The Tensor Product Theorem

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References

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Chapter 1

Prelimaries

For all the following Ω will be always a field.

Definition 1.1. *Let R be a ring and M an R-module. M is called simple if $M \neq 0$ and it has no proper nontrivial R-submodules.*

Definition 1.2. *An idempotented algebra is an ordered pair $(H, \mathcal{E})$ where H is an Ω-algebra (usually without unit) and $\mathcal{E}$ is a set of idempotents, which satisfies the following properties:*

1. $\forall e_1, e_2 \in \mathcal{E}, \exists e_0 \in \mathcal{E}$ *s.t.* $e_0 e_1 = e_1 e_0 = e_1$ *and* $e_0 e_2 = e_2 e_0 = e_2$.
2. $\forall f \in \mathcal{H}, \exists e \in \mathcal{E}$ *s.t.* $ef = fe = f$.

Remark 1.3. *There is a partial ordering $\geq$ on $\mathcal{E}$ defined as follows: if $e, f \in \mathcal{E}$,*

$$e \geq f \Leftrightarrow ef = fe = f.$$

Remark 1.4. *If $(H_1, \mathcal{E}_1)$ and $(H_2, \mathcal{E}_2)$ are idempotented Ω-algebras then the tensor product ring $(H_1 \otimes H_2, \mathcal{E})$, where*

$$\mathcal{E} = \{e_1 \otimes e_2 | e_1 \in \mathcal{E}_1, e_2 \in \mathcal{E}_2\},$$

is an idempotented Ω-algebra.

Definition 1.5. *Let Σ be some indexing set, and for all $v \in \Sigma$, let there be given a group G_v, and for almost all $v \in \Sigma$, let there be given a subgroup K_v of G_v. Then the restricted direct product of the G_v with respect to K_v is*

$$G = \{(a_v)_{v \in \Sigma} \in \prod_v G_v | a_v \in K_v, \text{ for almost all } v \in \Sigma\}.$$

Definition 1.6. *Let Σ be some indexing set, and for all $v \in \Sigma$, let there be given a vector space V_v, and for almost all $v \in \Sigma$, let there be given a nonzero $x_v^o \in V_v$. Let O be the set of all finite subsets S of Σ having the property if $v \notin S$ then x_v^o is defined. We order O by inclusion and then it is a directed set.*

$\forall S, S' \in O, S \subseteq S'$, we define a homomorphism

$$\lambda_{S,S'} : \bigotimes_{v\in S} V_v \to \bigotimes_{v\in S'} V_v,$$

namely, $\lambda_{S,S'}(x)$ is obtained by tensoring $x \in \bigotimes_{v\in S} V_v$ with $\otimes_{v\in S'-S} x_v^o$.

We form the direct limit of this family of maps

$$\otimes_v V_v := \lim_{\rightarrow} \bigotimes_{v\in S} V_v.$$

This product is called the restricted tensor product of the V_v.

Proposition 1.7. *Let R_v, where $v \in \Sigma$, be a family of rings, each with unit $e_v \in R_v$, and let R be the restricted tensor product of the R_v with respect to e_v. Let $\gamma : R \to \Omega$ be a ring homomorphism. Then there exists ring homomorphisms*

$$\gamma_v : R_v \to \Omega$$

s.t. $\gamma(\otimes_v r_v) = \prod_v \gamma_v(r_v)$.

Remark 1.8. *R is a ring.*

Remark 1.9. *We note that given a family of ring homomorphisms γ_v, if*

$$\otimes_v r_v \in \bigotimes_v R_v,$$

then $\gamma_v(r_v)$ is 1 for almost all v, so $\prod_v \gamma_v(r_v)$ is well defined.

Proof. Let 1_v denote the unit element in R_v. We have a ring homomorphism

$$i_v : R_v \to R,$$

defined by

$$i_v(x_v) = x_v \otimes (\otimes_{w\neq v} 1_w).$$

Let $\gamma_v = \gamma \circ i_v$. Then if $r \in R$, we can write $r = \prod_v i_v(r)$, where all but finitely many terms on the right are 1. Then is clear that $\gamma(r) = \prod_v \gamma_v(r_v)$. □

Definition 1.10. *Let $(H, \mathcal{E})$ be an idempotented Ω-algebra and let $e \in \mathcal{E}$. We call M smooth if $M = \cup_{e\in\mathcal{E}} eMe$ and admissible if it is smooth and $\dim_\Omega(eM) < \infty, \forall e \in \mathcal{E}$*

Chapter 2

The proof of the Tensor Product Theorem

We'll prove the Tensor Product Theorem, which asserts that if F is a global field, A its adele ring, v the places of F, and G is a redactive algebraic group over F then every irreducible admissible representation of $G(A)$ decomposes into a restricted tensor product of representations of the groups $G(F_v)$.

Theorem 2.1. (Burnside) *Let Ω be algebraically closed. Let R be an Ω-algebra and M be a simple R-module, finite dimensional over Ω. Let $\phi : R \to End_\Omega(M)$ an homomorphism. Then $R/\ker(\phi) \cong End_\Omega(M)$. Morevover, $End_R(M)$ is one dimensional over Ω, and consists of exactly the scalar endomorphisms $m \to \lambda m$ of M, where $\lambda \in \Omega$.*

Proof. Omitted. □

Proposition 2.2. (Bourbaki) *Let A and B be Ω-algebras (with unit). Let $R = A \otimes B$ and P a simple R-module that is finite dimensional over Ω. There exists a simple A-module M and a simple B-module N such that $P \cong M \otimes N$. Moreover, the isomorphism classes of M and N are uniquely determined.*

Proof. Omitted. □

Proposition 2.3. (Bourbaki) *Let Ω be algebraically closed. Let A and B be Ω-algebras, and let $R = A \otimes B$. Let M and N are A- and B- modules, respectively, that are finite dimensional over Ω. Then $M \otimes N$ is a simple R-module and every simple R-module that is finite dimensional over Ω has this form for uniquely determined M and N.*

Proof. We have homomorphisms

$$\phi_M : A \to End_\Omega(M)$$

and

$$\phi_N : A \to End_\Omega(N)$$

given by

$$\phi_M(a)m = a \cdot m \text{ for } a \in A, m \in M,$$

and

$$\phi_M(b)n = b \cdot n \text{ for } b \in B, n \in N.$$

These homomorphisms are surjective (Theorem 2.1). To show that $M \otimes N$ is a simple R-module, it is sufficient to show that it is a simple $End_\Omega(M) \otimes End_\Omega(N)$-module. It is easy to see that the natural map $End_\Omega(M) \otimes End_\Omega(N) \to End_\Omega(M \otimes N)$ is surjective. So it suffices to show that $M \otimes N$ is a simple $End_\Omega(M \otimes N)$-module, and this is clear. The rest follows easily from Proposition 2.2. □

Definition 2.4. *A group is called unimodular if the left and the right Haar measures coincide.*

Definition 2.5. *Let G be a unimodular locally compact totally disconnected group. We will denote with $\mathcal{H}_G$ the Hecke algebra of G, namely, the convolution algebra $C_c^\infty(G)$ of locally constant, compactly supported functions.*

Let K be a compact Lie group. We will denote with $\mathcal{H}_K$ the ring (under convolution) of smooth functions $\phi : K \to \mathbb{C}$ that are K-finite under both left and right translation by elements of K.

Proposition 2.6. *Let K be a compact Lie group. Let (π, V) be a representation of K that is an algebraic direct sum of finite-dimensional representations. Then we obtain a smooth representation $\pi : \mathcal{H}_K \to End(V)$ by*

$$\pi(\phi)v = \int_K \phi(k)\pi(k)v dk.$$

Conversely, if a smooth representation π of $\mathcal{H}_K$ is given, there exists a representation π of K such that the last equation is valid.

Proof. Omitted. □

Proposition 2.7. *Let G be a reductive Lie group, K is maximal compact subgroup and $\mathfrak{g}$ the Lie algebra of G. Let V be a $(\mathfrak{g}, K)$-module. Then V is naturally a smooth module for $\mathcal{H}_G$, and moreover every smooth module for $\mathcal{H}_G$ arises in this fashion.*

Proof. Omitted. □

Remark 2.8. *The content of the last proposition is that the $(\mathfrak{g}, K)$-modules are exactly the smooth modules over $\mathcal{H}_G$.*

Definition 2.9. *Let A be a partially ordered set and $B \subseteq A$.*

$$B \text{ is a cofinal subset of } A \Leftrightarrow \forall a \in A, \exists b \in B \text{ s.t. } a \leq b.$$

Proposition 2.10. *Let M be a nonzero module over the idempotented Ω-algebra $(H, \mathcal{E})$, and let $\mathcal{E}^o$ be a cofinal subset of $\mathcal{E}$. Then*

$$M \text{ is a simple } (H, \mathcal{E}) - \text{module} \iff eM = \begin{cases} 0 \\ \text{a simple } eHe - \text{module } \forall e \in \mathcal{E}^o \end{cases}$$

Proof. Omitted. □

Proposition 2.11. *Let M and N be simple admissible modules over the idempotented Ω-algebra $(H, \mathcal{E})$. Let $\mathcal{E}^o$ be a cofinal subset of $\mathcal{E}$. Then,*

$$M \cong N \iff eM \cong eN \text{ as } eHe - \text{modules } \forall e \in \mathcal{E}^o.$$

Proof. Omitted. □

Proposition 2.12. *Let R be a ring and let e, f be idempotents of R s.t. $ef = fe = e$. Then $f = e + e'$, where e' is idempotent, and $ee' = e'e = 0$. If M is any R-module, then $fM = eM \oplus e'M$. Suppose furthermore that Ω is algebraically closed, R is an Ω-algebra and that eMe is finite dimensional over Ω and simple as an eRe-module. Then $\dim(Hom_{eRe}(eM, fM)) = 1$.*

Proof. Omitted. □

Theorem 2.13. *Let Ω be algebraically closed. Let $(H_1, \mathcal{E}_1)$ and $(H_2, \mathcal{E}_2)$ be idempotented Ω-algebras and let $(H, \mathcal{E}) = (H_1, \mathcal{E}_1) \otimes (H_2, \mathcal{E}_2)$. If M_1 and M_2 are simple admissible H_1- and H_2- modules respectively, then $M_1 \otimes M_2$ is a simple admissible H-module, and every simple admissible H-module has this form. The isomorphism types of M_1 and M_2 are uniquely determined by that of M.*

Proof. Omitted. □

Proposition 2.14. *Let $\mathcal{H}_G$ be the Hecke algebra of a totally disconnected locally compact unimodular group G. Let V be a smooth module over $\mathcal{H}_G$. Then there exists a smooth representation*

$$\pi : G \to End_{\mathbb{C}}(V)$$

such that $\phi \cdot x = \pi(\phi)x$ for $\phi \in \mathcal{H}_G, x \in V$.

Proof. Omitted. □

Proposition 2.15. *Let G_1 and G_2 be locally compact totally disconnected groups. Let (π_i, M_i) be irreducible admissible representations of G_i, $(i = 1, 2)$. Then $(\pi_1 \otimes \pi_2, M_1 \otimes M_2)$ is an irreducible admissible representation of $G_1 \times G_2$, and every irreducible admissible represetation of $G_1 \times G_2$ is of this type.*

Proof. Since $\mathcal{H}_{G_1 \times G_2} \cong \mathcal{H}_{G_1} \otimes \mathcal{H}_{G_2}$, our result follows from Theorem 2.13 and Proposition 2.14. □

Definition 2.16. *Let H be an Ω-algebra. An linear map $\iota : H \to H$ is called antiinvolution if*

$$^{\iota}(xy) =^{\iota} y^{\iota}x.$$

Definition 2.17. *Let $(H, \mathcal{E})$ be an idempotented Ω-algebra. Let an idempotent $e^o \in \mathcal{E}$. We say that e^o is spherical if there exists an antiinvolution $\iota : H \to H$ s.t. $^{\iota}x = x \ \forall x \in e^o H e^o$.*

Note that the existence of such ι implies that $e^o H e^o$ is commutative, because if $x, y \in e^o H e^o$, then $xy =^{\iota} (xy) = yx$.

Theorem 2.18. *Let $(H, \mathcal{E})$ be an idempotented Ω-algebra, and e^o be a spherical idempotent. Let M and N be simple admissible H-modules s.t. $e^o M$ and $e^o N$ are nonzero. Then $e^o M \cong e^o N$ as $e^o H e^o$-modules, $\Rightarrow M \cong N$ as H-modules.*

Proof. Omitted. □

Theorem 2.19. *Let $(H_v, \mathcal{E}_v)$ $(v \in \Sigma)$ be an indexed family of idempotented Ω-algebras, and for almost all v, let $e_v^o \in \mathcal{E}_v$ be a spherical idempotent. Let $(H, \mathcal{E})$ be the restricted tensor product of the H_v, with respect to the e_v^o. (It is itself an idempotented Ω-algebra). For each $v \in \Sigma$ let there be specified a simple admissible module M_v and for almost all v let m_v^o be a nonzero element of $e_v^o M_v$. Let $M \otimes_v M_v$ with respect to the m_v^o. Then M is a simple admissible H-module. Moreover, every simple admissible module is of this type, with uniquely determined modules M_v.*

Proof. Let simple admissible modules M_v and non-zero elements $m_v^o \in e^o M_v \forall v \in \sigma$ be given. We will show that M is simple and admissible.

Let $e = \otimes e_v \in \mathcal{E}$ be given. Then there exists a finite subset S of Σ s.t. if $v \in \Sigma - S$, then $e_v = e_v^o$, and furthermore $e_v H e_v$ is commutative, so $\dim(e_v M) = 1$. Then

$$eM \cong \bigotimes_{v \in S} e_v M_v.$$

Indeed, this is because $\bigotimes_{v \notin S} e_v M_v$ is one dimensional, being spanned by the vector $\otimes_{v \notin S} m_v^o$. So tensoring with this vector is an isomorphism

$$\bigotimes_{u \in S} e_v M_v \to \bigotimes_{u \in \Sigma} e_v M_v = eM.$$

Now the left side (if nonzero) is simple by Theorem 2.13 and Proposition 2.7 (applied to M_v).

By Proposition 2.7 (applied to M), it follows that M is simple.

Now let M be a simple H-module. We must show that $M \cong \bigotimes_v M_v$, where the M_v are simple admissible modules for the H_v, and the tensor product is restricted with respect to $m_v^o \in M_v$. We'll prove this by combining two special cases.

Firstly, if the indexing set Σ is finite, the restricted tensor product is of course the same as the ordinary tensor product, and this result follows by iterated applications of Theorem 2.13.

We next consider another special case. We assume that e_v^o is spherical idempotented $\forall v$, and we also assume, with $e = \otimes_v e_v^o$, that $eM \neq 0$.

By irreducibility eMe has dimension 1, and if m denotes a generator, we obtain a ring homomorphism

$$\gamma : eHe \to \Omega,$$

by $hm = \gamma(h)m, h \in eHe$.

By Proposition 1.7, we may factor γ as

$$\gamma(\otimes_v h_v) = \prod_v \gamma_v(h_v)$$

when $h_v \in e_v^o H_v e_v^o$, where γ_v is a homomorphism $e_v^o H_v e_v^o \to \Omega$.

Now we claim that $\forall v : \exists$ a simple admissible module M_v of H_v and a nonzero element $m_v \in e_v^o M_v$ s.t.

$$h_v m_v = \gamma_v(h_v) m_v.$$

Indeed, we may see this by decomposing $H = H_v \otimes H_v'$, where $H_v' = \otimes_{w \in \Sigma, w \neq v} H_w$ (tensor product restricted by the e_v^o).

By Theorem 2.13, there exist simple admissible modules M_v and M_v' for H_v and H_v', respectively, s.t.

$$eM = e_v^o M_v \otimes e_v' M_v',$$

where $e_v' = \otimes_{w \neq v} e_v^o$.

Now consider $N = \otimes_v M_v$, with respect to the m_v. It is clear that $eN = \otimes_v e_v^o M_v \cong eM$, as eHe-modules, and therefore, by Theorem 2.18

$$M \cong \otimes_v M_v.$$

We deduce the general case from these two special cases.

Choose $e \in \mathcal{E}$ s.t. $v \in \Sigma - S$, then e_v^o is a spherical idempotent.

We represent H as a finite tensor product:

$$H = \bigotimes_{v \in S} H_v \otimes H',$$

where $H' = \bigotimes_{v \in \Sigma - S} H_v$.

Using the first special case proved above (Σ finite) we can write $M = \bigotimes_{v\in S} M_v \otimes M'$, where M_v is a simple admissible module for H_v, and M' is a simple admissible module for H'. By using the second "spherical" special case consider above, we obtain the further decomposition $M' = \bigotimes_{v\in\Sigma-S} M_v$. □

In the sequel F is a number field, A its adele ring and if v is a finite place of F then $\mathfrak{o}_v$ will be the ring of integers of F_v. We denote with S_∞ the set of the infinite places of F and we define

$$K_v = \begin{cases} O(n) & \text{if } v \text{ is a real place} \\ U(n) & \text{if } v \text{ is a complex place} \\ GL(n, \mathfrak{o}_v) & \text{if } v \text{ is a finite place} \end{cases}$$

We define the following

$$\mathfrak{g}_\infty = \prod_{v\in S_\infty} \mathfrak{gl}(n, F_v),$$

$$K_\infty = \prod_{v\in S_\infty} K_v.$$

We can now state the Tensor Product Theorem:

Theorem 2.20. (The Tensor Product Theorem) *Let (V,π) be an irred. admissible representation of $GL(n, A)$.*

- *$\forall$ infinite place v of F: $\exists$ an irred. admissible $(\mathfrak{g}_\infty, K_\infty)$-module (π_v, V_v),*
- *and $\forall$ finite place v : $\exists$ an irred. admissible representation (π_v, V_v) of $GL(n, F_v)$ s.t. for almost all v, V_v contains a nonzero K_v-fixed vector ξ_v^0*

s.t. $\pi = \otimes_v \pi_v$.

2.1 The proof

Let F be a global field, and let A be its adele ring. Let Σ be the set of all places of F.

If $v \in \Sigma$ we have defined a Hecke algebra $\mathcal{H}_{GL(n,F_v)}$ above.

If v is finite, let e_v^o be the characteristic function of $GL(n, \mathfrak{o}_v)$ i.e.

$$e_v^o(x) = \begin{cases} 1 & x \in GL(n, \mathfrak{o}_v) \\ 0 & x \notin GL(n, \mathfrak{o}_v) \end{cases}$$

We normalize the Haar measure on $\mathcal{H}_{GL(n,F_u)}$ so that the volume of $GL(n, \mathfrak{o}_v)$ is one. We claim that e_v^o is idempotent. Indeed,

$$e_v^o(x) \cdot e_v^o(x) = \int_{GL(n,F_v)} e_v^o(y) e_v^o(y^{-1}x) dy =$$

$$= \int_{GL(n,F_v)} e_v^o(x) e_v^o(y) dy = e_v^o(x) \int_{GL(n,F_v)} e_v^o(y) dy =$$

$$e_v^o(x) \int_{GL(n,\mathfrak{o}_v)} dy = e_v^o(x).$$

We claim that e_v^o is spherical.

Indeed, the transpose map on $GL(n, F_v)$ induces an antiinvolution ι on $\mathcal{H}_{GL(n,F_v)}$. It follows from the elementary divisor theorem that a complete set of double coset representatives for $GL(n, \mathfrak{o}_v) \setminus GL(n, F)/GL(n, \mathfrak{o}_v)$ consists of diagonal matrices.

This implies that the spherical Hecke algebra of $GL(n, \mathfrak{o}_v)$-biinvariant functions is commutative.

This spherical Hecke algebra is $e_v^o \mathcal{H}_{GL(n,F_v)} e_v^o$ (by definition).

We define the global Hecke algebra $\mathcal{H}_{GL(n,A)}$ to be the restricted tensor product of the local Hecke algebras $\mathcal{H}_{GL(n,F_v)}$. The tensor product is restricted with respect to the subalgebras $e_v^o \mathcal{H}_{GL(n,F_v)} e_v^o$.

In view of Propositions 2.7 and 2.14, we may reinterpret an irreducible representation of $GL(n, A)$ as a simple admissible module for $\mathcal{H}_{GL(n,A)}$.

With this reinterpretation, Theorem 2.20 follows immediately from Theorem 2.19

□