The Riemann Hypothesis

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Summary: In article the proof of well-known Riemann Hypothesis is given. The proof method is based on the zeta-function approach in the right half of the critical strip by pieces of products of Euler type.

MSC:11M26

1. Introduction.

Appearance of the zeta function and analytical methods in the Number Theory is connected with L.Euler's name (see [18, p. 54]). In 1748 Euler entered the zeta-function

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}, s > 1, \quad (1)$$

considering it as a function of real variable s. Using an identity

$$\zeta(s) = \prod_{p} \left(1 - p^{-s}\right)^{-1}, s > 1,$$

where the product is taken over all primes, he gave an analytical proof of Euclid's theorem about infinity of a set of prime numbers. Euler gave the relation the modern formulation of which is equivalent to the Riemann functional equation (see [13]).

In 1798 A.M. Legendre formulated for a quantity $\pi(x)$, denoting the number of primes, not exceeding x, the relationship $\lim_{x\to\infty} \frac{\pi(x)\ln x}{x} = 1$, and assumed that more exact representation $\pi(x) = x/(\ln x - B(x))$ holds, where $B(x)$ tends to the constant $B = 1.083...$ as $x \to \infty$.

Earlier K.Gauss has assumed that $\pi(x)$ can be approximated with a smaller error by using of a function $\int_2^x \frac{du}{\ln u}$. According to this assumption for B in the Legendre's formula can be written out the value $B = 1$.

In 1837 using and developing Euler's ideas, L.Dirichlet gave generalization of theorem of Euclid for arithmetic progressions, considering L -functions. Dirichlet tried to prove Legendre's formula entering the notion of an asymptotic law.

In 1851 and 1852 P.L. Tchebychev received exact results. He had shown that if the relation $\pi(x)\ln x/x$ tends to any limit then this limit will be 1 as well as it was assumed by Legendre. He, also, established that for a constant B the fair value can be only 1. In Tchebychev works search of Euler's function $\zeta(s)$ is lifted on higher level.

The great meaning of the zeta function for the Analytical Number Theory has been discovered by B. Riemann in 1859. Probably [41], Riemann was engaged in research of the zeta function under influence of Tchebychev's achievements. In the well known memoir [20] he had considered for the first time the zeta function as a function of a complex variable and had connected a problem of distribution of prime numbers with an arrangement of complex zeroes of the zeta function. Riemann proved the functional equation

$$\xi(s)=\xi(1-s);\xi(s)=\frac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

and formulated several hypotheses about the zeta function. One of them (further RH) was fated to stand a central problem for all of mathematics. This Hypothesis asserts, that all of complex zeroes of the zeta function, located in the critical strip $0<Res<1$, lies on the critical line $Res=0.5$.

D.Hilbert in the report at the International Paris Congress of 1900 included this Hypothesis into the list of 23 mathematical problems. Despite attempts of mathematicians several generations, it was remaining unsolved. To reach progress in the proof of RH, the following brunches of Analytical Number Theory have been developed:

1. Investigations of regions free from the zeroes of the zeta function;
2. Estimations of density of distribution of zeroes in the critical strip and their applications;
3. Studying of zeroes on the critical line;
4. Studying of distribution of values of the zeta function in the critical strip;
5. The computing problems connected with zeroes and so on.

These directions are classical and in the literature can be found full enough lighting of historical and other aspects of the questions connected with RH (see [3,6,12, 16,17,19,22,24,40-44]). Here, in introduction, we shall mention, in brief, works of 4th direction and some modern ideas connected with RH.

Studying of distribution of values of the zeta function has been begun by G. Bohr (see [24, p. 279]). In work [2], together with R. Courant, it was proved the theorem of everywhere density of values of $\zeta(\sigma+it), -\infty<t<\infty, \sigma\in(1/2,1)$.

The results of S.M.Voronin [25-32] connected with universality property of the zeta function have lifted on a new level the researches of the zeta function and other functions, defined by Dirichlet series. In S.M.Voronin's works the distributions of values of some Dirichlet series are studied and the new decision, in more general form, of D.Hilbert's problem about differential independence of the zeta function and L-functions is given. About other generalizations and improvements see ([1, 14-16]).

Last some years it has been begun studying of some families of Dirichlet series purpose of which was the consideration of questions of the zeta and L-functions' zeroes distribution (see [41-43]). B.Bagchi had considered (see [15 - 16]) a family of Dirichlet series defined by means of following product over all prime numbers

$$F(s,\theta)=\prod_p \left(1-\chi_p(\theta)p^{-s}\right)^{-1},$$

where, $Res>1$, and θ takes values from the topological product of the circles $|z_p|=1$ and $\chi_p(\theta)$ is a projection of θ into the circle $|z_p|=1$. He had shown, that this function can be analytically continued into the half plane $\mathrm{Re}\, s>1/2$ and has not there zeroes for almost all θ. Here the measure is a Haar measure. In the works [1,14-16] the questions, connected with property of joint universality of some Dirichlet series, are considered. By using of Ergodic methods, the special probability measures are constructed.

In the work [11] an equivalent variant of B.Bagchi's mentioned above result has been given by considering the function

$$F(s,\theta)=\prod_p \left(1-e^{2\pi i\theta_p}p^{-s}\right)^{-1}, 0\le\theta_p\le 1 \quad (2)$$

in $\Omega=[0,1]\times[0,1]\times\cdots$ with the product of Lebesgue measures.

In the works [33 - 38] questions on the distances of consecutive zeroes of the zeta function, located on the critical line, on numbers of zeroes in the circles of small radius at a close neighborhood of the critical line, and also, about multiple zeroes of the zeta function have been studied.

In the present work we study distribution of special curves of a kind $(\{t\lambda_n\})_{n\ge1}$ (the sign $\{\}$ means a fractional part and $\lambda_n>0, \lambda_n\to\infty$ as $n\to\infty$) in subsets of infinite dimensional unite cube on which some series is divergent. As a consequence, we prove justice of RH. For establishing the last, at first, we will approach $\zeta(s)$ in some circle located on the right half of the critical strip by means of partial products of a kind (2), using S.M.Voronin's lemma (see lemma 2). Further, we shall extend the received relationship to the all right half of the critical strip, using special structure of a set of divergence of some series (see section 5).

Definition 1. *Let* $\sigma : N \to N$ *be any one to one mapping of a set of natural numbers. If there will be a natural number* m *such that* $\sigma(n) = n$ *for any* $n > m$ *then we will say, that* σ *is finite permutation. A subset* $A \subset \Omega$ *will be called to be finite-symmetrical if for any element* $\theta = (\theta_n) \in A$ *and any finite permutation* σ *we have* $\sigma\ \sigma\theta = (\theta_{\sigma(n)}) \in A$.

Let Σ to denote the set of all finite permutations. We shall define on this set a product of two finite permutations as a composition of mappings. Then Σ becomes a group which contains each group of n degree permutations as a subgroup (we consider each n degree permutation σ as a finite permutation in the sense of definition 1 for which $\sigma(m) = m$ when $m > n$). The set Σ is enumerable set and we can arrange its elements in a sequence.

Theorem. *Let r be a real number* $0 < r < 1/4$. *Then there is a sequence* $(\overline{\theta}_n)_{n\geq 1}$ *of elements of* $\Omega\ (\overline{\theta}_n \in \Omega, n = 1,2,...)$ *and a sequence of integers* (m_n) *such that for any real* t *the relationship*

$$\lim_{n\to\infty} F_n(s+it, \overline{\theta}_n) = \zeta(s+it),$$

is satisfied uniformly by s in the circle $|s - 3/4| \leq r$ *; here*

$$F_n(s+it, \overline{\theta}_n) = \prod_{p\leq m_n} \left(1 - e^{2\pi i\theta_p^n} p^{-s}\right)^{-1}; \overline{\theta}_n = (\theta_p^n),$$

components of $\overline{\theta}_n$ *are indexed by prime numbers and the product is taken over all primes with an indicated inequality.*

It is necessary to notice, that the convergence speed in limiting process depends on t.

Consequence. *The Riemann Hypothesis is true, i.e.*

$$\zeta(s) \neq 0,$$

when $\sigma > 1/2$.

2. Auxliary statements.

Lemma 1. *Let a series of analytical functions*

$$\sum_{n=1}^{\infty} f_n(s)$$

be given in one-connected domain of a complex s*-plane* G *and be convergent absolutely almost everywhere in* G *in Lebesgue sense, and the function*

$$\Phi(\sigma,t)=\sum_{n=1}^{\infty}|\ f_n(s)\,|$$

is a summable function in G. Then the given series converges uniformly in any compact subdomain of G; in particular, the sum of this series is an analytical function in G.

Proof. It is enough to show that the theorem is true for any rectangular domain of the region G. Let C be a rectangle in G and C' be another rectangle inside C, moreover, their sides are parallel to co-ordinate axes. We can assume that on the contour of these rectangles given series converges almost everywhere according to the theorem of Fubini (see [7, p. 208]). Let $\Phi_0(s)=\Phi_0(\sigma,t)$ be the sum of the given series in the points of convergence. Under the theorem of Lebesgue on a bounded convergence (see [21, p. 293]):

$$(2\pi i)^{-1}\int_C \frac{\Phi_0(s)}{s-\xi}ds=\sum_{n=1}^{\infty}(2\pi i)^{-1}\int_C \frac{f_n(s)}{s-\xi}ds,$$

where the integrals are taken in the Lebesgue sense. As on the right part of the last equality integrals exist in the Riemann sense also, then, applying Cauchy formula, we receive

$$\Phi_1(\xi)=(2\pi i)^{-1}\int_C \frac{\Phi_0(s)}{s-\xi}ds=\sum_{n=1}^{\infty}f_n(s),$$

where $\Phi_1(\xi)=\Phi_0(\xi)$ almost everywhere and ξ is any point on or in a contour. Further, denoting by δ the minimal distance between the sides of C and C', we have

$$|\ f_n(\xi)\,|\le(2\pi)^{-1}\int_C \frac{|\ f_n(s)\,|}{|\ s-\xi\ |}|\ ds\,|\le(2\pi\delta)^{-1}\int_C|\ f_n(s)\,||\ ds\,|.$$

The series

$$\sum_{n=1}^{\infty}\int_C|\ f_n(s)\,||\ ds\,|$$

converges, in the consent with the Lebesgue theorem on monotonous convergence (see [21, p. 290]). Hence, the given series converges $\sum_{n=1}^{\infty}f_n(\xi)$ uniformly inside of C'. The lemma 2 is proved.

Let's enter now the notion of Hardy space.

Definition 2. *The set $H_2^{(R)}, R>0$, of functions $f(s)$ defined for $|s|<R$ and being analytical in this circle, is called a Hardy space if for any $f(s)\in H_2^{(R)}$ the following relationship holds*

$$\|f\|^2=\lim_{r\to R}\iint_{|s|<r}|f(s)|^2 d\sigma dt<\infty; s=\sigma+it.$$

It is obviously that the Hardy space is a linear space in which is possible to enter a scalar pro duct of functions by means of an equality

$$\left(f(s), g(s)\right) = \mathrm{Re}\iint_{|s|\le R} f(s)\overline{g(s)}d\sigma dt. \quad (3)$$

Considering $H_2^{(R)}$ as a linear space over the field of real numbers and using the entered scalar product, we transform this space into real Hilbert space.

Lemma 2. *The Hardy space $H_2^{(R)}$ with the entered scalar product (3) is a real Hilbert space.*

Proof. It is enough to prove, that any fundamental sequence $\left(f_n(s)\right)_{n\ge 1}$ converges to some analytical function $f(s) \in H_2^{(R)}$. As the sequence is fundamental, there will be found such a sequence of natural numbers $\left(n_k\right)_{k\ge 0}$ that for any natural k

$$\left\| f_{n_k} - f_{n_{k-1}} \right\| \le 2^{-k}.$$

Let's consider a series of analytical functions

$$f_{n_0} + \sum_{k=1}^{\infty}\left(f_{n_k} - f_{n_{k-1}}\right).$$

We shall prove that it converges uniformly in any closed circle contained in the circle $s < R$. Then, denoting $g(s) = \sum_{k=1}^{\infty}\left|f_{n_k}(s) - f_{n_{k-1}}(s)\right|$, we can write ($0 < r < R$):

$$\iint_{|s|<r} g(s)d\sigma dt \le \sum_{k=1}^{\infty}\left(\pi R^2 \lim_{r\to R}\iint_{|s|<r}\left|f_{n_k} - f_{n_{k-1}}\right|^2 d\sigma dt\right)^{1/2} \le \sqrt{\pi}R\sum_{k=1}^{\infty}2^{-k} < +\infty.$$

Hence, the function $g(s)$ is a summable function of variables σ, t, in the circle $s < R$, and the lemma 1 is applicable. Applying the lemma 1, we see that the series $f_{n_0} + \sum_{k=1}^{\infty}\left(f_{n_k} - f_{n_{k-1}}\right)$ converges uniformly in any circle $|s| \le r < R$. Then, the subsequence $\left(f_{n_k}(s)\right)_{k\ge 1}$ converges to some analytical function $\varphi(s)$. As the sequence is fundamental, for any $\varepsilon > 0$ there exist such n_0 that for any natural $m > n_0$ the inequality

$$\iint_{|s|<R}\left|\varphi(s) - f_m(s)\right|^2 d\sigma dt < \varepsilon$$

holds. Let $r < R$ be any real number. Then using the relationship of [23, p. 345], one receives

$$r^2\left|\varphi(s) - f_m(s)\right|^2 \le \pi^{-1}\iint_{|s|<R}\left|\varphi(s) - f_m(s)\right|^2 d\sigma dt < \varepsilon / \pi,$$

for any $s, |s| \le r$. As ε is arbitrary, the convergence of $\left(f_{n_k}(s)\right)_{k\ge 1}$ follows from here. Hence, the considered space is complete. The lemma 2 is proved.

The following lemma due to S. M.Voronin ([28]) (we bring the result in a little modified form).

Lemma 3. *Let $g(s)$ be an analytical function in the circle $|s|< r <1/4$ which is continuous and non vanishing in the closed circle $|s|\le r$. Then for any $\varepsilon>0$ and $y>2$ it is possible to find the finite set of prime numbers M, containing all of primes $p\le y$, and an element $\overline{\theta}=(\theta_p)_{p\in M}$ such that:*

1) $0\le\theta_p\le 1$ *for* $p\in M$;

2) $\theta_p=\theta_p^0$ *is set beforehand when* $p\le y$;

3) $\max_{|s|\le r}|g(s)-\zeta_M(s+3/4;\overline{\theta})|\le\varepsilon;$ *here* $\zeta_M(s+3/4;\overline{\theta})$ *is defined by the equality*

$$\zeta_M(s+3/4;\overline{\theta})=\prod_{p\in M}\left(1-e^{2\pi i\theta_p}p^{-s-3/4}\right)^{-1}.$$

Proof. We shall prove the lemma 3 by following S. M.Voronin's work [28]. As $g(s)$ is an analytical in the circle $|s|\le r$, we shall consider an auxiliary function $g(s/\gamma^2)$ $(\gamma>1,\gamma^2 r<1/4)$ which, for any $\varepsilon>0$, satisfies the inequality $\max\limits_{|s|\le r}\left|g(s)-g(s/\gamma^2)\right|<\varepsilon$ if γ is set by a suitable way. Therefore, it is enough to prove the statement of the lemma for function $g(s/\gamma^2)$ in the circle $|s|\le r$. An advantage consisted in that that the function $g(s/\gamma^2)$ belongs to the space $H_2^{(\gamma r)}$ (a circle has a radius greater than r which is important for the subsequent reasoning). Not breaking, therefore, the generality, we believe that the function $g(s)$ is an analytical in the circle $|s|\le r\gamma^2$, and we shall consider the space $H_2^{(\gamma r)}$.

The function $\log g(s)$, in the conditions of the theorem, has no singularities in the circle $|s|\le r\gamma$. Therefore, it is enough to prove existence of a such element $\overline{\theta}$, satisfying conditions of the lemma 3, that

$$\max_{|s|\le r}|\log g(s)-\log\zeta_M(s+3/4;\overline{\theta})|\le\varepsilon.$$

Let's put

$$u_k(s)=\log(1-e^{-2\pi i\theta_k}p_k^{-s-3/4}),$$

taking for the logarithm a principal brunch. Using expansion of logarithmic function into the power series, we can write

$$u_k(s)=-e^{-2\pi i\theta_k}p_k^{-s-3/4}+\nu(s),$$

where

$$\left|\nu(s)\right| \le \left|(1/2)e^{-4\pi i\theta_k} p_k^{-2s-3/2} + \cdots\right| = O(p_k^{2r-3/2}).$$

As $r < 1/4$, we can find $\delta > 0$ such that $2\delta + 2r - 3/2 < -1$. Then the definition of the function $u_k(s)$, together with the last inequality, shows that the series

$$\sum_{k=n+1}^{\infty} \eta_k(s); \eta_k(s) = -e^{-2\pi i\theta_k} p_k^{-s-3/4}; n = \pi(y), \quad (5)$$

differs from the series $\sum u_k(s)$ by an absolutely convergant series. At first, we shall prove that at an appropriate $\overline{\theta}$ for any function $\varphi(s) \in H_2^{(\gamma r)}$ of Hardy space there will be found some permutation of the series $\sum \eta_k(s)$, converging in the sence of the norm of the space, to the function $\varphi(s)$. The uniform convergence of this series, in the circle $|s| \le r$, to the function $\varphi(s)$, would follow from this according to the lemma 1. In particular, taking

$$\varphi(s) = \log g(s) - \sum_{k>n} (u_k(s) - \eta_k(s)) - \sum_{k \le n} u_k(s)$$

and considering the last remark we would find some permutation of the series $\sum_{k>n} \eta_k(s)$, converging to the $\varphi(s)$. Since, the corresponding permutation of the series $\sum_{k>n} (u_k(s) - \eta_k(s))$ converges to the previous its sum uniformly, then for any ε there will be found such a set of indexes M that

$$\max_{|s| \le r} \left| \varphi(s) - \sum_{k \in M, \log p_k > y} \eta_k(s) \right| \le \varepsilon / 2.$$

Let $q(s) = \sum_{k=n+1}^{\infty} (u_k(s) - \eta_k(s))$. As this series converges absolutely, the mentioned set M is possible to set by a such way that the following inequality was carried out

$$\left| q(s) - \sum_{k \in M, k > n} (u_k(s) - \eta_k(s)) \right| \le \varepsilon / 2.$$

Then we shall receive:

$$\left| \varphi(s) - \sum_{k \in M, \log p_k > y} \eta_k(s) \right| = \left| \log g(s) - \sum_{n \in M} u_n(s) \right| \le \varepsilon,$$

and, thereby, the proof of the lemma 3 will be finished.

Let's consider the series (4) and apply the theorem 1, §6 of Appendix of [24]. For this purpose, we shall prove feasibility of conditions of this theorem at the $\overline{\theta}$ chosen by a suitable way.

At first taking $R = \gamma r$, we shall consider the space $H_2^{(R)}$. We have:

$$\|\eta_k(s)\|^2 = \iint_{|s|\le R} \left| e^{-2\pi i\theta_k} p_k^{-s-3/4} \right|^2 d\sigma dt \le \pi R^2 p_k^{2r-3/2}.$$

Hence,

$$\sum_{k=1}^{\infty} \|\eta_k(s)\|^2 \le \pi R^2 \sum_{k=1}^{\infty} p_k^{2r-3/2} < +\infty,$$

i.e. the first condition of the theorem 1, mentioned above, is satisfied.

Let now $\varphi(s) \in H_2^{(R)}$ is any element with a condition $\|\varphi(s)\|^2 = 1$. Let $\varphi(s)$ has the following expansion into a power series in the circle $|s| \le R$:

$$\varphi(s) = \sum_{n=0}^{\infty} \alpha_n s^n.$$

Then,

$$1 = \iint_{|s|\le R} \left| \sum_{n=0}^{\infty} \alpha_n s^n \right|^2 d\sigma dt.$$

For exchanging of variables under the integral we put $\sigma = r\cos\varphi, t = r\sin\varphi, r \le R\ 0 \le \varphi < 2\pi$. Then,

$$1 = \sum_{n=0}^{\infty}\sum_{n=0}^{\infty} \alpha_n \overline{\alpha}_m \int_0^R r^{n+m+1} \int_0^{2\pi} (\cos 2\pi(n-m)\varphi + i\sin 2\pi(n-m)\varphi) d\varphi.$$

The interior integral is equal to 0 when $m = n$, and to 2π , otherwise. Hence,

$$\pi \sum_{n=0}^{\infty} |\alpha_n|^2 R^{2n+2} (n+1)^{-1} = 1. \quad (5)$$

Let's prove now, that there is a point $\overline{\theta}$, not dependent on the function $\varphi(s)$, such that the series $\sum_{k=1}^{\infty} (\eta_k(s), \varphi(s))$ converges after of some permutation of its members. We have

$$(\eta_k(s), \varphi(s)) = -Re \iint_{|s|\le R} e^{-2\pi i\theta_k} p_k^{-s-3/4} \overline{\varphi(s)} d\sigma dt = Re[-e^{-2\pi i\theta_k} \Delta(\log p_k)],$$

where

$$\Delta(x) = \iint_{|s|\le R} e^{-x(s+3/4)} \overline{\varphi(s)} d\sigma dt.$$

It is possible to present $\Delta(x)$ in a following form:

$$\Delta(x) = e^{-3x/4} \iint_{|s|\le R} \left(\sum_{n=0}^{\infty} (-sx)^n / n! \right) \left(\overline{\sum_{n=0}^{\infty} \alpha_n s^n} \right) d\sigma dt =$$

$$= \pi R^2 e^{-3x/4} \sum_{n=0}^{\infty} (-1)^n \overline{\alpha}_n x^n R^{2n} / (n+1)! = \pi R^2 e^{-3x/4} \sum_{n=0}^{\infty} \beta_n (xR)^n / n!,$$

by denoting $\beta_n = (-1)^n R^n \overline{\alpha}_n / (n+1)$. From (5) it is concluded:

$$\sum_{n=1}^{\infty} |\beta_n|^2 \leq 1.$$

Hence, $|\beta_n| \leq 1$ and, therefore, the function

$$F(u) = \sum_{m=0}^{\infty} \frac{\beta_m}{m!} u^m \quad (6)$$

will be an entire function. Thus,

$$\Delta(x) = \pi R^2 e^{-3x/4} F(xR).$$

Let's prove, that for any $\delta > 0$ there will be found a sequence $u_1, u_2, \ldots$ tending to infinity, and satisfying an inequality

$$|F(u_j)| > ce^{-(1+2\delta)u_j}. \quad (7)$$

Let's admit, for this purpose, an opposite by letting an existence a positive number $\delta < 1$ such that at an enough large $A > 0$ the following inequality

$$|F(u)| \leq Ae^{-(1+2\delta)u}.$$

holds for all $u \geq 0$; in this case we have:

$$\left|e^{(1+\delta)u} F(u)\right| \leq Ae^{-\delta|u|}\ ; u \geq 0.$$

From proved above we get:

$$|F(u)| \leq \sum_{n=0}^{\infty} |u|^n / n! = e^{-u}.$$

From this follows

$$\left|e^{(1+\delta)u} F(u)\right| \leq e^{\delta u} \leq e^{-\delta|u|}.$$

From the received estimations we conclude an existence of an integral

$$\int_{-\infty}^{\infty} \left|e^{(1+\delta)u} F(u)\right|^2 du.$$

As the function (6) is an entire function of exponential type, the function $e^{(1+\delta)u}F(u)$, also, will be so and the last belongs to the class E^{σ} with $\sigma < 3$ (see [4, p. 408]). Then under the theorem Paley and Wiener (see in the same place) will be found a finitary function $f(\xi) \in L_2(-3,3)$ such that

$$e^{(1+\delta)u} F(u) = \int_{-3}^{3} f(\xi) e^{iu\xi} d\xi.$$

Taking the convers Fourier transformation, we find:

$$f(\xi)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(e^{(1+\delta)u}F(u)\right)e^{-iu\xi}du.$$

From found above estimations it follows that this integral converges absolutely and uniformly in the strip $|\operatorname{Im}\xi|<\delta/2$ and, consequently, represents an analytical, in this strip, function. This contradicts finitaryness of $f(\xi)$. The received contradiction proves an existence of a sequence of points with the condition (7).

Denoting $x_j=u_j/R$, on the basis of (7) we can assert that

$$|\Delta(x_j)|>ce^{-3x_j/4}|F(x_jR)|\geq ce^{-3x_j/4}e^{-(1+2\delta)x_jR}=ce^{-x_j(R+2\delta R+3/4)}.$$

If $\delta>0$ is small enough then $R+2\delta R+3/4<1$. Hence, there exists $\delta_0>0$ such that

$$|\Delta(x_j)|>e^{-(1-\delta_0)x_j}. \quad (8)$$

Let's consider the function $\Delta(x)$ on the segment $[x_j-1,x_j+1]$. Following by [28], we put $N=\lfloor x_j\rfloor+1$. From an estimations for the coefficients β_n we get:

$$\left|\sum_{n=N^2+1}^{\infty}\frac{\beta_n}{n!}(xR)^n\right|\leq\sum_{n=N^2+1}^{\infty}\frac{(xR)^n}{n!}\leq\frac{(xR)^{N^2}}{(N^2)!}\sum_{n=0}^{\infty}\frac{(xR)^n}{n!}\leq\frac{(xR)^{N^2}}{(N^2)!}e^N,$$

since for integers $n,m\geq 0$ we have $(n+m)!=n!(n+1)\cdots(n+m)\geq n!m!$. If the natural number m is great enough then one has from the Stirling's formula:

$$m!=\Gamma(m+1)\geq e^{m\log m-m}=(m/e)^m.$$

Hence,

$$\left|\sum_{n=N^2+1}^{\infty}\frac{\beta_n}{n!}(xR)^n\right|\leq\frac{(xR)^{N^2}}{(N^2)!}e^N\leq N^{N^2}\left(\frac{N^2}{e}\right)^{N^2}e^N<<e^{-2x_j},$$

at $x\in[x_j-1,x_j+1]$. Further, $\sum_{n=0}^{N^2}\beta_n(xR)^n/n!<<e^{xR}$. Analogically,

$$\left|\sum_{n=N^2+1}^{\infty}\frac{(-3x/4)^n}{n!}\right|\leq\frac{(3x/4)^{N^2}}{(N^2)!}\sum_{n=0}^{\infty}\frac{(3x/4)^n}{n!}\leq\frac{(3x/4)^{N^2}}{(N^2)!}e^N<<e^{-2x_j}$$

and $\sum_{n=0}^{N^2}(-3x/4)^n/n!<<e^{3x/4}$ when $x\in[x_j-1,x_j+1]$. Thus,

$$\Delta(x)=\pi R^2\sum_{n=0}^{N^2}\frac{(-3x/4)^n}{n!}\sum_{n=0}^{N^2}\frac{\beta_n}{n!}(xR)^n+O(e^{-x_j})=\sum_{n=0}^{N^4}a_nx^n+O(e^{-x_j}).$$

According to (8), we receive the inequality

$$\max_{|x-x_j|\le 1} |\Delta(x)| > e^{-(1-\delta_0)x_j},$$

for any $j=1,2,\ldots$. Let $a_n = b_n + ic_n, b_n, c_n \in R$. Then,

$$\Delta(x) = \sum_{n=0}^{N^4} b_n x^n + i\sum_{n=0}^{N^4} c_n x^n + O(e^{x_j}),$$

therefore, for every j, at least, one of the following inequalities is executed:

$$\max_{|x-x_j|\le 1} \left|\sum_{n=0}^{N^4} b_n x^n\right| > 0.1e^{-(1-\delta_0)x_j},$$

or,

$$\max_{|x-x_j|\le 1} \left|\sum_{n=0}^{N^4} c_n x^n\right| > 0.1e^{-(1-\delta_0)x_j}.$$

Let's consider at first the first possibility. Let x_0 be a point where the maximum of modulus is reached. We shall denote by τ_j some interval, laying in the interval $[x_j-1, x_j+1]$, and containing such a point x_0 at which the mentioned above first possibility occurs. Let, for the definiteness, $g(x_0) < 0; g(x) = \sum_{n=0}^{N^4} b_n x^n$. If $\tau_j \ne [x_j-1, x_j+1]$ (the case of coincidence of intervals is trivial) then there will be found a point $x_1 \in \tau_j$ for which

$$|g(x_1)| \le 0.1|g(x_0)|.$$

Now we have:

$$|g(x_0) - g(x_1)| \ge 0.5|g(x_0)|.$$

Under the theorem of Lagrange there exist a point $y_j \in \tau_j$ such that

$$|g'(y_j)(x_1 - x_0)| \ge 0.5|g(x_0)|.$$

Applying the theorem 9, §2, [24], we find:

$$N^8|g(x_0)||x_1 - x_0| \ge |g'(y_j)(x_1 - x_0)| \ge 0.5|g(x_0)|.$$

So, the interval τ_j has a length not less than $0.5x_j^{-8}$. For definiteness, we shall put $\tau_j = [\alpha, \alpha+\beta]$. Under C.J. Valle-Poisson theorem the interval τ_j contains, at least

$$\int_{e^\alpha}^{e^{\alpha+\beta}} \frac{dx}{\log x} + O(e^{\alpha+\beta}e^{c\sqrt{\alpha}}) = \int_{\alpha}^{\alpha+\beta} \frac{e^u}{u}du + O(e^{\alpha+\beta}e^{c\sqrt{\alpha}}) \ge \frac{e^\alpha}{\alpha}\left[\left(e^\beta - 1\right) + O\left(\frac{e^\beta}{e^{c\sqrt{\alpha}}}\right)\right] >> \frac{\beta e^\alpha}{\alpha}$$

prime numbers. Selecting the primes p_k for which $p_k > y, k \equiv 0 (\mathrm{mod}\, 4)$, we put $\theta_k = 0$. In the case when $g(x_0) > 0$, we take the primes $p_k > y, k \equiv 2 (\mathrm{mod}\, 4)$ and put $\theta_k = 1/2$. Then,

$$\sum_{\log p_k \in \tau_j, k \equiv 0 (\mathrm{mod}\, 4)} (\eta_k(s), \varphi(s)) = \sum_{\log p_k \in \tau_j, k \equiv 0 (\mathrm{mod}\, 4)} Re[-e^{-2\pi i \theta_k} \Delta(\log p_k)] >> e^{x_j} e^{-(1-\delta_0) x_j} x_j^{-8} >> e^{\delta_0 x_j / 2}.$$

Further, at the second possibility, i.e. when the inequality

$$\max_{|x - x_j| \le 1} \left| \sum_{n=0}^{N^4} c_n x^n \right| > 0.1 e^{-(1-\delta_0) x_j}$$

is executed, we select the primes p_k, with $k \equiv 1 (\mathrm{mod}\, 4)$, taking $\theta_k = 1/4$, if the maximal value of a modulus of the polynomial in the point x_0 is negative; otherwise we take k with $k \equiv 3 (\mathrm{mod}\, 4)$ and put $\theta_k = 3/4$.

Thus, there exist an infinite set of indexes with the condition

$$\sum_{\log p_k \in \tau_j, k \equiv 0 \vee 2 (\mathrm{mod}\, 4)} (\eta_k(s), \varphi(s)) >> e^{\delta_0 x_j / 2},$$

and an infinite set of other values of j for which

$$- \sum_{\log p_k \in \tau_j, k \equiv 1 \vee 3 (\mathrm{mod}\, 4)} (\eta_k(s), \varphi(s)) >> e^{\delta_0 x_j / 2}.$$

From proved above estimations we conclude that

$$| \Delta(x) | \le \pi R^2 e^{-x/2}.$$

Further, we have $|(\eta_k(s), \varphi(s))| \to 0$ when $k \to \infty$. Hence, the series

$$\sum_{n=1}^{\infty} (\eta_k(s), \varphi(s))$$

contains subseries, having not common components and being divergent, accordingly, to $+\infty$ and to $-\infty$. Then, some permutation of the series

$$\sum_{n=1}^{\infty} (\eta_k(s), \varphi(s))$$

converges conditionally. Therefore, on the theorem 1, §6, [24], there is a permutation of the series $\sum_{p_n > y} u_n(s)$ converging uniformly to the function $\varphi(s) - \sum_{p_n \le y} u_n(s)$. Taking a long enough partial sum, we receive the necessary result. The lemma 2 is proved.

3. The basic auxiliary result.

Let, $\omega \in \Omega$, $\Sigma(\omega) = \{\sigma\omega \mid \sigma \in \Sigma\}$ and $\Sigma'(\omega)$ means the closed set of all limit points of the sequence $\Sigma(\omega)$. For real t we denote $\{t\Lambda\} = (\{t\lambda_n\})$ where $\Lambda = (\lambda_n)$. Let μ to denote the product of linear Lebesgue measures m given on the interval $[0,1]$: $\mu = m \times m \times \cdots$.

Lemma 3. *Let $A \subset \Omega$ be a finite-symmetric subset of a measure of zero and $\Lambda = (\lambda_n)$ is an unbounded, monotonically increasing sequence of positive real numbers any finite subfamily of elements of which is linearly independent over the field of rational numbers. Let $B \supset A$ be any open subset with $\mu(B) < \varepsilon$,*

$$E_0 = \{0 \le t \le 1 \mid \{t\Lambda\} \in A \wedge \Sigma'\{t\Lambda\} \subset B\}.$$

Then, we have $m(E_0) \le 6c\varepsilon$ where c is an absolute constant, m designates the Lebesgue measure.

Proof. Let ε be any small positive number. As numbers λ_n are linearly independent, then for any finite permutation σ we have $(\{t_1\lambda_n\}) \ne (\{t_2\lambda_{\sigma(n)}\})$ when $t_1 \ne t_2$. Really, otherwise we would receive equality $\{t_1\lambda_s\} = \{t_2\lambda_s\}$ for enough large natural s, i. e. $(t_1 - t_2)\lambda_s = k, k \in Z$. Writing down the same equality for some other natural $r > m$ we have the relationship

$$k_1/\lambda_r - k/\lambda_s = \frac{k_1\lambda_s - k\lambda_r}{\lambda_r\lambda_s} = 0$$

which contradicts the linear independence of numbers λ_n. Hence, for any pair of various numbers t_1 and t_2 one has $(\{t_1\lambda_n\}) \notin \{(\{t_2\lambda_{\sigma(n)}\}) \mid \sigma \in \Sigma\}$. On the lemma's condition there exist a family of open spheres $B_1, B_2, ...$ (in Tikhonov's topology) such that each sphere does not contain any other sphere from this family (the sphere, containing in other one, can be deleted), thus

$$A \subset B \subset \bigcup_{j=1}^{\infty} B_j, \sum \mu(B_j) < 1.5\varepsilon.$$

Now we take some finite permutation $\sigma \in \Sigma$ defined by equalities $\sigma(1) = n_1,\ ..., \sigma(k) = n_k$ where natural numbers are picked up as follows. At first we take such N that

$$\mu(B'_N) < 2\varepsilon_1,$$

where B'_N is a projection of the sphere B_1 into the subspace of first N co-ordinate axes and $\mu(B_1) = \varepsilon_1$. We shall cover B'_N with cubes with an edge of δ and with a total measure not exceeding $3\varepsilon_1$. We put $k = N$ and define the numbers $n_1, ..., n_k$, using following inequalities

$$\lambda_{n_1} > 1, \lambda_{n_2}^{-1} < (1/4)\delta\lambda_{n_1}^{-1}, \lambda_{n_3}^{-1} < (1/4)\delta\lambda_{n_2}^{-1}, ..., \lambda_{n_k}^{-1} < (1/4)\delta\lambda_{n_{k-1}}^{-1}, \delta < 1. \quad (9)$$

Now we take any cube with an edge of δ and with the centre in some point $(\alpha_m)_{1 \le m \le k}$. Then, the point $(\{t\lambda_{n_m}\})$ will belong to this cube if

$$| \{t\lambda_{n_m}\} - \alpha_m | \le \frac{\delta}{2}.$$

From definition of a fractional part at $m = 1$ for some whole r one has:

$$\frac{r + \alpha_1 - \delta/2}{\lambda_{n_1}} \le t \le \frac{r + \alpha_1 + \delta/2}{\lambda_{n_1}}. \quad (10)$$

The measure of a set of the such t does not exceed the value $\delta\lambda_{n_1}^{-1}$. The number of such intervals corresponding to different values of $r = [t\lambda_{n_1}] + 1 \le \lambda_{n_1} + 1$ does not exceed

$$[\lambda_{n_1}] + 2 \le \lambda_{n_1} + 2.$$

The total measure of corresponding intervals is

$$\le (\lambda_{n_1} + 2)\delta\lambda_{n_1}^{-1} \le (1 + 2\lambda_{n_1}^{-1})\delta.$$

Now we consider one of intervals (10); taking $m = 2$, we shall have

$$\frac{s + \alpha_2 - \delta/2}{\lambda_{n_2}} \le t \le \frac{s + \alpha_2 + \delta/2}{\lambda_{n_2}} \quad (11)$$

with $s = [t\lambda_{n_2}] + 1 \le \lambda_{n_2} + 1$. As we consider conditions (10) and (11) simultaneously, we must estimate a total measure of intervals (11) which have nonempty intersections with intervals of a kind (10), using conditions (9). The number of intervals of a kind (11), with the length $\lambda_{n_2}^{-1}$, having with one interval of a kind (10) nonempty intersection, does not exceed the value

$$[\delta\lambda_{n_1}^{-1}\lambda_{n_2}] + 2 \le \delta\lambda_{n_1}^{-1}\lambda_{n_2} + 2.$$

Then the measure of a set of values, t for which the conditions (10) and (11) are satisfied simultaneously, does not exceed

$$(\lambda_{n_1} + 2)(2 + \delta\lambda_{n_1}^{-1}\lambda_{n_2})\delta\lambda_{n_2}^{-1}.$$

It is possible to continue these reasoning considering all conditions of a kind

$$\frac{l + \alpha - \delta/2}{\lambda_{n_m}} \le t \le \frac{l + \alpha + \delta/2}{\lambda_{n_m}}, m = 1, ..., k.$$

Then we find the following estimation for a measure of a set $m(\delta)$ of such t for which the points $(\{t\lambda_{n_m}\})$ contained by a cub with an edge δ:

$$m(\delta) \le (2 + \lambda_{n_1})(2 + \delta\lambda_{n_1}^{-1}\lambda_{n_2}) \ldots (2 + \delta\lambda_{n_{k-1}}^{-1}\lambda_{n_k}).$$

Making simple transformations, we find, using the conditions (9):

$$m(\delta) \le (2+\lambda_{n_1})(2+\delta\lambda_{n_1}^{-1}\lambda_{n_2})\cdots(2+\delta\lambda_{n_{k-1}}^{-1}\lambda_{n_k})\delta\lambda_{n_k}^{-1} \le \delta^k \prod_{m=1}^{\infty}(1+2m^{-2})$$

Summing over all such cubes, for a measure of a set of such t for which $(\{t\lambda_{n_m}\}) \in B_1$, we receive the final estimation of a kind $\le 3c\varepsilon, c > 0$.

We must notice that the sequence $\Lambda = (\lambda_n)$, defined above, depends on δ. We for each sphere B_k shall fix some sequence Λ_k, using the conditions (9). Considering all such spheres, we denote $\Sigma_0 = \{\Lambda_k \mid k = 1,2,...\}$. As the set A is a finite-symmetrical, a measure of a set of values of t, interesting us, is possible to estimate by using of any sequence Λ_k because, as it has been shown above, the sets $\Sigma(\{t\Lambda\})$ for different values of t have empty intersections.

Let's prove that for any point $t \in E_0$ the set $\Sigma(\{t\Lambda\})$ contained in the union $\bigcup_{k\le n} B_k$ for some n. Really, let at some $t \in E_0$ all members of the sequence $\Sigma(\{t\Lambda\})$ does not contained in the union $\bigcup_{k\le n} B_k$, for any natural n. Two cases are possible: 1) there will be a point $\overline{\theta} \in \Sigma(\{t\Lambda\})$ belonging to infinite number of spheres B_k; 2) there will be a sequence of elements $\overline{\theta}_j, \overline{\theta}_j \in \Sigma(\{t\Lambda\})$ which does not contained in any finite union of spheres B_k. We shall consider both possibilities separately and shall prove that they lead to the contradiction.

1) Let $\overline{\theta} \in B_{k_1}, B_{k_2}, B_{k_3}, ...$ are all spheres to which the element $\overline{\theta}$ belongs. We shall denote d the distance from $\overline{\theta}$ to the bound of B_{k_1}. As B_{k_1} is open set, then $d > 0$. Let B_k be any sphere of radius $< d/2$ from the list above, containing the point $\overline{\theta}$. From the told it follows that the sphere B_k should contained in the sphere B_{k_1}. But it contradicts the agreement accepted above.

2) Let $\overline{\theta}$ be some limit point of the sequence $(\overline{\theta}_j)$. According to the condition of the lemma 3 $\overline{\theta} \in B_s$ for some s. Let d denotes the distance from $\overline{\theta}$ to the bound of B_s. As $\overline{\theta}$ is a limit point, then a sphere with the centre in the point $\overline{\theta}$ and radius $d/4$ contains an infinite set of members of the sequence $(\overline{\theta}_j)$, say members $\overline{\theta}_{j_1}, \overline{\theta}_{j_2}, ...$. According to 1), each point of this sequence can belong only to finite number of spheres. So the specified sequence will be contained in a union of infinite subfamily of spheres B_k. Among them will be found infinitely many number of spheres having radius $< d/4$. All of them, then, should contained in the sphere B_s. The received contradiction excludes the case 2) also.

So, for any $t \in E_0$ it will be found such n for which $\Sigma(\{t\Lambda\}) \subset \bigcup_{k\le n} B_k$. From here it follows that the set E_0 can be represented as a union of subsets $E_k, k = 1,2,\ldots$, where

$$E_k = \{t \in E_0 \mid \Sigma(t\Lambda) \subset \bigcup_{s\le k} B_s\}.$$

Then,

$$E_0 = \bigcup_{k=1}^{\infty} E_k;\ \ E_k \subset E_{k+1} (k \ge 1).$$

Further, $m(E_0) = \lim_{k\to\infty} m(E_k)$, in agree with [23, p. 368]. As it has been noted, $m(E_k)$ is possible to estimate using any sequence $\Lambda' \in \Sigma_0$:

$$m(E_k) \le \limsup_{\Lambda'\in\Sigma_0} m(E_k(\Lambda')),$$

where $E_k(\Lambda') = \{t \in E_k \mid (\{t\Lambda'\}) \in \bigcup_{s\le k} B_s\}$. Hence,

$$m(E_k(\Lambda')) \le \sum_{s\le k} m(E^{(s)}(\Lambda')),$$

where $E^{(k)}(\Lambda') = \{t \in E_0 \mid (\{t\Lambda'\}) \in B_k\}$. Applying the lemma 3, we find (by choosing suitable Λ'):

$$m(E(\Lambda')) \le 6c(\varepsilon_1 + \cdots + \varepsilon_k).$$

Passing to the limit, as $k \to \infty$, we receive a demanded result. The proof of the lemma 3 is finished.

4. Local approximation.

Lemma 4. *There is a sequence of points* $(\overline{\theta}_k)$ $(\overline{\theta}_k \in \Omega)$ *and natural numbers* (m_k) *such that*

$$\lim_{k\to\infty} F_k(s + 3/4, \overline{\theta}_k) = \zeta(s + 3/4)$$

as $\overline{\theta}_k \to 0$ *in the circle* $|s| \le r, 0 < r < 1/4$ *uniformly by* s .

Proof. Let $y > 2$ to denote a positive integer which more precisely will be defined below. We put

$$y_0 = y, y_1 = 2y_0, \ldots, y_m = 2y_{m-1} = 2^m y_0, \ldots.$$

From the lemma 1 it follows that for a given positive number ε and $y > 2$ there will be found a set M_1 of primes and a point $\overline{\theta}_1 = (\theta_p^0)_{p\in M_1}$ such that M_1 contains all primes $p \le y$ with $\theta_p^0 = 0$ and

$$\max_{|s|\le r} |\zeta(s + 3/4) - \eta_1(s + 3/4)| \le \varepsilon; \eta_1(s + 3/4) = \prod_{p\in M_1} \left(1 - e^{2\pi i\theta_p^0} p^{-s-3/4}\right)^{-1}.$$

Now, denoting $m_1 = \max_{m\in M_1} m$, we put

$$F_1(s+3/4;\overline{\theta}) = \prod_{p\le m_1}\left(1-e^{2\pi i\theta_p^0}p^{-s-3/4}\right)^{-1}$$

and

$$h_1(s+3/4;\overline{\theta}) = F_1(s+3/4;\overline{\theta})\prod_{p\in M_1}\left(1-e^{2\pi i\theta_p^0}p^{-s-3/4}\right)-1;$$

here $\theta_p = \theta_p^0$ for $p \in M_1$. Let n to denote the natural number which cononical factorization contains only primes p, $p \overline{\in} M_1, p \le m_1$ and

$$a_n(\overline{\theta}) = e^{2\pi i\sum_{p\backslash n}\alpha_p\theta_p}; n = \prod p^{\alpha_p}.$$

If $r+\delta < 1/4$ we have

$$\int_{\Omega_1}\left(\iint_{|s|\le r+\delta} |h_1(s+3/4;\overline{\theta})|^2\, d\sigma dt\right)d\overline{\theta} \le$$

$$\le \iint_{|s|\le r+\delta}\left(\int_{\Omega_1} |h_1(s+3/4;\overline{\theta})|^2\, d\overline{\theta}\right)d\sigma dt \le$$

$$\le \pi(r+\delta)^2 \max_{|s|\le r+\delta}\int_{\Omega_1} |\sum_{n>y} a_n(\overline{\theta})n^{-s+3/4}|^2\, d\overline{\theta} \le \frac{4\pi(r+\delta)^2}{1-4r-4\delta}y^{-1/2+2r+2\delta};$$

here Ω_1 means a projection of Ω into the subspace of co-ordinate axes $\theta_p, p \overline{\in} M_1$. Then from the inequality received above follows an existence of a point $\overline{\theta_1'} = (\theta_p)_{p\overline{\in} M_1}$ such that

$$\iint_{|s|\le r+\delta} |h_1(s+3/4;\overline{\theta_1'})|^2\, d\sigma dt \le \frac{4\pi(r+\delta)^2}{1-4r-4\delta}y^{-1/2+2r+2\delta},$$

or

$$\max_{|s|\le r} |h_1(s+3/4;\overline{\theta_1'})| \le \sqrt{2}\delta^{-1}\left(\frac{1}{2\pi}\iint_{|s|\le r} |h_1(s+3/4;\overline{\theta_1'})|^2\, d\sigma dt\right)^{1/2} \le c(\delta)y^{\delta+r-1/4}$$

(see [22 p. 345]) where $c(\delta) > 0$ is a constant. Hence, taking $\overline{\theta}_0 = (\theta_p^0)_{p\in M_1}$ and $\overline{\theta}_1 = (\overline{\theta}_0, \overline{\theta_1'})$ define $y = y_0$, satisfying the condition

$$(A+1)c(\delta)y_0^{r+\delta-1/4} \le \varepsilon; A = \max_{|s|\le r} |\zeta(3/4+s)|.$$

Then we have

$$\max_{|s|\le r}\left\{|\zeta(3/4+s) - F_1(3/4+s;\overline{\theta}_1)|\right\} \le$$

$$\le \max_{|s|\le r}\left\{|\zeta(3/4+s) - \eta_1(3/4+s)| + |\eta_1(3/4+s)|\cdot|h_1(3/4+s;\overline{\theta_1'})|\right\} \le$$

$$\leq \varepsilon + (A+1)c(\delta) y_0^{r+\delta-1/4} \leq 2\varepsilon.$$

Now, we replace ε by $\varepsilon/2$. There is a set of prime numbers M_2, containing all of prime numbers $p \leq 2y_0 = y_1$ and satisfying, by the lemma 1, the condition

$$\max_{|s|\leq r} | \zeta(3/4+s) - \eta_2(3/4+s) | \leq \varepsilon/2,$$

where

$$\eta_2(s+3/4) = \prod_{p\in M_2} \left(1 - e^{2\pi i \theta_p^{(1)}} p^{-s-3/4}\right)^{-1},$$

and $\theta_p^{(1)} = 0$ when $p \leq y_1$. Analogically, as above, we define functions

$$F_2(s+3/4;\overline{\theta}) = \prod_{p\leq m_2} \left(1 - e^{2\pi i \theta_p} p^{-s-3/4}\right)^{-1}; m_2 = \max_{m\in M_2} m$$

and

$$h_2(s+3/4;\overline{\theta}) = F_2(s+3/4;\overline{\theta}) \prod_{p\in M_1} \left(1 - e^{2\pi i \theta_p} p^{-s-3/4}\right) - 1;$$

in a similar way, we find a point $\overline{\theta_2'} \in \Omega_2$ (Ω_2 is a projection of Ω into the subspace of co-ordinate axes of $\theta_p, p\overline{\in} M_2$) such that

$$\max_{|s|\leq r} | \zeta(3/4+s) - F_2(3/4+s;\overline{\theta}_2) | \leq 2^{1+(r+\delta-1/4)} \varepsilon, \overline{\theta}_2 = (\overline{\theta}_1, \overline{\theta_2'}).$$

Really,

$$| F_2(3/4+s) - \eta_2(3/4+s) | = | \eta_2(3/4+s) | \cdot | h_2(3/4+s;\overline{\theta_2'}) |.$$

Now, taking mean values as above, we receive

$$\max_{|s|\leq r} | h_2(s+3/4;\overline{\theta_2'}) | \leq \sqrt{2}\delta^{-1} \left(\frac{1}{2\pi} \iint_{|s|\leq r} | h_2(s+3/4;\overline{\theta_2'}) |^2 \, d\sigma dt \right)^{1/2} \leq c(\delta)(2y_0)^{\delta+r-1/4}.$$

Hence,

$$\max_{|s|\leq r} | \zeta(3/4+s) - F_2(3/4+s;\theta_2) | \leq \varepsilon/2 + 2^{1+(r+\delta-1/4)} \varepsilon, \overline{\theta}_2 = (\theta_1, \overline{\theta_2'}).$$

Repeating reasonings, for every $k > 1$, it can be found $\overline{\theta}_{k+1} = (\overline{\theta}_k, \overline{\theta'}_{k+1}) \in \Omega, \overline{\theta}_k = (\overline{\theta}_p^{\,k})_{p\in M_{k+1}}$ such that $\theta_p^k = 0$ when $p \leq y_k$, and

$$\max_{|s|\leq r} | \zeta(3/4+s) - F_{k+1}(3/4+s;\overline{\theta}_{k+1}) | \leq 2^{1+k(r+\delta-1/4)} \varepsilon;$$

here

$$F_{k+1}(s+3/4;\overline{\theta})=\prod_{p\le m_{k+1}}\left(1-e^{2\pi i\theta_p^0}p^{-s-3/4}\right)^{-1};m_{k+1}=\max_{m\in M_{k+1}} m.$$

Therefore, uniformly by $s,|s|\le r$ we have

$$\lim_{k\to\infty}F_k(3/4+s,\overline{\theta}_k)=\zeta(3/4+s).$$

The lemma 4 is proved.

5. Proof of the theorem.

Now we shall consider an integral

$$B_k=\int_\Omega\left(\iint_{|s|\le r}|F_{k+1}(3/4+s;\overline{\theta}_{k+1}+\overline{\theta})-F_k(3/4+s;\overline{\theta}_k+\overline{\theta})|\,d\sigma d\tau\right)d\overline{\theta}$$

where $k=0,1,...$, and if $k=0$ then one receives $F_0(3/4+s;\overline{\theta}_0+\overline{\theta})=0$. Applying Schwartz's inequality and changing an integration order, we find as above:

$$B_k^2\le 4\pi r^2\iint_{|s|\le r}d\sigma d\tau\int_\Omega|\prod_{p\le 2^{k-1}y_0}\left(1-e^{-2\pi i(\theta_p^n+\theta_p)}p^{-s-3/4}\right)^{-1}|^2\prod_{p\le 2^{k-1}y_0}d\theta_p\times$$

$$\times\sum_{n>2^{k-1}y_0}n^{2r+2\delta-3/2}\le c_\delta\left(2^{k-1}y_0\right)^{2r+2\delta+1-1/2};c_\delta>0.$$

Since $2r+2\delta-1/2<0$, then from this estimation it follows a convergence of the series below

$$\sum_{k=1}^{\infty}\iint_{|s|\le r}|F_k(3/4+s;\overline{\theta}_k+\overline{\theta})-F_{k-1}(3/4+s;\overline{\theta}_{k-1}+\overline{\theta})|\,d\sigma d\tau \quad (12)$$

almost everywhere (for every $\overline{\theta}\in\Omega_0$ from the subset Ω_0 of a measure 1 and the set $A=\Omega\setminus\Omega_0$ is finite-symmetrical). According to Yegorov's theorem (see [10, p. 166]) this series converges almost uniformly. It means that this series converges uniformly in the outside of some set $\Omega(\varepsilon),\mu(\Omega(\varepsilon))\le\varepsilon$ for every given $\varepsilon>0$. Put $\Omega_1'=\bigcap_\varepsilon\Omega(\varepsilon)$ we can assume, that $\mu(\Omega_1')=0$ and the set $A\cup\Omega_1'$ is finite-symmetrical (otherwise it is possible to take the set of all finite permutations of all its elements). There will be found some enumarable family of spheres B_r with a total measure, not exseeding ε, the union of which contains the set $A\cup\Omega_1'$. For every natural *n* we define the set $\Sigma_n'(t\Lambda)$ as a set of all limit points of the sequence $\Sigma_n(\overline{\omega})=\{\sigma\overline{\omega}\mid\sigma\in\Sigma\wedge\sigma(1)=1\wedge\cdots\wedge\sigma(n)=n\}$. Let

$$B^{(n)}=\{t\mid\{t\Lambda\}\in A\wedge\Sigma'_n(\{t\Lambda\})\subset\bigcup_{r=1}^{\infty}B_r\},\lambda_n=(1/2\pi)\log p_n\ n=1,2,....$$

For every t the sequence $\sum_{n+1}(\{t\Lambda\})$ is a subsequence of the sequence $\sum_n(\{t\Lambda\})$. Therefore, $\sum'_{n+1}(\{t\Lambda\})\subset\sum'_n(\{t\Lambda\})$ and we have $B^{(n)}\subset B^{(n+1)}$. If we denote $B=\bigcup_n B^{(n)}$, then we shall receive $m(B)\le\sup_n m(B^{(n)})$.

Let's estimate $m(B^{(n)})$. The set $\sum'_n(\{t\Lambda\})$ is a closed set. Clearly, if we shall "truncate" sequences $\{t\Lambda\}$, leaving only components $\{t\lambda_n\}$ with indexes greater than n and shall denote the truncated sequence as $\{t\Lambda\}'\in\Omega$, then the set $\sum'(\{t\Lambda\}')$ also will be closed. Now we consider the products $[0,1]^n\times\{\{t\Lambda\}'\}$ (external brackets designate the set of one element) for every t. We have

$$\{t\Lambda\}\in[0,1]^n\times\{\{t\Lambda\}'\}\subset A.$$

The example below shows that from feasibility of last relationship it does not follow the equality $A=\Omega$. Let $I=[0,1];U=[0;1/2];V=[1/2;1]$ and

$$X_0=U\times U\times\ldots,X_1=V\times U\times\ldots,$$

$$X_2=I\times V\times U\times\ldots,X_{s+1}=I^s\times V\times U\times\ldots,\ldots.$$

Clearly, that for $\mu(X_s)=0$ for all s. Then, $\mu(X)=0$ where

$$X=\bigcup_{s=0}^{\infty}X_s.$$

So, we have $X=[0,1]^s\times X$ for any natural s.

As the set $[0,1]^n\times\{\{t\Lambda\}'\}$ is closed, then there is only a finite set R of natural numbers such that $[0,1]^n\times\{\{t\Lambda\}'\}\subset\bigcup_{r\in R}B_r$. We shall consider the set of all "truncated" points $\bar{\theta}'$ of spheres B_r. Let $B'_r=\{\bar{\theta}'\,|\,\bar{\theta}\in B_r\}$. From the previous relationship we get $\{t\Lambda\}'\in B'_r$ for all $r\in R$. Then the intersection $\bigcap_{r\in R}B'_r$, being an open set, contains the point $\{t\Lambda\}'$. Thus, we have

$$[0,1]^n\times\{\{t\Lambda\}'\}\subset[0,1]^n\times\bigcap_{r\in R}B'_r\subset\bigcup_{r\in R}B_r,\quad(13)$$

for each considered point t. The similar relationship is fair in a case when the point $\{t\Lambda\}$ will be replaced by any limit point ϖ of the sequence $\Sigma(\{t\Lambda\})$ also, because $\varpi\in B_r$. If to denote by B' the union of all open sets of a kind $\bigcap_{r\in R}B'_r$, corresponding to every possible values of t and of a limit point ϖ, we shall receive the relation

$$\{t\Lambda\}\in[0,1]^n\times\{\{t\Lambda\}'\}\subset A\subset[0,1]^n\times B'\subset\bigcup_{r=1}^{\infty}B_r,$$

for each considered values of t and

$$\{\overline{\omega}\} \in [0,1]^n \times \{\overline{\omega}\}' \subset A \subset [0,1]^n \times B' \subset \bigcup_{r=1}^{\infty} B_r,$$

for each limit point $\overline{\omega}$. From this it follows the inequality $\mu^*(B') \le \varepsilon$ where μ^* means an external measure. The set B' is open and $\Sigma'(\{t\Lambda\}') \in B'$. Now we can apply the lemma 3 and receive an esti - mation $m(B^{(n)}) \le 6c\varepsilon$. Thus, we have $m(B) \le 6c\varepsilon$.

Let $t \notin B$. Then, $t \notin B^{(n)}$ for every $n = y_k, k = 1,2,3,...$. Consequently, for every k, there is a such limit point $\overline{\omega}_k \in \Omega \setminus \bigcup_r B_r$ of the sequence $\sum_n(\{t\Lambda\})$ for which the series

$$\sum_{l=1}^{\infty} \iint_{|s|\le r} | F_l(3/4+s;\overline{\theta}_l + \overline{\omega}_k) - F_{l-1}(3/4+s;\overline{\theta}_{l-1} + \overline{\omega}_k) | \, d\sigma d\tau$$

converges. As the set $\Omega \setminus \bigcup_r B_r$ is closed, the limit point $\overline{\omega} = (\{t\Lambda\})$ of the sequence $(\overline{\omega}_k)$ will belong to the set $\Omega \setminus \bigcup_r B_r$. Therefore, the series

$$\sum_{l=1}^{\infty} \iint_{|s|\le r} | F_l(3/4+s;\overline{\theta}_l + i\{t\Lambda\}) - F_{l-1}(3/4+s;\overline{\theta}_{l-1} + i\{t\Lambda\}) | \, d\sigma d\tau \quad (14)$$

converges, because it converges on this set uniformly.

So, the series (14) converges for every t with exception of values of t from some set of a measure, not exceeding $12c\varepsilon$. Owing to randomness of ε, last result shows a convergence of (14) for almost all t (clearly, the condition $0 \le t \le 1$ can be omitted now). Then, on a lemma 2, for any given $\delta_0 < 1$ the sequence

$$F_k(3/4+s;\overline{\theta}_k + i\{t\Lambda\}), \quad (15)$$

for all such t converges uniformly in the circle $|s| \le r\delta_0 (\delta_0 < 1)$ to some analytical function $f(s+3/4;t)$:

$$\lim_{k\to\infty} F_k(3/4+s+it;\overline{\theta}_k) = f(s+3/4;t).$$

Despite the received result we cannot use t as a variable becaus the left and right parts of this equality can differ in their arguments (the right part is defined as a limit of the sequence (15) where t enters into the expression of a discontinuous function $\{t\Lambda\}$). Hence, the principle of analytical continuation cannot be applied. To finish the theorem proof, we take any great real number T. As, considered values of t are everywhere dense in an interval $[-T,T]$, the union of the circles

$$C(t) = \{3/4 + it + s : |s| \le r\delta_0\}$$

contains the rectangle $3/4 - r\delta_0^2 \le \mathrm{Re}(s+3/4) \le 3/4 + r\delta_0^2, -T \le \mathrm{Im}(s+3/4) \le T$ in which the conditions of the lemma 2 are executed for the series

$$F_1(s+3/4;\overline{\theta}_1) + (F_2(s+3/4;\overline{\theta}_2) - F_1(s+3/4;\overline{\theta}_1)) + \ldots . \quad (16)$$

Hence, by the lemma 2, this series defines an analytical function in the considered rectangle which coincides with $\zeta(3/4+s)$ in the inside of the circle $C(0)$. To apply a principle of analytical continuation we take one-coneted open domain where both functions $\log F_*(s)$ and $\log\zeta(s)$ are regular (here function $F_*(s)$ is a sum of the series (16)). Let $\rho_1,...,\rho_L$ to designate all possible zeroes of the function $\zeta(s)$ in the considered rectangle a contour of which does not contain zeroes of the function $\zeta(s)$. We take cross-cuts along segments $1/2 \le \mathrm{Re}\, s \le \mathrm{Re}\,\rho_l, \mathrm{Im}\, s = \mathrm{Im}\,\rho_l, l = 1,...,L.$ In the open domain of the considered rectangle, not containing specified segments, the functions $\log F_*(s)$ and $\log\zeta(s)$ are regular. Then, by the principle of analytical continuation, the equality $F_*(s) = \zeta(s)$ is executed in all open domain defined above. Now we receive justice of an equality $F_*(s) = \zeta(s)$ in the all rectangle (without cross-cats) where both functions are regular. The theorem's proof is finished.

6. Proof of the consequence.

The conclusion of the consequence based on the theorem of Rouch'e (see [19, p. 137]). Let t be any real number. We shall prove that for any $0 < r < 3/4$ in the circle $C = \{s \,||\, s - 3/4 - it \,|\le r\}$ the function $\zeta(s)$ has no zeroes. Let

$$m = \min_{s\in C} |\zeta(s)|.$$

Under the theorem there exist $n = n(t)$ such that on and in a contour C the following inequality is executed:

$$|\zeta(s) - F_n(s;\overline{\theta}_n)| \le 0.25m.$$

Then on the C the inequality:

$$|\zeta(s) - F_n(s;\overline{\theta}_n)| \le |\zeta(s)|$$

is satisfied. From the theorem of Rouch'e it follows that the functions $\zeta(s)$ and $F_n(s;\overline{\theta}_n)$ have an identical number of zeroes inside C. But, the function $F_n(s;\overline{\theta}_n)$ has no zeroes in the circle C. Hence, $\zeta(s)$ also has no zeroes in the circle C. As t is any, from the last we conclude that the strip $-r < \mathrm{Re}\, s - 3/4 < r$ for any $0 < r < 1/4$ is free from the zeroes of the function $\zeta(s)$. The consequence is proved.

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