Special relativity and steps towards general relativity: ϵGR

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= vector space (e.g. 4-momentum vectors)

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= dual vector space (think: contour map, gradients)

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duality in a basis of $T_{\mathbf{x}}M$ and a basis of $T^*_{\mathbf{x}}M$ usually defined using δ^{μ}_{ν}

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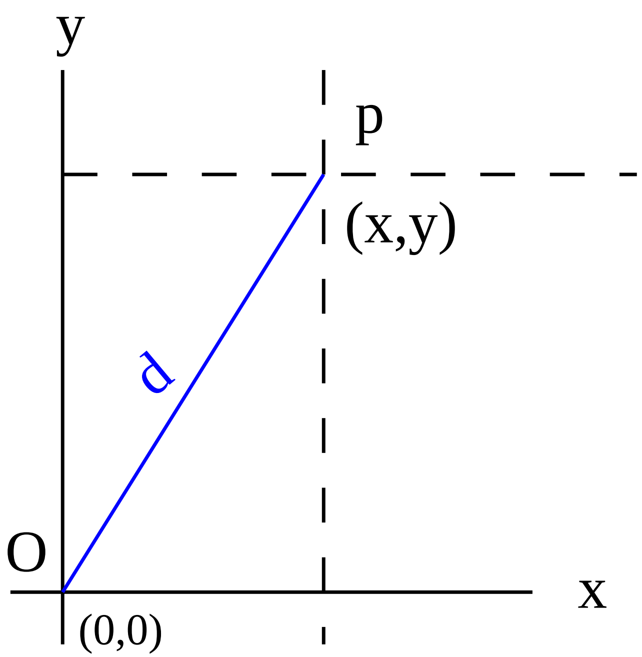
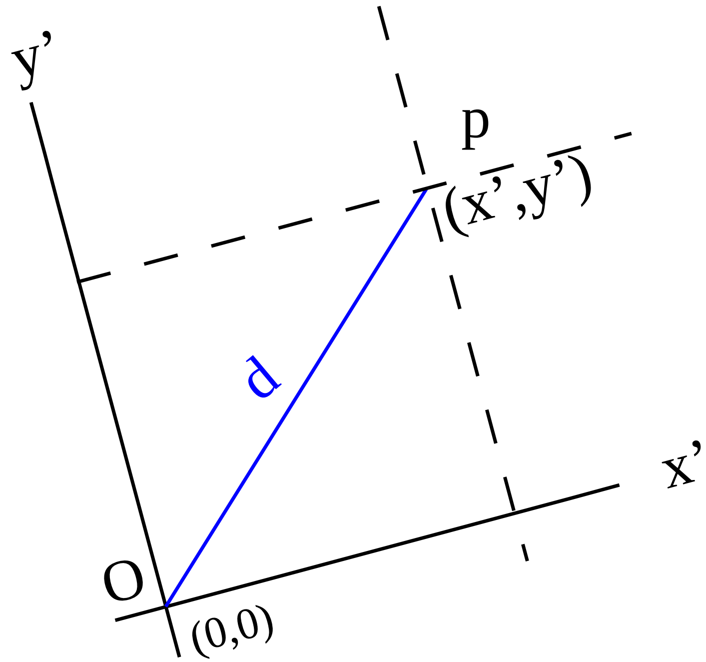
5. metric $\Leftarrow$ Einstein field equations

GR: coordinate transformations



GR: coordinate transformations



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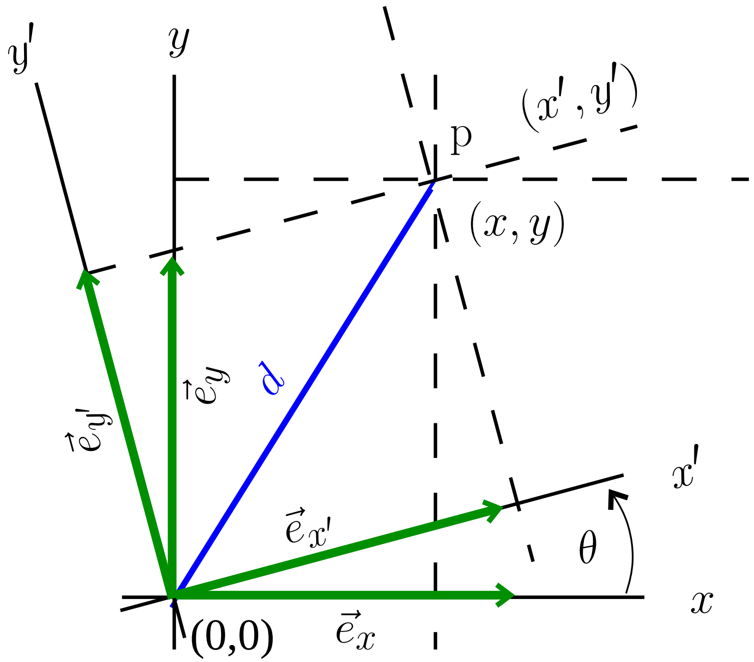

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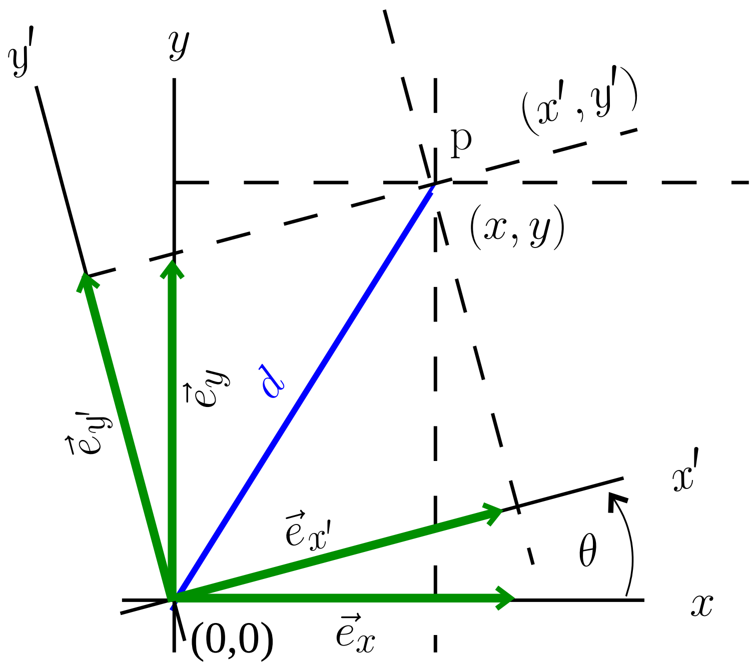
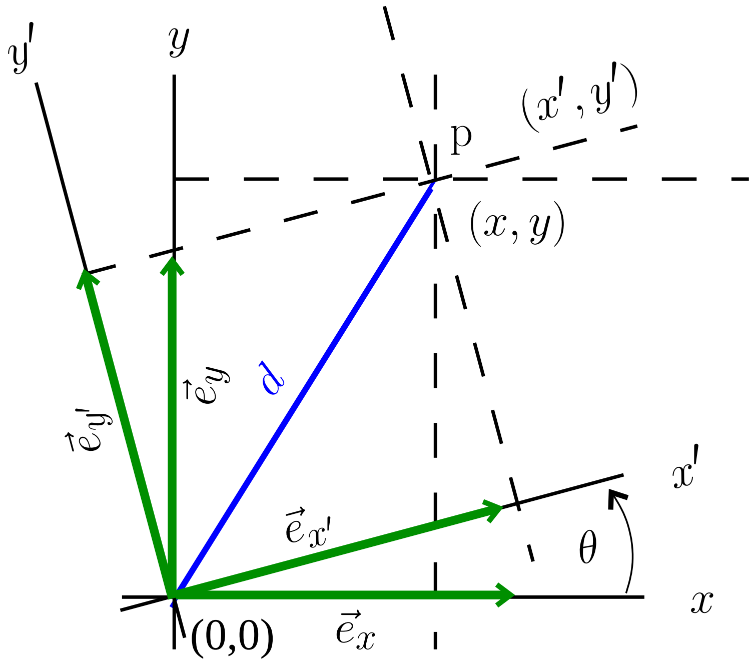


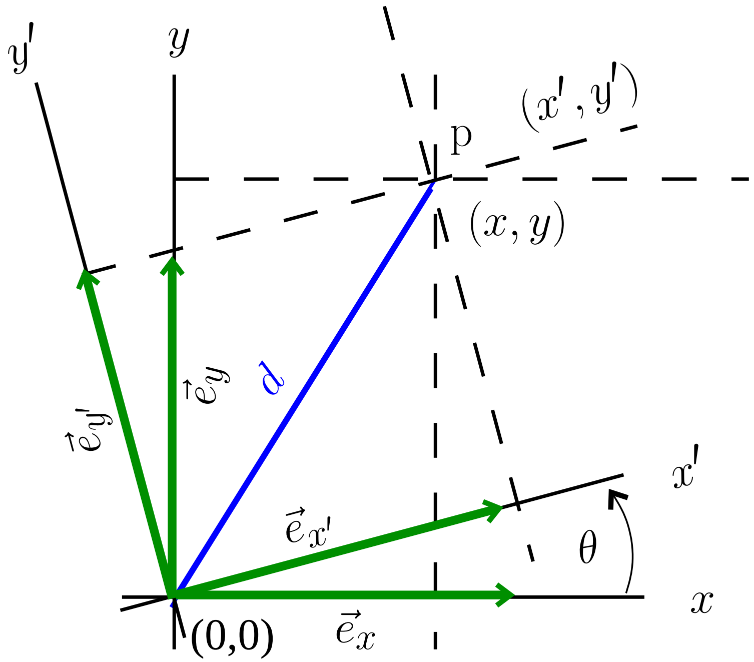
$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

GR: coordinate transformations



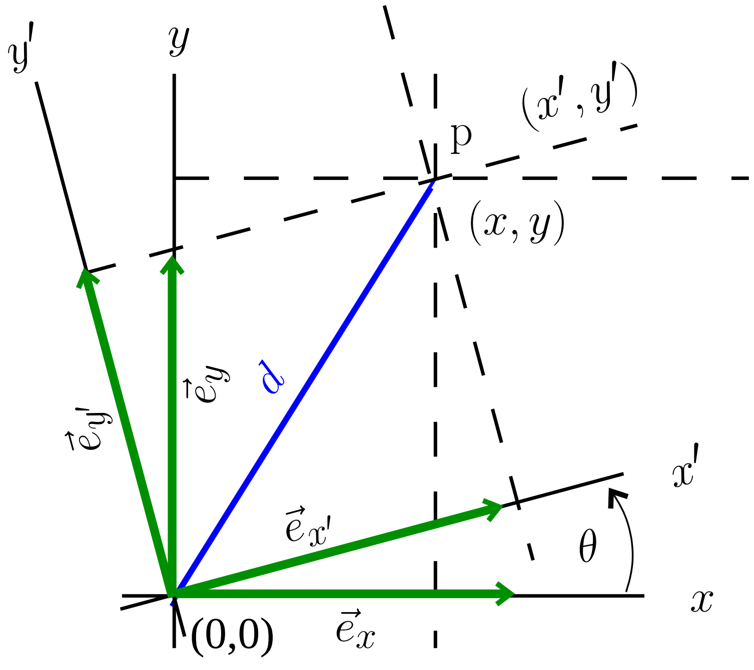
$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \Lambda \begin{pmatrix} x \\ y \end{pmatrix}$$

GR: coordinate transformations



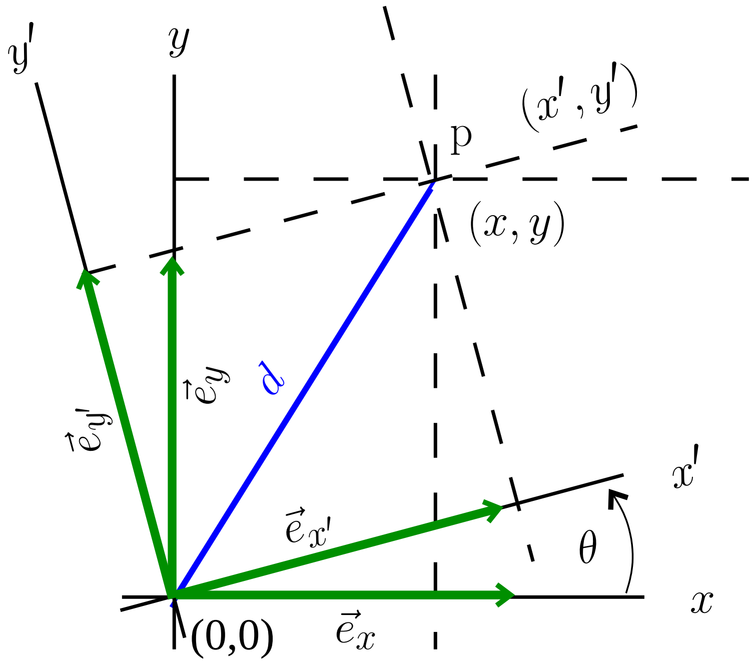
but $\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

GR: coordinate transformations



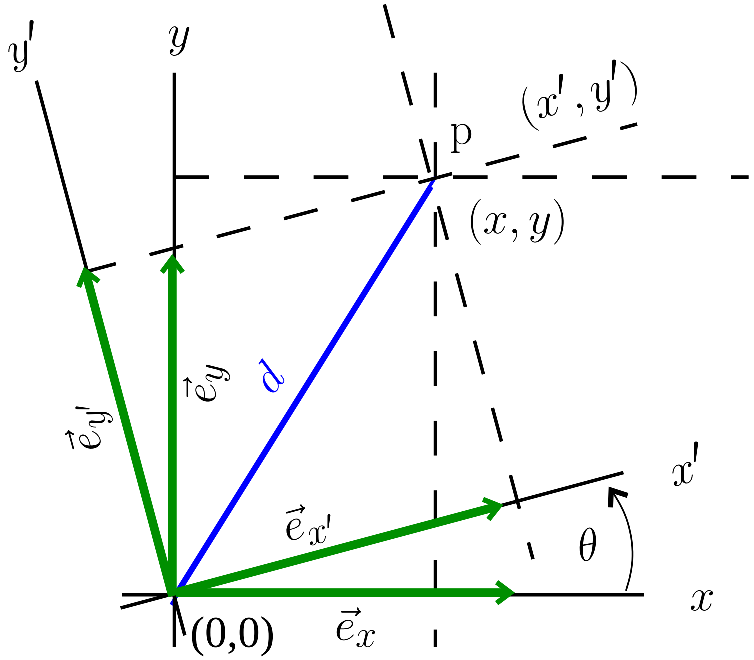
$$\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

GR: coordinate transformations



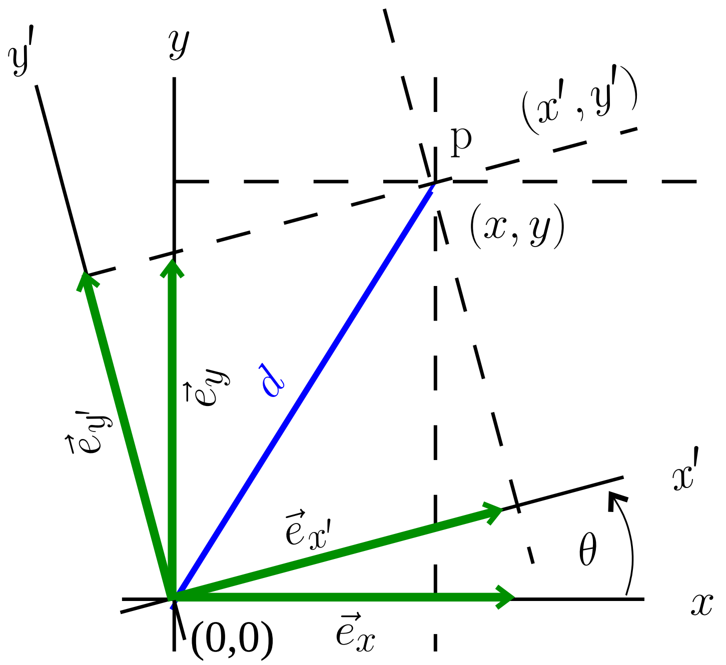
$$\vec{e}_{x'} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \vec{e}_x + \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \vec{e}_y$$

GR: coordinate transformations



$$\vec{e}_{x'} = \Lambda^x_{x'}\,\vec{e}_x + \Lambda^y_{x'}\,\vec{e}_y$$

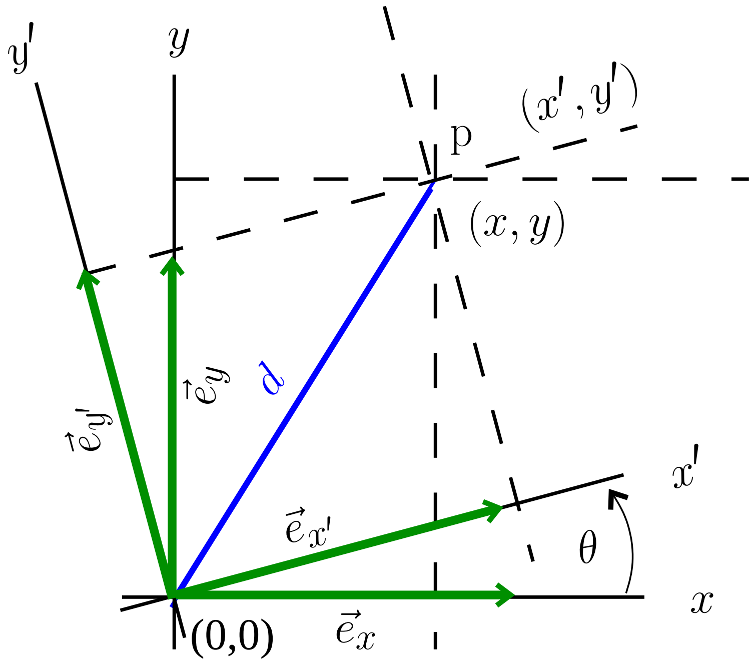
GR: coordinate transformations



also:

$$\begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

GR: coordinate transformations



$$\vec{e}_{y'} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \vec{e}_x + \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \vec{e}_y$$

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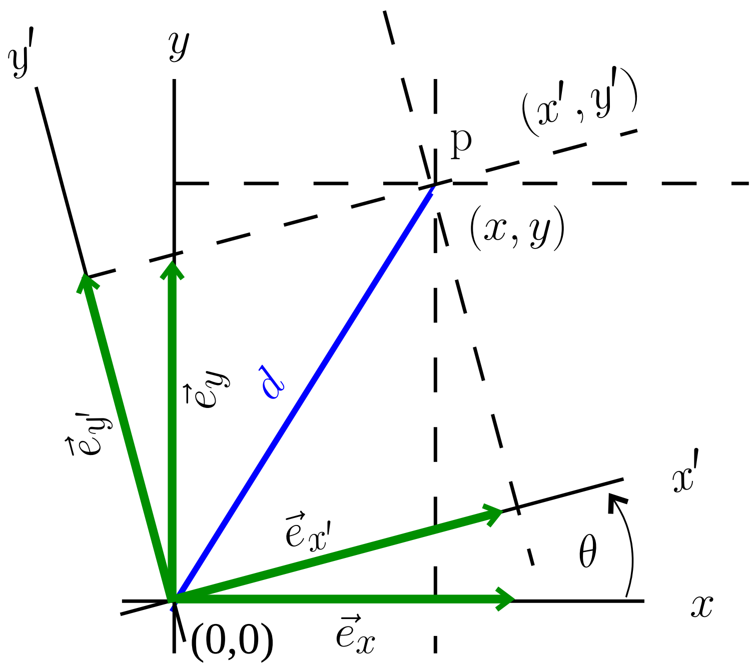

summary:

$\vec{e}_{x'} = \Lambda^x_{x'}\,\vec{e}_x + \Lambda^y_{x'}\,\vec{e}_y,$

$\vec{e}_{y'} = \Lambda^x_{y'}\,\vec{e}_x + \Lambda^y_{y'}\,\vec{e}_y,$

where $\Lambda^\alpha_{\beta'} :=$ element of inverse of $\Lambda^{\alpha'}_{\beta}$,

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \Lambda \begin{pmatrix} x \\ y \end{pmatrix}$$

GR: coordinate transformations

$$\vec{e}_{x'} = \Lambda^{x}_{x'}\,\vec{e}_x + \Lambda^{y}_{x'}\,\vec{e}_y,$$
$$\vec{e}_{y'} = \Lambda^{x}_{y'}\,\vec{e}_x + \Lambda^{y}_{y'}\,\vec{e}_y,$$

$$\vec{p} \to_{\mathcal{O}'} \begin{pmatrix} x' \\ y' \end{pmatrix} = \Lambda \begin{pmatrix} x \\ y \end{pmatrix}$$

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$$\vec{p} = \sum_i p^i \vec{e}_i$$

GR: coordinate transformations

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$\vec{p} = p^i \vec{e}_i$ (w:Einstein summation)

GR: coordinate transformations

$$\begin{aligned}\vec{e}_{x'} &= \Lambda^x_{x'}\,\vec{e}_x + \Lambda^y_{x'}\,\vec{e}_y,\\ \vec{e}_{y'} &= \Lambda^x_{y'}\,\vec{e}_x + \Lambda^y_{y'}\,\vec{e}_y,\end{aligned} \qquad \vec{p} \to_{\mathcal{O}'} \begin{pmatrix} x' \\ y' \end{pmatrix} = \Lambda \begin{pmatrix} x \\ y \end{pmatrix}$$

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Einstein summation:

- **coordinates** like r, θ, x, y:

not a sum: $\Lambda^x_{y'}\,\vec{e}_x$

- repeated up-down coordinate **indices** like $i, j \in \{0, 1, 2\}$ or $\alpha, \beta, \gamma, \lambda, \mu, \nu \in \{0, 1, 2, 3\}$:

sum: $\Lambda^i_{j'}\,\vec{e}_i := \Lambda^x_{y'}\,\vec{e}_x + \Lambda^y_{y'}\,\vec{e}_y$ for a 2D manifold, coords x, y

GR: coordinate transformations

$$\vec{e}_{x'} = \Lambda^{x}_{x'}\,\vec{e}_x + \Lambda^{y}_{x'}\,\vec{e}_y, \qquad \vec{e}_{y'} = \Lambda^{x}_{y'}\,\vec{e}_x + \Lambda^{y}_{y'}\,\vec{e}_y,$$

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new basis vectors = sum of inverse $\Lambda\times$ old vectors

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$\vec{e}_{\mu'} = \sum_\nu \Lambda^\nu{}_{\mu'} \vec{e}_\nu$

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new coords of vector $\vec{p}$ = $\Lambda\times$ old coords of same vector $\vec{p}$

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vector invariance requires contravariance of its coords
“contra" = inverse of change of basis vectors

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vector invariance requires contravariance of its coords
“contra" = inverse of change of basis vectors

- $\vec{p}$ is invariant: no dependence on coords
- $\vec{p}$ is contravariant: p^i change inversely to $\vec{e}_i$

GR: coord. transf.: 1-forms

ϕ = scalar field = $\phi(x, y) \equiv \phi(x', y')$

write $\phi_{,x} := \frac{\partial \phi}{\partial x} =: (\tilde{\mathrm{d}}\phi)_x$

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What is the relation between $(\phi_{,x'}, \phi_{,y'})$

and $(\phi_{,x}, \phi_{,y})$?

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ϕ depends either on x and y, or on x' and y'

$$\Rightarrow \frac{\partial\phi}{\partial x'} = \frac{\partial\phi}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial\phi}{\partial y}\frac{\partial y}{\partial x'}$$

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$(\phi_{,x'}, \phi_{,y'}) =$

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$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}$$ (example: rotation)

$x_{,x'} = \frac{\partial x}{\partial x'} = \cos\theta$

$x_{,y'} = \frac{\partial x}{\partial y'} = -\sin\theta \ldots$

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$$\begin{pmatrix} x \\ y \end{pmatrix} = \Lambda^{-1} \begin{pmatrix} x' \\ y' \end{pmatrix} \quad \text{(general)}$$

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$$\tilde{\mathrm{d}}\phi = \left((\tilde{\mathrm{d}}\phi)_{x'}, (\tilde{\mathrm{d}}\phi)_{y'}\right) = \left((\tilde{\mathrm{d}}\phi)_x, (\tilde{\mathrm{d}}\phi)_y\right) \Lambda^{-1}$$

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$$\tilde{\mathrm{d}}\phi = \left((\tilde{\mathrm{d}}\phi)_{x'}, (\tilde{\mathrm{d}}\phi)_{y'}\right) = \left((\tilde{\mathrm{d}}\phi)_x, (\tilde{\mathrm{d}}\phi)_y\right) \Lambda^{-1}$$

$$(\tilde{\mathrm{d}}\phi)_{\mu'} = (\tilde{\mathrm{d}}\phi)_\nu \Lambda^\nu{}_{\mu'}$$

GR: coord. transf.: 1-forms

basis vectors of different bases: $\vec{e}_{\mu'} = \Lambda^{\nu}{}_{\mu'} \vec{e}_{\nu}$

same vector: $(\vec{p})^{\mu'} = \Lambda^{\mu'}{}_{\nu} (\vec{p})^{\nu}$

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same vector: $p^{\mu'} = \Lambda^{\mu'}{}_{\nu} p^{\nu}$

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w:Covariance and contravariance of vectors

GR: $\vec{p}, \tilde{q}, \langle \vec{p}, \tilde{q} \rangle$, **g**

GR tensors: two different scalar products

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can be called I with components δ^μ_ν in a coordinate basis

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think: vector $\rightarrow$ column vector
1-form $\rightarrow$ row vector

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$\langle \; , \; \rangle$ = (1,1)-tensor = “row-column" matrix I with $I^\mu_\nu = \delta^\mu_\nu$

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GR tensors: two different scalar products

ordinary linear algebra: column vectors, row vectors, matrices

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(m,n)-tensor algebra: m column n row $m+n$-arrays

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(m, n)-tensor algebra: m column n row $m+n$-arrays

e.g.: $(0, 2)$-tensor: metric $g_{\mu\nu}$

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using $\langle\ ,\ \rangle$, $(1,0)$-tensor = vector = function of 1-forms

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loosely speaking, the second $\otimes$ means “function of two vectors" (or 1-forms, or a vector and a 1-form) in *that particular left-to-right order*

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warning: the “rank" of tensors has two different meanings: w:Tensor_(intrinsic_definition)#Tensor_rank

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dimension of $V^* \otimes V^*$ = 16 (for $V =$ spacetime)

GR: g

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also written: $\vec{A} \cdot \vec{B}$ “dot product"

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$$\mathbf{g}(\vec{A}, \vec{B}) = \left[\begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \begin{pmatrix} A^r \\ A^\theta \end{pmatrix} \right]^{\mathrm{T}} \begin{pmatrix} B^r \\ B^\theta \end{pmatrix}$$

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$$\mathbf{g}(\vec{A}, \vec{B}) \;= \left(A^r, A^\theta r^2\right) \begin{pmatrix} B^r \\ B^\theta \end{pmatrix}$$

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$\mathbf{g}(\vec{A}, \vec{B}) \quad = A^r B^r + A^\theta B^\theta r^2 \quad = A^x B^x + A^y B^y$

in general, for a 2-form **T**, $\mathbf{T}(\vec{A}, \vec{B}) \neq \mathbf{T}(\vec{B}, \vec{A})$

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$\mathbf{g} = g_{\mu\nu}\tilde{e}^\mu \otimes \tilde{e}^\nu$

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$\mathbf{g}(\vec{A}, \vec{B}) \;\; = A^r B^r + A^\theta B^\theta r^2 \;\; = A^x B^x + A^y B^y$

$\mathbf{g} = \sum_{\mu\in\{r,\theta\},\nu\in\{r,\theta\}} g_{\mu\nu}\tilde{e}^\mu \otimes \tilde{e}^\nu$

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g can be applied to basis vectors $\vec{e}_\mu$

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$\mathbf{g}$ can be applied to basis vectors $\vec{e}_\mu$

we can define components (used earlier): $g_{\mu\nu} := \mathbf{g}(\vec{e}_\mu, \vec{e}_\nu)$

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$\Rightarrow \mathbf{g} = g_{\mu\nu}\tilde{e}^\mu \otimes \tilde{e}^\nu$

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e.g. $\mathbf{g} = g_{rr}\tilde{e}^r \otimes \tilde{e}^r + g_{r\theta}\tilde{e}^r \otimes \tilde{e}^\theta + g_{\theta r}\tilde{e}^\theta \otimes \tilde{e}^r + g_{\theta\theta}\tilde{e}^\theta \otimes \tilde{e}^\theta$

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e.g. $\mathbf{g} = g_{rr}\tilde{e}^r \otimes \tilde{e}^r + g_{\theta\theta}\tilde{e}^\theta \otimes \tilde{e}^\theta$

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GR: what is a coordinate?

a coordinate, e.g. x^0 or x^1 is a scalar field on the 4-manifold

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a coordinate system x^μ = set of four scalar fields on the 4-manifold

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(Bertschinger writes $x^\mu_{\mathbf{x}}$ to show dependence on position $\mathbf{x}$ in manifold $\neq$ vector space)

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x^μ are differentiable *almost everywhere*

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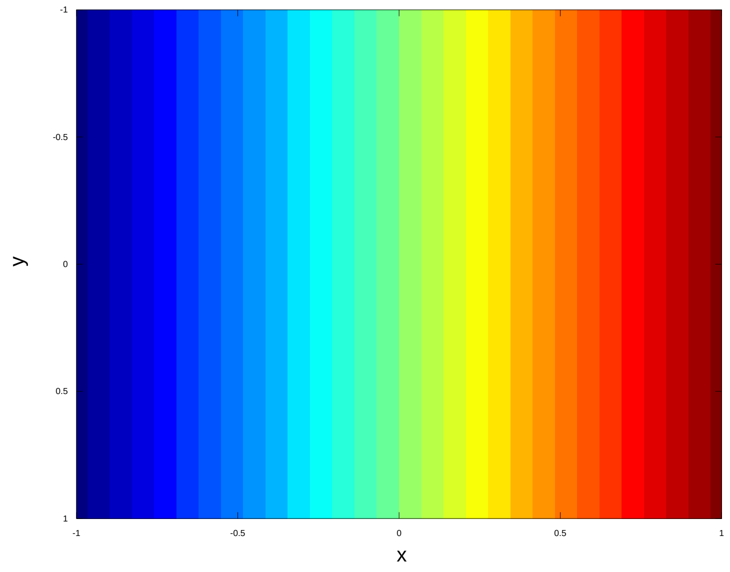

e.g. on $\mathbb{R}^2$

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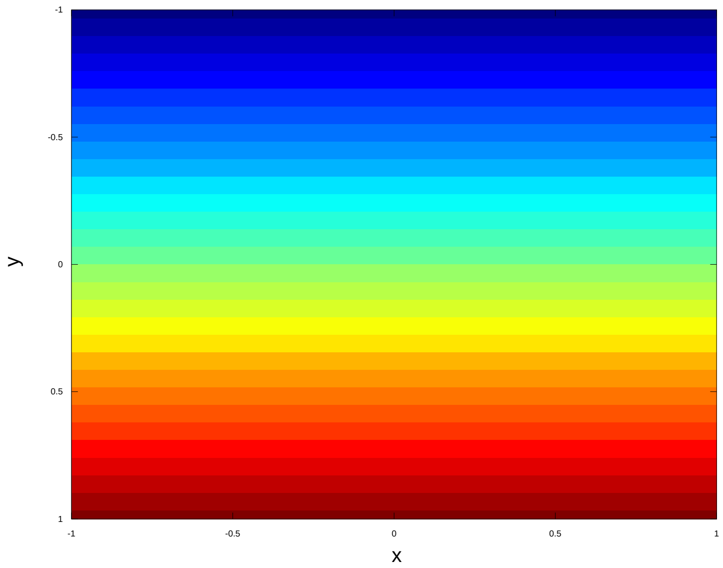

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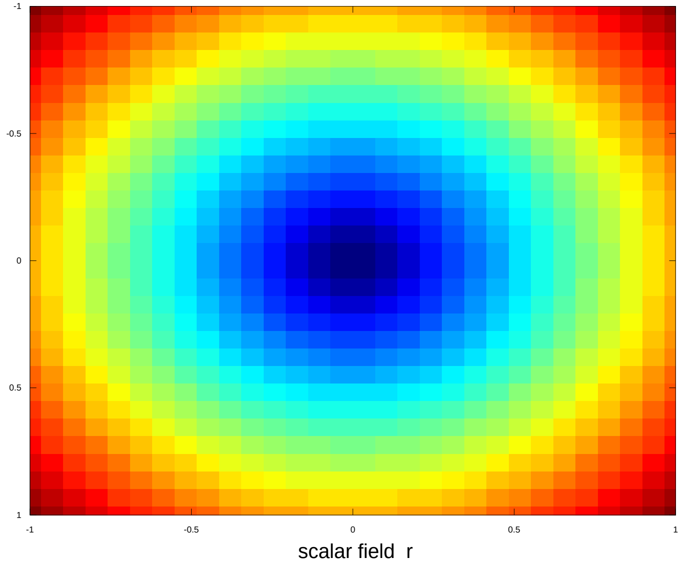

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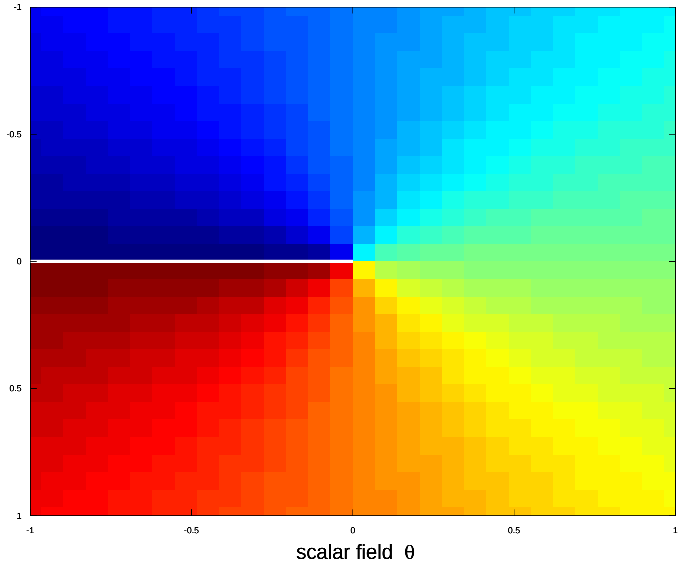

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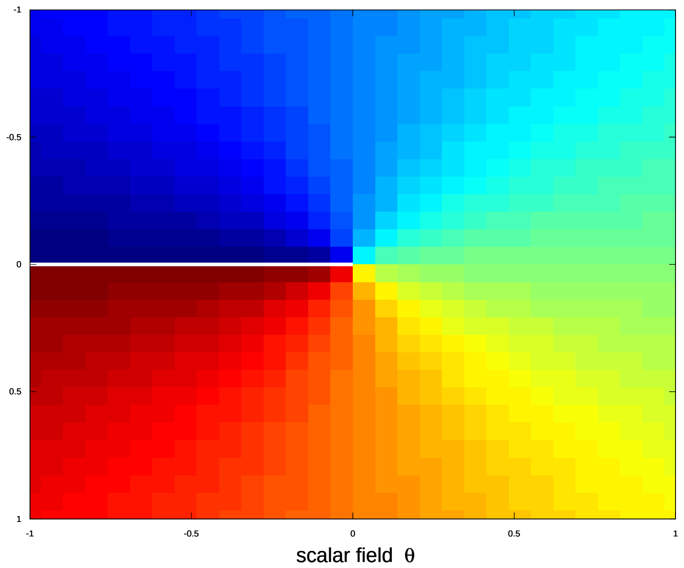

e.g. on $\mathbb{R}^2$

coordinate singularity $\neq$ singularity in manifold

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(Bertschinger writes $\widetilde{\nabla}$ for the gradient $\tilde{\mathrm{d}}$)

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$\mathrm{d}s^2 = g_{\mu\nu} \mathrm{d}x^\mu x^\nu$ if x^μ are a coordinate basis

GR: e.g. Euclidean g on $\mathbb{R}^2$

$g_{r\theta}$ and g_{xy}

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 = \mathrm{d}r^2 + r^2 \mathrm{d}\theta^2$$

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$g_{r\theta}$ and g_{xy}

$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 = \mathrm{d}r^2 + r^2\mathrm{d}\theta^2$

$\vec{e}_x \cdot \vec{e}_x = 1 = \vec{e}_y \cdot \vec{e}_y$, others zero

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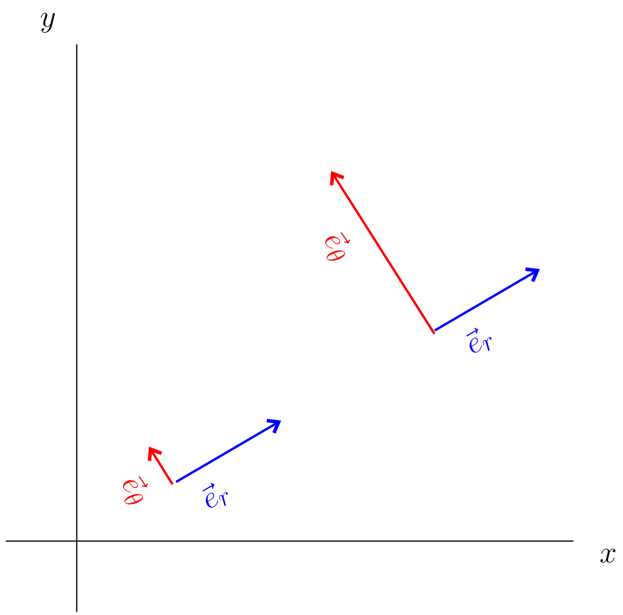
$\vec{e}_r \cdot \vec{e}_r = 1$, $\vec{e}_\theta \cdot \vec{e}_\theta = r^2 \neq 1$



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$g^{\mu\alpha} g_{\alpha\nu} = \delta^\mu_\nu \Rightarrow g^{xx} = 1 = g^{yy}, g^{xy} = 0 = g^{yx}$

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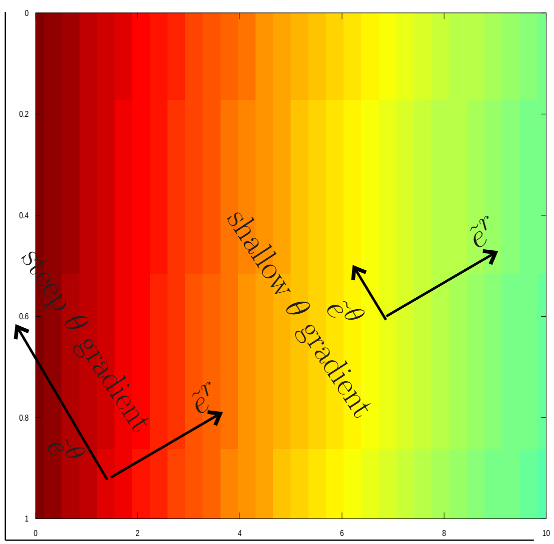

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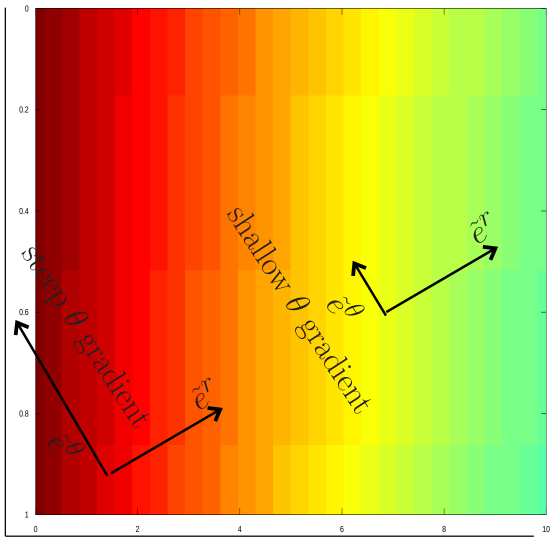

so $\tilde{e}^r \cdot \tilde{e}^r = 1$, $\tilde{e}^\theta \cdot \tilde{e}^\theta = r^{-2} \neq 1$

GR: gradient of a vector: $\nabla \vec{A}$

gradient of scalar field: $\tilde{\mathrm{d}}\phi \equiv \widetilde{\nabla}\phi$

GR: gradient of a vector: $\nabla\vec{A}$

what is gradient of vector field $\widetilde{\nabla}\vec{A}$?

GR: gradient of a vector: $\nabla\vec{A}$

$\widetilde{\nabla}\vec{A} = \widetilde{\nabla}(A^{\nu}\vec{e}_{\nu})$

GR: gradient of a vector: $\nabla\vec{A}$

$$\widetilde{\nabla}\vec{A} = \widetilde{\nabla}(A^\nu \vec{e}_\nu)$$

$$= \tilde{e}^\mu \partial_\mu (A^\nu \vec{e}_\nu)$$

GR: gradient of a vector: $\nabla \vec{A}$

$\widetilde{\nabla} \vec{A} = \widetilde{\nabla}(A^\nu \vec{e}_\nu)$

$= \tilde{e}^\mu \partial_\mu (A^\nu \vec{e}_\nu)$

$= \tilde{e}^\mu \otimes [(\partial_\mu A^\nu)\vec{e}_\nu + A^\nu \partial_\mu \vec{e}_\nu]$ by product rule and linearity

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$$\widetilde{\nabla}\vec{A} = \widetilde{\nabla}(A^\nu \vec{e}_\nu)$$

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$$= \partial_\mu A^\nu \tilde{e}^\mu \otimes \vec{e}_\nu + A^\nu \tilde{e}^\mu \otimes \partial_\mu \vec{e}_\nu$$

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give a name to the second part: it must be a linear combination of basis vectors $\vec{e}_\lambda$

GR: gradient of a vector: $\nabla\vec{A}$

$\widetilde{\nabla}\vec{A} = \widetilde{\nabla}(A^\nu\vec{e}_\nu)$

$= \tilde{e}^\mu\partial_\mu(A^\nu\vec{e}_\nu)$

$= \partial_\mu A^\nu\tilde{e}^\mu \otimes \vec{e}_\nu + A^\nu\tilde{e}^\mu \otimes \partial_\mu\vec{e}_\nu$

define $\Gamma^\lambda{}_{\nu\mu}\vec{e}_\lambda := \partial_\mu\vec{e}_\nu$ Christoffel symbols of second kind (symmetric defn)

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$= \partial_\mu A^\nu\tilde{e}^\mu \otimes \vec{e}_\nu + A^\nu\Gamma^\lambda{}_{\nu\mu}\tilde{e}^\mu \otimes \vec{e}_\lambda$ since any $\Gamma^\lambda{}_{\nu\mu}$ is a scalar

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since name of summation index is arbitrary, e.g.
$\sum_\lambda x^{-2\lambda} = \sum_\mu x^{-2\mu} = \sum_\nu x^{-2\nu}$

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$= \partial_\mu A^\nu \tilde{e}^\mu \otimes \vec{e}_\nu + A^\lambda \Gamma^\nu{}_{\lambda\mu} \tilde{e}^\mu \otimes \vec{e}_\nu$

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$= (\partial_\mu A^\nu + A^\lambda\Gamma^\nu{}_{\lambda\mu})\tilde{e}^\mu \otimes \vec{e}_\nu$

$$\boxed{\nabla_\mu A^\nu := A^\nu{}_{;\mu} := \partial_\mu A^\nu + A^\lambda\Gamma^\nu{}_{\lambda\mu}}$$

w:covariant derivative of vector

GR: gradient of a vector: $\nabla\vec{A}$

$\widetilde{\nabla}\vec{A} = \widetilde{\nabla}(A^\nu\vec{e}_\nu)$

$= \tilde{e}^\mu\partial_\mu(A^\nu\vec{e}_\nu)$

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w:covariant derivative of vector

GR: $\tilde{\nabla}$ and $\Gamma^{\nu}_{\lambda\mu}$

mathematically deeper: $\tilde{\nabla}$, usually written just as ∇, is the w:Levi-Civita connection

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and $\tilde{\nabla}$ on a $(1,0)$-tensor field = vector field $\vec{A}$ gives a $(1,1)$-tensor with components $\nabla_\mu A^\nu = \partial_\mu A^\nu + A^\lambda {\Gamma^\nu}_{\lambda\mu}$

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- not components of tensor: $\Gamma^{\nu}{}_{\lambda\mu}$

GR: gradient of one-form $\widetilde{\nabla}\tilde{A}$

how does a one-form change with position? $\widetilde{\nabla}\tilde{A} =$?

GR: gradient of one-form $\widetilde{\nabla}\tilde{A}$

evaluating $\widetilde{\nabla}\tilde{A}$ as we did $\widetilde{\nabla}\vec{A}$ shows that we again need $\partial_\mu \tilde{e}^\nu = F^\nu{}_{\lambda\mu}\tilde{e}^\lambda$ for some coefficients $F^\nu{}_{\lambda\mu}$

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relation between vectors and one-forms: $\langle \tilde{e}^\nu, \vec{e}_\lambda \rangle = \delta^\nu{}_\lambda$

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$\partial_\mu \delta^\nu{}_\lambda = 0$ (obviously)

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can we use the product rule with this scalar product?

$\partial_\mu \left(\left\langle \tilde{A}, \vec{B} \right\rangle \right) =$?

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can we use the product rule with this scalar product?

$\partial_\mu \left(\left\langle \tilde{A}, \vec{B} \right\rangle \right) = \partial_\mu \left(A_\nu B^\nu \right)$ in some coordinate basis

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$\partial_\mu \left(\left\langle \tilde{A}, \vec{B} \right\rangle\right) = \partial_\mu \left(A_\nu B^\nu\right)$

$= (\partial_\mu A_\nu) B^\nu + A_\nu (\partial_\mu B^\nu)$ by product rule on functions

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relation between vectors and one-forms: $\langle \tilde{e}^\nu, \vec{e}_\lambda \rangle = \delta^\nu{}_\lambda$

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$= \left\langle \partial_\mu \tilde{A}, \vec{B} \right\rangle + \left\langle \tilde{A}, \partial_\mu \vec{B} \right\rangle$ since

$\partial_\mu \tilde{A} = (\partial_\mu A_0, \partial_\mu A_1, \partial_\mu A_2, \partial_\mu A_3)$

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product rule holds: $\partial_\mu \left(\left\langle \tilde{A}, \vec{B} \right\rangle \right) = \left\langle \partial_\mu \tilde{A}, \vec{B} \right\rangle + \left\langle \tilde{A}, \partial_\mu \vec{B} \right\rangle$

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$0 = \partial_\mu \delta^\nu{}_\lambda = \partial_\mu \left(\langle \tilde{e}^\nu, \vec{e}_\lambda \rangle \right)$

product rule holds: $\partial_\mu \left(\left\langle \tilde{A}, \vec{B} \right\rangle \right) = \left\langle \partial_\mu \tilde{A}, \vec{B} \right\rangle + \left\langle \tilde{A}, \partial_\mu \vec{B} \right\rangle$

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GR: gradient of one-form $\widetilde{\nabla}\tilde{A}$

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$= \left\langle F^\nu{}_{\kappa\mu}\tilde{e}^\kappa, \vec{e}_\lambda \right\rangle + \left\langle \tilde{e}^\nu, \Gamma^\kappa{}_{\lambda\mu}\vec{e}_\kappa \right\rangle$

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$$\boxed{\nabla_\mu A^\nu = \partial_\mu A^\nu + A^\lambda \Gamma^\nu{}_{\lambda\mu} \quad , \quad \nabla_\mu A_\nu = \partial_\mu A_\nu - A_\lambda \Gamma^\lambda{}_{\mu\nu}}$$

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$$\boxed{A^\nu{}_{;\mu} = A^\nu{}_{,\mu} + A^\lambda \Gamma^\nu{}_{\lambda\mu} \quad , \quad A_{\nu;\mu} = A_{\nu,\mu} - A_\lambda \Gamma^\lambda{}_{\mu\nu}}$$

GR: smooth manifold and $\widetilde{\nabla}\mathbf{g}$

similarly, we can write the $(0,3)$-tensor

$\widetilde{\nabla}\mathbf{g} = (\nabla_\lambda g_{\mu\nu})\tilde{e}^\lambda \otimes \tilde{e}^\mu \otimes \tilde{e}^\nu$

giving $\nabla_\lambda g_{\mu\nu} = \partial_\lambda g_{\mu\nu} - \Gamma^\kappa{}_{\mu\lambda} g_{\kappa\nu} - \Gamma^\kappa{}_{\nu\lambda} g_{\mu\kappa}$

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also $\widetilde{\nabla}\mathbf{g}^{-1} = (\nabla_\lambda g^{\mu\nu})\tilde{e}^\lambda \otimes \vec{e}_\mu \otimes \vec{e}_\nu$

and $\nabla_\lambda g^{\mu\nu} = \partial_\lambda g^{\mu\nu} + \Gamma^\mu{}_{\kappa\lambda} g_{\kappa\nu} + \Gamma^\nu{}_{\kappa\lambda} g_{\mu\kappa}$

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Do we know anything interesting about $\widetilde{\nabla}\mathbf{g}$ for the manifolds of interest to GR?

GR: smooth manifold and $\widetilde{\nabla}\mathbf{g}$

similarly, we can write the $(0,3)$-tensor

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Do we know anything interesting about $\widetilde{\nabla}\mathbf{g}$ for the manifolds of interest to GR?

First, we need a rough description of the manifolds we need for GR.

GR: smooth manifold and $\widetilde{\nabla}\mathbf{g}$

topological manifold M
w:Manifold#Mathematical_definition

- only topological properties needed

GR: smooth manifold and $\widetilde{\nabla}$**g**

topological manifold M
w:Manifold#Mathematical_definition

- only topological properties needed
- no differentiability, no metric needed

GR: smooth manifold and $\widetilde{\nabla}\mathbf{g}$

topological manifold M

w:Manifold#Mathematical_definition

- only topological properties needed

next: relation with $\mathbb{R}^4$ (or M^4)

GR: smooth manifold and $\widetilde{\nabla}\mathbf{g}$

topological manifold M
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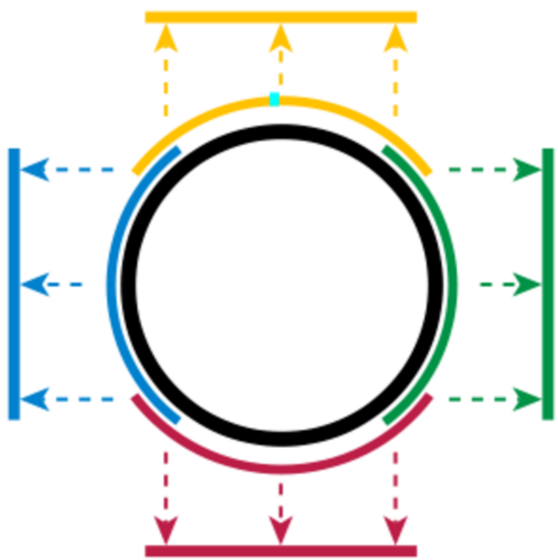
next: relation with $\mathbb{R}^4$ (or M^4)

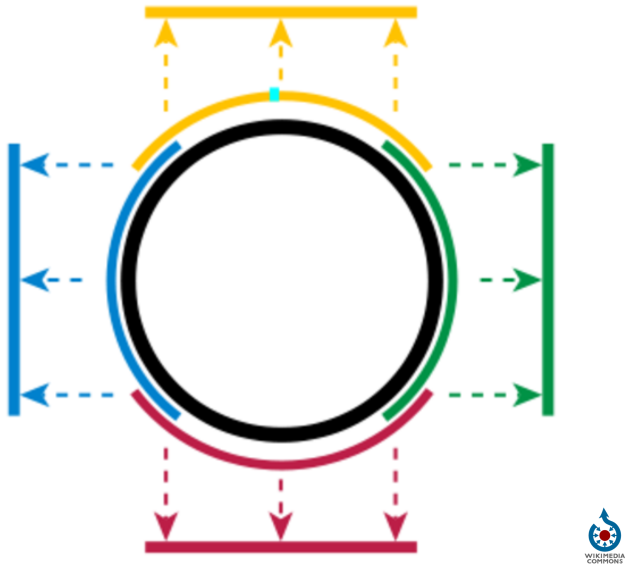


GR: smooth manifold and $\widetilde{\nabla}\mathbf{g}$

topological manifold M
w:Manifold#Mathematical_definition

- only topological properties needed

next: relation with $\mathbb{R}^4$ (or M^4)



w:Manifold

- chart := function ϕ_α from part of pseudo-4-manifold M to part of M^4 (Minkowski)
- atlas := set of overlapping charts that cover M

GR: smooth manifold and $\widetilde{\nabla}\mathbf{g}$

if every transition chart $:= \phi_\beta \circ \phi_\alpha^{-1}$ in an atlas for M is differentiable on $\mathbb{R}^4$ (or M^4), then M is a w:differentiable 4-(pseudo-)manifold

GR: smooth manifold and $\widetilde{\nabla}\mathbf{g}$

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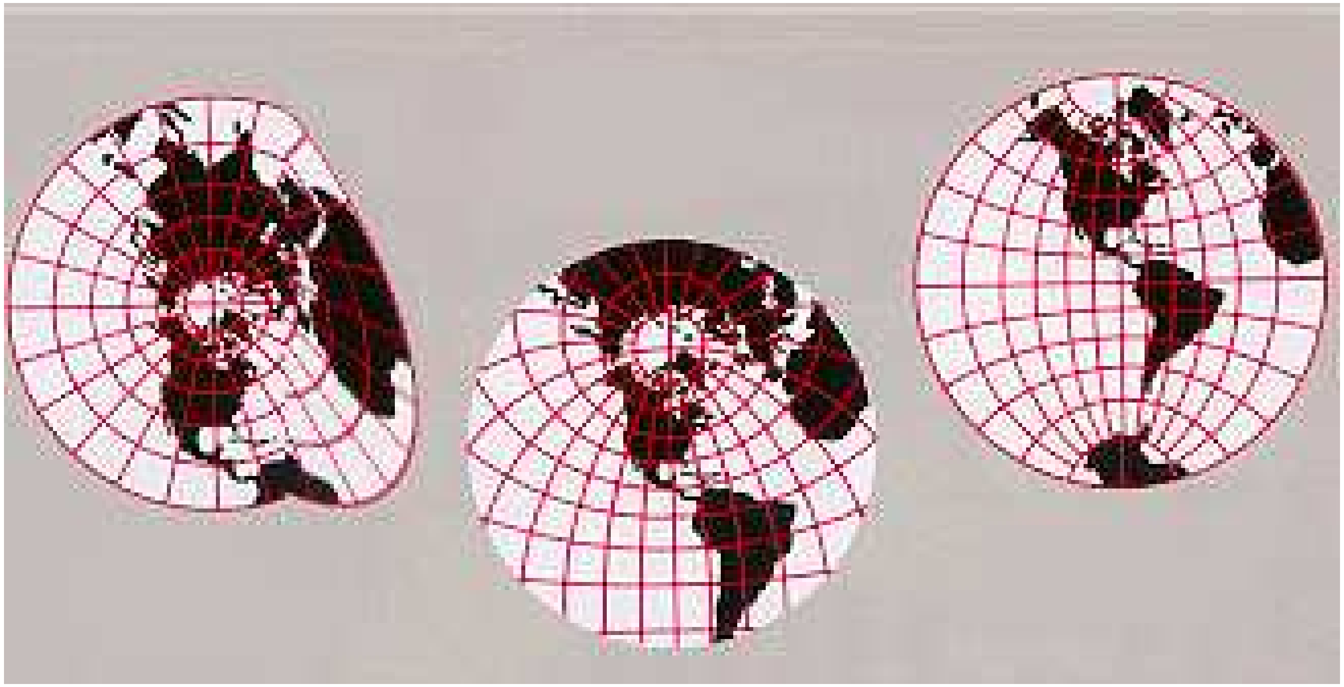

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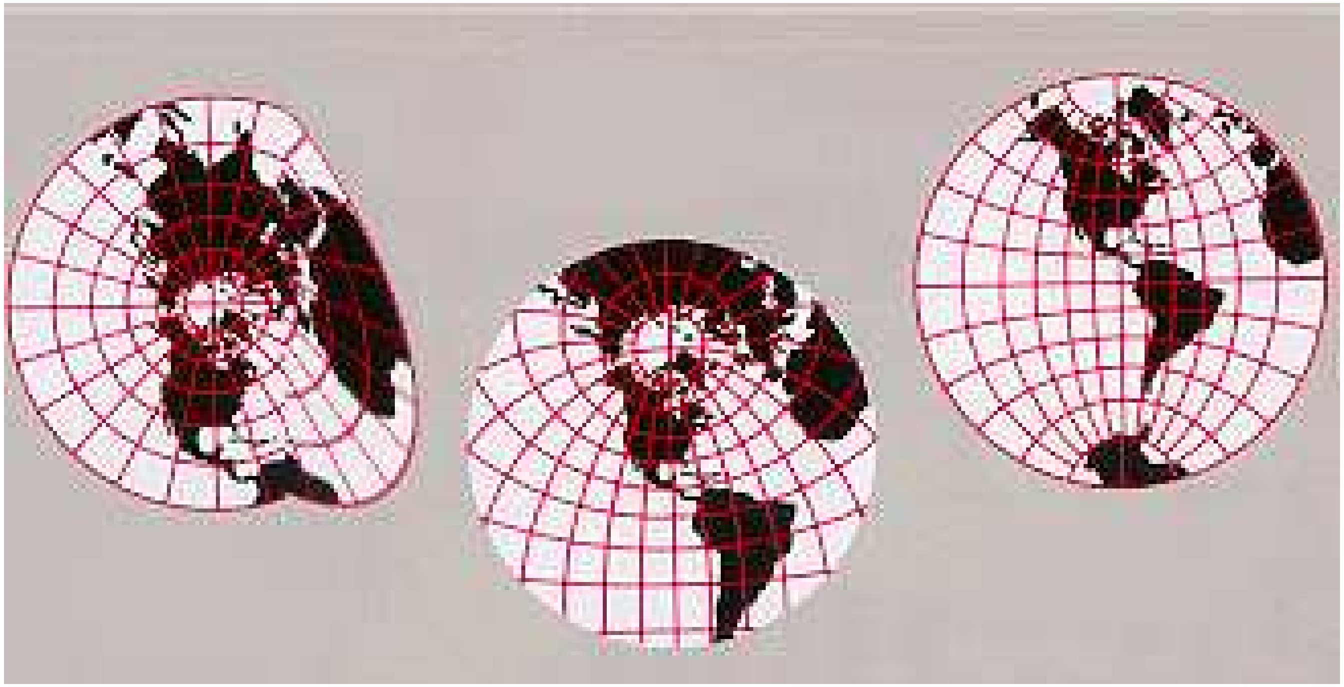

w:

projections (left-to-right) ϕ_1, ϕ_2, ϕ_3 from S^2 to $\mathbb{R}^2$

GR: smooth manifold and $\widetilde{\nabla}$g

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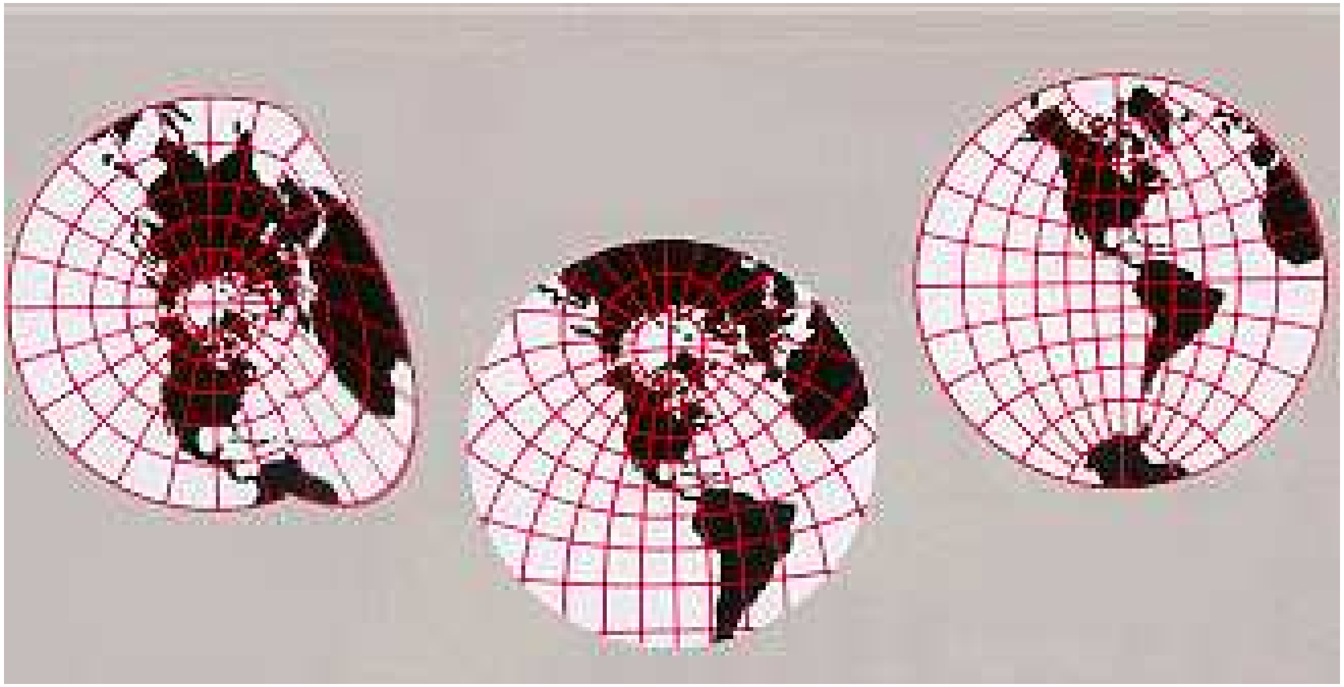

w:

ϕ_1 is not differentiable, so $\phi_1 \circ \phi_2^{-1}$ is not differentiable

GR: smooth manifold and $\widetilde{\nabla}\mathbf{g}$

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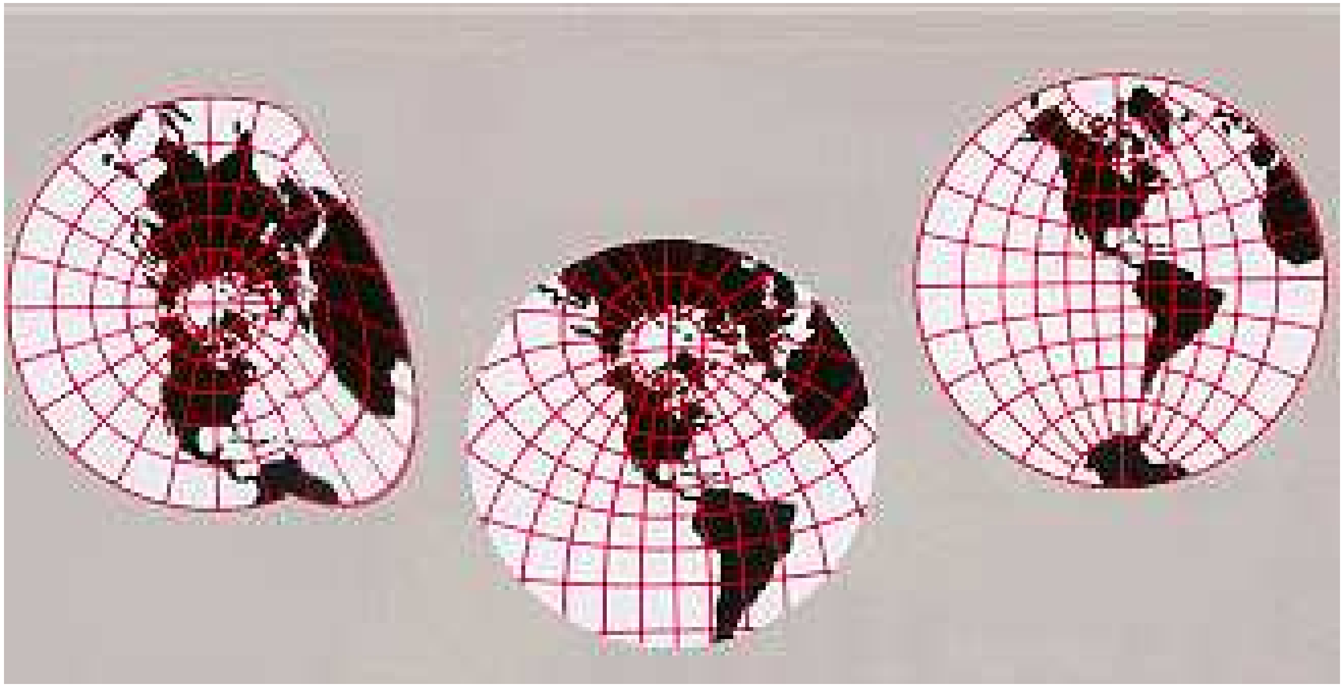

w:

atlas not enough to show that $S^2 =$ differentiable 2-manifold

GR: smooth manifold and $\widetilde{\nabla} \mathbf{g}$

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GR: smooth manifold and $\widetilde{\nabla \mathbf{g}}$

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if $\forall k \geq 1$, $\exists k$-th derivatives, then M is a smooth 4-(pseudo-)manifold

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if a (pseudo-)w:Riemannian metric $\mathbf{g}$ can be added to M, then $(M,\mathbf{g})$ is a (pseudo-)Riemannian 4-manifold

GR: smooth manifold and $\widetilde{\nabla \mathbf{g}}$

- if every transition chart $:= \phi_\beta \circ \phi_\alpha^{-1}$ in an atlas for M is differentiable on $\mathbb{R}^4$ (or M^4), then M is a w:differentiable 4-(pseudo-)manifold
- if $\forall k \geq 1$, $\exists k$-th derivatives, then M is a smooth 4-(pseudo-)manifold
- if a (pseudo-)w:Riemannian metric **g** can be added to M, then $(M,\mathbf{g})$ is a (pseudo-)Riemannian 4-manifold
- if **g** has signature $(1, n-1)$ (i.e. $(-,+,+,+)$ or $(+,-,-,-)$, etc.), then $(M,\mathbf{g})$ is a Lorentzian n-manifold

GR: smooth manifold and $\widetilde{\nabla}$g

$\boxed{\text{topological manifolds}}$

GR: smooth manifold and $\widetilde{\nabla \mathbf{g}}$

topological manifolds

differentiable (pseudo-)manifolds

GR: smooth manifold and $\widetilde{\nabla}\mathbf{g}$

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differentiable (pseudo-)manifolds

smooth (pseudo-)manifolds

GR: smooth manifold and $\widetilde{\nabla}\mathbf{g}$

topological manifolds

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smooth (pseudo-)manifolds

(pseudo-)Riemannian manifolds

GR: smooth manifold and $\widetilde{\nabla}\mathbf{g}$

topological manifolds

differentiable (pseudo-)manifolds

smooth (pseudo-)manifolds

(pseudo-)Riemannian manifolds

Lorentzian manifolds

GR: smooth manifold and $\widetilde{\nabla}\mathbf{g}$

- topological manifolds
  - differentiable (pseudo-)manifolds
    - smooth (pseudo-)manifolds
      - (pseudo-)Riemannian manifolds
        - Lorentzian manifolds
          - Lorentzian 4-manifolds

GR: smooth manifold and $\widetilde{\nabla}\mathbf{g}$

- topological manifolds
  - differentiable (pseudo-)manifolds
    - smooth (pseudo-)manifolds
      - (pseudo-)Riemannian manifolds
        - Lorentzian manifolds
          - Lorentzian 4-manifolds

GR: assume that spacetime is a Lorentzian 4-manifold

GR: smooth manifold and $\widetilde{\nabla}$**g**

from above:

$$\nabla_\lambda g_{\mu\nu} = \partial_\lambda g_{\mu\nu} - \Gamma^\kappa{}_{\mu\lambda} g_{\kappa\nu} - \Gamma^\kappa{}_{\nu\lambda} g_{\mu\kappa}$$

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in the tangent space at $\mathbf{x}$, $\exists$ coordinate basis $\vec{e}_{\bar{\mu}}$ with

$$g_{\bar{\mu}\bar{\nu}} = \eta_{\bar{\mu}\bar{\nu}} = \mathrm{diag}(-1, 1, 1, 1) = g^{\bar{\mu}\bar{\nu}}$$

$$\Rightarrow \partial_{\bar{\lambda}} g_{\bar{\mu}\bar{\nu}} = \partial_{\bar{\lambda}} \eta_{\bar{\mu}\bar{\nu}}$$

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also, $\Gamma^{\bar{\lambda}}{}_{\bar{\nu}\bar{\mu}} \vec{e}_{\bar{\lambda}} := \vec{e}_{\bar{\nu},\bar{\mu}} = \partial_{\bar{\mu}} \vec{e}_{\bar{\nu}}$

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but in a Cartesian or Minkowski (vector) space, the basis vectors always point in the same direction and their lengths are fixed

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M$^4 \Rightarrow \Gamma^{\bar{\lambda}}{}_{\bar{\nu}\bar{\mu}} \vec{e}_{\bar{\lambda}} = 0$

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so $\nabla_{\bar{\lambda}} g_{\bar{\mu}\bar{\nu}} = 0$

so $\widetilde{\nabla}\mathbf{g} = \mathbf{0}$ (also $\widetilde{\nabla}\mathbf{g}^{-1} = 0$) on the tangent space, since if true in one coord system, also true in others

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also, $\Gamma^{\bar{\lambda}}{}_{\bar{\nu}\bar{\mu}} \vec{e}_{\bar{\lambda}} := \vec{e}_{\bar{\nu},\bar{\mu}} = \partial_{\bar{\mu}} \vec{e}_{\bar{\nu}}$

so $\nabla_{\bar{\lambda}} g_{\bar{\mu}\bar{\nu}} = 0$

so $\widetilde{\nabla}\mathbf{g} = \mathbf{0} = \widetilde{\nabla}\mathbf{g}^{-1}$ on tangent space

$\ldots\,\widetilde{\nabla}\mathbf{g} = \mathbf{0} = \widetilde{\nabla}\mathbf{g}^{-1}$ on M

GR: smooth manifold and $\widetilde{\nabla}$g

from above:

$\nabla_\lambda g_{\mu\nu} = \partial_\lambda g_{\mu\nu} - \Gamma^\kappa{}_{\mu\lambda} g_{\kappa\nu} - \Gamma^\kappa{}_{\nu\lambda} g_{\mu\kappa}$

in the tangent space at $\mathbf{x}$, $\exists$ coordinate basis $\vec{e}_{\bar\mu}$ with

$g_{\bar\mu\bar\nu} = \eta_{\bar\mu\bar\nu} = \mathrm{diag}(-1,1,1,1) = g^{\bar\mu\bar\nu}$

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...

$\boxed{\Gamma^\lambda{}_{\mu\nu} = \frac{1}{2} g^{\lambda\kappa}(\partial_\mu g_{\nu\kappa} + \partial_\nu g_{\mu\kappa} - \partial_\kappa g_{\mu\nu}) \text{ in a coordinate basis}}$

GR: directional deriv.: $< \widetilde{\nabla}\vec{A}, \vec{V} >$

- $\widetilde{\nabla}\phi, \widetilde{\nabla}\vec{A}, \widetilde{\nabla}\tilde{A}$ gave how the fields ϕ, $\vec{A}$, or $\tilde{A}$ change around the manifold in general

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warning: $\{x^\mu(\lambda)\}$ at some λ on the manifold is a point on the manifold but NOT a vector; while $\mathrm{d}\vec{x}$ — in the tangent space — IS a vector

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using $\vec{V}(\lambda) := \frac{\mathrm{d}\vec{x}}{\mathrm{d}\lambda}$, project covariant derivative to curve
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∇_V written by Bertschinger without $^{\rightarrow}$or $\tilde{}$ because ∇_V **T** of tensor **T** has the same tensor order as **T**

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so in a coord basis,

$$\boxed{\nabla_V \vec{A} = \left(\frac{\mathrm{d}A^\nu}{\mathrm{d}\lambda} + V^\mu A^\kappa \Gamma^\nu{}_{\kappa\mu} \right) \vec{e}_\nu}$$

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special (interesting) case: vector field $\vec{A}$ and curve with tangents $\vec{V} := \frac{\mathrm{d}\vec{x}}{\mathrm{d}\lambda}$ where $\vec{A}$ "locally does not change direction"

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i.e. $\nabla_V \vec{A} = 0$

$\boxed{\nabla_V \vec{A} = 0 \text{ defn: parallel transport of } \vec{A} \text{ along path } \mathbf{x}(\lambda)}$

where $\vec{V} := \frac{\mathrm{d}\vec{x}}{\mathrm{d}\lambda}$

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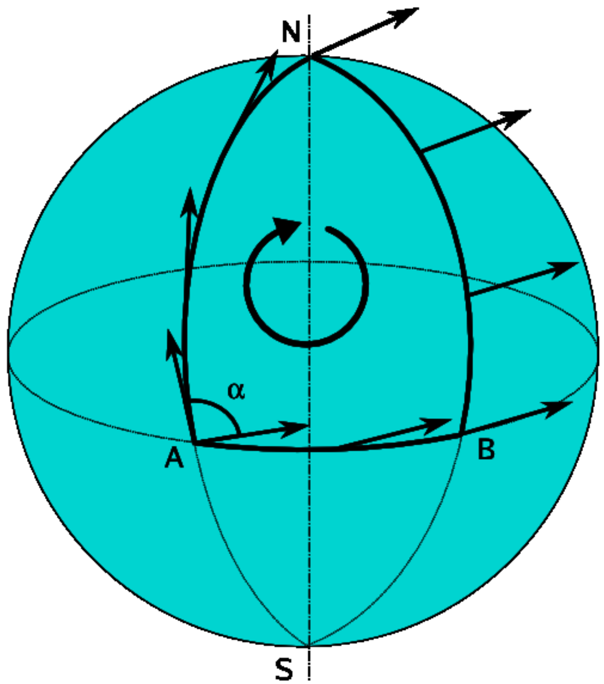
example:



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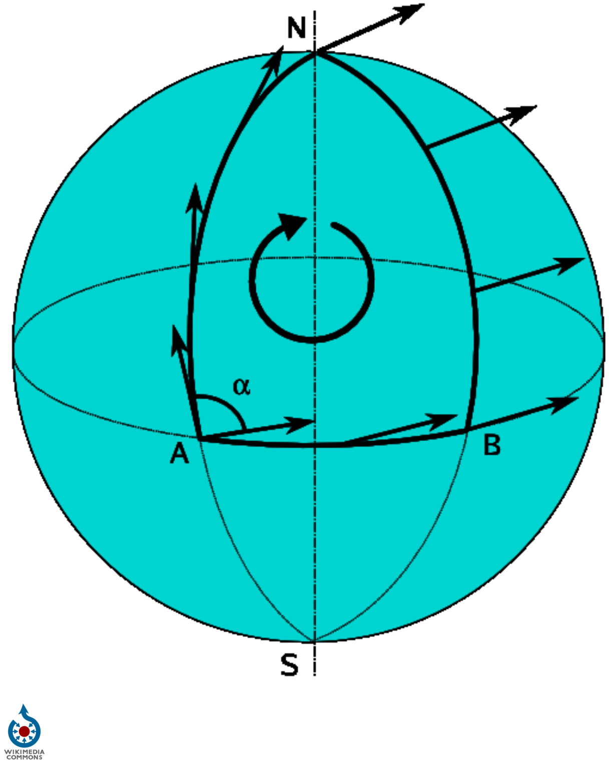

on S^2, parallel transport of $\vec{A}$ around a closed loop does not conserve $\vec{A}$'s direction

GR: directional deriv.: $< \widetilde{\nabla} \vec{A}, \vec{V} >$

$\boxed{\nabla_V \vec{V} = 0}$ defn: w:geodesic

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- more general definition of “straight line" than “shortest distance between two points"
- tensorial definition — independent of coordinate basis
- allows more than one “straight line" between two points a and b in a manifold — consider S^2, T^3

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a and b in a manifold — consider S^2, T^3

i.e. $(\frac{\mathrm{d}V^\nu}{\mathrm{d}\lambda} + V^\mu V^\kappa \Gamma^\nu{}_{\kappa\mu})\vec{e}_\nu = \vec{0}$

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i.e. $\frac{\mathrm{d}V^\nu}{\mathrm{d}\lambda} + V^\mu V^\kappa \Gamma^\nu_{\kappa\mu} = 0 \; \forall \nu$

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a and b in a manifold — consider S^2, T^3

i.e. $\frac{\mathrm{d}^2 x^\nu}{\mathrm{d}\lambda^2} + \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\kappa}{\mathrm{d}\lambda}\Gamma^\nu{}_{\kappa\mu} = 0 \ \forall \nu$

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cf w:Euler-Lagrange equation

GR: parallel transp. @closed curve

parallel transport around “small" parallelogram in two directions $\mathrm{d}\vec{x}_1$, $\mathrm{d}\vec{x}_2$,

(“1" and “2" are not component indices here)

GR: parallel transp. @closed curve

parallel transport around “small" parallelogram in two directions $\mathrm{d}\vec{x}_1$, $\mathrm{d}\vec{x}_2$,

What is the change in $\vec{A}$ after parallel transport around the closed loop $\mathrm{d}\vec{x}_1$, $\mathrm{d}\vec{x}_2$, $-\mathrm{d}\vec{x}_1$, $-\mathrm{d}\vec{x}_2$?

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$\Rightarrow$ must exist a tensor **R** that is a function of 3 vectors (“inputs"),

i.e. is a $\otimes$ of 3 one-forms

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defn: $\boxed{-\mathbf{R}(\cdot, \vec{A}, \mathrm{d}\vec{x}_1, \mathrm{d}\vec{x}_2) := \mathrm{d}\vec{A}(\cdot)}$

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(minus sign convention: MTW1973, Bertschinger)

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$$\mathrm{d}\vec{A}(\cdot) = -R^{\mu}{}_{\nu\alpha\beta} A^{\nu}\, \mathrm{d}x_1^{\alpha}\, \mathrm{d}x_2^{\beta}\, \vec{e}_{\mu}(\cdot)$$

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i.e. $-\mathbf{R} = R^{\mu}{}_{\nu\alpha\beta}\vec{e}_{\mu} \otimes \tilde{e}^{\nu} \otimes \tilde{e}^{\alpha} \otimes \tilde{e}^{\beta}$

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i.e. $-\mathbf{R} = \sum_\mu \sum_\nu \sum_\alpha \sum_\beta R^{\mu}_{\ \nu\alpha\beta} \vec{e}_\mu \otimes \tilde{e}^{\nu} \otimes \tilde{e}^{\alpha} \otimes \tilde{e}^{\beta}$

GR: Riemann tensor

how can **R** be evaluated?

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use covariant derivatives of covariant derivatives . . .

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Ricci identity:

$$\boxed{(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha) A^\mu = R^\mu{}_{\nu\alpha\beta} A^\nu \text{ in a coord. basis}}$$

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how can **R** be evaluated?

use covariant derivatives of covariant derivatives . . .

Ricci identity:

$$\boxed{(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha) A^\mu = R^\mu{}_{\nu\alpha\beta} A^\nu \text{ in a coord. basis}}$$

also written with commutator $[\,,\,]$

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- so **R** has second order partial derivatives of $g_{\nu\kappa}, \ldots$

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- first order ∂:

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- second order ∂:

(pseudo-)manifold globally like $\mathbb{R}^3$ (M^4) $\Leftrightarrow R^{\mu}{}_{\nu\alpha\beta}(\mathbf{x}) = 0 \; \forall \mathbf{x}$

GR: Bianchi, Ricci, Einstein

…second Bianchi identity:

$$\nabla_\sigma R^\mu{}_{\nu\kappa\lambda} + \nabla_\kappa R^\mu{}_{\nu\lambda\sigma} + \nabla_\lambda R^\mu{}_{\nu\sigma\kappa} = 0$$

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warning: “R" written coordinate-free may mean:

- an order 4, dimension 64 tensor **R**;
- an order 2, dimension 16 tensor **R** or R; or
- an order 0, dimension 1 tensor $\equiv$ scalar R
- all three are fields over a spacetime 4-manifold

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defn Einstein tensor (by components):

$$\boxed{G^{\mu\nu} := R^{\mu\nu} - \tfrac{1}{2} g^{\mu\nu} R}$$

$$\Rightarrow \nabla_\nu G^{\mu\nu} = 0$$

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Cactus - http://cactuscode.org