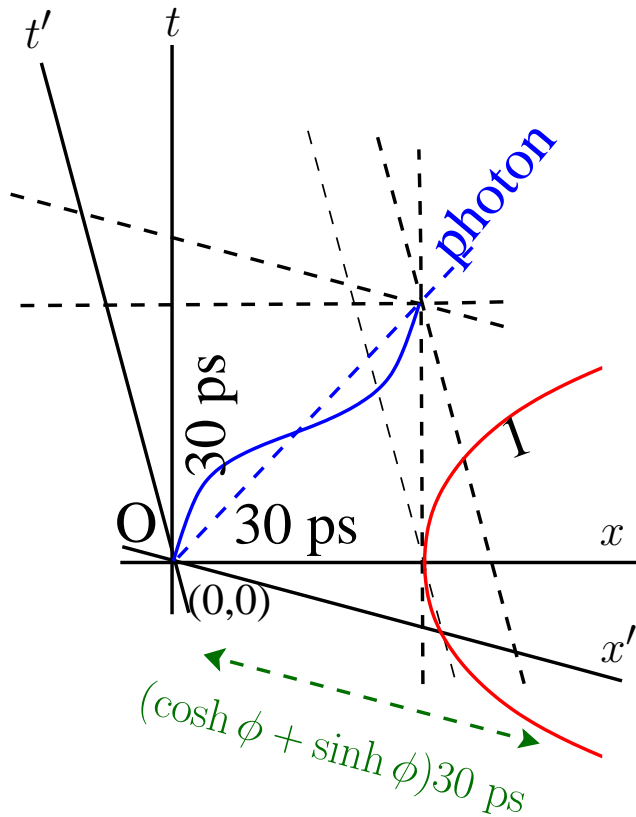
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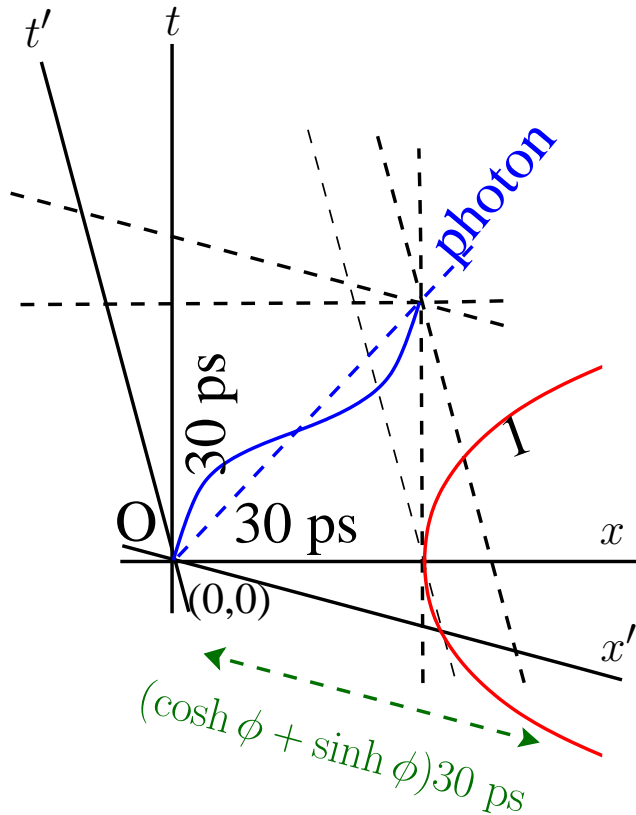


# SR: Doppler shift





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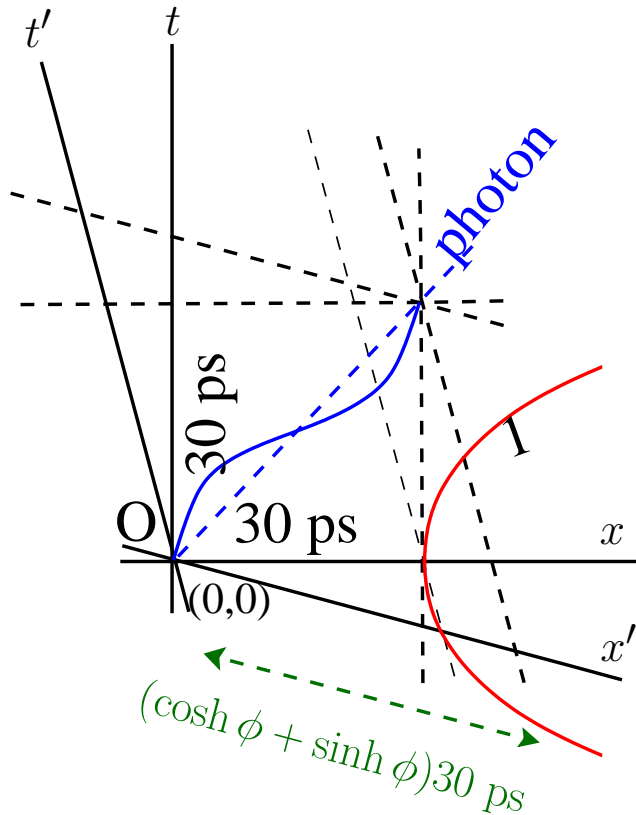


see photon worldline calculation





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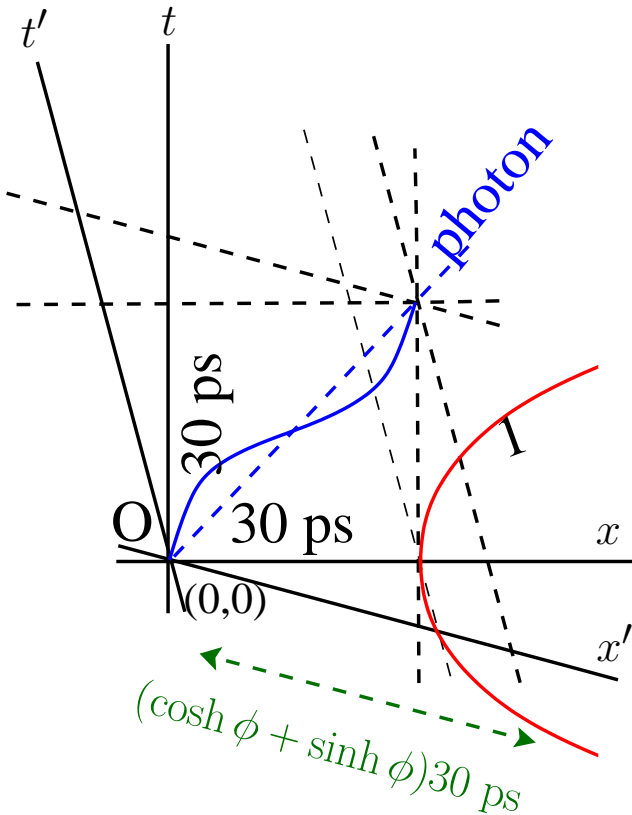
see photon worldline calculation

$$x' = (\cosh \phi + \sinh \phi)t$$





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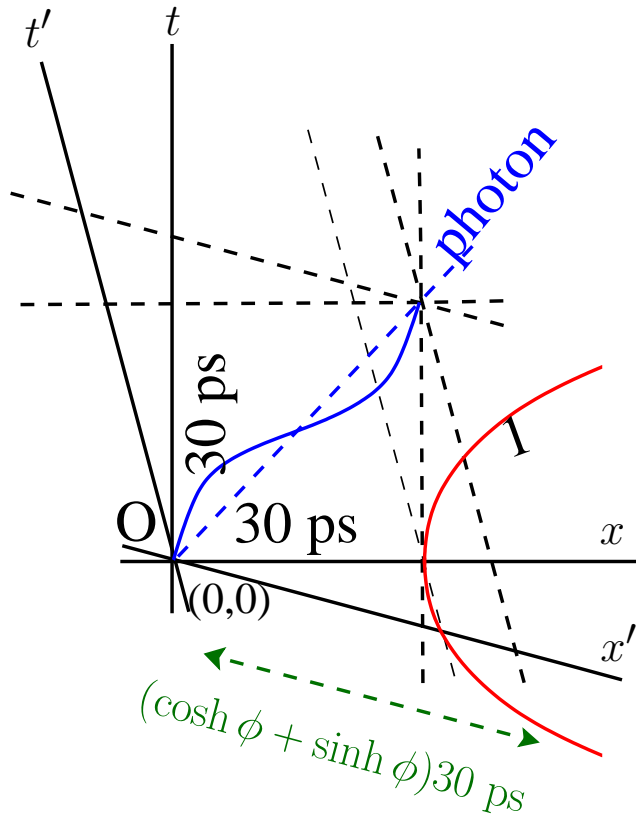


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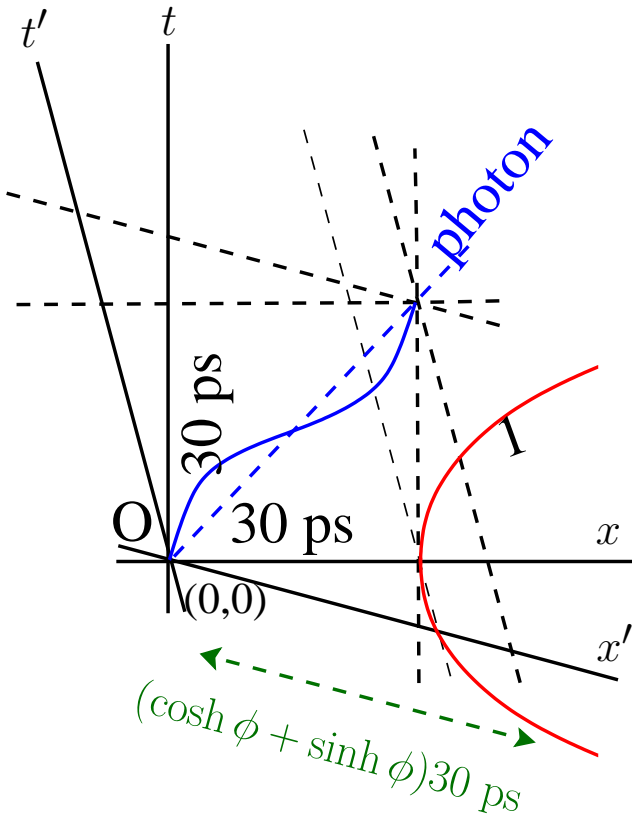


see photon worldline calculation

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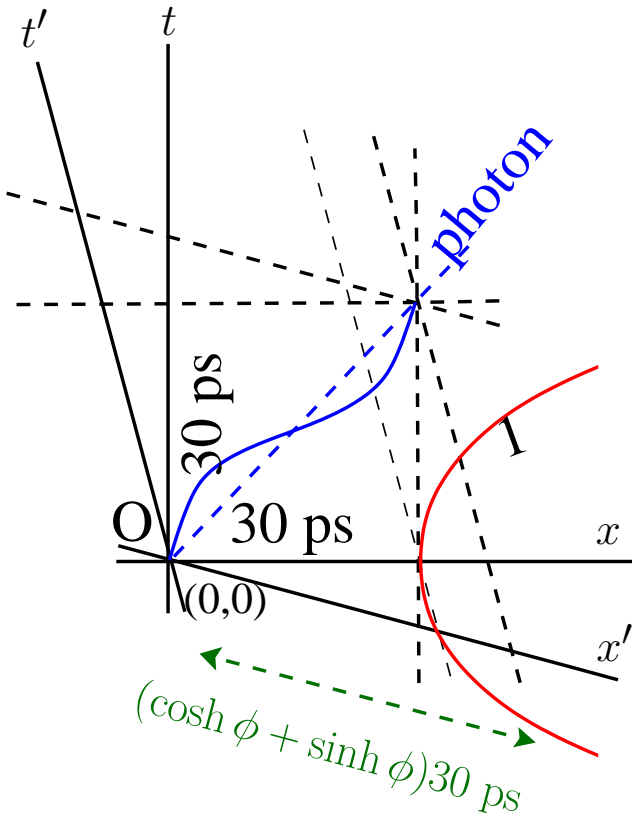
# SR: Doppler shift



see photon worldline calculation

$$x'/x = \gamma + \beta\gamma$$

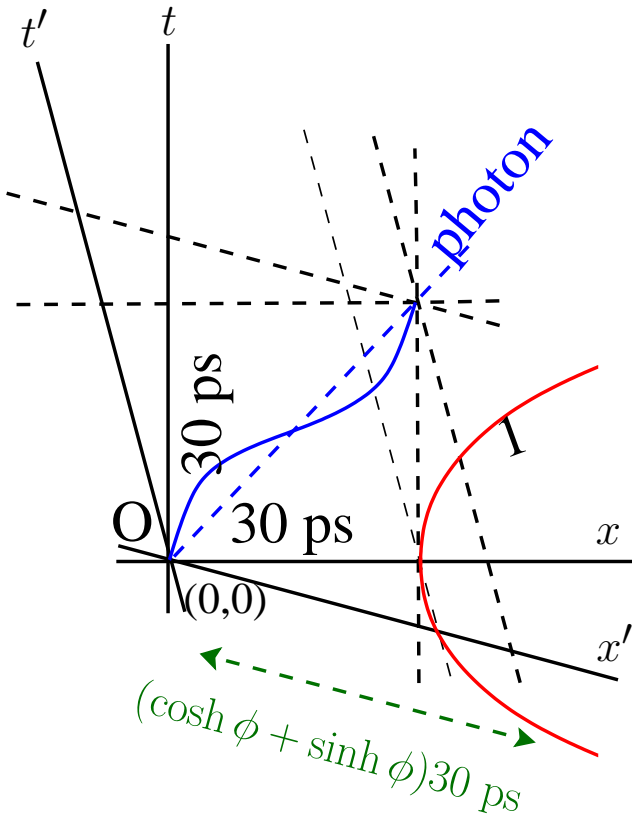
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see photon worldline calculation

$$x'/x = \gamma(1 + \beta)$$

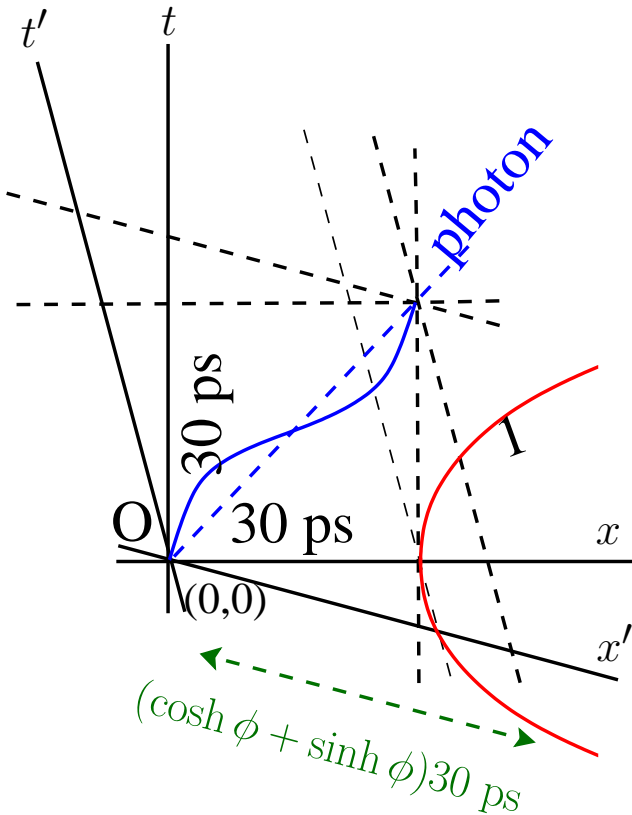
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see photon worldline calculation

$$x'/x = \gamma(1 + \beta) = \frac{1 + \beta}{\sqrt{1 - \beta^2}}$$

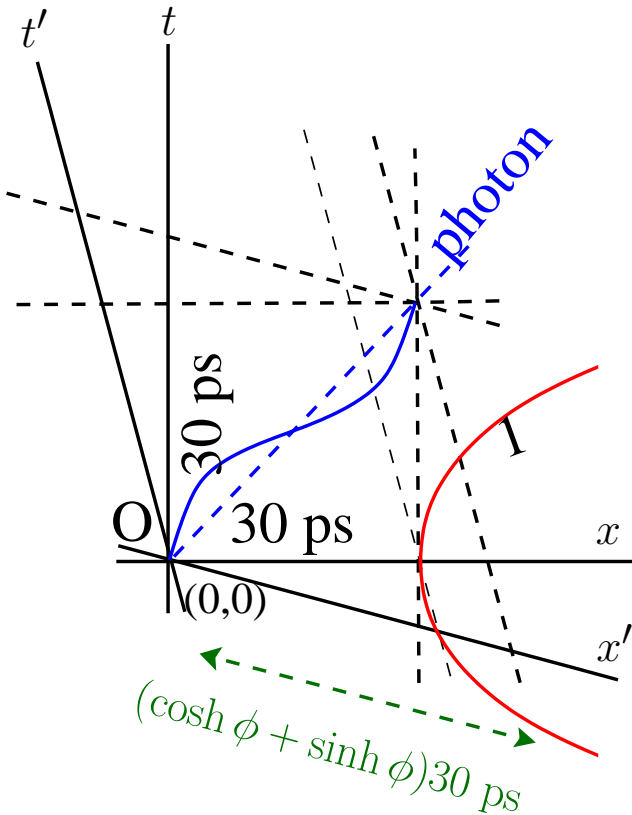
# SR: Doppler shift



see photon worldline calculation

$$x'/x = \gamma(1 + \beta) = \frac{\sqrt{(1+\beta)^2}}{\sqrt{(1-\beta)(1+\beta)}}$$

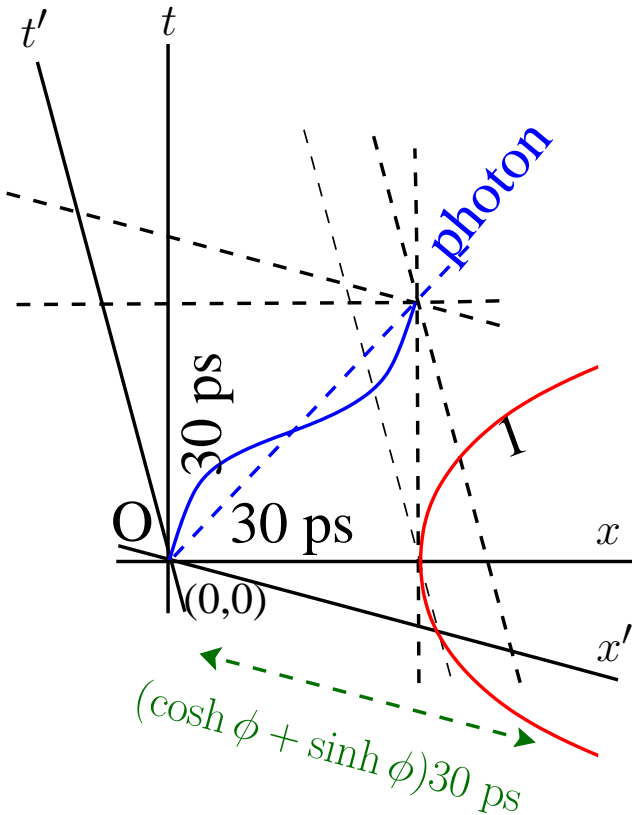
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see photon worldline calculation

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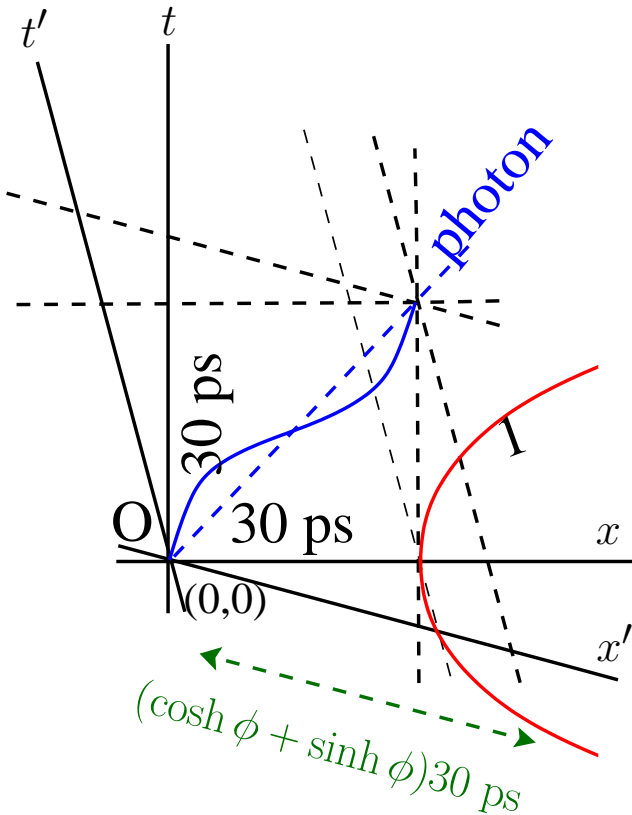
see photon worldline calculation

$$1 + z := \lambda' / \lambda = \gamma(1 + \beta) \equiv \sqrt{\frac{1+\beta}{1-\beta}}$$

redshift



# SR: Doppler shift



see photon worldline calculation

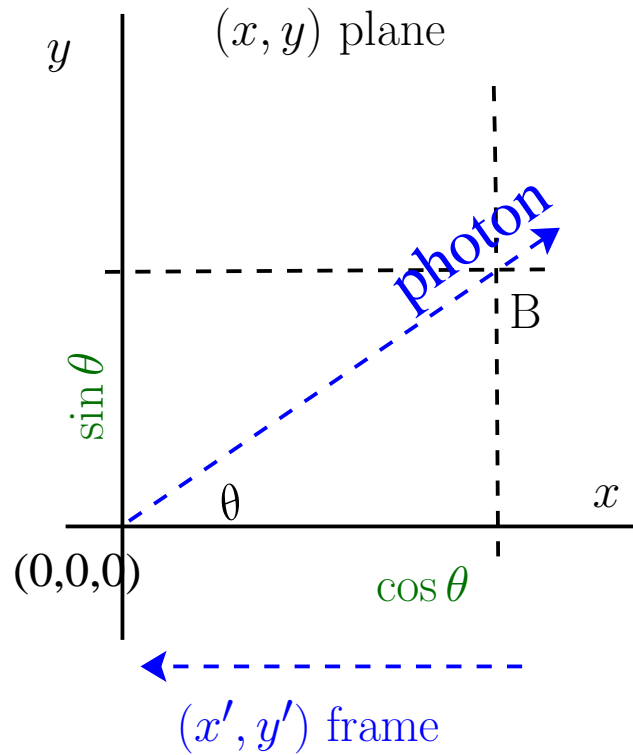
$$1 + z := \lambda' / \lambda = \gamma(1 + \beta) \equiv \sqrt{\frac{1+\beta}{1-\beta}}$$

redshift

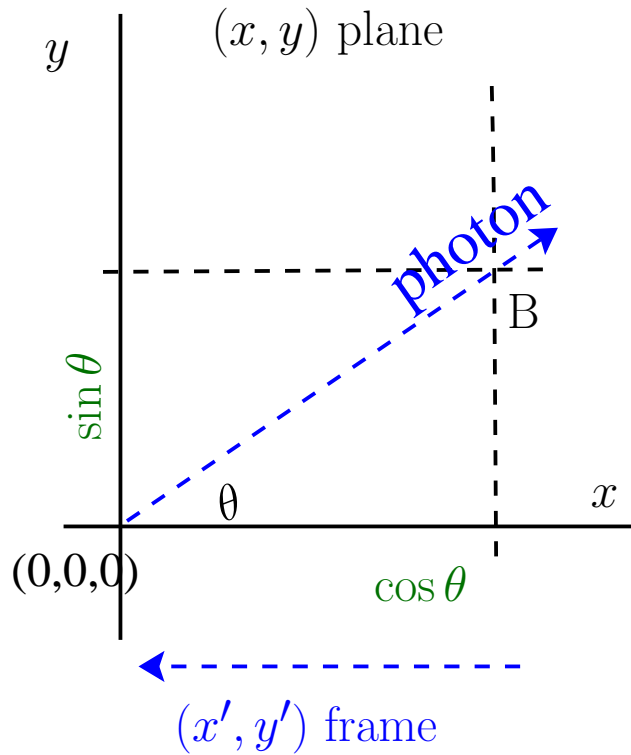
$\Rightarrow$  when  $\beta \ll 1$ ,  $z \approx \beta$



# SR: relativistic aberration



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event B:  $(x, y, t) = (\cos \theta, \sin \theta, 1)$



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$$\Lambda^{-1} \begin{pmatrix} \cos \theta \\ \sin \theta \\ 1 \end{pmatrix} = \begin{pmatrix} \gamma & 0 & \beta\gamma \\ 0 & 1 & 0 \\ \beta\gamma & 0 & \gamma \end{pmatrix} \begin{pmatrix} \cos \theta \\ \sin \theta \\ 1 \end{pmatrix}$$

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$$\Lambda^{-1} \begin{pmatrix} \cos \theta \\ \sin \theta \\ 1 \end{pmatrix} = \begin{pmatrix} \gamma \cos \theta + \beta \gamma \\ \sin \theta \\ \beta \gamma \cos \theta + \gamma \end{pmatrix}$$

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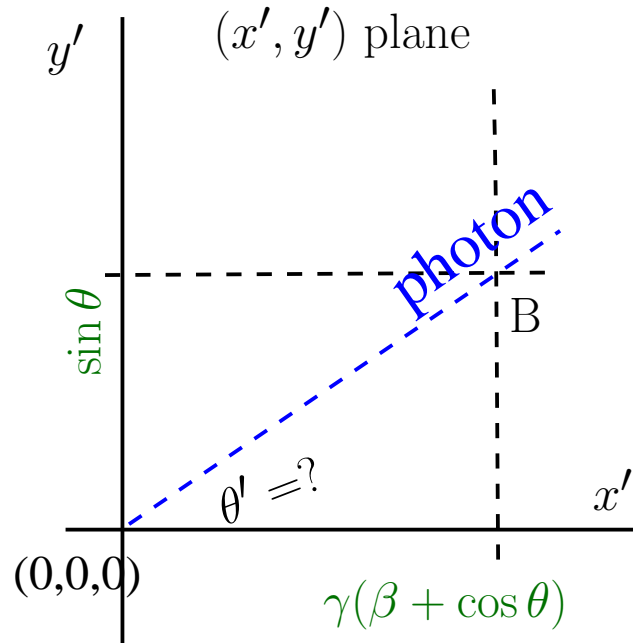
$$\Lambda^{-1} \begin{pmatrix} \cos \theta \\ \sin \theta \\ 1 \end{pmatrix} = \begin{pmatrix} \gamma(\cos \theta + \beta) \\ \sin \theta \\ \gamma(1 + \beta \cos \theta) \end{pmatrix}$$





# SR: relativistic aberration

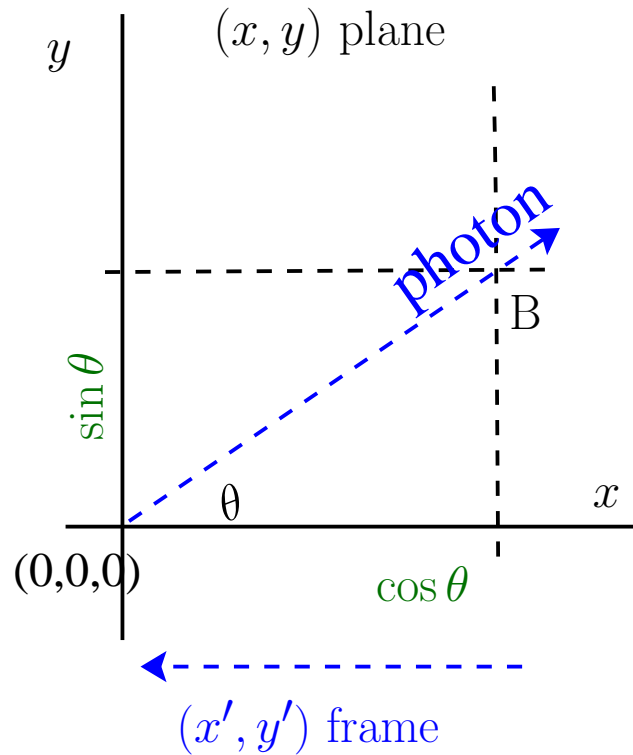
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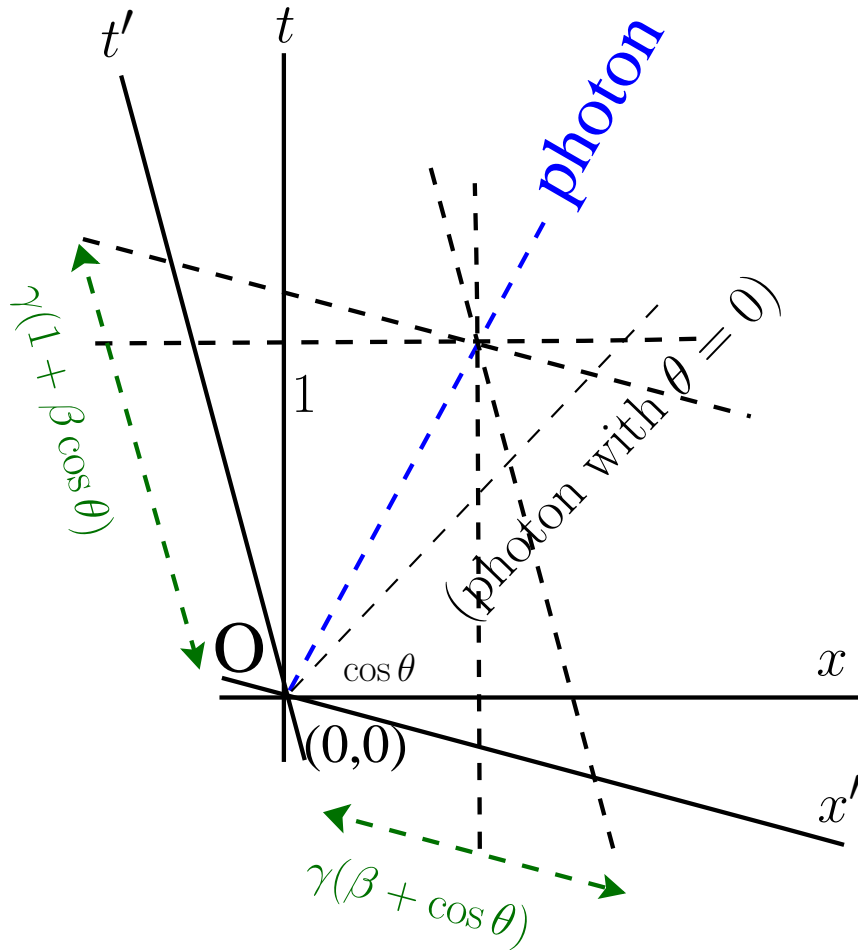
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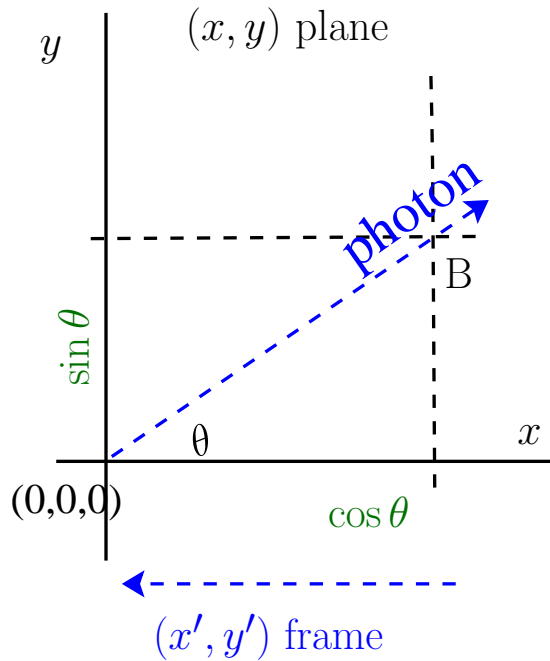
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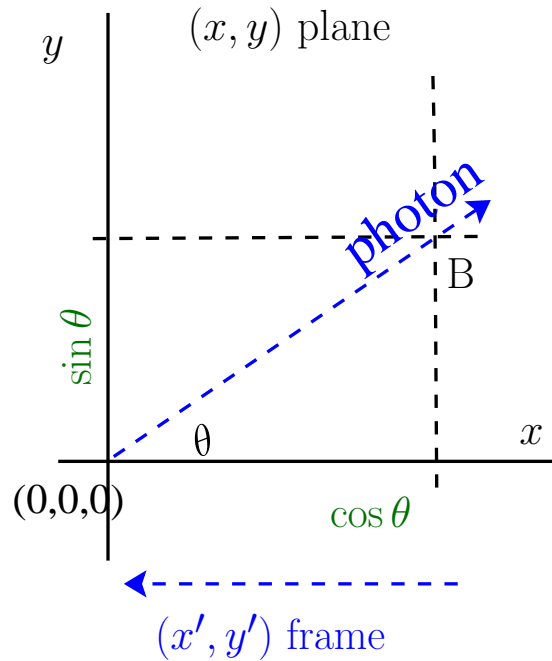
$$\tan \theta' = \frac{\sin \theta}{\gamma(\beta + \cos \theta)}$$





# SR: relativistic aberration

event B:  $(x, y, t) = (\cos \theta, \sin \theta, 1)$



$$\tan \theta' = \frac{\sin \theta}{\gamma(\beta + \cos \theta)} < \tan \theta \text{ if } 0 < \beta < 1$$

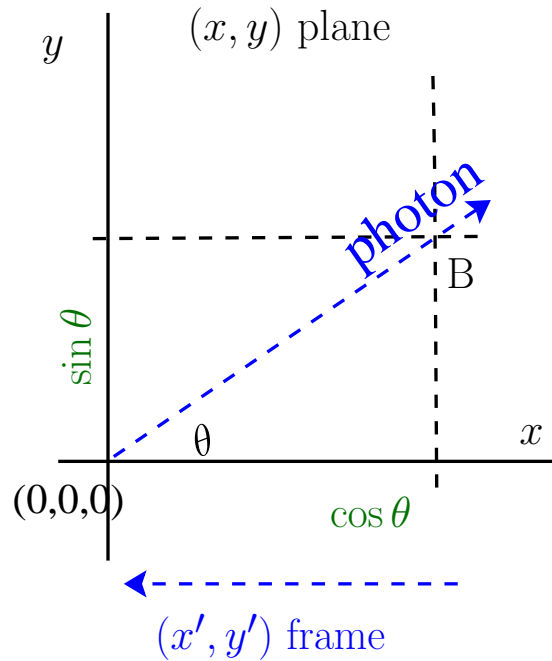
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# SR: relativistic aberration

event B:  $(x, y, t) = (\cos \theta, \sin \theta, 1)$



$$\tan \theta' = \frac{\sin \theta}{\gamma(\beta + \cos \theta)} < \tan \theta \text{ if } 0 < \beta < 1$$

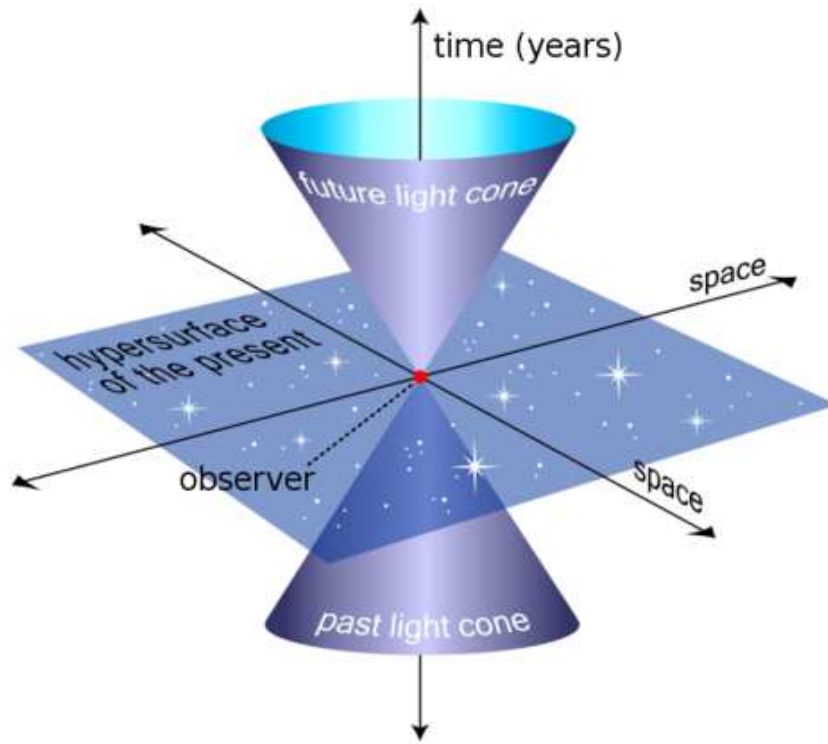
w: Relativistic aberration

$\Rightarrow$  relativistic beaming, e.g. AGN jets

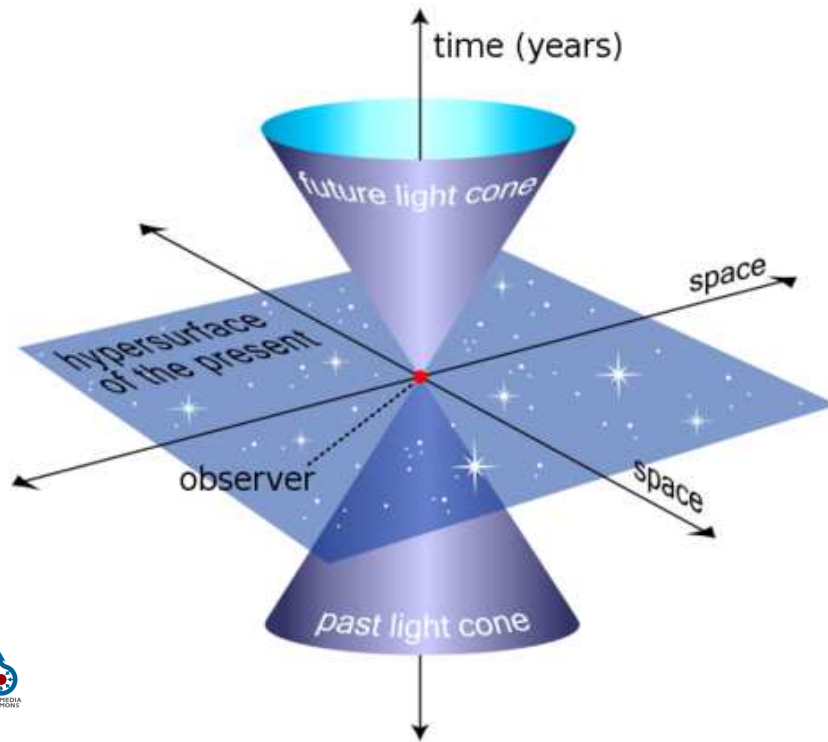




# SR: world line



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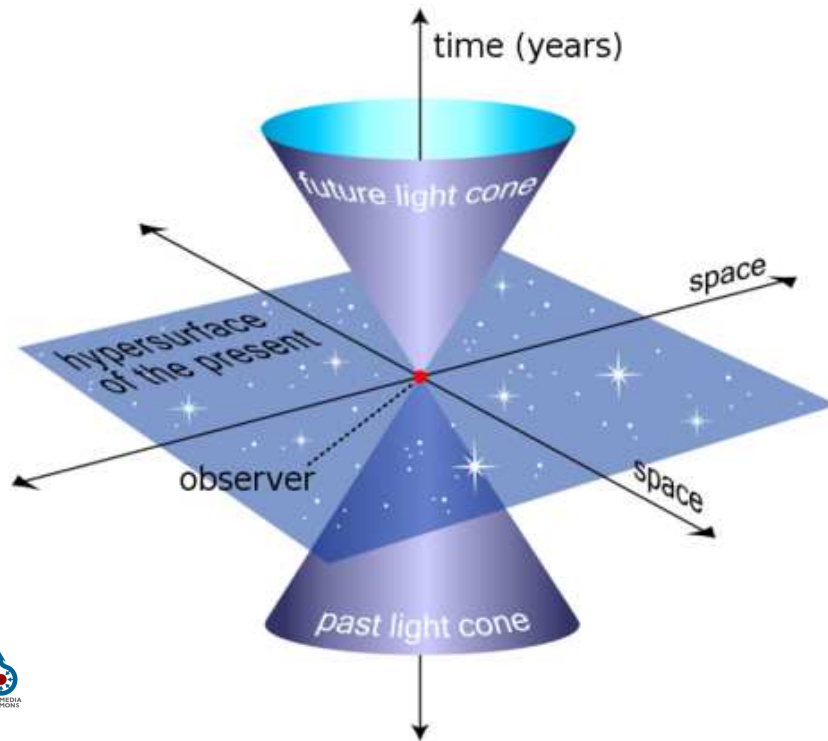


lightlike interval = null interval:  $(\Delta s)^2 = 0$   
 spacetime =





# SR: world line

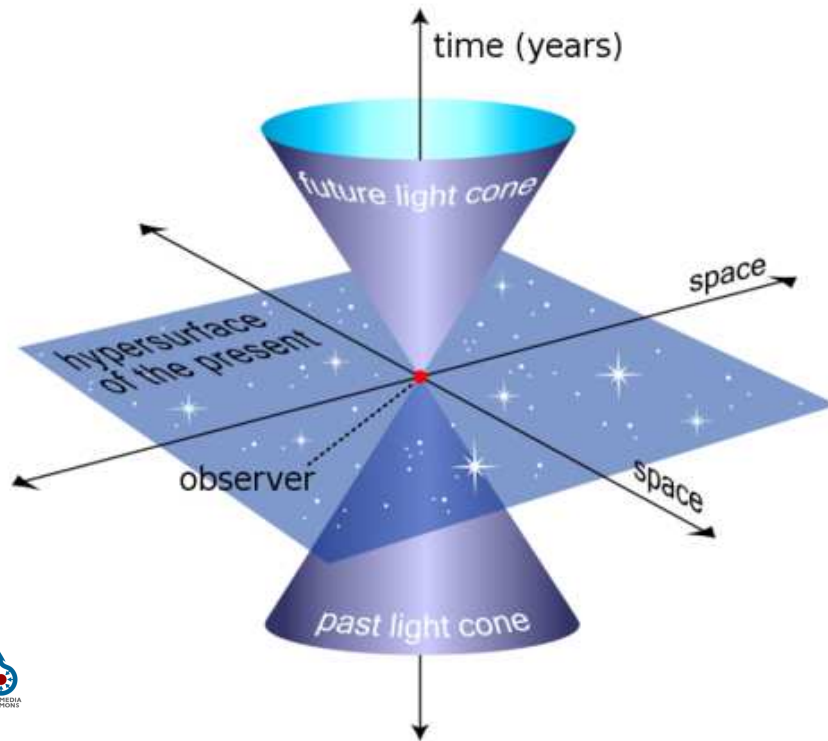


lightlike interval = null interval:  $(\Delta s)^2 = 0$

spacetime = on past w:light cone + inside past light cone



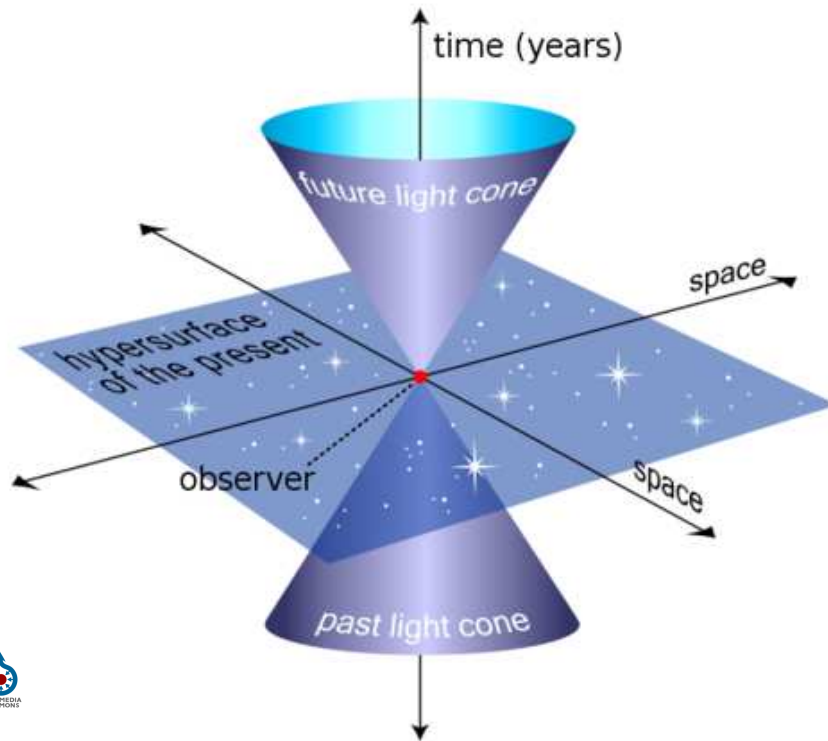
# SR: world line



lightlike interval = null interval:  $(\Delta s)^2 = 0$

spacetime = on past world light cone + inside past light cone  
 + on future light cone + inside future light cone

# SR: world line



lightlike interval = null interval:  $(\Delta s)^2 = 0$

spacetime = on past world light cone + inside past light cone

+ on future light cone + inside future light cone

+ elsewhere



# SR: world line

Lorentz transform of world line





# SR: world line

## Lorentz transform of world line



# SR: world line



Lorentz transform of world line



- coordinate time in spacetime model  $\neq$  time in your brain (thinking)



# SR: world line



## Lorentz transform of world line



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- $\frac{dt}{dt_{\text{thinking}}}$  can be positive or negative



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- $\frac{dt}{d\lambda}$  can be positive or negative,  $\lambda$  arbitrary real parameter





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## Lorentz transform of world line



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- “elsewhere” spacetime events can change from past to future even though  $\frac{dt}{d\lambda} > 0$



# SR: world line

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- $\frac{dt}{d\lambda}$  can be positive or negative,  $\lambda$  arbitrary real parameter
- “elsewhere” spacetime events can change from past to future even though  $\frac{dt}{d\lambda} > 0$
- w:proper time  $\tau :=$  time along a worldline measured by clock following that worldline

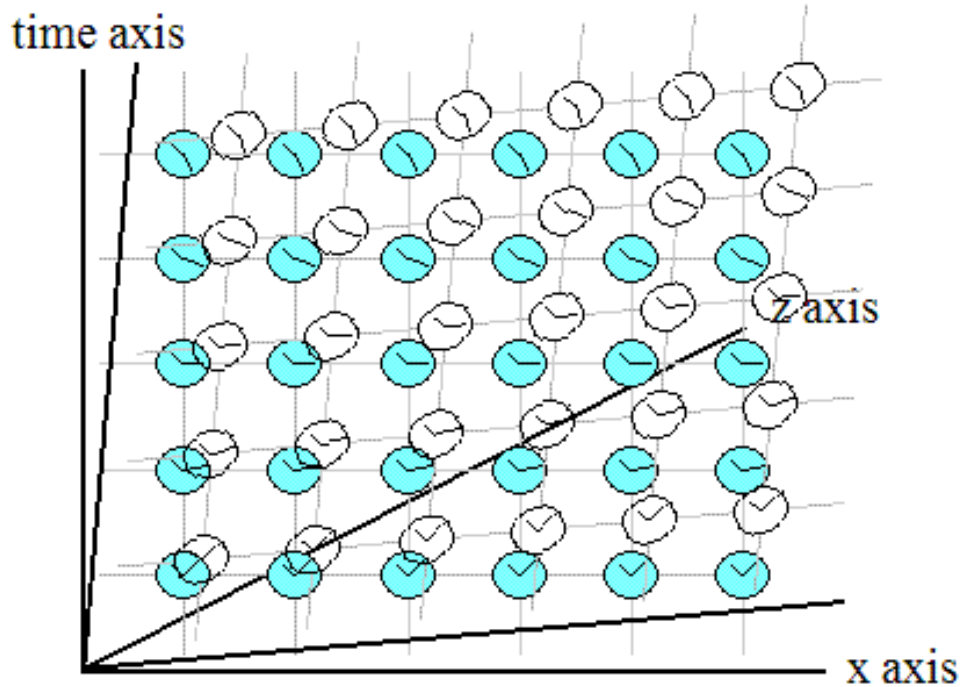
# SR: world line

## Lorentz transform of world line



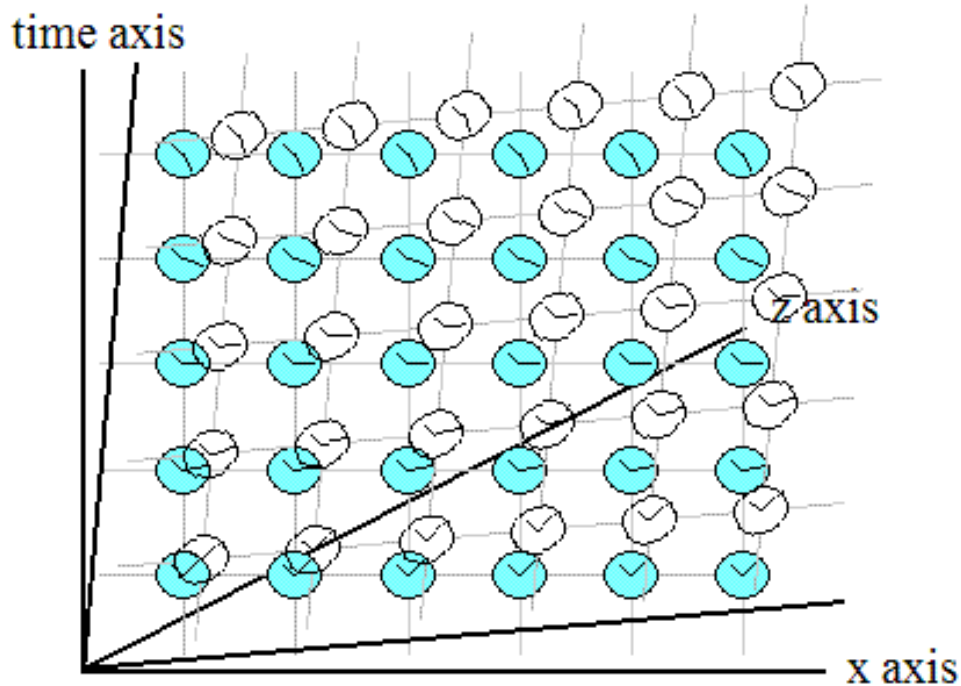
- coordinate time in spacetime model  $\neq$  time in your brain (thinking)
- $\frac{dt}{d\lambda}$  can be positive or negative,  $\lambda$  arbitrary real parameter
- “elsewhere” spacetime events can change from past to future even though  $\frac{dt}{d\lambda} > 0$
- w:proper time  $\tau :=$  time along a worldline measured by clock following that worldline
- often  $d\tau$  is useful for integrating

# SR: Rietdijk–Putnam–Penrose p.



Relativity shows that the inertial frames of reference of relatively moving objects do not overlie each other.

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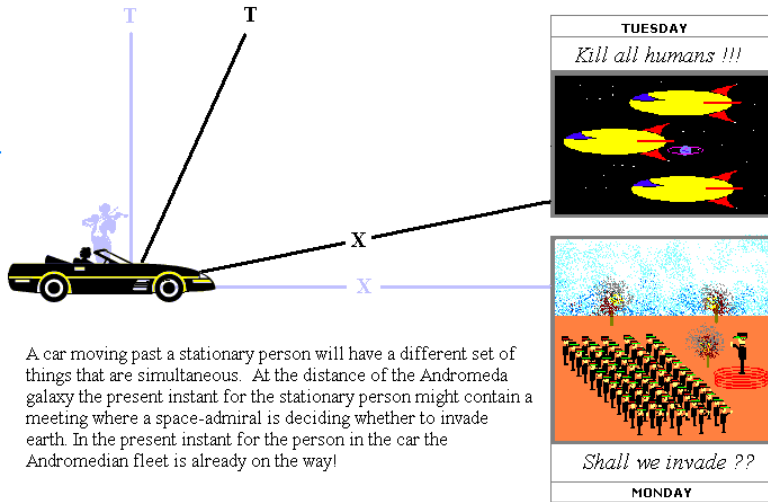
b:Inertialoverlay.GIF

- each observer can synchronise clocks + rods



# SR: Rietdijk–Putnam–Penrose p.

## The Andromeda Paradox

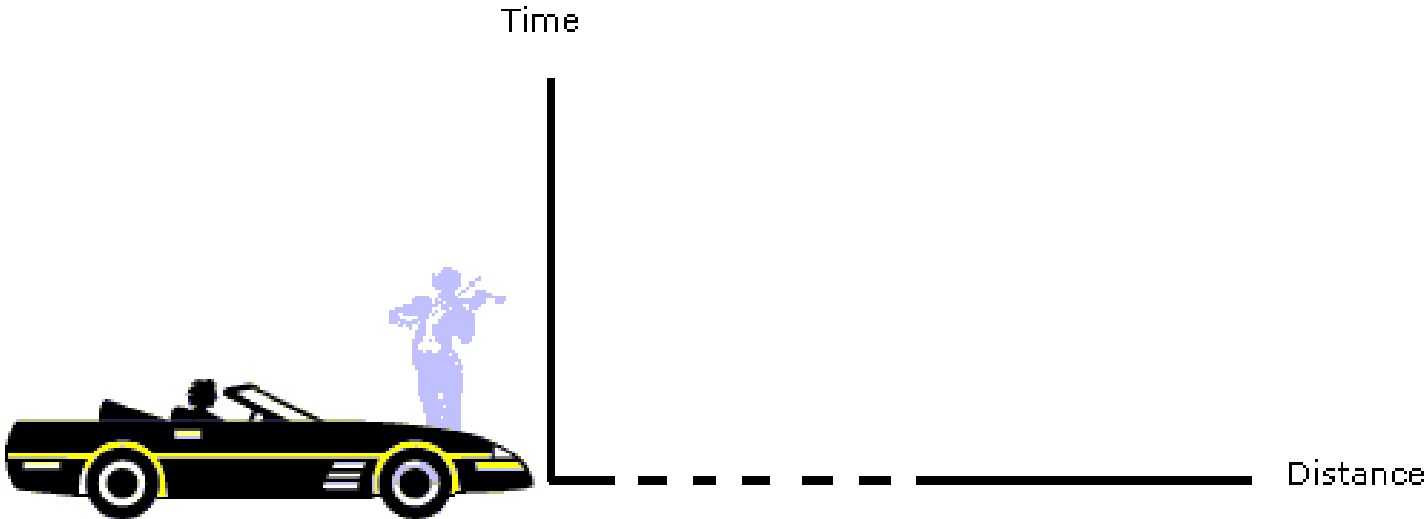


w:Rietdijk-Putnam argument b:Rel2.gif





# SR: Rietdijk–Putnam–Penrose p.



For the car driver the stationary man and the invasion fleet are all events in the present moment.

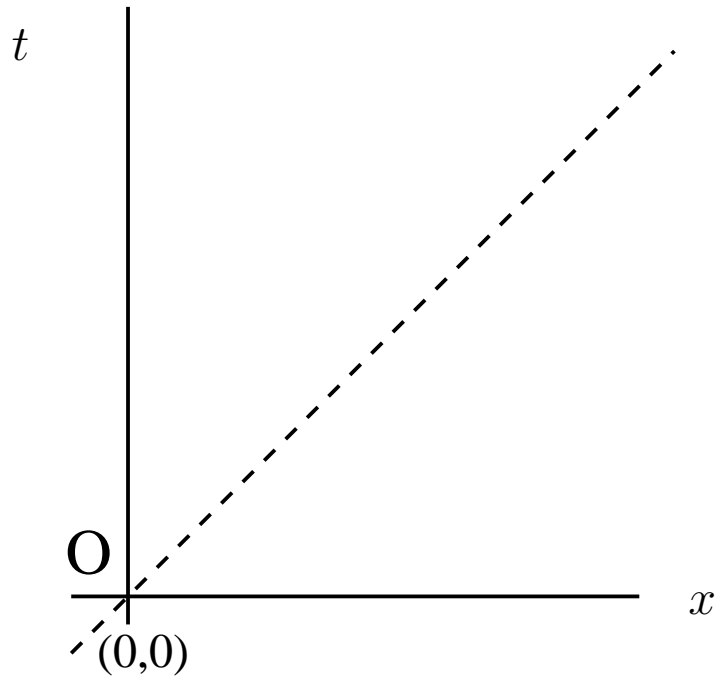


[w:Rietdijk-Putnam argument](#) [b:Rel3.gif](#)





# SR: tachyons and causality



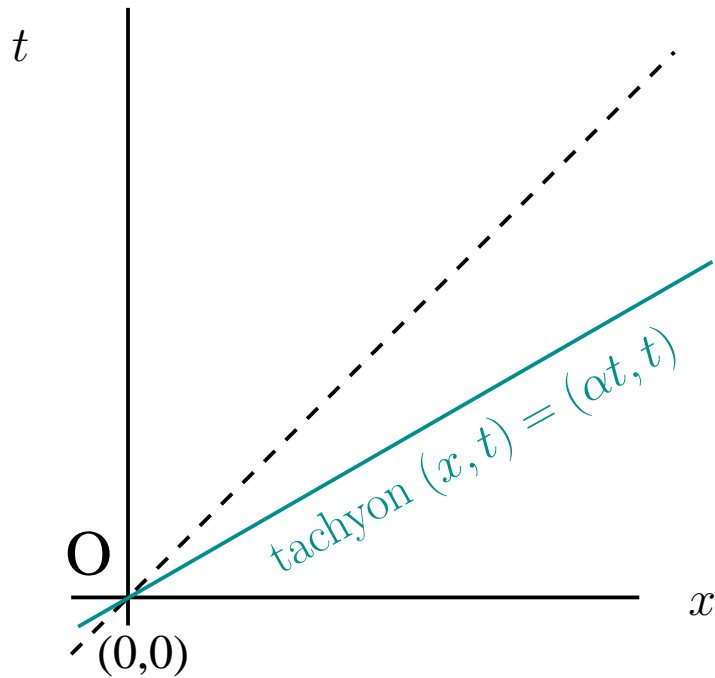
observer "at rest"







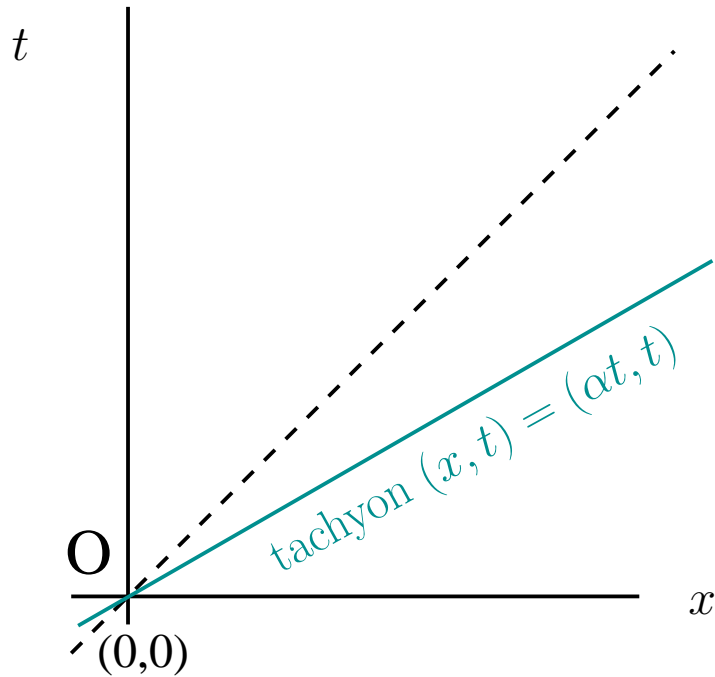
# SR: tachyons and causality



add a tachyon with speed  $\alpha > 1$



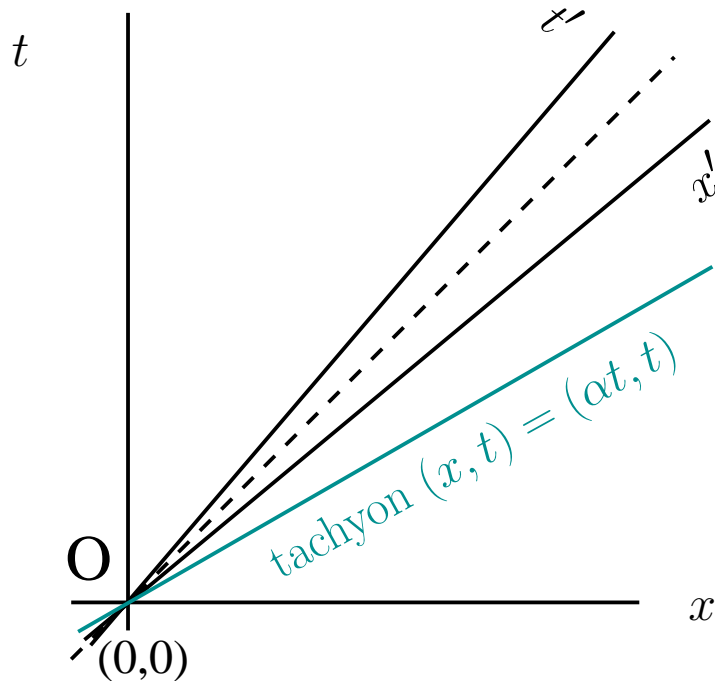
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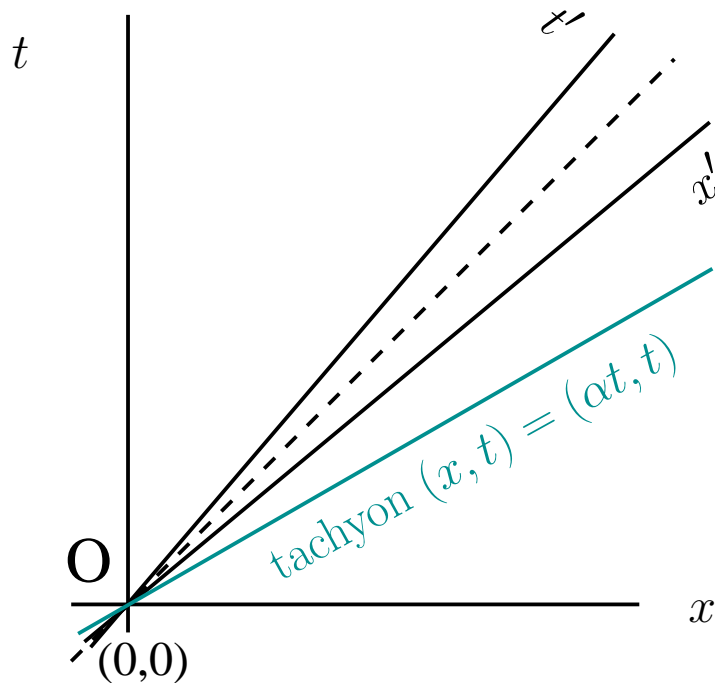


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rocket frame:  $(\alpha t, t)$  becomes  $\Lambda (\alpha t, t)^T$



# SR: tachyons and causality

$$\begin{pmatrix} x' \\ t' \end{pmatrix} = \Lambda \begin{pmatrix} \alpha t \\ t \end{pmatrix} =$$





# SR: tachyons and causality

$$\begin{pmatrix} x' \\ t' \end{pmatrix} = \Lambda \begin{pmatrix} \alpha t \\ t \end{pmatrix} = \begin{pmatrix} \gamma \alpha t - \beta \gamma t \\ -\alpha \beta \gamma t + \gamma t \end{pmatrix}$$



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same sequence of spacetime events = tachyon worldline:

$t$  increases for observer “at rest”,

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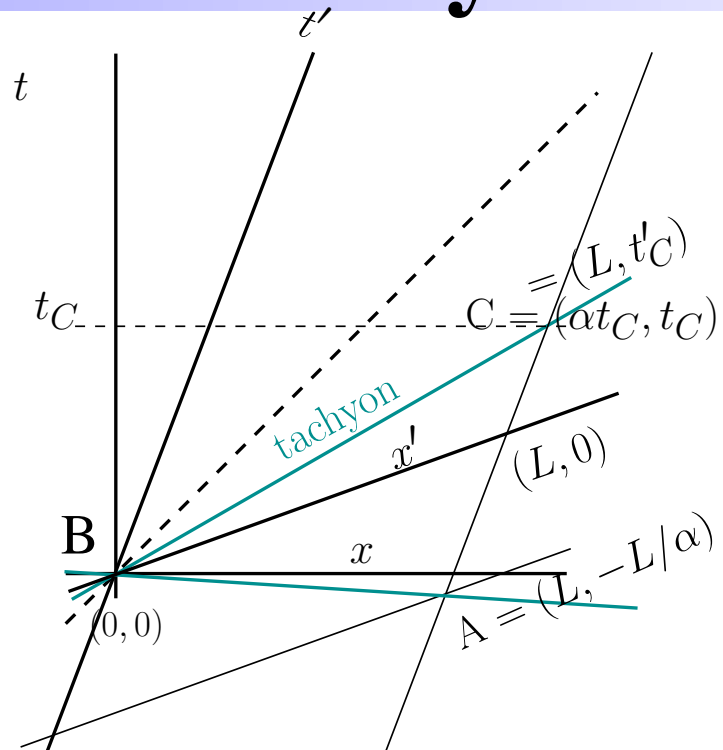
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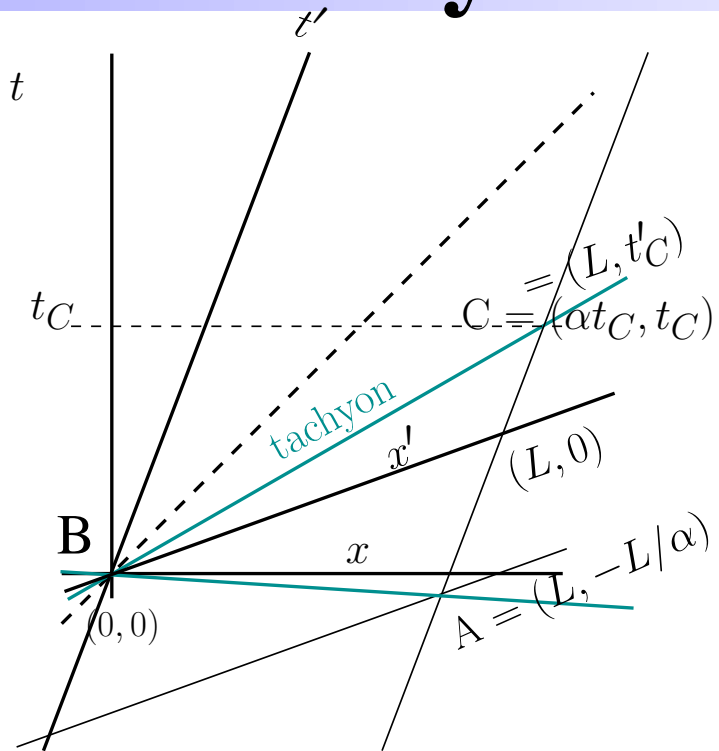
- observer at rest: tachyon emitted at origin
- rocket: tachyon absorbed at origin



# SR: tachyonic antitelephone



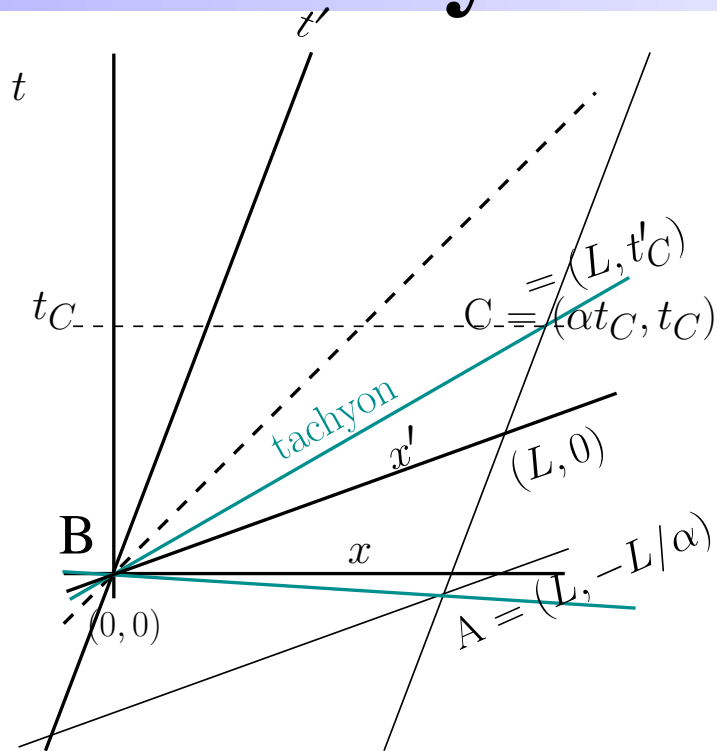
# SR: tachyonic antitelephone



B stationary:  $(x, t)$   
frame



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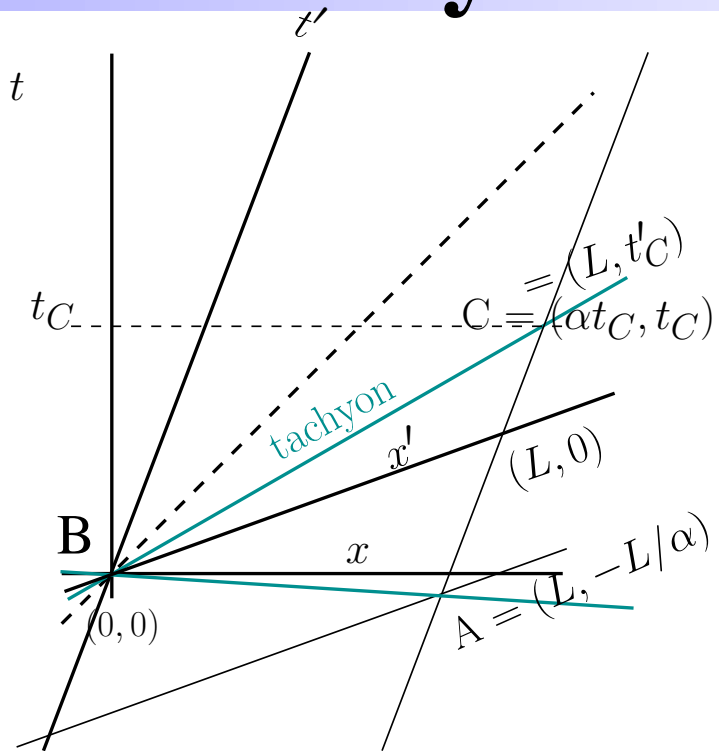


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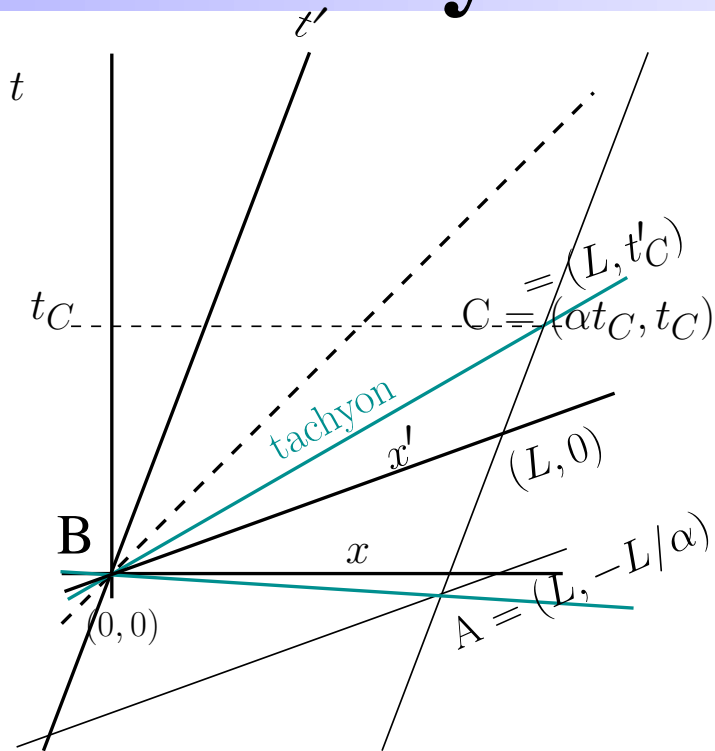


A: tachyon at  $\alpha > 1$  to B

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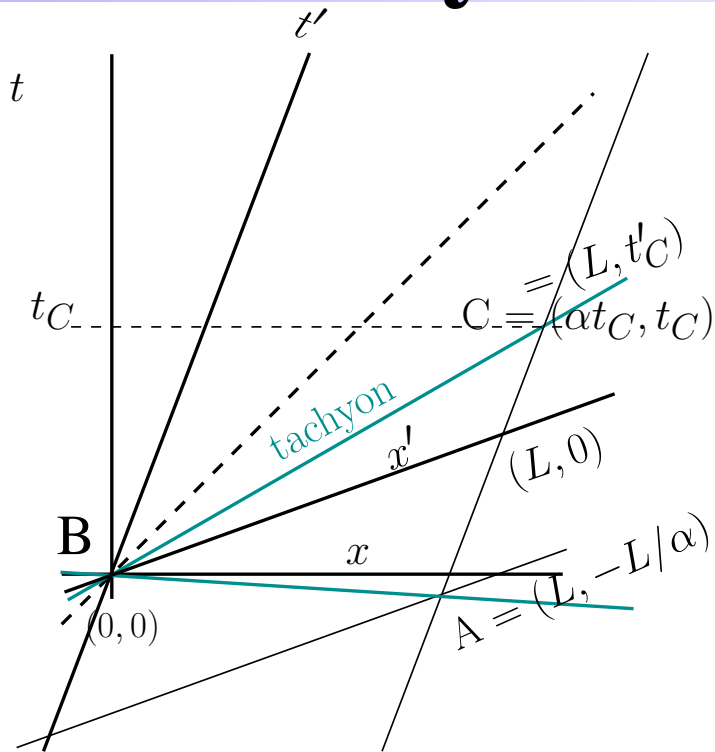
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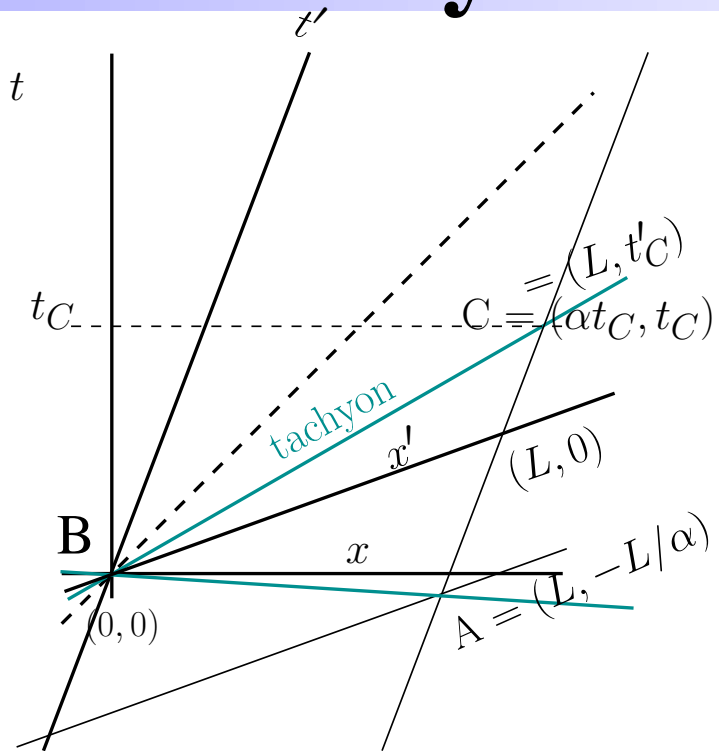


$$C: \begin{pmatrix} L \\ t'_C \end{pmatrix} = \Lambda \begin{pmatrix} \alpha t_C \\ t_C \end{pmatrix}$$

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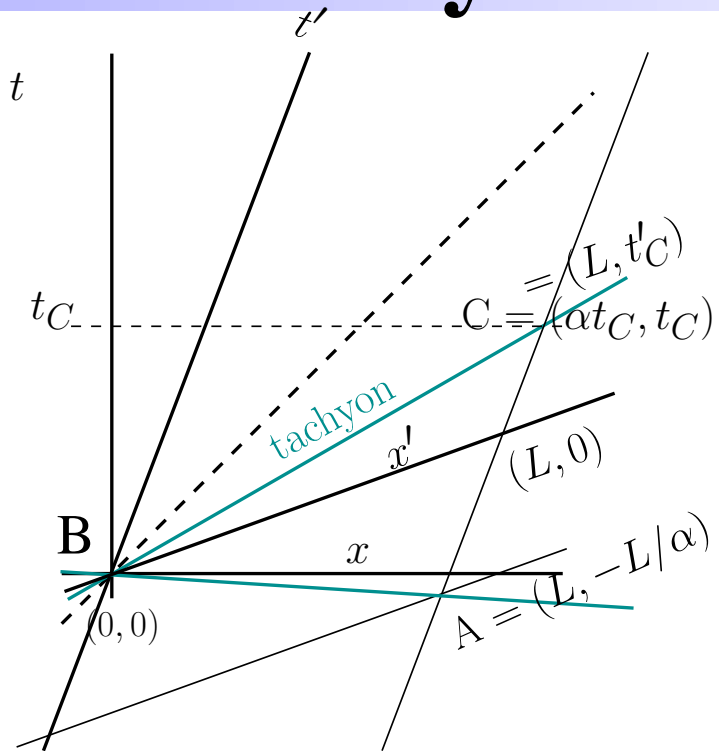


$$\begin{pmatrix} L \\ t'_C \end{pmatrix} = \gamma t_C \begin{pmatrix} \alpha - \beta \\ -\alpha\beta + 1 \end{pmatrix}$$

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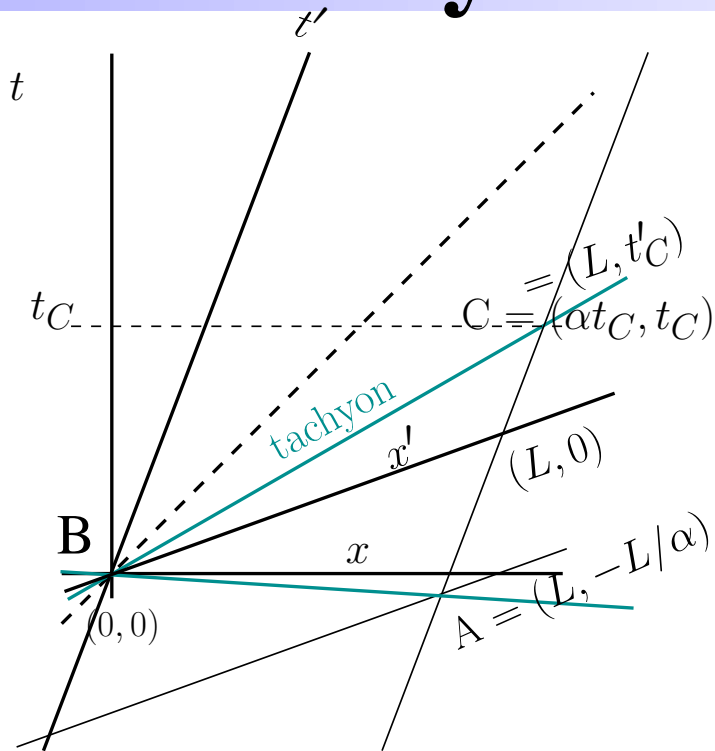
$$\begin{pmatrix} L \\ t'_C \end{pmatrix} = \gamma t_C \begin{pmatrix} \alpha - \beta \\ -\alpha\beta + 1 \end{pmatrix}$$

$$t'_C = \gamma t_C (1 - \alpha\beta)$$

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A moving at speed  $\beta$ :  
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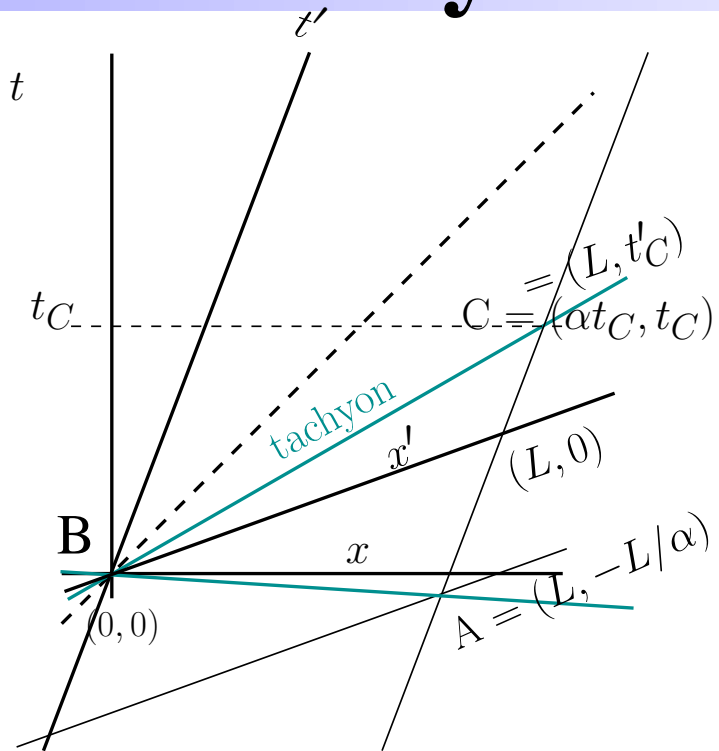
$$\begin{pmatrix} L \\ t'_C \end{pmatrix} = \gamma t_C \begin{pmatrix} \alpha - \beta \\ -\alpha\beta + 1 \end{pmatrix}$$

$$t'_C = \gamma \frac{L}{\gamma(\alpha - \beta)} (1 - \alpha\beta)$$

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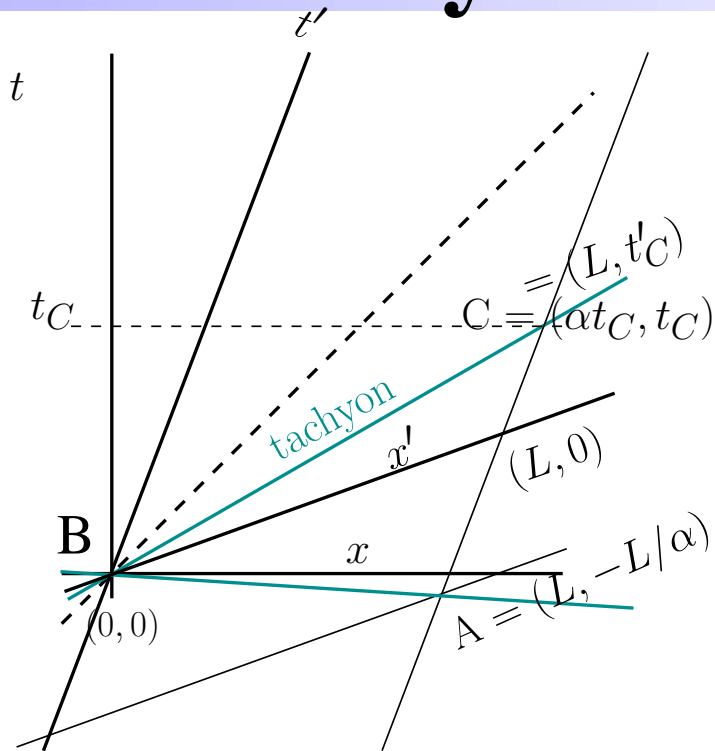
$$\begin{pmatrix} L \\ t'_C \end{pmatrix} = \gamma t_C \begin{pmatrix} \alpha - \beta \\ -\alpha\beta + 1 \end{pmatrix}$$

$$t'_C = L \frac{1 - \alpha\beta}{\alpha - \beta}$$

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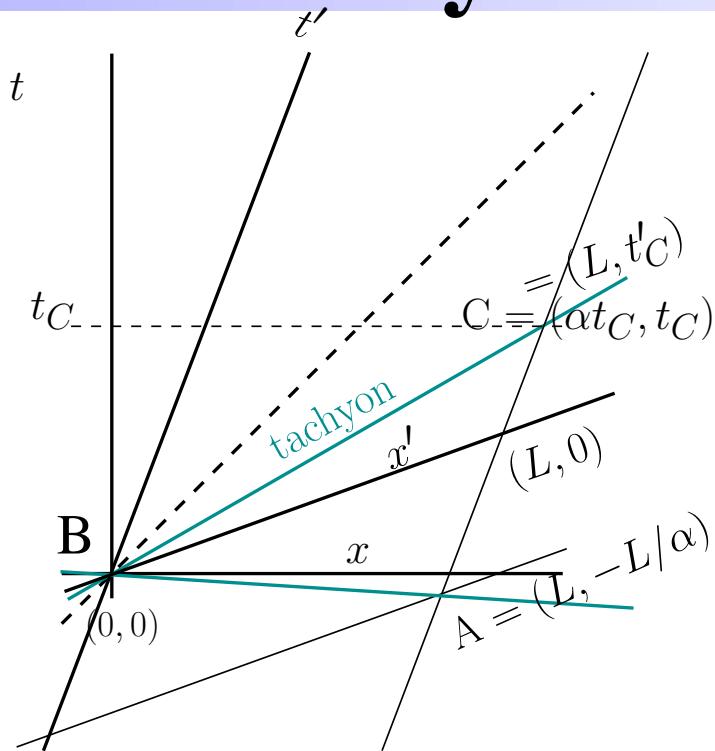
$$t'_C = L \frac{1 - \alpha\beta}{\alpha - \beta}$$

$$t'_C - t'_A = L \left( \frac{1 - \alpha\beta}{\alpha - \beta} + \frac{1}{\alpha} \right)$$

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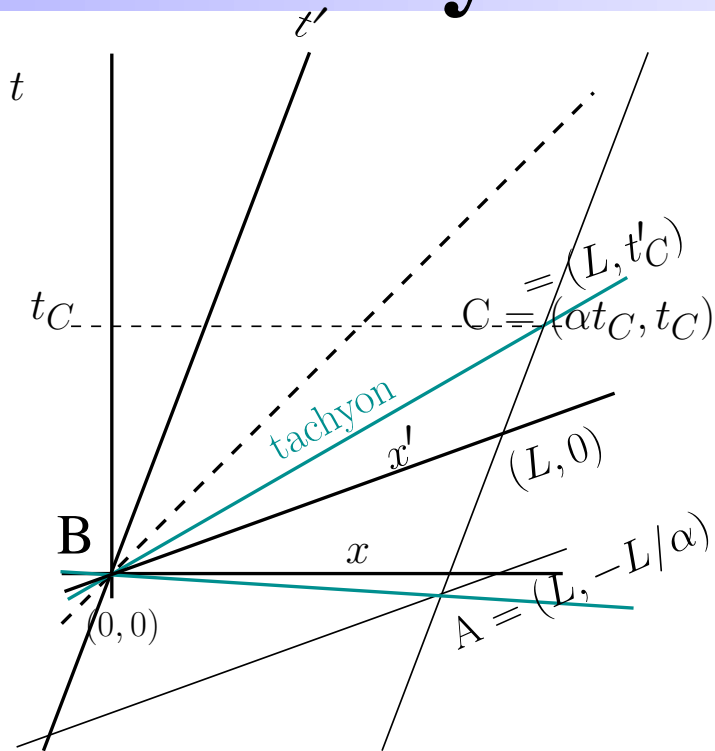
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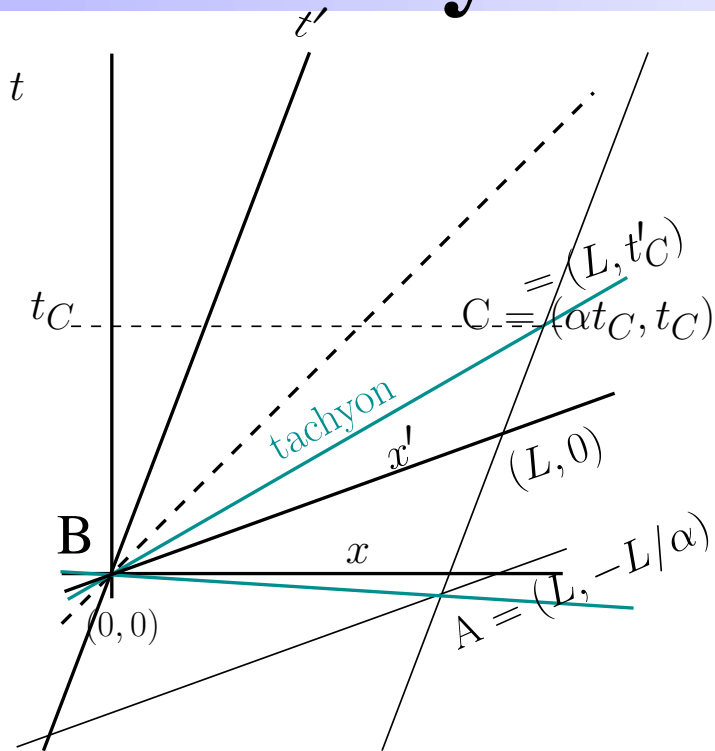
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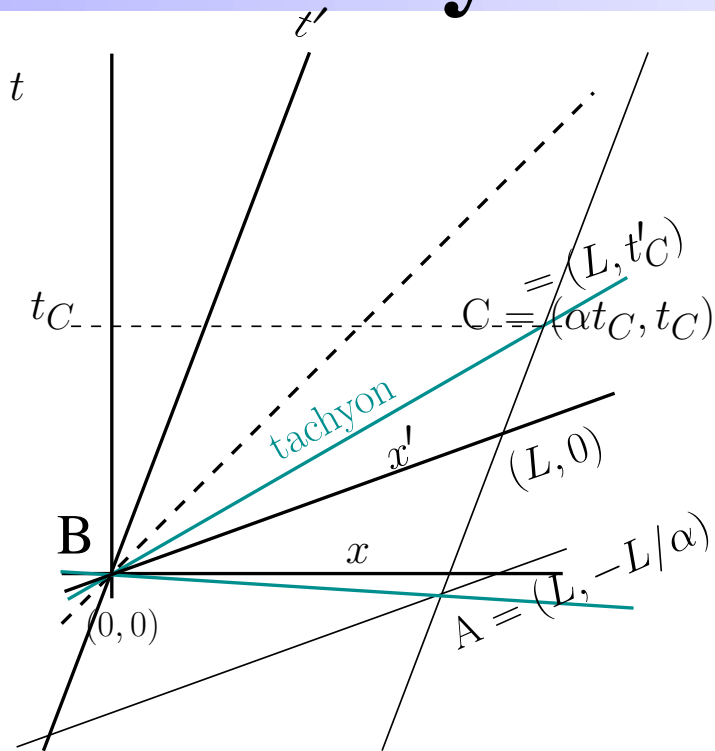
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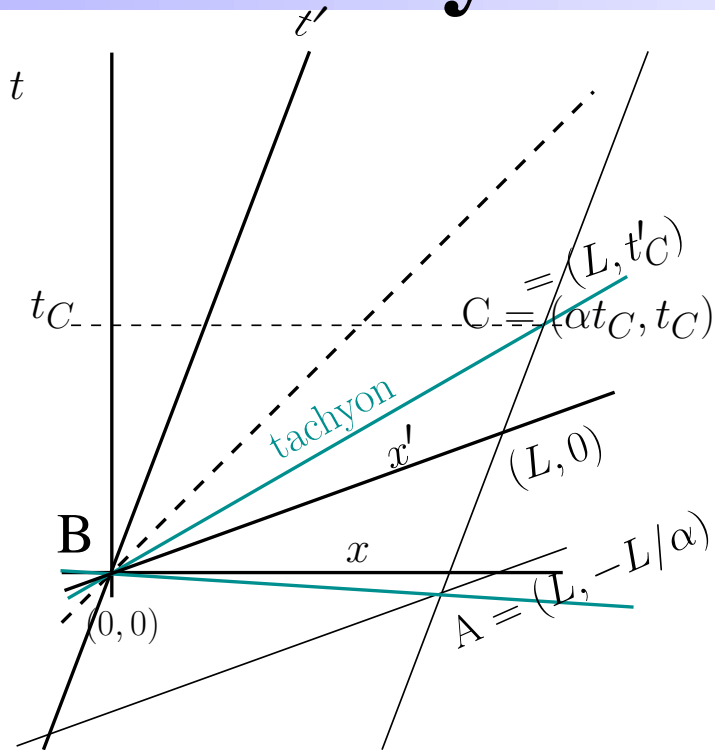
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A receives tachyonic response at C before sending it

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w: tachyonic antitelephone



# SR: pole-barn/ladder paradox



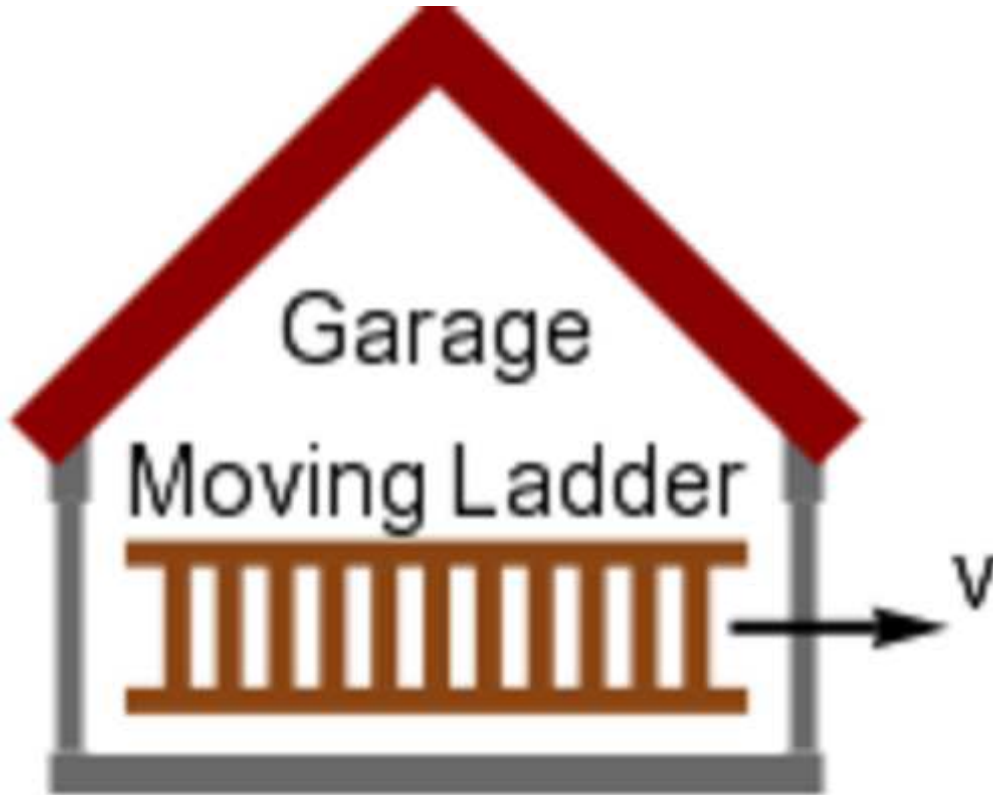
# SR: pole-barn/ladder paradox



- ladder of length  $29.9\gamma$  ns, garage length 30 ns

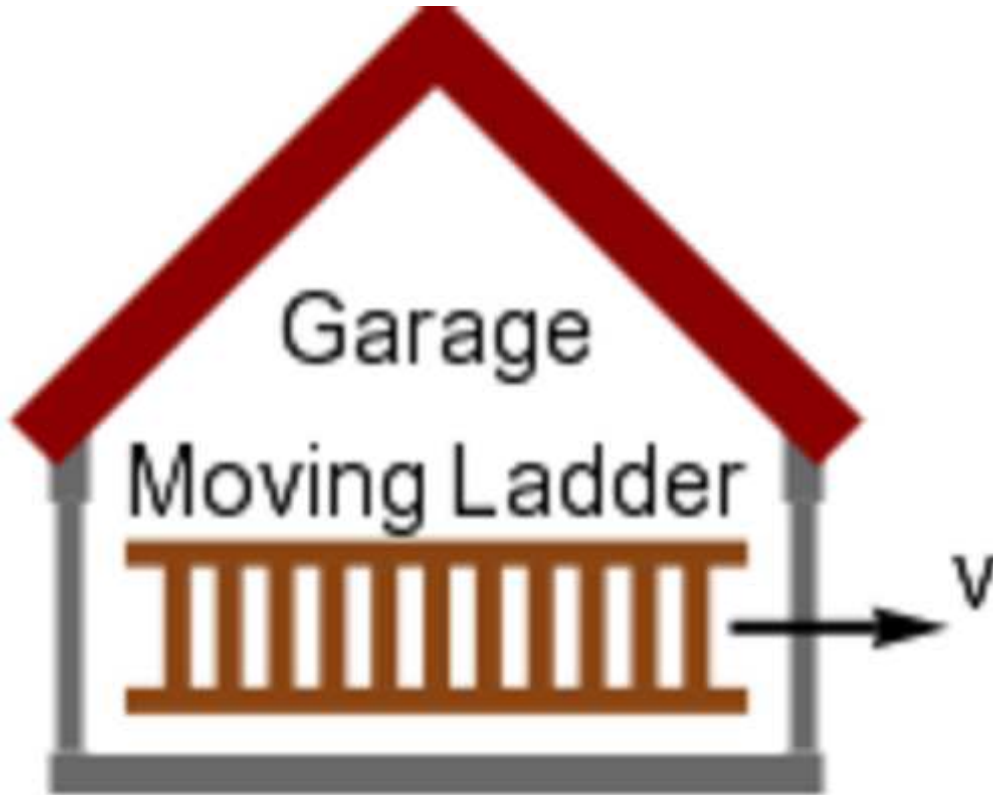


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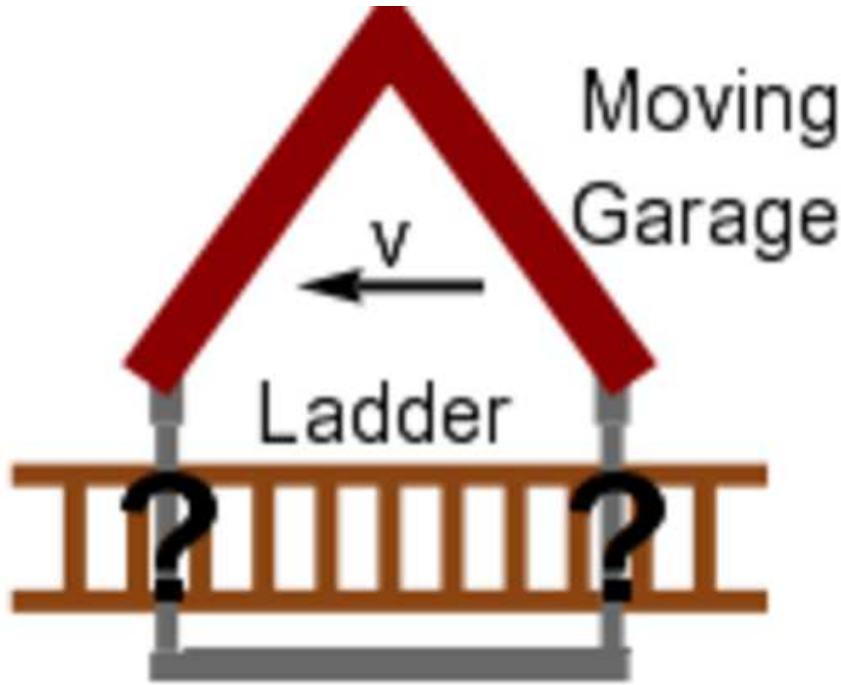
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- instantaneously close front + back doors
- $29.9\gamma$  ns /  $\gamma < 30$  ns  $\Rightarrow$  OK

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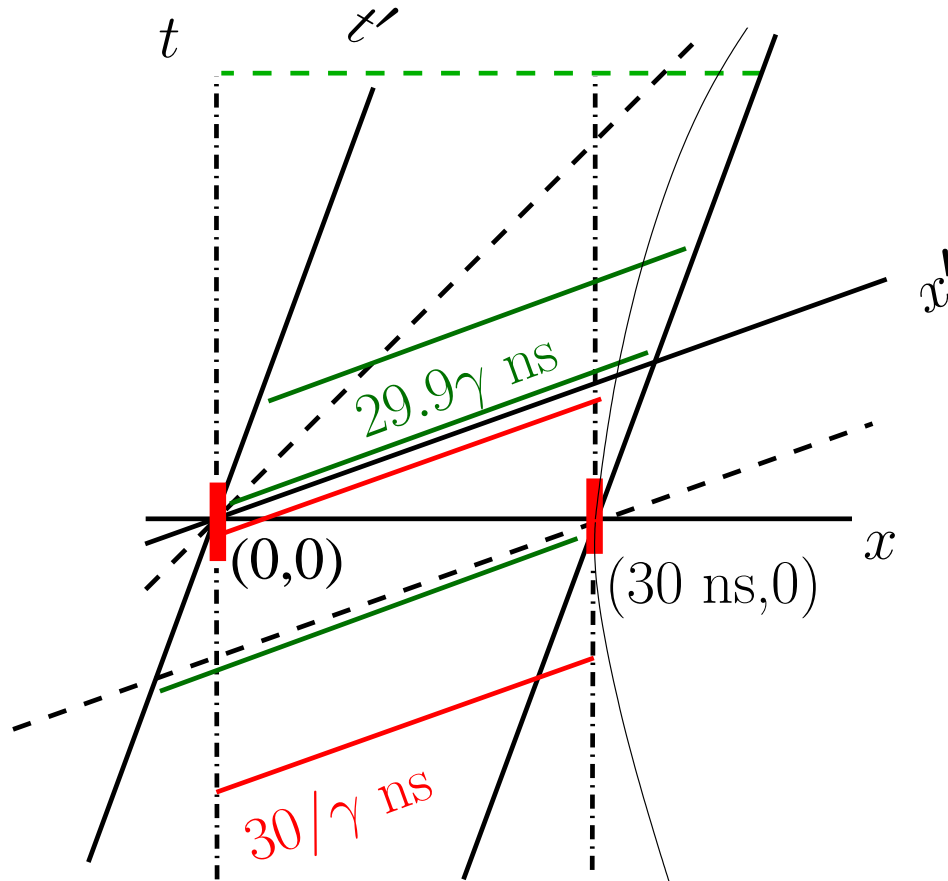


- ladder of length  $29.9\gamma$  ns, garage length 30 ns
  - instantaneously close front + back doors
  - ladder frame: garage  $30/\gamma$  ns long  $\ll 29.9\gamma$  ns!!
- Is this possible or not? Make a spacetime diagram.



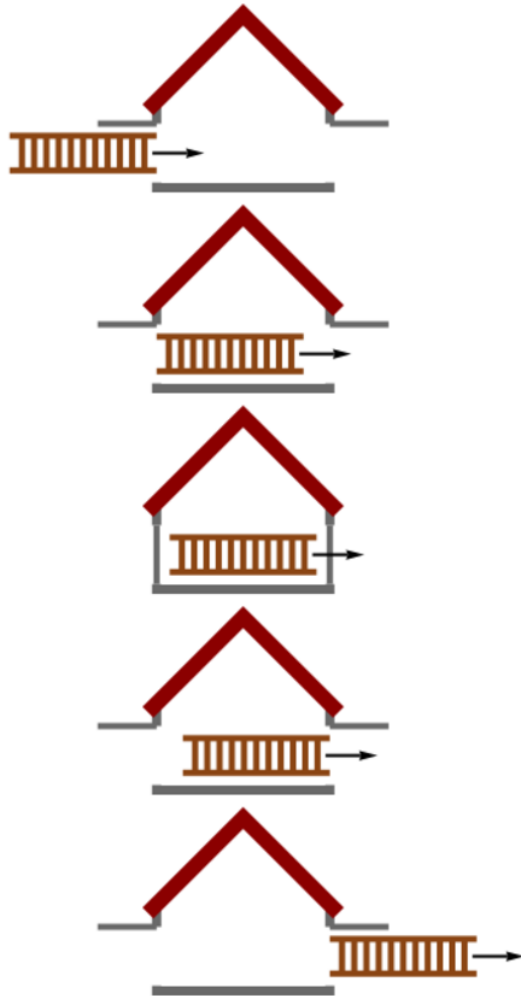


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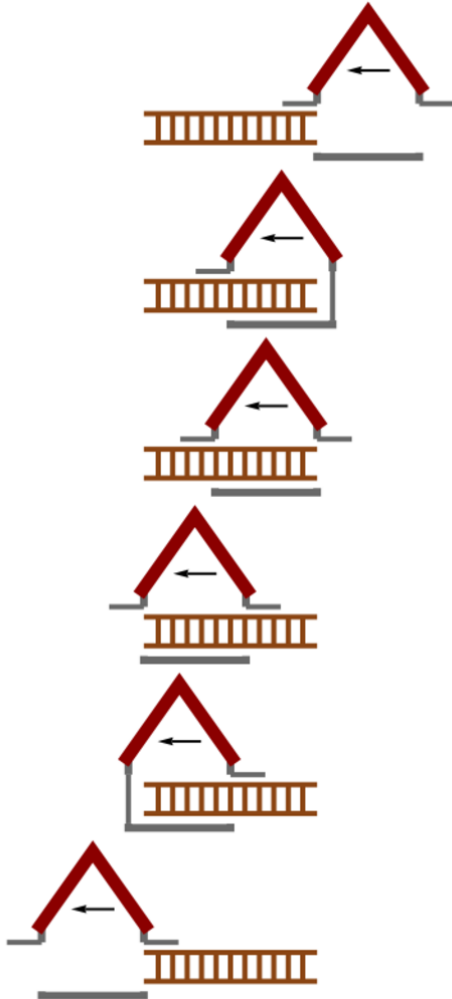


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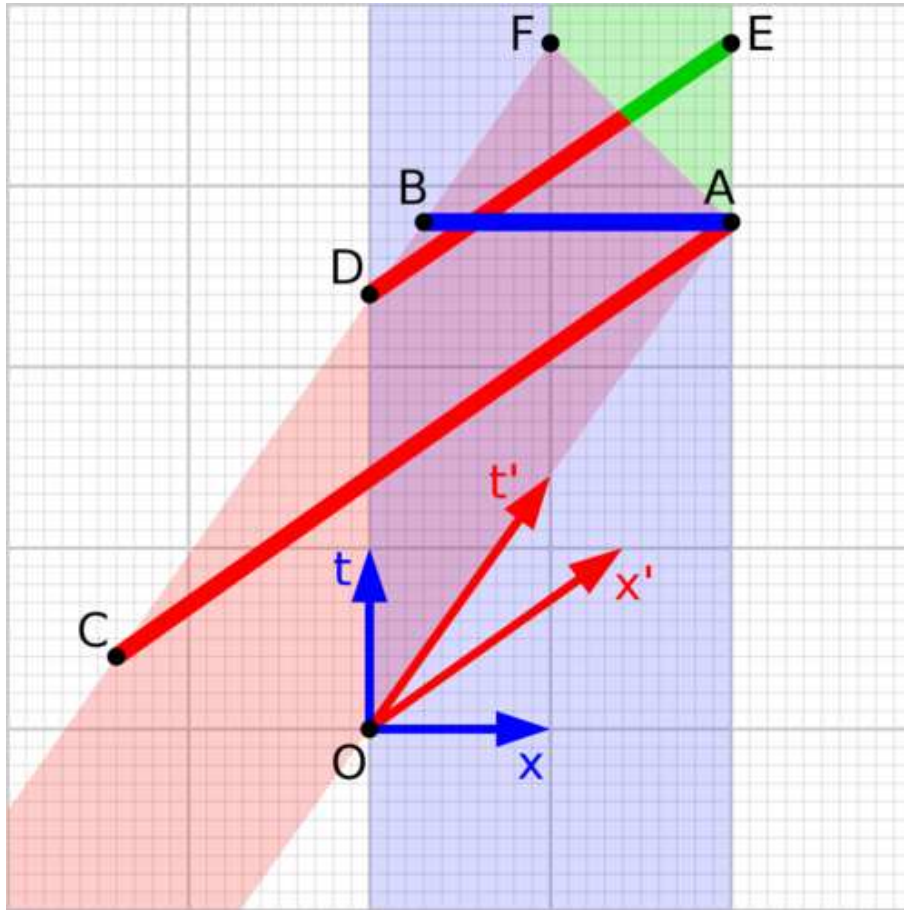


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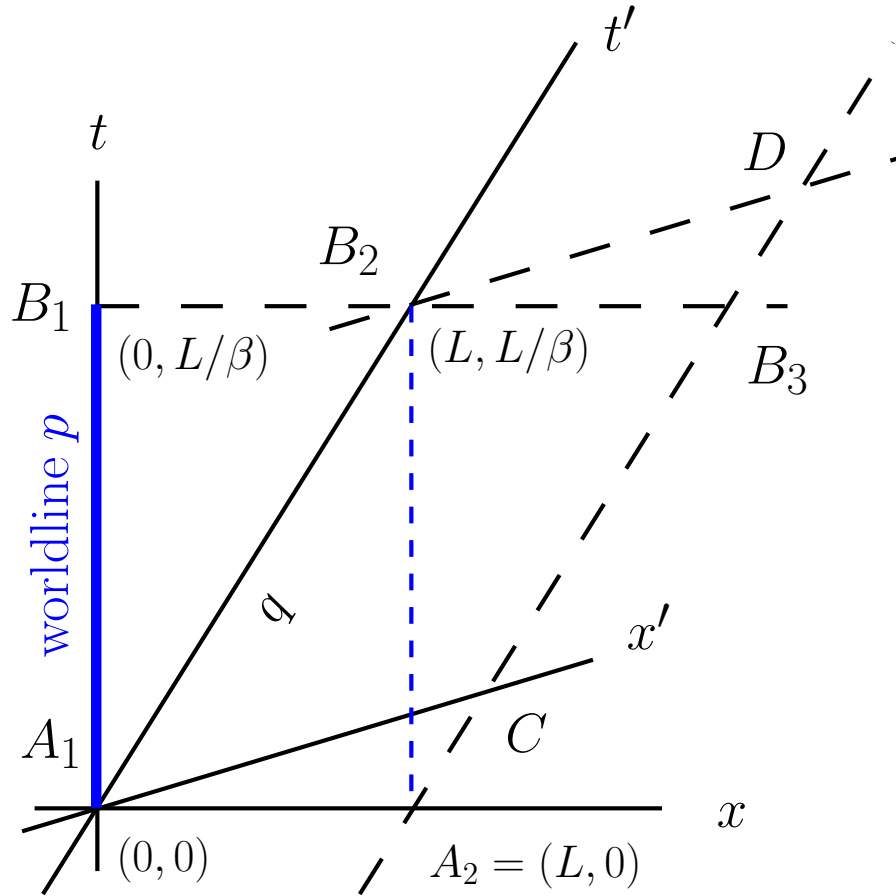


w:Ladder paradox





# SR: twins paradox

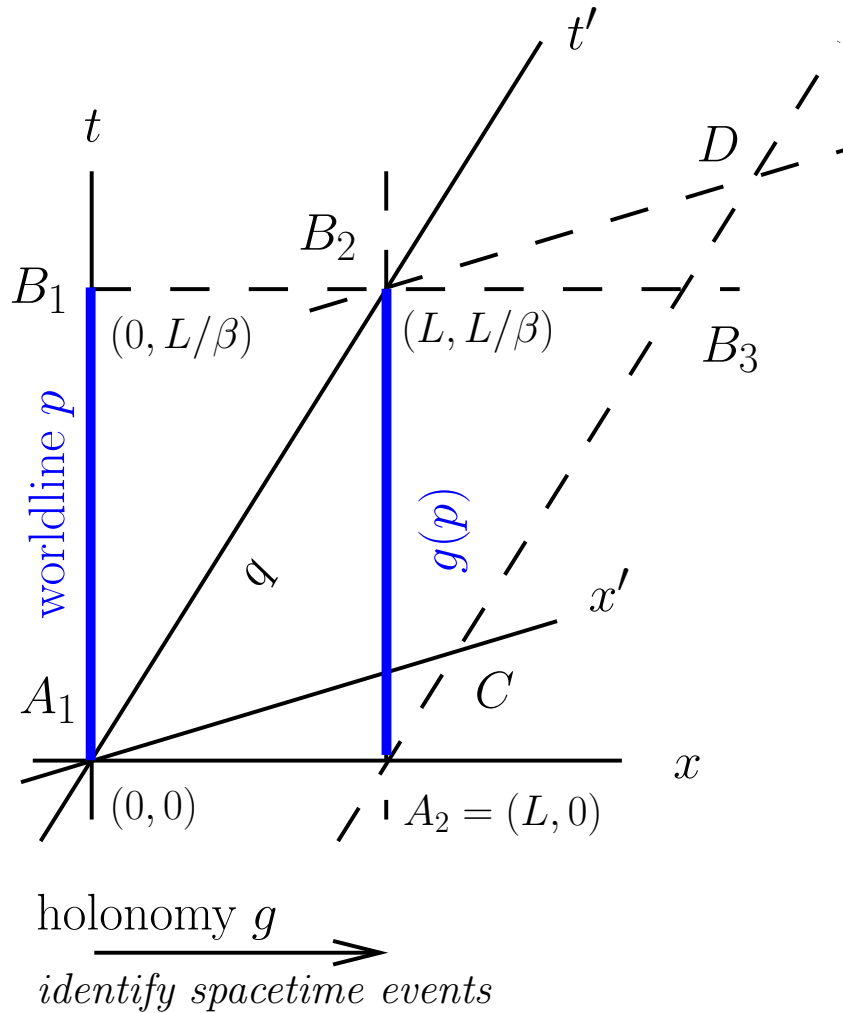


simply connected Minkowski



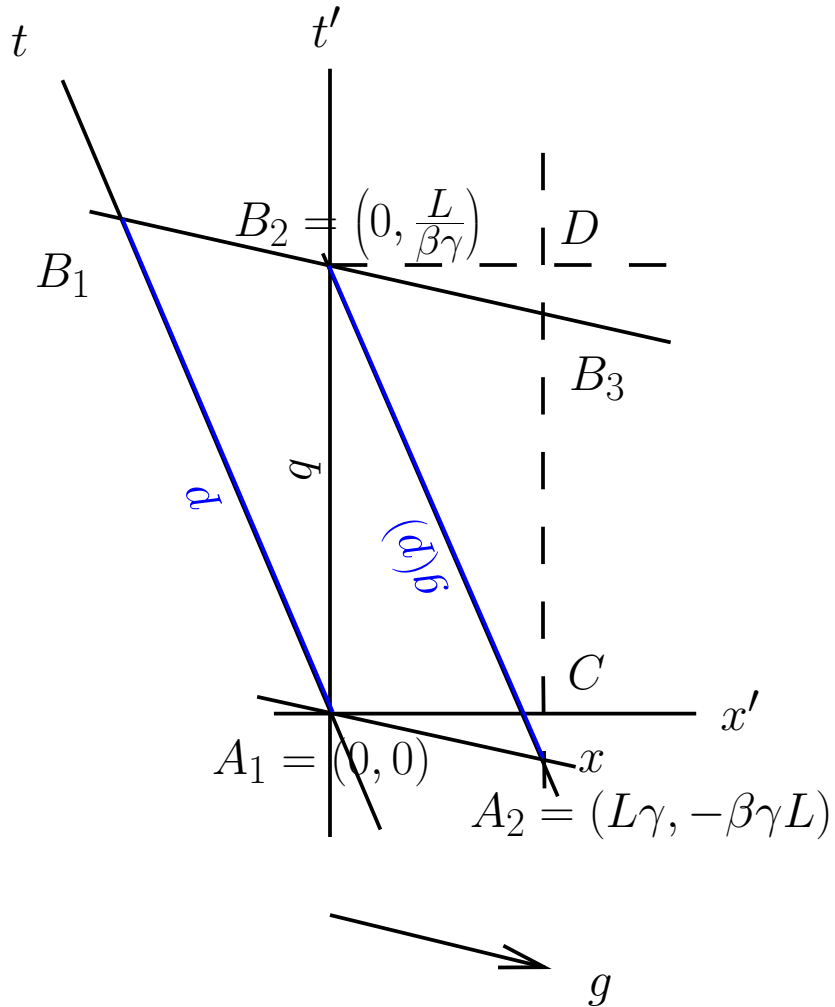


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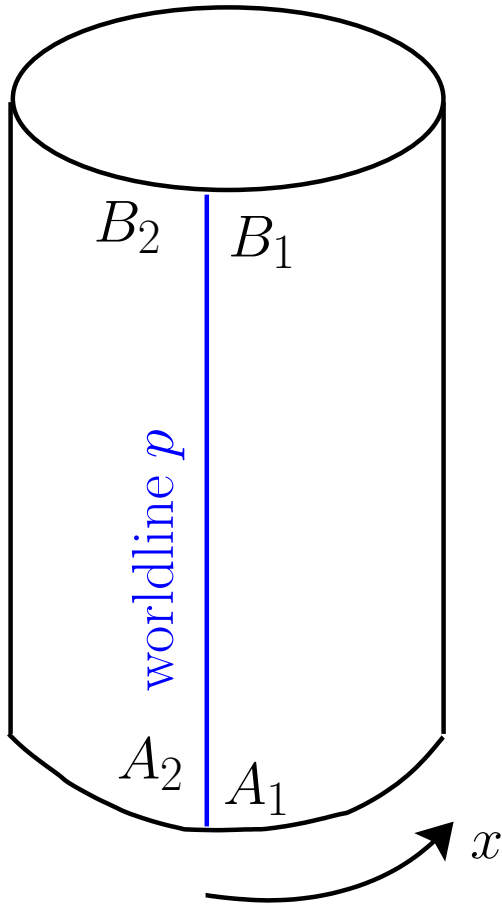


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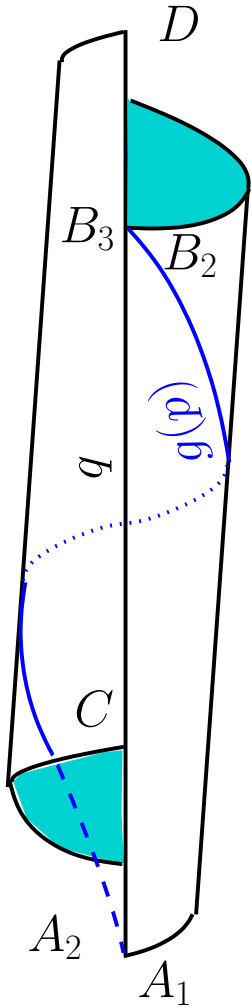
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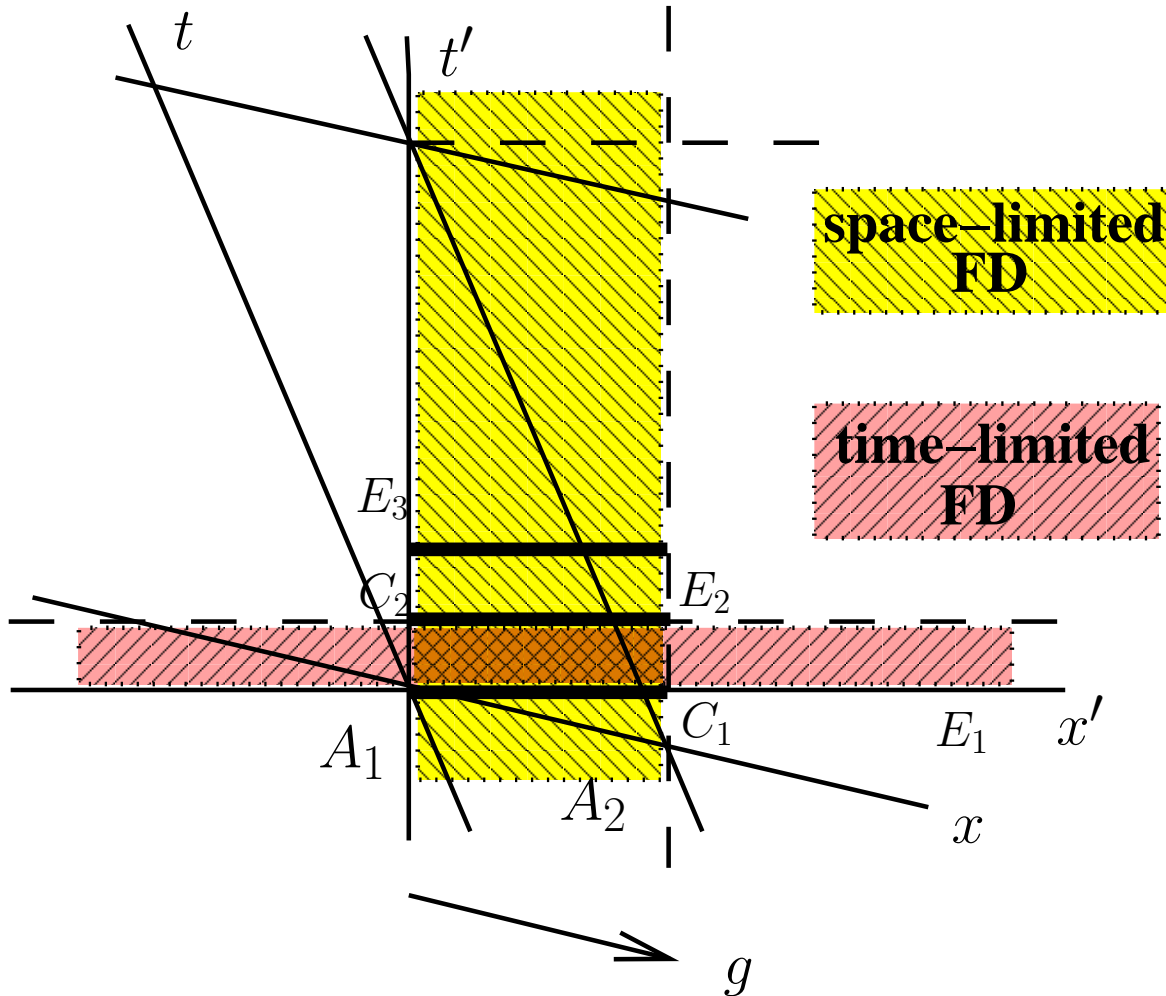


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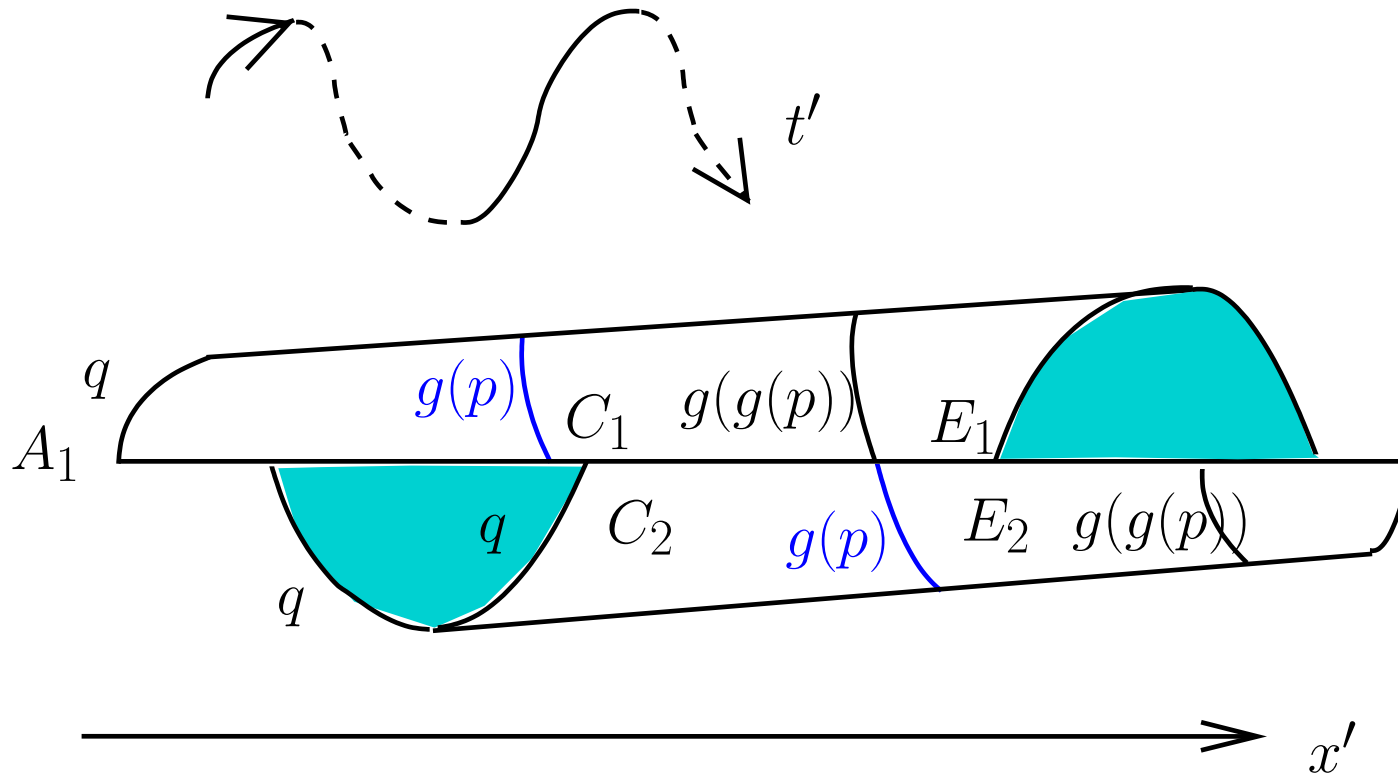




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Roukema & Bajtlik 2008, MNRAS, 390, 655  
[arXiv:astro-ph/0612155](https://arxiv.org/abs/astro-ph/0612155)

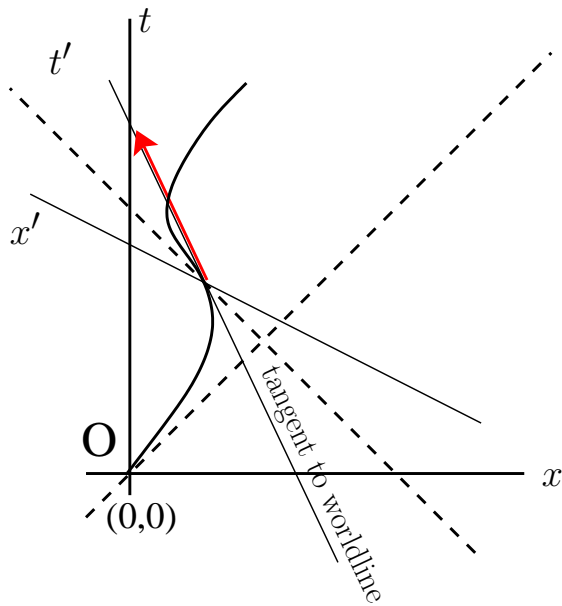
- helps understand w:Ehrenfest paradox

# SR: four-velocity, four-momentum

choose  $x$  axis so that 3-velocity  $u_{\text{Galilean}} = (\beta, 0, 0)^T$  for observer with  $(t, x, y, z)^T$  coord system

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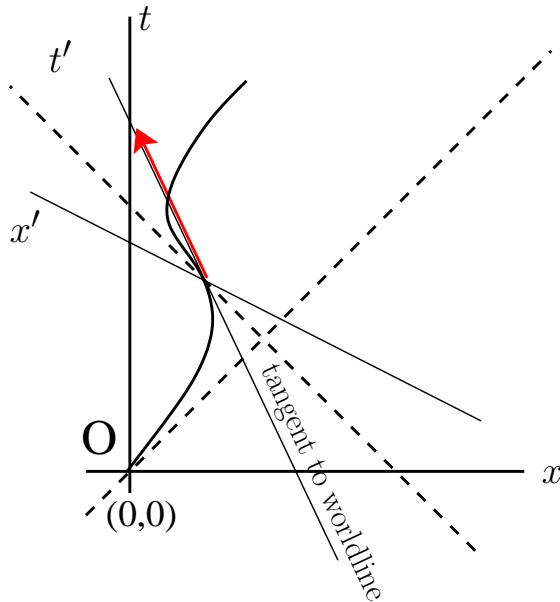
- in  $(t, x)$  spacetime 2-plane, extend from scalar speed  $\beta$  to spacetime vector = tangent to worldline

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$$(u^t, u^x) := \left( \frac{d}{d\tau} t(\tau), \frac{d}{d\tau} x(\tau) \right)$$

w:four-velocity



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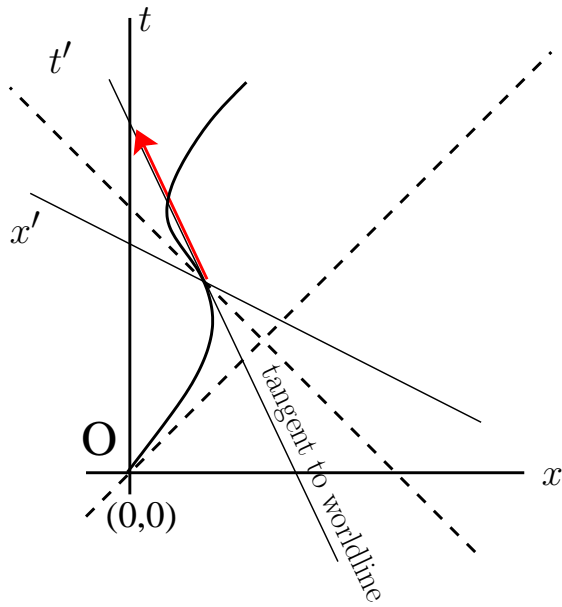
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similarly  $(u^{t'}, u^{x'}) =$

$$\left( \frac{d}{d\tau} t'(\tau), \frac{d}{d\tau} x'(\tau) \right)$$

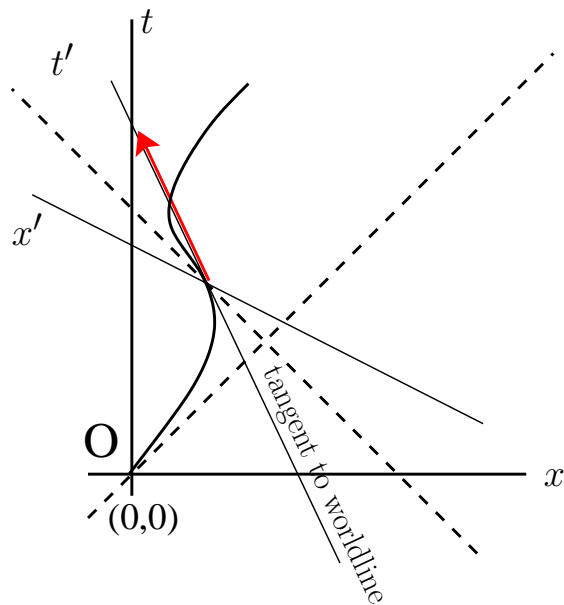


• in  $(t, x)$  spacetime

2-plane, extend from

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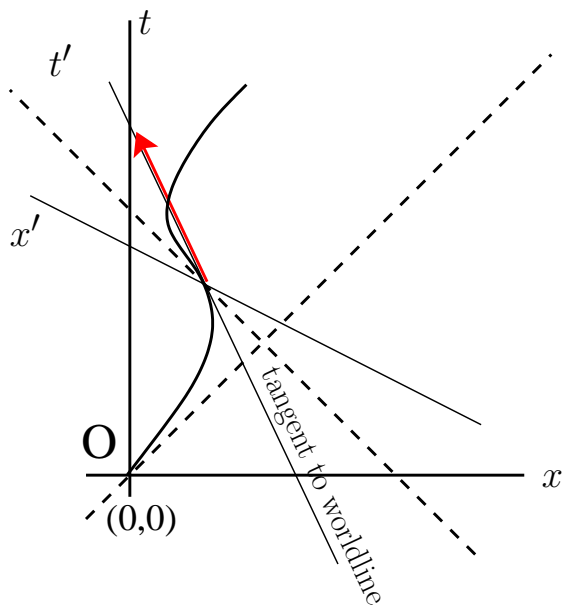
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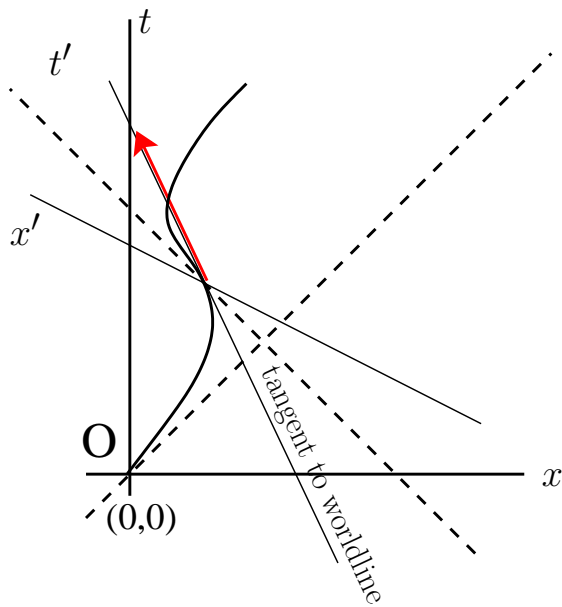
want  $\vec{u}$  Lorentz invariant  $\Rightarrow (u^t, u^x)^T = \Lambda^{-1}(1, 0)^T$



- in  $(t, x)$  spacetime 2-plane, extend from scalar speed  $\beta$  to spacetime vector

# SR: four-velocity, four-momentum

choose  $x$  axis so that 3-velocity  $u_{\text{Galilean}} = (\beta, 0, 0)^T$  for observer with  $(t, x, y, z)$  coord system



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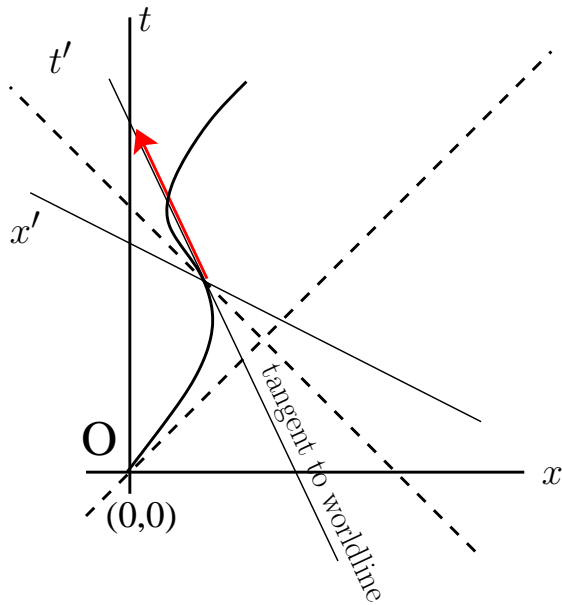
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4D:  $\vec{u} = \gamma(1, \beta^x, \beta^y, \beta^z)^T$

notation in this pdf:

$\vec{u} = 4\text{-vector}$ ,  $^{(3)}\vec{u} = \text{spatial component}$

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momentum:  $\vec{p} := m\vec{u} = m\gamma(1, \beta^x, \beta^y, \beta^z)^T$ , where  $m =$   
constant w:invariant mass

$x$  ... = tensor-style component notation, not powers



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What does the time component of momentum =  $p^0 = m\gamma$  mean physically?

- first look at spatial component in a given ref. frame

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let us define 4-acceleration, 4-force

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# SR: invariance of ${}^{(4)}u$ , ${}^{(4)}a$ , ${}^{(4)}f$



Euclidean norm:  $\|\vec{x}\|^2 = \sum_{\mu} (x^{\mu})^2$



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Minkowski pseudo-norm:  $\|\vec{x}\|^2 = \sum_{\mu,\nu} \eta_{\mu\nu} x^\mu x^\nu$



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w:Einstein summation sum is implicit





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similarly:  $\|\vec{a}\|^2$ ,  $\|\vec{f}\|^2$  **invariant**



# SR: energy: varies with ref frame



Newtonian  $K = (1/2)m\beta^2 = 0$  in rest frame



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(assume  ${}^{(3)}\vec{f}/\gamma \parallel \vec{x}$ )

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so  $p^0 = \text{kinetic energy} + \text{rest mass}$

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Does small  $\beta$  limit agree with Newtonian  $K$ ?



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$$p^0 = m\gamma = m(1 - \beta^2)^{-1/2}$$

$$= m[1 - (1/2)(-\beta^2) + \mathcal{O}(\beta^4)] \text{ if } \beta \ll 1$$





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Yes.



# SR: $\vec{p} \dots$ : invariant or not?

momentum:  $\vec{p} = m\gamma(1, \beta^x, \beta^y, \beta^z)^T$

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**WARNING:** assume that 4-momentum vectors at different space-time positions can be parallel-transported; not the case in curved spacetime



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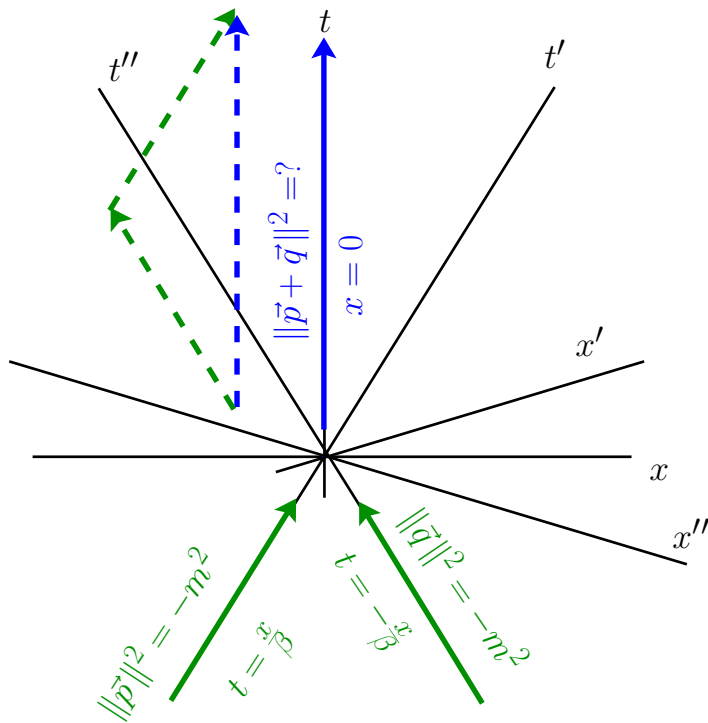
vector space  $\Rightarrow p^0 + q^0 = r^0$

= conservation of (relativistic) “total energy” =  $m + K$   
(Newtonian:  $m$  conserved,  $K$  not conserved,  $K +$   
potential energy conserved)

but NOT invariant (Newtonian:  $m$  invariant,  $K$  not invariant)

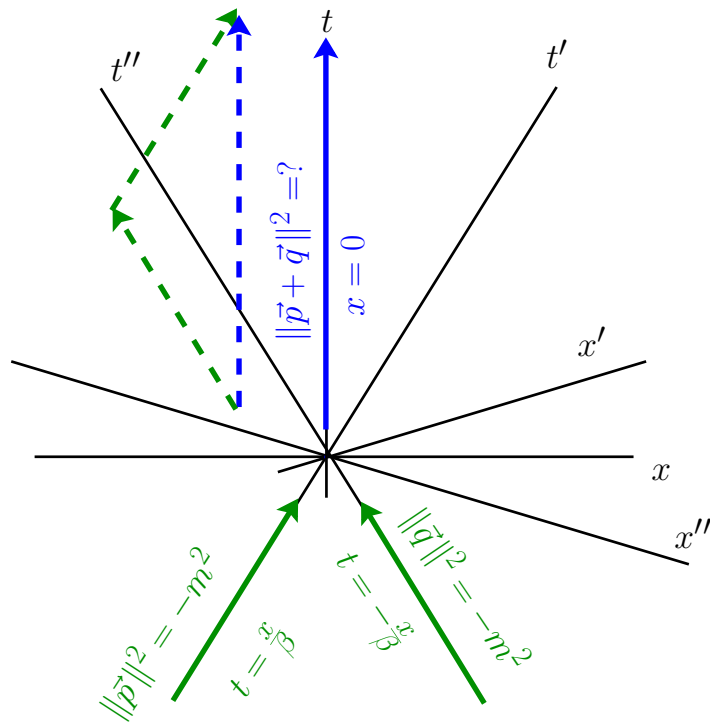


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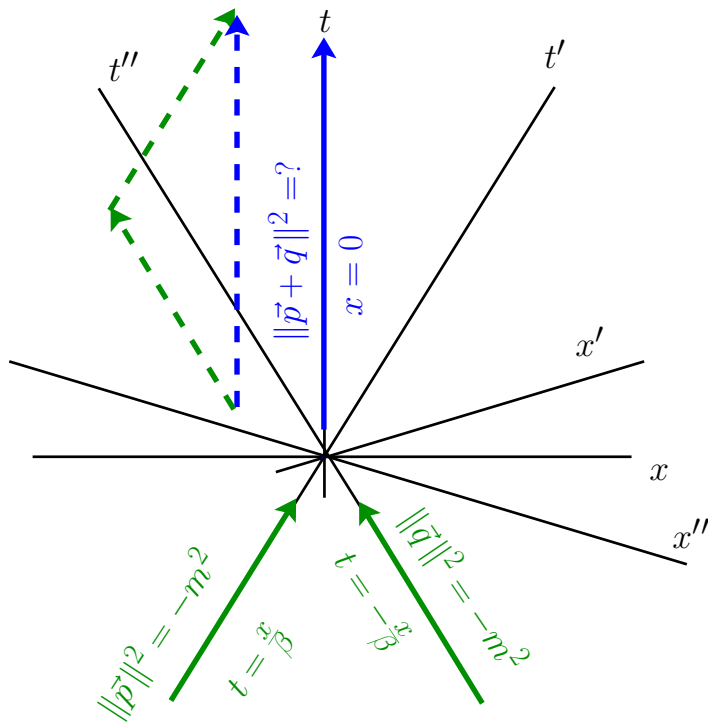
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$$\vec{p} = m(\gamma, +\beta\gamma, 0, 0)^T$$



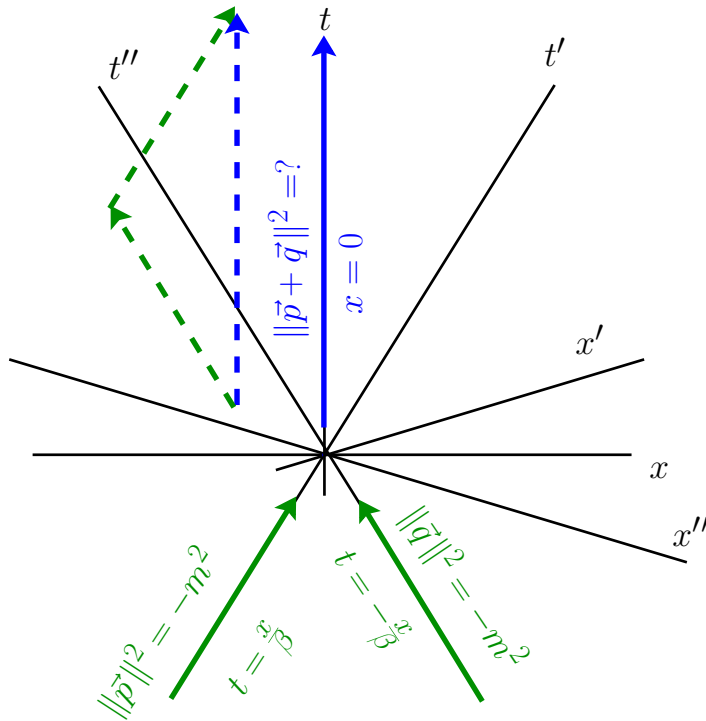
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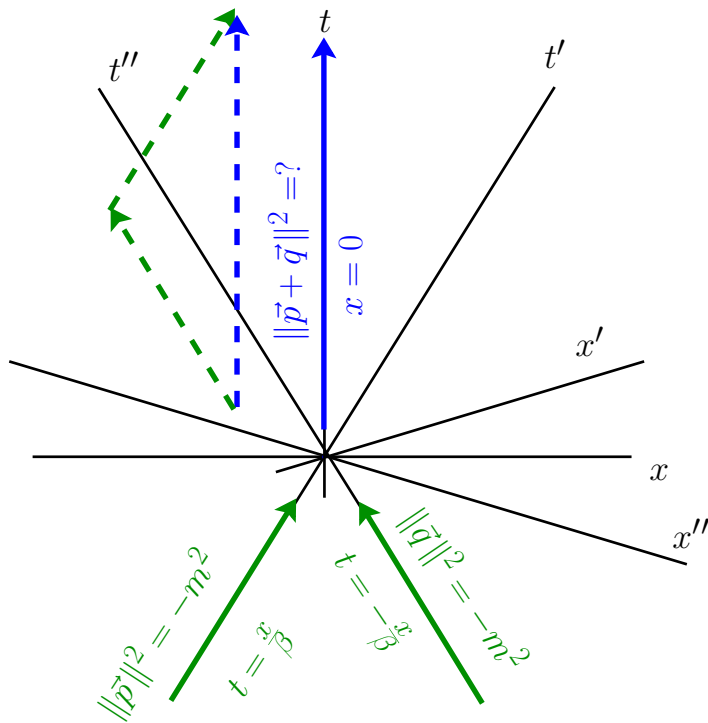
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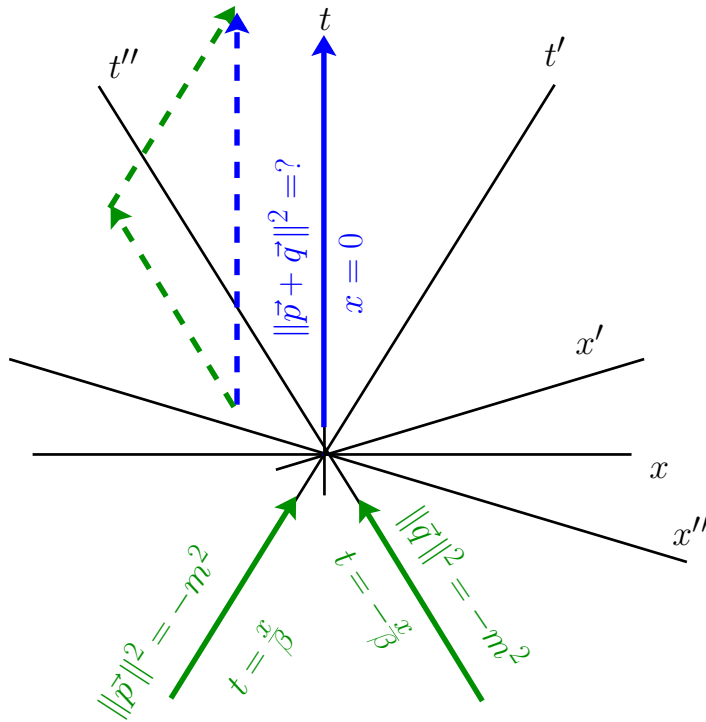
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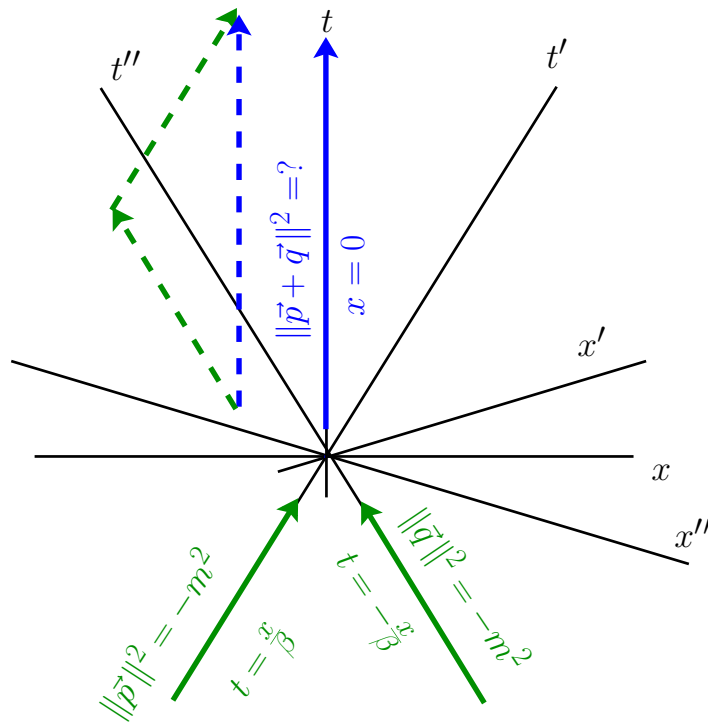


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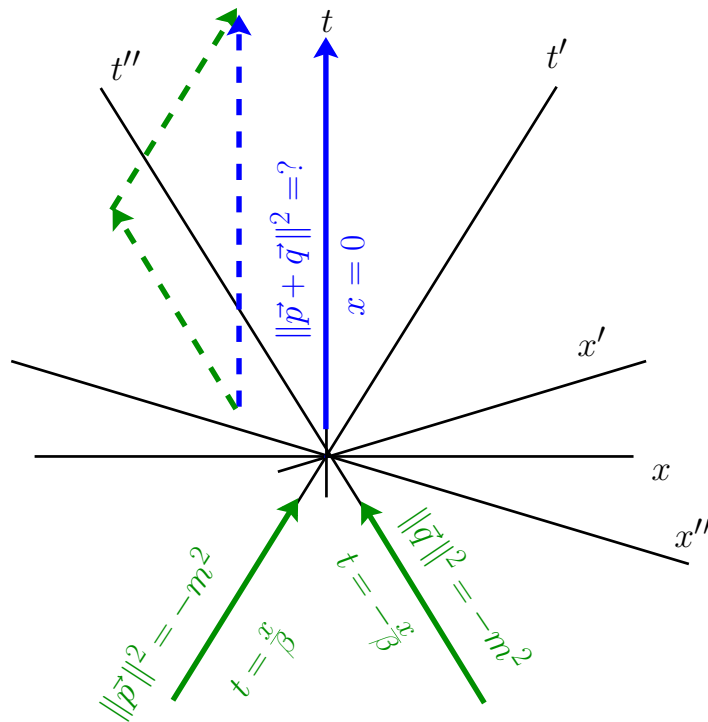
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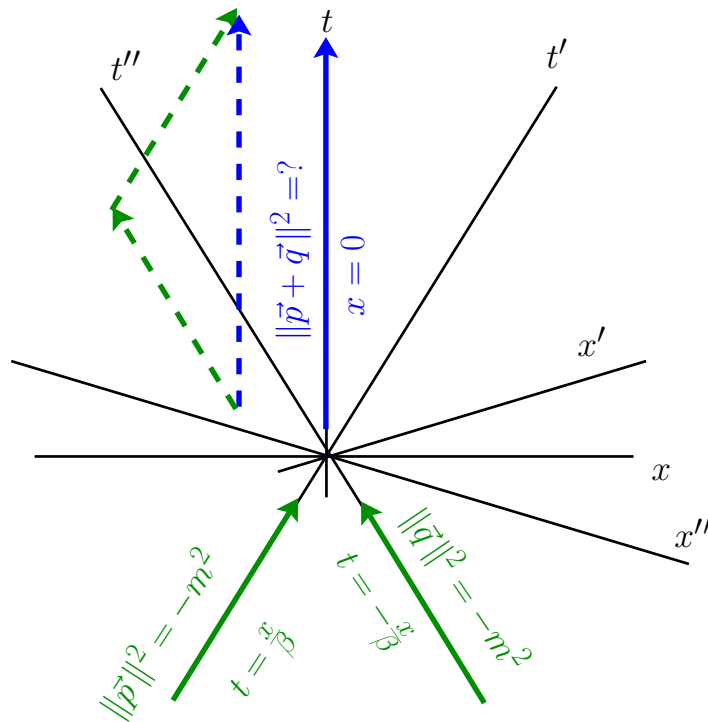
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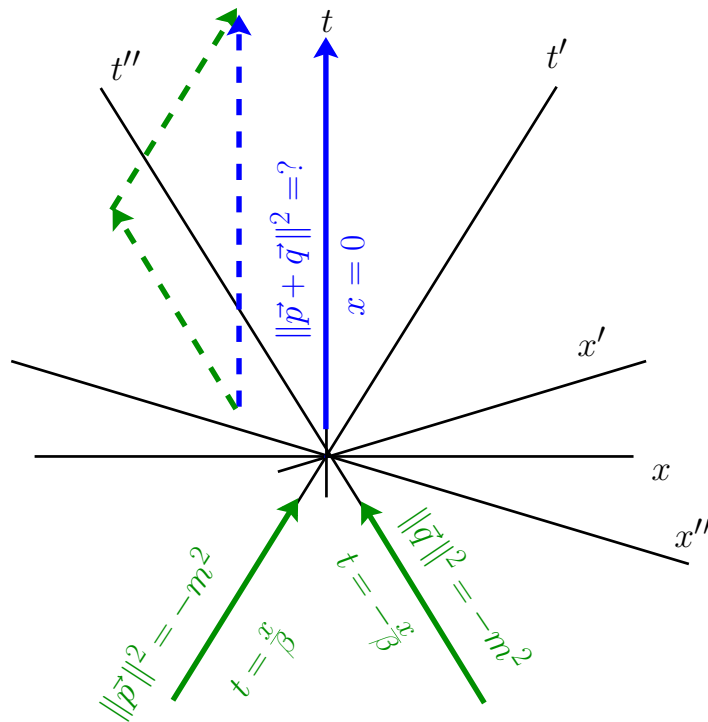
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system mass is invariant, but can be divided into  $p^0$  and  $p^i, i \in \{1, 2, 3\}$  components in many different ways



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refuse the assumption of absolute simultaneity (time)

