A Study of Hitting Times for Undirected Cayley Graphs[1]

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Abstract

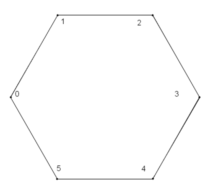
This paper deals with mean hitting times for random walks on unweighted Cayley graphs of $\mathbb{Z}_n$. In particular, we investigate the notion of the fundamental matrix of a graph, and use it to quantify hitting time values. We then seek to relate the fundamental matrix to the graph's transition matrix such that we need only use the transition matrix to generate hitting time values. We end with a discussion of roots of unity and how they play a role in determining hitting times for Cayley graphs of $\mathbb{Z}_n$.

Introduction

We start by defining a *random walk* on an n-vertex graph G as the following process:

(1) Start at an arbitrary vertex v_i of G.
(2) Define A as the set of all vertices of G adjacent to the vertex the walk is currently at. Choose an element v_j of A at random.
(3) Move to v_j, and repeat step 2.

Now, a *Cayley graph* is a visual representation of a group. The vertices of Cayley graph G represent the elements of the group. We choose a set $\mathcal{S}$ of elements in the group $\mathcal{G}$, which we call the *alphabet.* Let $\mathsf{i}, \mathsf{j} \in \mathcal{G}$ such that vertex v_i of G represents i and v_j of G represents j. Then a directed edge connects v_i to v_j if and only if $\mathsf{ia} = \mathsf{j}$.



As an example, consider $G =$ the undirected (or bi-directed) 6-cycle, pictured above. G is equivalent to the Cayley graph of $\mathbb{Z}_6$ on generators 1 and -1 (or 5). That is, $G = Cay(\mathbb{Z}_6, \{\pm 1\})$.

Finally, a *mean hitting time* is the expected number of steps to reach a given vertex j of a graph G starting from a vertex i of G. We denote this value as $E_i(T_j)$. Just as with any other expected value, we can intuitively define $E_i(T_j)$ as a weighted average. Thus, $E_i(T_j) = \sum_{n=0}^{\infty} n \cdot \mathbb{P}(walk\ first\ reaches\ j\ starting\ from\ i\ in\ n\ steps)$. In practice, calculating $E_i(T_j)$ values from this probabilistic definition appears to be a very difficult task to carry out. So, we seek an easier method for determining mean hitting times on Cayley graphs.

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Defining Irreducibility of a Transition Matrix

The following definitions and background allow us to find such a method. First of all, the *transition matrix* of an n-vertex graph is the $n \times n$ matrix whose ij-th entry describes the probability of a random walk moving from state i to state j. We refer to a graph's transition matrix as P. The undirected 6-cycle has the following transition matrix:

$$P = \begin{bmatrix} 0 & 1/2 & 0 & 0 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 0 & 0 & 1/2 & 0 \end{bmatrix}$$

Note that any transition matrix is *stochastic*, meaning that the entries in each row sum to 1. This makes sense because the entries in row i describe transition probabilities for a random walk at vertex v_i, and we know that any random walk will move from v_i to some other vertex with probability 1. Now, we call a graph G *strongly connected* if, for each vertex v_i of G there exist paths from v_i to any other vertex in G. That is, we can get from any vertex in G to any other vertex in G. It is easy to show that if $G = Cay(\mathbb{Z}_n, \mathcal{S})$ such that the greatest common divisor of the elements of $\mathcal{S}$ is 1, then G is strongly connected. If this is the case, then we call the alphabet $\mathcal{S}$ a *generating set*. We say that G's transition matrix P is *irreducible* if G is strongly connected. Finally, if P is irreducible, then there exists a unique stable probability distribution π on the vertices of G such that $\pi P = \pi$. Thus, π is the $1 \times n$ row vector $[\pi_1 \ \cdots \ \pi_n]$, where each π_i describes the portion of the stable distribution present at vertex v_i. With these definitions in mind, we arrive at our first result, albeit a trivial one.

Theorem 1. Consider the Cayley graph G on $\mathbb{Z}_n$ with generating set $\mathcal{S}$. Then $G = Cay(\mathbb{Z}_n, \mathcal{S})$ has irreducible transition matrix P. Furthermore, the stable distribution π is uniform.

Proof. First of all, since the elements of $\mathcal{S}$ have greatest common divisor 1, G is strongly connected, and therefore P is irreducible. This also implies the existence of a stable distribution π. To show π is uniform, assume $\mathcal{S}$ has cardinality m. Then, since G is a Cayley graph, there are exactly m edges leaving each vertex. We assign each edge leaving each vertex v_i weight $\frac{1}{m}$ (that is, the graph is essentially unweighted). This yields a transition matrix P, which we know is irreducible. Furthermore, since G is undirected, P is symmetric, and thus $P = P^T$. Hence P^T must also be stochastic, in addition to P. This is equivalent to saying that P is stochastic along its columns. Now, consider the row vector $\pi = [\pi_1 \ \cdots \ \pi_n]$, where $\pi_j = \frac{1}{n}$ for all j. Consider an arbitrary column i of P. This column has exactly m nonzero terms $a_1, \ldots, a_m$ summing to 1. Then $(\pi P)_i = \frac{a_1}{n} + \frac{a_2}{n} + \cdots + \frac{a_m}{n} = \frac{1}{n} = \pi_i$. Therefore, $\pi P = \pi$, and so the stable distribution is uniform.

The Fundamental Matrix

Now we can introduce the concept that is crucial to our discussion of hitting times. The *fundamental matrix* Z of an n-vertex graph G with irreducible transition matrix P is defined as follows:

$$Z_{ij} = \sum_{t=0}^{\infty} (p_{ij}^{(t)} - \pi_j).$$

Using a matrix-wise definition instead of an entry-wise one, this becomes:

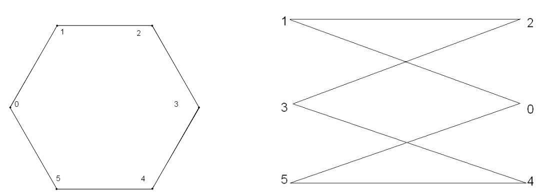
$$Z = \sum_{t=0}^{\infty} \left(P^t - \begin{bmatrix} \pi \\ \vdots \\ \pi \end{bmatrix} \right).$$

It can be verified that Z has constant diagonal entries and rows summing to 0. These properties will come into play later in our discussion. Now, if we know the fundamental matrix and the stable distribution of a graph G, the following formulas give us mean hitting time values:

Formula 1. $\pi_i E_\pi(T_i) = Z_{ii}$. This formula gives the expected number of steps to hit vertex i starting from an arbitrary vertex given the stable distribution.

Formula 2. $\pi_j E_i(T_j) = Z_{jj} - Z_{ij}$. This formula describes the mean hitting time we defined in probabilistic terms earlier.

Now, an even stronger condition than irreducibility for transition matrices is regularity. We call a transition matrix *regular* if there exists some positive integer k for which P^k has all positive entries. If this is the case, then any starting distribution ρ on the vertices of G will converge to the stable distribution π. Note that not all strongly connected graphs have regular transition matrices. For example, the undirected 6-cycle does not have regular P, for it is bipartite, as shown below:



Thus, it is impossible to reach vertex 1 starting from vertex 0 in an even number steps, and it is impossible to reach vertex 2 starting from vertex 0 in an odd number of steps. So, note that if the distribution starts entirely at vertex 0, in the limit, the distribution will oscillate between the vertex subset $\{1, 3, 5\}$ and the subset $\{2, 0, 4\}$. We call such a graph *periodic.* No periodic graphs have regular transition matrices. Formally, the Perron-Frobenius theorem[3] tells us that a transition matrix is regular if and only if it has exactly one eigenvalue with absolute value 1; and the transition matrix of the undirected 6-cycle has both 1 and -1 as eigenvalues. To study convergence times, we need regularity, but all we need to study hitting times is irreducibility.

[3] See Saloff-Coste, for instance.

Thus, the fundamental matrix is a powerful tool for our purposes, so long as we can determine it. However, for most graphs it seems difficult to calculate this infinite sum, just as it seems dificult to calculate a hitting time from its probabilistic definition. We therefore seek to equate this definition of Z with a more useable one. Specifically, we wish to show the following:

Theorem 2.

$$\sum_{t=0}^{\infty}\left(P^t - \begin{bmatrix}\pi\\ \vdots\\ \pi\end{bmatrix}\right) = Z = (I-(P-P_\infty))^{-1} - P_\infty.$$

We define P_∞ as the $n\times n$ matrix representing the stable distribution. Thus, each row of P_∞ is equal to the distribution π; that is,

$$P_\infty = \begin{bmatrix}\pi\\ \vdots\\ \pi\end{bmatrix}.$$

Lemma 1. P acts like an identity with respect to P_∞; that is

$$PP_\infty = P_\infty = P_\infty P.$$

Furthermore, $P_\infty P_\infty = P_\infty$.

Proof. Define J_n as the n-row column vector of all ones. Thus, we have $PP_\infty = PJ_n\pi$. Now, by stochasticity of P, $PJ_n = J_n$, so $PP_\infty = J_n\pi = P_\infty$. Similarly, $P_\infty P = J_n\pi P$. Since P is irreducible, $\pi P = \pi$. So, $P_\infty P = J_n\pi = P_\infty$. To prove $P_\infty P_\infty = P_\infty$, recall that the stable distribution on unweighted, undirected Cayley graphs of $\mathbb{Z}_n$ converge is uniform. Thus, $P_\infty =$

$$\begin{bmatrix}\frac{1}{n} & \cdots & \frac{1}{n}\\ \vdots & \ddots & \vdots\\ \frac{1}{n} & \cdots & \frac{1}{n}\end{bmatrix}$$

So, $(P_\infty P_\infty)_{ij} = \frac{1}{n^2} + \cdots + \frac{1}{n^2} = \frac{n}{n^2} = \frac{1}{n} = (P_\infty)_{ij}$. Therefore, $P_\infty P_\infty = P_\infty$.

Proof of Theorem. Note from the above definition of π that

$$\sum_{t=0}^{\infty}\left(P^t - \begin{bmatrix}\pi\\ \vdots\\ \pi\end{bmatrix}\right) = \sum_{t=0}^{\infty}\left(P^t - P_\infty\right).$$

Consider the case where $t=0$. In this case, $P^0 - P_\infty = I - P_\infty = (P-P_\infty)^0 - P_\infty$. Now, for $t\geq 1$, we appeal to the binomial theorem. From the binomial theorem, we know that $(1+x)^t = \sum_{i=0}^{t}\binom{t}{i}x^i$. Now, let $x=-1$. Clearly,

$$0 = (1+(-1))^t = \sum_{i=0}^{t}\binom{t}{i}(-1)^i.$$

So, for $t \geq 1$, if we negate every other term of the expansion of $(P - P_\infty)^t$, the coefficients of the terms will sum to 0. This yields

$$\begin{aligned}(P - P_\infty)^t &= P^t - \binom{t}{1} P^{t-1} P_\infty + \binom{t}{2} P^{t-2} P_\infty^2 - \cdots \pm P_\infty^t \\ &= P^t - \binom{t}{1} P_\infty + \binom{t}{2} P_\infty - \cdots \pm P_\infty \\ &= P^t - P_\infty.\end{aligned}$$

Our lemma enables us to get from the first line to the second line. We get from the second line to the third line by knowing that since the first term of the sequence, P^t, has coefficient 1, the remaining terms of the sequence must sum to -1. So, now we know that

$$\sum_{t=0}^{\infty} (P^t - P_\infty) = \sum_{t=0}^{\infty} (P - P_\infty)^t - P_\infty.$$

Remember that the zeroth term of the sequence on the left yields an extra $-P_\infty$ that we must tack on to the end of the sequence on the right. Consider $\sum_{t=0}^{\infty} (P - P_\infty)^t$. The first term of this series is I, and the common ratio is $(P - P_\infty)$. Since $0 \leq P_{ij} < 1 \ \forall \ i, j \in (1, \ldots, n)$, and every term of P_∞ is equal to $1/n$, every term of $(P - P_\infty)$ has absolute value less than 1. Thus, the infinite series has a finite sum equal to

$$\frac{I}{I - (P - P_\infty)} = (I - (P - P_\infty))^{-1}.$$

Therefore,

$$\sum_{t=0}^{\infty} (P^t - P_\infty) = \sum_{t=0}^{\infty} (P - P_\infty)^t - P_\infty = (I - (P - P_\infty))^{-1} - P_\infty = Z.$$

Example 1. Let G be the undirected 6-cycle; that is, $G = Cay(\mathbb{Z}_6, \{\pm 1\})$. Using G's transition matrix, which we showed above, the identity matrix, and P_∞ (the 6×6 matrix where each entry is equal to $\frac{1}{6}$), we construct Z (via Mathematica) according to the formula above:

$$Z = \begin{bmatrix} 35/36 & 5/36 & -13/36 & -19/36 & -13/36 & 5/36 \\ 5/36 & 35/36 & 5/36 & -13/36 & -19/36 & -13/36 \\ -13/36 & 5/36 & 35/36 & 5/36 & -13/36 & -19/36 \\ -19/36 & -13/36 & 5/36 & 35/36 & 5/36 & -13/36 \\ -13/36 & -19/36 & -13/36 & 5/36 & 35/36 & 5/36 \\ 5/36 & -13/36 & -19/36 & -13/36 & 5/36 & 35/36 \end{bmatrix}$$

Labeling the the vertices from 0 to 5, we can calculate the hitting times as follows:

$E_0(T_1) = \frac{1}{\pi_1}(Z_{11} - Z_{01}) = 6(\frac{35}{36} - \frac{5}{36}) = 5.$

$E_0(T_2) = \frac{1}{\pi_2}(Z_{22} - Z_{02}) = 6(\frac{35}{36} + \frac{13}{36}) = 8.$

$E_0(T_3) = \frac{1}{\pi_1}(Z_{33} - Z_{03}) = 6(\frac{35}{36} + \frac{19}{36}) = 9.$

$E_\pi(T_4) = \frac{1}{\pi_4}(Z_{44}) = 6(\frac{35}{36}) = \frac{35}{6}.$

Quantifying Hitting Time Values Using Only P

So, using Theorem 2 and Definitions 1 and 2, we can quantify hitting times on any finite graph with irreducible transition matrix. However, the fundamental matrix is an abstract concept that is hard to visualize. It is hard to tell exactly where the actual values for the hitting times are coming from. Thus, it would be nice if we could determine hitting times straight from P, for we can get P directly from the graph itself. We work toward this result in the next section of this paper.

Now, we call a tranisition matrix *circulant* if each row is equal to a cyclic shift of the first row. That is, the i-th row is equal to the first row, except that each entry is shifted, in a cyclic manner, $i-1$ places to the right. Looking back at the transition matrix of the undirected 6-cycle, we see that it is circulant. Hence, the unidirected 6-cycle is a *circulant graph.* Indeed, all Cayley graphs on $\mathbb{Z}_n$ are circulant, for $\mathbb{Z}_n$ is always a cyclic group. Now, recall that $\pi_i E_\pi(T_i) = Z_{ii}$, where $E_\pi(T_i)$ represents the expected number of steps to reach vertex i when starting from an arbitrary vertex in the stable distribution. Consider $G = Cay(\mathbb{Z}_n, \mathcal{S})$ as defined above. G, then, has the stable probability distribution π, where $\pi_i = \frac{1}{n}\ \forall\ i \in \{1, \ldots, n\}$. Since the distribution is uniform, and all diagonal entries of Z are equal by definition, note that $E_\pi(T_i)$ does not depend on i; it is uniform as well. We shall refer to this uniform $E_\pi(T_i)$ value for Cayley graphs as $E_\pi(T_{\mathbf{n}})$.

Lemma 2. $E_\pi(T_{\mathbf{n}})$ is equal to the average of the mean hitting times starting from an arbitrary vertex j and going to an arbitrary vertex i. That is,

$$E_\pi(T_{\mathbf{n}}) = \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} E_i(T_j).$$

Proof. First, note that since G is undirected and strongly connected,

$$E_i(T_j) = E_j(T_i) \quad \textit{for all} \quad i, j \in \{1, \ldots, n\}.$$

Furthermore, since G is circulant, for some arbitrary $k \in \{1, \ldots, n\}$,

$$E_i(T_j) = E_{i\pm k}(T_{j\pm k}) \quad \textit{for all} \quad i, j \in \{1, \ldots, n\}.$$

Thus, in order to show that $E_\pi(T_{\mathbf{n}})$ is the average of all the mean hitting times, it is sufficient to show that

$$E_\pi(T_{\mathbf{n}}) = \frac{1}{n}\sum_{j=1}^{n} E_i(T_j).$$

By Formula 1, $\pi_j E_\pi(T_j) = Z_{jj}$ for some j,

$$E_\pi(T_j) = \frac{1}{\pi_j} Z_{jj} = nZ_{jj}.$$

Since the rows of Z sum to 0, we have

$$E_\pi(T_j) = nZ_{jj} - \sum_{j=1}^{n} Z_{ij}.$$

And since the diagonal entries of Z are constant, this becomes

$$E_\pi(T_j) = \sum_{j=1}^{n} (Z_{jj} - Z_{ij}).$$

By Formula 2,

$$\sum_{j=1}^{n}(Z_{jj} - Z_{ij}) = \sum_{j=1}^{n}(\pi_j E_i(T_j)) = \frac{1}{n}\sum_{j=1}^{n} E_i(T_j).$$

And therefore,

$$E_\pi(T_j) = E_\pi(T_{\mathbf{n}}) = \frac{1}{n}\sum_{j=1}^{n} E_i(T_j) = \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} E_i(T_j).$$

Theorem 3.

$$\sum_{j=1}^{n} E_i(T_j) = \sum_{m=2}^{n}(1 - \lambda_m(P))^{-1}.$$

Proof. By Lemma 2, $\sum_{j=1}^{n} E_i(T_j) = E_\pi(T_{\mathbf{n}}) = E_\pi(T_i)$ for any i. Once again, Formula 1 gives us $E_\pi(T_i) = \frac{1}{\pi_i} Z_{ii} = nZ_{ii}$. Now, by the definition of Z, $nZ_{ii} = Tr(Z)$. From linear algebra, we know that if two matrices commute with each other and are symmetric, then they are simultaneously diagonalizable. First of all, since G is undirected, P is symmetric. And since Z is written in terms of I, P, and P_∞, all symmetric matrices, Z must also be symmetric. Furthermore, we know that P commutes with Z, since symmetric matrices always commute. Thus, the two are simultaneously diagonalizable. And since there exists a natural formula relating Z to P, the same formula relates eigenvalues of Z to eigenvalues of P. That is, just as $Z = I - (P - P_\infty)^{-1} - P_\infty$, we can write

$$\begin{aligned} Tr(Z) = \sum_{m=1}^{n} \lambda_m(Z) &= \sum_{m=1}^{n} \lambda_m\left((I - (P - P_\infty))^{-1} - P_\infty\right) \\ &= \sum_{m=1}^{n}\left([\lambda_m(I) - (\lambda_m(P) - \lambda_m(P_\infty))]^{-1} - \lambda_m(P_\infty)\right). \end{aligned}$$

Now, P_∞ has rank 1 and null space of dimension $n-1$. Thus, it must have eigenvalue 0 of multiplicity $n-1$. And, since it is a stochastic matrix, its only nonzero eigenvalue must be 1, of multiplicity 1. Thus, for $m \geq 2$, P_∞ makes no contribution to the sum. Furthermore, the value of the summand when $m = 1$ is $(1 - (1-1))^{-1} - 1 = 0$. So,

$$\sum_{m=1}^{n}\left([\lambda_m(I) - (\lambda_m(P) - \lambda_m(P_\infty))]^{-1} - \lambda_m(P_\infty)\right)$$

$$= \sum_{m=2}^{n}[\lambda_m(I) - \lambda_m(P)]^{-1} = \sum_{m=2}^{n}(1 - \lambda_m(P))^{-1}.$$

Therefore, we have

$$\sum_{j=1}^{n} E_i(T_j) = \sum_{m=2}^{n}(1 - \lambda_m(P))^{-1}.$$

Example 2. Consider $G = Cay(\mathbb{Z}_6, \{\pm 1\})$: the undirected 6-cycle, as before. The eigenvalues of transition matrix P of G are $\left\{1, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -1\right\}$, and the eigenvalues of the fundamental matrix Z of G are $\left\{2, 2, \frac{2}{3}, \frac{2}{3}, \frac{1}{2}, 0\right\}$. Let us verify that

$$E_\pi(T_{\mathbf{n}}) = \sum_{m=2}^{n} (1-\lambda_m)^{-1}.$$

In this case, $Z_{ii} = \frac{35}{60}$ for all i and $\pi_i = \frac{1}{6}$ for all i, so $E_\pi(T_{\mathbf{n}}) = 6\frac{35}{60} = \frac{35}{6}$. And indeed,

$$\sum_{m=2}^{n}(1-\lambda_m(P))^{-1} = \frac{1}{1-1/2}+\frac{1}{1-1/2}+\frac{1}{1+1/2}+\frac{1}{1+1/2}+\frac{1}{1+1} = 2+2+\frac{2}{3}+\frac{2}{3}+\frac{1}{2} = \frac{35}{6}$$

as well. Note that each term of the sum is an eigenvalue of the Z-matrix, such that every Z-eigenvalue shows up in the sum except 0 (which, of course, would make no contribution to the sum anyway). So, we can find $E_\pi(T_{\mathbf{n}})$ values for Cayley graphs right from the transition matrix. We will now try to determine $E_i(T_j)$ values using only P. Spectral decomposition of P allows us to do so.

Theorem 4.

$$E_i(T_j) = n\sum_{m=2}^{n}(1-\lambda_m(P))^{-1}u_{jm}(u_{jm}-u_{im}).$$

Proof. Recall that P is symmetric becuase G is undirected. From linear algebra, we know that symmetric matrices can be diagonalized by an orthogonal transformation of their eigenvectors and eigenvalues. Now, since P is stochastic, we diagonalize it by an orthonormal transformation; that is, $P = U\Lambda U^T$, where U is an $n \times n$ matrix whose columns are the unit eigenvectors of P, and Λ is the diagonal matrix consisting of P's eigenvalues. Thus, $P_{ij} = (U\Lambda U^T)_{ij}$. By the rules of matrix multiplication,

$$(U\Lambda U^T)_{ij} = \sum_{m=1}^{n}\sum_{k=1}^{n}(U_{im}\Lambda_{mk}U^T_{kj}).$$

Now, Λ_{mk} is nonzero only where $m = k$, so we have

$$(U\Lambda U^T)_{ij} = \sum_{m=1}^{n} U_{im}\Lambda_{mm}U^T_{mj} = \sum_{m=1}^{n} u_{im}\lambda_m u^T_{mj} = \sum_{m=1}^{n}\lambda_m u_{im}u_{jm}.$$

Thus, $P_{ij} = \sum_{m=1}^{n}\lambda_m u_{im}u_{jm}$. Defining P exactly in terms of its eigenvalues and unit eigenvectors brings us one step closer to quantifying hitting times on G.

In Theorem 3, we proved that

$$\sum_{m=1}^{n}\lambda_m(Z) = \sum_{m=2}^{n}(1-\lambda_m(P))^{-1}.$$

We also showed that Z and P share the same eigenspace. So, we can write

$$Z_{ij} = \sum_{m=2}^{n}(1-\lambda_m(P))^{-1}u_{im}u_{jm}.$$

Note that the largest eigenvalue of P is 1, which has no inverse under the defined expression relating eigenvalues between the two matrices. Instead, 1 has a "quasi-inverse of 0", which shows up as an eigenvalue in the Z-matrix. Thus, we do need to alter the summand when we change the lower bound on the summation from 1 to 2, since the only term we lose in doing so has value $0u_{i1}u_{j1} = 0$.

We are now ready to define hitting times in terms of eigenvectors and eigenvalues of P. From above,

$$\begin{aligned} E_i(T_j) &= n(Z_{jj} - Z_{ij}) \\ &= n\left(\sum_{m=2}^{n}(1-\lambda_m(P))^{-1}u_{jm}u_{jm} - \sum_{m=2}^{n}(1-\lambda_m(P))^{-1}u_{im}u_{jm}\right) \\ &= n\sum_{m=2}^{n}(1-\lambda_m(P))^{-1}u_{jm}(u_{jm}-u_{im}). \end{aligned}$$

Example 3. Let us calculate hitting times on the undirected 6-cycle again and verify that they match up with the values we got in Example 2. Now, the eigenvalues and orthonormal eigenvectors of P are as follows:

$\lambda_1 = 1$, with eigenvector $\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}/\sqrt{6}$;

$\lambda_2 = \frac{1}{2}$, with eigenvector $\begin{bmatrix} 1 & 0 & -1 & -1 & 0 & 1 \end{bmatrix}/2$;

$\lambda_3 = \frac{1}{2}$, with eigenvector $\begin{bmatrix} -1/2 & -1 & -1/2 & 1/2 & 1 & 1/2 \end{bmatrix}/\sqrt{3}$;

$\lambda_4 = -\frac{1}{2}$, with eigenvector $\begin{bmatrix} -1 & 0 & 1 & -1 & 0 & 1 \end{bmatrix}/2$;

$\lambda_5 = -\frac{1}{2}$, with eigenvector $\begin{bmatrix} -1/2 & 1 & -1/2 & -1/2 & 1 & -1/2 \end{bmatrix}/\sqrt{3}$;

and $\lambda_6 = -1$, with eigenvector $\begin{bmatrix} -1 & 1 & -1 & 1 & -1 & 1 \end{bmatrix}/\sqrt{6}$.

Note: Mathematica did not give me orthogonal eigenvectors; I had to orthogonalize the non-orthogonal ones using the Gram-Schmidt process.

$$\begin{aligned} E_0(T_1) &= 6\sum_{m=2}^{n}\left(1-\lambda_m(P)\right)^{-1}u_{1m}(u_{1m}-u_{0m}) \\ &= 6\Big[2\cdot 0(0-1/2) + 2\cdot -1/\sqrt{3}(-1/\sqrt{3}+1/2\sqrt{3}) + 2/3\cdot 0(0+1/2) \\ &\quad +2/3\cdot 1/\sqrt{3}(1/\sqrt{3}+1/2\sqrt{3}) + 1/2\cdot 1/\sqrt{6}(1/\sqrt{6}+1/\sqrt{6})\Big] \\ &= 6\Big[2\cdot 0 + 2\cdot -1/\sqrt{3}\cdot -1/2\sqrt{3} + 2/3\cdot 0 + 2/3\cdot 1/\sqrt{3}\cdot 1/\sqrt{3} + 1/2\cdot 1/\sqrt{6}\cdot 1/\sqrt{6}\Big] \\ &= 6\left[0+1/3+0+1/3+1/6\right] = 5 = 6(Z_{11}-Z_{01}). \end{aligned}$$

$$
\begin{aligned}
E_0(T_2) &= 6\sum_{m=2}^{n}\left(1-\lambda_m(P)\right)^{-1}u_{2m}(u_{2m}-u_{0m})\\
&= 6\Big[2\cdot -1/2(-1/2-1/2) + 2\cdot -1/2\sqrt{3}(-1/2\sqrt{3}+1/2\sqrt{3}) + 2/3\cdot 1/2(1/2+1/2)\\
&\quad +2/3\cdot -1/2\sqrt{3}(-1/2\sqrt{3}+1/2\sqrt{3}) + 1/2\cdot -1/\sqrt{6}(-1/\sqrt{6}+1/\sqrt{6})\Big]\\
&= 6\Big[2\cdot -1/2\cdot -1 + 2\cdot -1/2\sqrt{3}\cdot 0 + 2/3\cdot 1/2\cdot 1 + 2/3\cdot -1/2\sqrt{3}\cdot 0 + 1/2\cdot -1/\sqrt{6}\cdot 0\Big]\\
&= 6\left[1+0+1/3+0+0\right] = 8 = 6(Z_{22}-Z_{02}).
\end{aligned}
$$

We can verify that the other hitting times work as well using this formula. So, we can now determine hitting times directly from the transition matrix.

Using Roots of Unity to Determine Hitting Times

Almost everything we have done up to this point is generalizable to any graph with irreducible transition matrix. The only property Cayley graphs have that not all other graphs with irreducible P have is that π is uniform. This makes proving Theorem 1 a little easier, and replaces all $\frac{1}{\pi_i}$ coefficients in hitting time calculations with the uniform value n. Also, working with undirected Cayley graphs means that P is always symmetric, which, admittedly, makes proving Theorem 1, and especially Theorem 4, easier. If P was not symmetric, we would have to use powers of π to symmetrize it before diagonalizing with the orthonormal transformation as shown above.

However, there is a special property unique to Cayley graphs of $\mathbb{Z}_n$: they are circulant. Therefore, we can use primitive roots of unity to generate eigenvalues and orthonormal eigenvectors for their transition matrices. The following theorem is taken from Julia Lazenby's thesis, *Circulant Graphs and Their Spectra*:

Theorem 5. If $X = Cay(\mathbb{Z}_n, \mathcal{S})$, then $Spec(X) = \{\lambda_x | x \in \mathbb{Z}_n\}$ where

$$\lambda_x = \sum_{\mathsf{s}\in\mathcal{S}} exp\left(\frac{2\pi i x s}{n}\right).$$

Proof. Let T be a linear operator corresponding to the adjacency matrix of a circulant graph $X = Cay(\mathbb{Z}_n, \{a_1, a_2, \ldots, a_m\})$. If f is any real function on the vertices of X we have

$$T(f)(x) = f(x+a_1) + f(x+a_2) + \cdots + f(x+a_m).$$

Let ω be a primitive n^{th} root of unity and let $g(x) = \omega^{kx}$ for some $k \in \mathbb{Z}_n$. Then,

$$
\begin{aligned}
T(g)(x) &= \omega^{kx+ka_1} + \omega^{kx+ka_2} + \cdots + \omega^{kx+ka_m}\\
&= \omega^{kx}\left(\omega^{ka_1} + \omega^{ka_2} + \cdots + \omega^{ka_m}\right).
\end{aligned}
$$

Thus, g is an eigenfunction and $\omega^{ka_1} + \omega^{ka_2} + \cdots + \omega^{ka_m}$ is an eigenvalue.

This theorem concerns the spectrum of the adjacency matrix of a Cayley graph. However, our transition matrices are simply the adjacency matrix divided by the graph's regularity. That is, assuming the cardinality of $\mathcal{S}$ is m, we divide each entry of a graph's adjacency matrix by m to come up with the graph's transition matrix. So, if we substitute P in for T in the above proof, we come up with

$$P(f)(x) = \frac{f(x+a_1) + f(x+a_2) + \cdots + f(x+a_m)}{m}.$$

Defining ω and g exactly the same as in the proof,

$$\begin{aligned} P(g)(x) &= \frac{\omega^{kx+ka_1} + \omega^{kx+ka_2} + \cdots + \omega^{kx+ka_m}}{m} \\ &= \omega^{kx}\left(\frac{\omega^{ka_1} + \omega^{ka_2} + \cdots + \omega^{ka_m}}{m}\right). \end{aligned}$$

So, this new g is an eigenfunction of P, and $\frac{\omega^{ka_1}+\omega^{ka_2}+\cdots+\omega^{ka_m}}{m}$ is an eigenvalue of P.

Example 4. Now, if we once again consider the undirected 6-cycle, $\omega = exp(\frac{2\pi i}{6})$, k ranges from 0 to 5, $m = 2$, $a_1 = -1$, and $a_2 = a_m = 1$. So, we compute eigenvectors and eigenvalues as follows:

$$\lambda_1 = \frac{\omega^{0\cdot 1} + \omega^{0\cdot -1}}{2} = \frac{1+1}{2} = 1.$$

$u_{x1} = \omega^{0x}$ for x ranging from 0 to 5. Thus,

$$u_1 = \begin{bmatrix} \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

$$\begin{aligned} \lambda_2 = \frac{\omega^{1\cdot 1} + \omega^{1\cdot -1}}{2} &= \frac{exp(\frac{2\pi i}{6}) + exp(\frac{-2\pi i}{6})}{2} \\ &= \frac{\cos\frac{\pi}{3} + i\sin\frac{\pi}{3} + \cos\frac{-\pi}{3} + i\sin\frac{-\pi}{3}}{2} \\ &= \frac{1 + i\left(\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right)}{2} = \frac{1}{2}. \end{aligned}$$

$$u_2 = [\ \omega^{1\cdot 0} \quad \omega^{1\cdot 1} \quad \omega^{1\cdot 2} \quad \omega^{1\cdot 3} \quad \omega^{1\cdot 4} \quad \omega^{1\cdot 5}\ ] = [\ \omega^0 \quad \omega^1 \quad \omega^2 \quad \omega^3 \quad \omega^4 \quad \omega^5\ ].$$

Now, all but one of these entries will be a complex number. To come up with eigenvectors without complex entries, we add this eigenvector to the one generated when $k = 5$, and subtract from this one the one generated when $k = 5$. That is, the new u_2 will be

$$[\ \omega^0 \quad \omega^1 \quad \omega^2 \quad \omega^3 \quad \omega^4 \quad \omega^5\ ] + [\ \omega^0 \quad \omega^5 \quad \omega^{10} \quad \omega^{15} \quad \omega^{20} \quad \omega^{25}\ ].$$

Similarly, the new u_3 will be

$$[\ \omega^0 \quad \omega^1 \quad \omega^2 \quad \omega^3 \quad \omega^4 \quad \omega^5\ ] - [\ \omega^0 \quad \omega^5 \quad \omega^{10} \quad \omega^{15} \quad \omega^{20} \quad \omega^{25}\ ].$$

The eigenvectors generated when $k = 2$ and when $k = 4$ also have some complex entries, so we use the same process to make their entries real.

Using the above method, we calculate all eigenvalues and eigenvectors, normalize the eigenvectors, and come up with the following:

$\lambda_1 = 1$, with unit eigenvector $\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}/\sqrt{6}$.

$\lambda_2 = \frac{1}{2}$, with unit eigenvector $\begin{bmatrix} 1 & 1/2 & -1/2 & -1 & -1/2 & 1/2 \end{bmatrix}/\sqrt{3}$.

$\lambda_3 = \frac{1}{2}$, with unit eigenvector $\left[\begin{array}{cccccc} 0 & 1 & 1 & 0 & -1 & -1 \end{array}\right]/2$.

$\lambda_4 = -\frac{1}{2}$, with unit eigenvector $\left[\begin{array}{cccccc} 1 & -1/2 & -1/2 & 1 & -1/2 & -1/2 \end{array}\right]/\sqrt{3}$.

$\lambda_5 = -\frac{1}{2}$, with unit eigenvector $\left[\begin{array}{cccccc} 0 & 1 & -1 & 0 & 1 & -1 \end{array}\right]/2$.

$\lambda_6 = -1$, with unit eigenvector $\left[\begin{array}{cccccc} 1 & -1 & 1 & -1 & 1 & -1 \end{array}\right]/\sqrt{6}$.

Note that these eigenvectors are not all the same as they were in Example 3. However, it can be verified that all these eigenvectors are orthogonal, and so we can use Theorem 4 to generate hitting times. Once again, we compute $E_0(T_1)$.

$$\begin{aligned}
E_0(T_1) &= 6\sum_{m=2}^{n}\left(1-\lambda_m(P)\right)^{-1}u_{1m}(u_{1m}-u_{0m}) \\
&= 6\Big[2\cdot 1/2\sqrt{3}(1/2\sqrt{3}-1/\sqrt{3}) + 2\cdot 1/2(1/2-0) + 2/3\cdot -1/2\sqrt{3}(-1/2\sqrt{3}-1/\sqrt{3}) \\
&\qquad +2/3\cdot 1/2(1/2-0) + 1/2\cdot -1/\sqrt{6}(-1/\sqrt{6}-1/\sqrt{6})\Big] \\
&= 6\Big[2\cdot 1/2\sqrt{3}\cdot -1/2\sqrt{3} + 2\cdot 1/4 + 2/3\cdot -1/2\sqrt{3}\cdot -3/2\sqrt{3} \\
&\qquad +2/3\cdot 1/4 + 1/2\cdot -1/\sqrt{6}\cdot -2/\sqrt{6}\Big] \\
&= 6\left[-1/6+1/2+1/6+1/6+1/6\right] = 5 = 6(Z_{11}-Z_{01}).
\end{aligned}$$

So, using roots of unity to generate hitting times on undirected Cayley graphs of $\mathbb{Z}_n$ seems to be a very natural process, for it automatically constructs orthogonal eigenvectors, unlike Mathematica. While we do need to correct for complex entries, we do not need to orthogonalize the eigenvectors using the Gram-Schmidt process like we did in Example 3. Thus, as long as we have a symmetric generating set $\mathcal{S}$ on a finite cyclic group, the mean hitting times on the resulting graph are coming directly from these roots of unity. With some more manipulation, it appears that we can apply this method to Cayley graphs of any finite abelian group with any generating set.

References

(1) D. Aldous, J. Fill. *Reversible Markov Chains and Random Walks on Graphs.* Ch. 2, 1999. Ch. 3, 2002.

(2) N. Biggs. *Algebraic Graph Theory*, 2^{nd} ed. Cambridge University Press. pages 16-17, 1993.

(3) J. Lazenby. *Circulant Graphs and Their Spectra*, Reed College Thesis. pages 5-6, 2008.

(4) J. Norris. *Markov Chains*. Cambridge University Press. pages 1-56, 1997.

(5) L. Saloff-Coste. *Lectures on Finite Markov Chains*. *Lecture Notes in Math*, 1665, Springer, Berlin. pages 301-413, 1997.