

IX. *The Construction and Properties of a new Quadratrix to the Hyperbola, By Mr . . . Perks. Communicated by Mr Abr. de Moivre, F. R. S.*

**T**He *Circle*, *Ellipsis* and *Hyperbola* being not Geometrically Quadrable ( as infinite others ) there have been two ways made use of to find their *Area's*. 1. By *Converging Series*, whereby Approaches are made nearer and nearer, according to the exactness desir'd. 2. By *Quadratices*, that is, Mechanical Curves, which determine the Length of certain Lines, whose Squares or Rectangles give the *Area* of the Figure desir'd. Of this sort is the old *Quadratrix of Dinostratus*, by which the Circle and Ellipse are squared; and another sort (for the same purpose) I inserted in the *Transactions* about 5 years ago. Since that, having found the Construction of a Curve, from whence (besides its own *Quadrature* and *Rectification*) the *Quadrature* of the *Hyperbola* is deriv'd, I thought the following Account might not ( to some ) be unacceptable.

Let *AB*, *CD*, be two straight Rulers joyned at *B*, and there making a right Angle. ( Their length according to the largeness of the Figure you will describe. ) *EF* is another Ruler somewhat longer than *AB*. Near the one end *E*, let a little *Truckle-wheel* ( represented edge-wise by *gh*, and made of a thin Plate of Brass or Iron ) be fastned to the Ruler by a Pin ( *i*, ) thorew its Center, so that the Wheel may turn about upon the Pin ( *i* ) tight to the Ruler without joggling.

On the under side, of this Rular ( the side from the Eye in the Scheme ) let there be pinn'd or glewed a little piece of Wood ( in the form of a Quadrant, the part which is seen being mark'd  $kl$  ) whose edge ( or limb )  $kl$ , is an arch of a Circle of Center ( $i$ ) and Radius  $ib$  ( the same with the little Wheel. ) The design of this piece of Wood is, that in the several Positions of the Rular  $EF$ , the circular Limb  $kl$  always touching and sliding by the edge of the Rular  $AB$ , the Center of the Wheel may be always in a line ( $im$ ) parallel to the Rular  $AB$ .

In the Rular  $CD$  make  $MB = ib$  or  $ik$ , and at  $M$  fasten a little Pin, and another to the Rular  $EF$  near the Wheel, as at  $p$ . To these two Pins let be fastned the two ends of a String  $MR$ , so that its whole length ( from Pin to Pin )  $+ pi$ , be equal to the intended Axis of the Curve  $TW$ .

The Instrument being thus prepar'd, let a strong Rular  $SO$ , be fastned ( or held fast ) upon the Paper or Plain that the Curve is to be drawn upon. Lay the Rular  $EF$  from  $M$  towards  $A$ , and parallel to  $AB$ , so that the String lye all straight along the edge of the Rular  $EF$  from  $M$  to  $p$ , the point  $Sk$  of the Quadrantal piece of Wood resting upon the edge of the Rular  $AB$ . Then with a small Pin at  $M$  keeping the String close to the edge of the Rular  $EF$ , and with your other hand upon the end  $E$ , keeping the Wheel tight to the Paper or Plain, move the Pin String and Rular  $EF$  from  $M$  towards  $O$ , the Rular  $CD$  sliding along by the fastned Rular  $SO$  in a right line, the Wheel  $gb$  will by its motion describe the desired Curve  $TV$ .

Note,

*Note,* The Semi-diameter of the little Wheel must be about the *Sum* of the thickneses of the two Rulers EF and AB, that it may touch the Paper. Also it will be convenient that its edge be thin, and a little rough, that it may not slide flat-ways, and that it may leave a visible impressiion.

From this Construction the following Properties are demonstrable.

I. It is evident from the Construction, that the *Sum of the Tangent and Subtangent* is every where equal to the same given Line =  $MR + Ri = TW$ .) for the String ( first straight at TW, afterwards making an Angle at R ) being every where the same, the Line Ri ( or Rp + pi ) is always the Tangent, and the Remainder RM the Subtangent; the Contact of the Wheel with the Plain, being the point of the Curve to which they belong.

II. It hence follows, that any assignable part of the Curve is *Rectifiable*, or equal to any assignable straight Line: In Fig. 2. Let FAE be a part of the Curve, its Vertex F. HD *d* is the Line described by the motion of the Pin R ( in Fig. 1. ) and may be shewn to be Asymptote to the Curve. FH a perpendicular to HD: Let A be given point in the Curve, AD the Tangent, and BD the Subtangent to the same point A. Let *a* be another point in the Curve infinitely near to A. to which let *a* *d* be the Tangent, and *b* *d* the Subtangent. Draw AG, *a* *g* perpendicular to FH and AB, *a* *b* perpendicular to HD. By the Construction  $AD + DB = a d + d b$ . Let *a* *s* be made equal to *a* D, and draw D*s*. Then because  $a d + b d = AD + DB$ . Subtract *b* D and *a* D ( or *a* *s* ) from both Sums ( Equals from Equals ) there remains  $s d + d D = A a + B b$  ( or C a ) A a C,

$Dd$  & are like Triangles ( or differing infinitely little from such ) therefore  $C a ( B b ) : A a :: d : D d$ . and compounding  $B b + A a : A a :: d + D d : D d$ . Alternating  $B b + A a : d + D d :: A a : D d$ . But  $B b + A a = d + D d$  ( as is shewn above ) therefore  $A a = D d$ .  $A a$  is the fluxional Particle of the Curve  $F A$ , and  $D d$  is the fluxional Particle of the Line  $H D$ : These Fluxions or Augments, being equal, and their flowing quantities beginning together, are themselves therefore equal, *viz.*  $F A = H D$ :

Let  $F G = x$ .  $G A ( = H B ) = y$ .  $A D = t$ .  $B D = S$ . So is the Curve  $F A = H D = y + S$ : that is, the Curve from the Vertex to any given point therein, is equal to the Sum of its Ordinate, and Subtangent to the same point which is its second Property.

III. The next Property ( and whereupon I call it the *Hyperbolic Quadratrix* ) is this, In Fig. 2. let  $F A E$  be a part of the Curve, ( &c. as before. )  $F I K H$  is a Square upon the line  $F H$ .  $\Delta I L$  is an Equilateral Hyperbola whose Vertex is  $I$ , its Asymptotes  $H O, H R$ . its Ax  $H I \mu$ . From a given point  $L$  in the Hyperbola ( below its Vertex  $I$  ) draw  $L A$  parallel to the Asymptote  $R H$ , intersecting the Diagonal  $I H$  in  $M$ ,  $F H$  in  $G$ , and touching the Quadratrix in  $A$ . I say, that the Hyperbolic Area  $I L M$  is equal to a Rectangle, whose sides are the Ordinate  $G A$ , and twice  $F H$ , the Ax to the Quadratrix, that is,  $I L M = 2 F H \times G A$ .

Let  $F H = a$ ,  $F G = x$ ,  $G A = y$ . because of the Hyperbola  $G L X G H ( L S ) = F H q$ . therefore  $G L = \frac{F H q}{G H}$ ; and  $L M = \frac{F H q}{G H} - G H ( M G )$  that is,

$$I L M = \frac{a a}{a - x} - a + x = \frac{2 a x - x x}{a - x}$$
, and consequently

the fluxion of the Area  $I L M = \frac{2 a x - x x}{a - x} \times \frac{a - x - x}{a - x}$

In the Rectangle triangle A D B, A B = a - x, B D = S, A D = t = a - S; then is A D q = A B q + B D q: or a a 2 a S + S S = a a - 2 a x + x x + S S, which being reduced, gives  $S = \frac{2 a x - x x}{2 a}$

Let l a be a right line supposed infinitely near and parallel to L A, and intersecting A B in C. Because of like triangles A C a, A B D; A B : B D :: A C : C a that is a - x : S ( =  $\frac{2 a x - x x}{2 a}$  ) :: x : y . therefore y =

$\frac{2 a x - x x}{2 a a - 2 a x} x$  . Multiply each by 2 a, and 'tis 2 a y

$\frac{2 a x - x x}{a - x} x$  . The *Flowing quantity* of 2 a y is 2 a y

and the *flowing quantity* of  $\frac{2 a x - x x}{a} x$  is the Hyperbolic

Area I L M ( as is shewn before. ) These two Area's beginning together at F and I, and having every where equal *Fluxions*, or Augments, are therefore themselves every where equal.

N. The Quadrature of the Trilinear Figure I L M being thus found, any other Area bounded with the Curve line I L. and any other Right Lines is also given.

IV. Supposing the same things as in the precedent Proposition, I say, that the Area of the Quadratrix F a b H F is equal to half the square of F g, wanting the Cube of

F g divided by six F H, or F a b H F =  $\frac{x x - x x x}{x} \frac{1}{6 a}$ . The

Fluxion of this Area is the Rectangle C a b B = a - x x y

= a - x x  $\frac{2 a x - x x}{2 a a - a x} x = x x - \frac{x x}{2 a} x$  . The

flowing quantity of x x is  $\frac{1}{2} x x$  : And the flowing quantity

city of  $\frac{x x}{2 a}$  is  $\frac{x x x}{6 a}$  [as is easily shewn by bringing back these flowing quantities to their respective Fluxions.] And hence also it follows, that the whole Area continued on infinitely towards E, is *one third of the Square F I K H*; or  $\frac{1}{3} a a$ . For supposing  $x = a$  the Area above becomes  $\frac{a a}{2} - \frac{a a}{6} = \frac{a a}{3}$ .

While I was considering the other Properties of this Curve, and had given some account of them to my Ingenious Friend Mr *John Colson*, he returned me a Letter with the Addition of the Quadrature of the Curves Area, which I had not then enquired into:

V. Supposing still the same things, I say that the Solid made by the conversion of the Area F a b H F about the Line H b as an Axis, is equal to a Cylinder whose Radius is F H = a, and height equal to  $\frac{x x}{2 a} - \frac{x^3}{2 a a} + \frac{x^4}{8 a^3}$ . And the whole Solid made by conversion of the whole Figure infinitely continued, is equal to an eighth part of a Cylinder, whose Radius and Height are each equal to F H or a.

Let  $\frac{P}{D}$  express the Proportion of the Periforie and Diameter of a Circle. Then is  $\frac{P}{D} a b$  quad. the Area of a Circle whose Radius is a b. And because  $C a = \dot{y} = x - \frac{x x}{2 a \dot{x}}$  the fluxion of the Solid is  $\frac{P}{D} a b \cdot q. \cdot \frac{x - x x}{2 a \dot{x}}$   
 $\frac{P}{D} a b \cdot q. \cdot \frac{x - x x}{2 a \dot{x}}$  or

$$\text{of } \frac{P}{D} a - x^2 \cdot \frac{x - \frac{x x}{2 a}}{a - x} = \frac{P}{D} a x - \frac{3}{2} x x + \frac{x^3}{2 a} x$$

whose flowing quantity is  $\frac{P}{D} a x x - \frac{x x x}{2} +$

$\frac{x^4}{8 a}$ . Which Solid being divided by  $\frac{P}{D} a a$  ( the Area

of a Circle whose Radius is  $a$  ) gives  $\frac{x x - x x x}{2 a} + \frac{x^4}{8 a a}$

for the height of a Cylinder on the said circular Base, and equal to the Solid made by conversion of the Area  $F a b H F$  about the Line  $H b$  as an Axis. When  $x = a$  ( that is when the whole Figure is turn'd about its Asymptote ) the height  $\frac{x x}{2 a} - \frac{x^3}{2 a a} + \frac{x^4}{8 a a}$  become  $\frac{1}{8} a$

VI. The Curve surface of the Solid generated by the Conversion of the Figure  $F a b H F$  about  $H B$ , is equal to the Curve surface of a Cylinder whose Radius is  $a$ , and height equal to  $\frac{x}{2} - \frac{x x}{4 a} + \frac{x x x}{12 a a}$ . And the whole Curve Surface of the Solid infinitely continued, is equal to *one third part of the Curve Surface of a Cylinder whose Radius and Height are equal to  $F H$  or  $a$ .* Which may be demonstrated after the manner of the precedent Proposition.

VII. The Radius of the Curvature of any Partiele of the Quadratrix is  $\frac{r^2}{a - x}$  and this found Geometrically.

In Fig. 3;  $F A E$  is the *Quadratrix*,  $H D$  the Asymptote,  $A D$  the Tangent,  $B D$  the Subtangent to a given point  $A$ . Make  $B V = A D$ . Upon  $V$  rise the perpendicular  $V W$ . from  $A$  draw  $A W$  perpendicular to the Tangent  $A D$ ; till

it meet A W in W. So is A W the Radius of the Curvature at A.

VIII. This Curve may be continued on infinitely above the point F ( but by a different and more operose way of Construction ) whose Properties will be these. 1. The *Difference* of its Tangent and Subtangent ( taking the Subtangent in the Line H S ) will be always equal to the same given Line F H or  $a$ . That is, as  $t + s = a$ , below F, so  $t - s = a$  above F. 2. As below F the Curve-Line is equal to the *Sum* of its Ordinate and Subtangent, so above, it is equal to their *Difference* or  $-S - y$ . 3. As below F,  $2ay = ILM$ , so above  $2ay = I\lambda\mu$ . All which (and its other Properties) may be demonstrated as the Precedent *mutatis mutandis*.

IX. With a little variation in the precedent Construction may the *Logarithmick Curve* be constructed, which is also a *Quadratrix* to the Hyperbola. In Fig. 1. omitting the String M R P, let the distance M R be equal to the *Subtangent* of the intended Logarithmick Curve (which, as 'tis known, is invariable.) Stick a Pin at R in the Rular C D, to which apply the Rular E F, so that the edge of the little Quadrant  $k l$ , resting upon the Rular A B, the distance M i be equal to M R. Then keeping the Rular E F tight to the Pin R and Rular A B, slide the Rular C D along in a straight Line (by the Rular or Line S O.) So will the Wheel  $g h$  describe a part of the Logarithmick Curve T V, whose *Subtangent* is every where M R.

X. Fig. 2. Let F A E represent the *Logarithmick Curve*, whose Subtangent is equal to F H. L I  $\Delta$  is an Equilateral Hyperbola (*Ec.* as before § III.) Let F G =  $x$ , G a =  $y$ . F H (= B D) =  $a$ . G H (= L S) =  $a - x$ . A C =  $x$  C a =  $y$ . Then A C : C a :: A B : B D. that is  $x : y :: a$   
— x



$x : a :: a : \frac{a^2}{a - x}$ . therefore a  $\dot{y} = \frac{a^2}{a - x} \dot{x}$ . The

*Flowing quantity* of a  $\dot{y}$  is a  $y$ ; and the *Flowing quantity* of  $\frac{a^2}{a - x} \dot{x}$  is the Hyperbolick Area F I L G (for by the

nature of the Hyperbola  $GL = \frac{a^2}{a - x}$ ) therefore is the Hyperbolick Area F I L G equal to a  $y$ , a Rectangle whose sides are the Subtangent (B D = F H) and Ordinate G A (as here accounted) of the Logarithmick Curve.

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