The Sum of Divisors Function and the Riemann Hypothesis

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Abstruct

The Riemann hypothesis (RH) is well known. In the RH, the Robin criterion is one of the most famous theorems. In this paper we first obtain a new condition equivalent to the RH on the sum of divisors function. This condition is a generalization of the Robin criterion. Next, we prove that the new condition holds unconditionally.

Keywords; Riemann hypothesis; Sum of divisors function; Robin criterion; Hardy-Ramanujan number.

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1. Introduction

The function $\zeta(s)$ defined by an absolute convergent Dirichlet's series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{1}$$

in complex half-plane $\operatorname{Re} s > 1$ is called the Riemann's zeta function ([3]). The Riemann's zeta function has a simple pole with the residue 1 at $s = 1$ and except the point $s = 1$ the function $\zeta(s)$ is analytically continued to whole complex plane. And $\zeta(s)$ is expressed for $\operatorname{Re} s > 1$ as

$$\zeta(s) = \prod_p \left(1 - p^{-s}\right)^{-1}, \tag{2}$$

where infinite product runs over all the prime numbers. Also for $\operatorname{Re} s > 1$ the function $\zeta(s)$ satisfies the functional equation

$$\zeta(s) = 2 \cdot (2\pi)^{s-1} \cdot \Gamma(1-s) \cdot \sin\left(\frac{\pi s}{2}\right) \cdot \zeta(1-s), \tag{3}$$

where $\Gamma(s)$ is the gamma function ([3])

$$\Gamma(s) = \int_0^{+\infty} e^{-s} x^{s-1} dx . \tag{4}$$

From the infinite product of $\zeta(s)$ the Riemann's zeta function has no zeros in $\operatorname{Re} s > 1$ and from the functional equation of $\zeta(s)$ it has trivial zeros $-2, -4, -6, \cdots$ in $\operatorname{Re} s < 0$. The zeros of $\zeta(s)$ in $0 \le \operatorname{Re} s \le 1$ are called the nontrivial zeros of $\zeta(s)$ ([3]). In 1859 G. Riemann conjectured that all the nontrivial zeros of $\zeta(s)$ would lie on the line $\operatorname{Re} s = 1/2$. This is just the Riemenn's hypothesis. There have been published many research results on the RH. But the RH is unsolved until now ([1~8]).

To study the RH we will here consider the sum of divisors function. The sum of divisors function is one of the important arithmetical functions, but its properties are not well known in the RH. In the past, the study of the sum of divisors function had been mostly limited to the relation with the Euler's function and to the relation with the perfect numbers, but for the RH it has been studied after Robin criterion in 1984 ([6]). The Robin criterion is one of the most famous theorems for the RH. Recently, The Robin criterion has been studied in many papers, but it is not still unsolved completely ([1,2]).

In this paper we **first** obtain a new condition equivalent to the RH, which would be called an equivalence condition (EC). The EC is closely related with the RC and it is a generalization of the Robin criterion. And it is easy to consider rather than Robin criterion. We have also a new idea to prove that the EC holds unconditionally. The idea is to introduce a notion, which would be called a sigma-index of the natural number (see [15]). **Next,** in this paper we show that the new condition holds unconditionally. To do this, we work with a new standpoint that any natural number has the three dimensional structure. On the basis of the standpoint, we pass three steps for the completion of the proof. The every step is accompanied with the process reducing the dimension of the natural number or of the sigma-index, and it needs a new method corresponding with that. **The first step** is the relation of the sum of divisors function and the Hardy-Ramanujan number. This relation is also one of some important properties of the sigma-index. **The second step** is an optimization problem of a certain exponential function with the sun of divisors function. This problem is related with the existence of the optimum points of the given exponential function under the constraint of the inequalities. From the result of the second step, we get an estimate on the difference between consecutive primes. **The third step** is related with an inequality on the sum of divisors function. This inequality is also generalized than the Robin criterion, and it is a new one equivalent to the RH. Consequently, we would prove that the EC holds unconditionally. Our proof has the **three steps** and our result consists of **five papers**. This result would give us a firm possibility that the RH is true. We are sure that our result would be right. But we need an objective verification on the result. Our result is of all of our. So we would like to contribute our result to the **INTERNET**. We hope that our result would give the valuable help whom would like to prove the RH.

Let N be the set of the natural numbers. The function $\sigma(n)=\sum_{d|n} d$ is called the sum of divisors function of n ([3,5]). Then the function $\sigma(n)$ is multiplicative on the coprime numbers.

It is well known that the RH is true if and only if it holds that, for any $n \geq 5041$,

$$\sigma(n) \leq e^{\gamma} \cdot n \cdot \log\log n, \tag{5}$$

where $\gamma = 0.577\cdots$ is Euler's constant. This (5) is called the Robin criterion or the Robin inequality. In the paper [2], they showed that the Robin inequality holds for any odd $n \geq 5041$. But the Robin inequality is determinately related with the even numbers. In particular, it is essentially related with the Hardy-Ramanujan number ([2]). In deed, the Robin inequality in the case of all odd numbers is only a corollary of the result on the case of all Hardy-Ramanujan numbers.

2. Main results of the paper

In this section we show the main results of the paper without the proofs.

We have

Theorem 1. The RH is true if and only if there exist constants $c_0 \geq 1$, $c_1 \geq 0$ and $c_2 \geq 0$ such that, for any $n \geq 2$, it holds that

$$\sigma(n) \leq e^{\gamma} \cdot n \cdot \log\log\left(c_0 \cdot n \cdot \exp\left(c_1 \cdot \sqrt{\log n} \cdot \exp\left(c_2 \cdot \sqrt{\log\log(n+1)}\right)\right)\right). \quad (6)$$

Theorem 2. There exists constant $c_0 \geq 1$ such that, for any $n \geq 2$, it holds unconditionally that

$$\sigma(n) \leq e^{\gamma} \cdot n \cdot \log\log\left(c_0 \cdot n \cdot \exp\left(\sqrt{\log n} \cdot \exp\left(\sqrt{\log\log(n+1)}\right)\right)\right). \quad (7)$$

3. The proof of the theorem 1

In this section we will prove the theorem 1. The main idea of the proof is the Robin theorems in the papers [5,6].

We could

Proof of the theorem 1. Suppose that the RH is true. Then, by the Robin's theorem ([5,6]), for any $n \geq 5041$, it holds that

$$\sigma(n) \leq e^{\gamma} \cdot n \cdot \log\log n . \tag{8}$$

Hence there exists a constant $c_0 \geq 1$ such that, for any $n \geq 2$, we have

$$\sigma(n) \leq e^{\gamma} \cdot n \cdot \log\log(c_0 \cdot n) .$$

On the other hand, for any constants $c_1 \geq 0$ and $c_2 \geq 0$, it is clear that

$$\exp\left(c_1 \cdot \sqrt{\log n} \cdot \exp\left(c_2 \cdot \sqrt{\log\log(n+1)}\right)\right) \geq 1 . \tag{9}$$

Therefore we have (6). Suppose that inequality (6) holds, but the RH is not true. Then also by the Robin's theorem ([5,6]) there exist a constant $c > 0$ and a constant $0 < \beta < 1/2$ such that, for infinitely many number n, it holds that

$$e^{\gamma} \cdot n \cdot \log\log n + c \cdot \frac{n \cdot \log\log n}{(\log n)^{\beta}} \leq \sigma(n) . \tag{10}$$

On the other hand, it is clear that

$$\begin{aligned} &\log\log\left(c_0 \cdot n \cdot \exp\left(c_1 \cdot \sqrt{\log n} \cdot \exp\left(c_2 \cdot \sqrt{\log\log(n+1)}\right)\right)\right) = \\ &= \log\left(\log c_0 + \log n + c_1 \cdot \sqrt{\log n} \cdot \exp\left(c_2 \cdot \sqrt{\log\log(n+1)}\right)\right) = \\ &= \log\left(\log n\left(1 + \frac{\log c_0}{\log n} + \frac{c_1 \cdot \sqrt{\log n} \cdot \exp\left(c_2 \cdot \sqrt{\log\log(n+1)}\right)}{\log n}\right)\right) = \end{aligned} \tag{11}$$

$$= \log\log n + \log\left(1 + \frac{\log c_0}{\log n} + \frac{c_1 \cdot \exp\left(c_2 \cdot \sqrt{\log\log(n+1)}\right)}{\sqrt{\log n}}\right) \le$$

$$\le \log\log n + \frac{\log c_0}{\log n} + \frac{c_1 \cdot \exp\left(c_2 \cdot \sqrt{\log\log(n+1)}\right)}{\sqrt{\log n}}.$$

Hence, from (6) and (10), for infinitely many numbers n, we have

$$\begin{aligned} & e^{\gamma} \cdot n \cdot \log\log n + c \cdot \frac{n \cdot \log\log n}{(\log n)^{\beta}} \le \sigma(n) \le \\ & \le e^{\gamma} \cdot n \cdot \left(\log\log n + \frac{\log c_0}{\log n} + \frac{c_1 \cdot \exp\left(c_2 \cdot \sqrt{\log\log(n+1)}\right)}{\sqrt{\log n}}\right) \end{aligned} \tag{12}$$

and

$$c \cdot e^{-\gamma} \cdot \frac{\log\log n}{(\log n)^{\beta}} \le \frac{\log c_0}{\log n} + \frac{c_1 \cdot \exp\left(c_2 \cdot \sqrt{\log\log(n+1)}\right)}{\sqrt{\log n}}. \tag{13}$$

If $c_0 = 1$ or $c_1 = 0$, then (13) is impossible. So suppose that $c_0 > 1$ and $c_1 > 0$. Then since $(1/2 - \beta) > 0$ we have

$$0 < c \cdot e^{-\gamma} \le \left(\frac{\log c_0}{(\log n)^{1-\beta} \cdot \log\log n} + \frac{c_1 \cdot \exp\left(c_2 \cdot \sqrt{\log\log(n+1)}\right)}{(\log n)^{1/2-\beta} \cdot \log\log n}\right) \to 0 \ (n \to \infty)$$

(14)

This is also a contradiction. □

By using the method of the proof of the theorem 1, we have more.

Proposition 3.1. The below statements are equivalent to each other.

a) The RH is true.

b) There exists a constant $c_0 \ge 1$ such that, for any $n \ge 2$, it holds that

$$\sigma(n) \le e^{\gamma} \cdot n \cdot \log\log(c_0 \cdot n). \tag{15}$$

c) There exist constants $c_0 \ge 1$, $c_1 \ge 0$ and $c_2 \ge 0$ such that, for any $n \ge 2$, it holds that

$$\sigma(n) \le e^{\gamma} \cdot n \cdot \log\log\left(c_0 \cdot n \cdot \exp\left(c_1 \cdot \sqrt{\log n} \cdot \exp\left(c_2 \cdot \left(\log\log(n+1)\right)^{\alpha}\right)\right)\right), \tag{16}$$

where $0 < \alpha < 1$.

d) It holds that

$$\limsup_{n\to\infty}\left(\frac{\sigma(n)}{n} - e^{\gamma} \cdot \log\log n\right) \cdot \frac{\sqrt{\log n}}{\exp\left(\sqrt{\log\log(n+1)}\right)} < +\infty . \tag{17}$$

It is not difficult to check the proof of the proposition 3.1. Here the impotant is the meaning of every expression. We note that (15) is one showing the most simple and clear relation with the Robin inequality. We could prove that the (15) holds unconditionally under $c_0 = 6$. In fact, by the concrete calculation, we are able to get $c_0 = \exp\left(\exp\left(e^{-\gamma} \cdot 3/2\right)\right)/2 = 5.0951\cdots$. And (16) is the most generalized type of the Robin ineqyality. We note that $\sqrt{\log n}$ in (16) is unable to change into $(\log n)^{\mu}$ with $\mu > 1/2$. In his paper (see [4]), Ramanujan showed under the RH it holds that

$$\limsup_{n\to\infty}\left(\frac{\sigma(n)}{n} - e^{\gamma} \cdot \log\log n\right) \cdot \sqrt{\log n} \le$$
$$\le e^{\gamma} \cdot \left(4 - 2\sqrt{2} + \gamma - \log 4\pi\right) = -1.39\cdots . \tag{18}$$

It is easy to see that (18) is more weak that (17). Therefore the proposition 3.1 shows that the Ramanujan's formula (18) is a condition equivalent to the RH. Of course, we are able to change $\sqrt{\log\log(n+1)}$ in (17) into $\left(\log\log(n+1)\right)^{\alpha}$ with $0<\alpha<1$.

Note. See more the paper [13,14] for the proof of the theorem 1.

4. The proof of the theorem 2

To prove the theorem 2 we need to take three steps. That reason is explained as follows. As noted in the introduction, we are able to say that any natural number has three-dimentional structure. In fact, suppose that $n = q_1^{\lambda_1} \cdot q_2^{\lambda_2} \cdots q_m^{\lambda_m}$ is a prime factorization of $n \in N$, where $q_1, q_2, \cdots q_m$ are distinct primes and $\lambda_1, \lambda_2, \cdots, \lambda_m$ are non-negative integers. We assume that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \geq 1$. And we put $\omega(n) = m$, $\bar{\lambda}(n) = (\lambda_1, \lambda_2, \cdots, \lambda_m)$ and $\bar{q}(n) = (q_1, q_2, \cdots, q_m)$, which would be called an exponent length, an exponent pattern and a prime factor pattern, respectively. Here $\omega(n) = \sum_{p|n} 1$ ([4]) is the number of the prime factors of a given n. Then we could write any natural number n and the set N as

$$n = n\left(\bar{q}(n),\ \bar{\lambda}(n),\ \omega(n)\right) \tag{19}$$

and

$$N = \bigcup_{\omega(n)} \bigcup_{\bar{\lambda}(n)} \bigcup_{\bar{q}(n)} n\left(\bar{q}(n), \bar{\lambda}(n), \omega(n)\right) \tag{20}$$

respectively. Hence we can say that any natural number n has the three-parameters. Of course, both $\bar{\lambda}(n)$ and $\bar{q}(n)$ are dependent on $\omega(n) = m$. But, if we take such the standpoint at the consideration of the theorem 2, then we would be able to prove it more easily.

Let $p_1 = 2,\ p_2 = 3,\ p_3 = 5, \cdots, p_n, \cdots$ be the first primes ([4]). Here then p_n is n-th prime number. If $\bar{\lambda}(n)$ and $\omega(n)$ are fixed in a given number $n = q_1^{\lambda_1} \cdot q_2^{\lambda_2} \cdots q_m^{\lambda_m}$, then we put $r_0(n) = p_1^{\lambda_1} \cdot p_2^{\lambda_2} \cdots p_m^{\lambda_m}$ to n. If $n = r_0(n)$ then the number n is called a Hardy-Ramanujan number (HRN) ([2]). In other words, the HRN is just a natural number of such forms as $p_1^{\lambda_1} \cdot p_2^{\lambda_2} \cdots p_m^{\lambda_m}$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \geq 1$. The HRN has special properties. In particular, the HRN has the close relations with the sum of divisors function. We put

$$S\left(\overline{\lambda}, m\right) = \left\{ n \in N \mid \overline{\lambda} = \overline{\lambda}(n) = \left(\lambda_1, \lambda_2, \cdots, \lambda_m\right),\ \omega(n) = m \right\}, \tag{21}$$

and

$$HR(m) = \left\{ n \in N \mid n = r_0(n), \omega(n) = m \right\}. \tag{22}$$

Then $S\left(\overline{\lambda}, m\right)$ consists of the natural numbers with $\overline{\lambda}(n) = \left(\lambda_1, \lambda_2, \cdots, \lambda_m\right)$ and $\omega(n) = m$. And the set $HR(m)$ consists of the HRN with $\omega(n) = m$. And for $n = q_1^{\lambda_1} \cdot q_2^{\lambda_2} \cdots q_m^{\lambda_m}$ it holds that

$$\frac{\sigma(n)}{n} = \prod_{i=1}^{m} \frac{1 - q_i^{-\lambda_i - 1}}{1 - q_i^{-1}}. \tag{23}$$

4.1. The sum of divisors function and the Hardy-Ramanujan Number

In this section we will show a relation with the sum of divisors function and the Hardy-Ramanujan number. This relation says that one can reduce the dimension of the natural number at the proof of (7) or the Robin inequality.

4.1.1 The Hardy-Ramanujan number

In this section we will show a property of the HRN. The HRN is a unique minimum element in the set $S\left(\overline{\lambda}, m\right)$. We have

Theorem 4.1.1. For any $n \in S(\bar{\lambda}, m)$ we have $r_0(n) \le n$, that is,

$$r_0(n) = \min_{(q_1, q_2, \cdots q_m)} S(\bar{\lambda}, m). \tag{24}$$

The proof of this theorem 4.1.1 is in the paper [9].

4.1.2. The sum of divisors function

In this section we will show a relation between the sun of divisors function and the HRN. This relation is one of many interesting properties of the sum of divisors function. We have

Theorem 4.1.2. For any $n \in S(\bar{\lambda}, m)$ we have $\frac{\sigma(n)}{n} \le \frac{\sigma(r_0(n))}{r_0(n)}$, i.e.

$$\frac{\sigma(r_0(n))}{r_0(n)} = \max_{(q_1, q_2, \cdots q_m)} \left\{ \frac{\sigma(n)}{n} \right\}. \tag{25}$$

The proof of this theorem 4.1.2 is also in the paper [9].

4.1.3. Some note

We put

$$H(n) = \frac{\exp\left(\exp\left(e^{-\gamma} \cdot \sigma(n)/n\right)\right)}{n}. \tag{26}$$

This $H(n)$ we would like to call a sigma-index of the natural number. Our aim in the proof of the theorem 2 is to obtain upper estimate of the sigma-index $H(n)$. By above the theorem 4.1.1 and the theorem 4.1.2, for any $n \in S(\bar{\lambda}, m)$ we have $H(n) \le H(r_0(n))$. Here $H(r_0(n))$ is related only with the exponent pattern $\bar{\lambda}(n)$ and the exponent length $\omega(n)$. Thus

$H\left(r_0\left(n\right)\right)$ has two-parameters. Hence the consideration of the sigma-index $H\left(n\right)$ on any $n \geq 2$ is reduced to one on the set $\bigcup_m HR\left(m\right)$ of the HRN.

4.2. The sum of divisors function and a certain optimization problem

In this section we will consider the sum of divisors function and the opimization problem of a certain exponential function. By this consideration, we have obtained an estimate for the difference between the consecutive primes. This is a new result at the distribution of the prime numbers. This section is the second step for the proof of the theorem 2. We here assume that $\overline{\lambda} = \left(\lambda_1, \lambda_2, \cdots, \lambda_m\right)$ are real numbers and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \geq 1$. Of course, let $p_1 = 2,\ p_2 = 3, \cdots, p_m, \cdots$ be consecutive primes. We will choose $p_m \geq 5$ arbitrarily and fix it. We define functions $F\left(\overline{\lambda}\right)$ and $H\left(\overline{\lambda}\right)$ respectively by

$$F\left(\overline{\lambda}\right) = F\left(\lambda_1,\ \lambda_2, \cdots, \lambda_m\right) = \prod_{i=1}^{m} \frac{1 - p_i^{-\lambda_i - 1}}{1 - p_i^{-1}}, \tag{27}$$

$$H\left(\overline{\lambda}\right) = H\left(\lambda_1,\ \lambda_2, \cdots, \lambda_m\right) = \frac{\exp\left(\exp\left(e^{-\gamma} \cdot F\left(\overline{\lambda}\right)\right)\right)}{p_1^{\lambda_1} \cdot p_2^{\lambda_2} \cdots p_m^{\lambda_m}}, \tag{28}$$

where $\gamma = 0.577\cdots$ is Euler's constant ([3,4]). The aim of this section is to show an existence of the optimum points of the function $H\left(\overline{\lambda}\right)$ in the m-dimensional real space R^m and to estimate the optimum points.

4.2.1. An existence of the optimum points of the function $H\left(\overline{\lambda}\right)$

Here we will show that the function $H\left(\overline{\lambda}\right)$ has an optimum point in the space R^m. The maximum value theorem of the continuous function is used here. We get

Theorem 4.2.1 There exist $\overline{\lambda}_0=\left(\lambda_1^0,\, \lambda_2^0, \cdots, \lambda_m^0\right)\in R^m$ such that for any $\left(\lambda_1,\, \lambda_2, \cdots, \lambda_m\right)\in R^m$ we have $H\left(\overline{\lambda}\right)\le H\left(\overline{\lambda}_0\right)$, that is,

$$H\left(\lambda_1^0,\, \lambda_2^0, \cdots, \lambda_m^0\right)=\max_{(\lambda_1,\,\lambda_2,\cdots,\lambda_m)\in R^m}\frac{\exp\left(\exp\left(e^{-\gamma}\cdot\prod_{i=1}^{m}\frac{1-p_i^{-\lambda_i-1}}{1-p_i^{-1}}\right)\right)}{p_1^{\lambda_1}\cdot p_2^{\lambda_2}\cdots p_m^{\lambda_m}}. \qquad (29)$$

There is the proof of this theorem 4.2.1 in the paper [10].

4.2.2. The estimate of the optimum points of the function $H\left(\overline{\lambda}\right)$

Here we will estimate the optimum points $\overline{\lambda}_0=\left(\lambda_1^0,\, \lambda_2^0, \cdots, \lambda_m^0\right)$ of the function $H\left(\overline{\lambda}\right)$ obtained from the theorem 4.2.1. The optimization problem of the function $H\left(\overline{\lambda}\right)$ with the constraints of the certain inequalities is discussed here. We obtain

Theorem 4.2.2 Assume that $p_m\ge 5$. Then for the optimum points $\overline{\lambda}_0=\left(\lambda_1^0,\, \lambda_2^0, \cdots, \lambda_m^0\right)\in R^m$ of the function $H\left(\overline{\lambda}\right)$ in the space R^m we have;

① There exist some λ_i^0 in $\left\{\lambda_1^0,\, \lambda_2^0, \cdots \lambda_m^0\right\}$ such that $\lambda_i^0=1$. In particular, we have $\lambda_m^0=1$.

② There exist some λ_i^0 in $\left\{\lambda_1^0,\, \lambda_2^0, \cdots \lambda_m^0\right\}$ such that $\lambda_i^0>1$. In particular, we have $\lambda_1^0>1$.

③ There exists a number k such that

$$\lambda_1^0 > \lambda_2^0 > \cdots > \lambda_k^0 > \lambda_{k+1}^0 = \cdots = \lambda_m^0 = 1. \qquad (30)$$

In particular, for any $i\left(1 \le i \le k\right)$ we have

$$\lambda_i^0 = \left(\frac{\log p_m}{\log p_i} + \frac{\log\log p_m}{\log p_i} - 1\right) + \mathrm{O}\left(\frac{1}{\log p_i \cdot \log p_m}\right). \qquad (31)$$

There is also the proof of this theorem 4.2.2 in the paper [10].

The last bigger number k than 1 in the optimum points $\left\{\lambda_1^0, \lambda_2^0, \cdots, \lambda_m^0\right\}$ of the function $H\left(\bar{\lambda}\right)$ is especially important. We will here discuss λ_k, p_k and k in detail. In the furture, we assume that $p_m \ge 5$. We have

Theorem 4.2.3. For the number k such that $\lambda_1^0 > \lambda_k^0 > \lambda_{k+1}^0 = 1$ we have;

$$① \quad \lambda_k^0 = 1 + \mathrm{O}\left(\frac{1}{\log p_m}\right), \qquad (32)$$

$$② \quad p_k = \sqrt{p_m \cdot \log p_m} \cdot \left(1 + \mathrm{O}\left(\frac{1}{\log p_m}\right)\right), \qquad (33)$$

$$③ \quad k = 2\sqrt{m} \cdot \left(1 + \mathrm{O}\left(\frac{\log\log p_m}{\log p_m}\right)\right). \qquad (34)$$

There is also the proof of this theorem 4.2.3 in the paper [10].

Note. In the proof of the theorem 4.2.3, we have taken a certain suitable constant $a > 1$ determining the region $\prod \subset R_+^m$ such that there exist the optimum points $\bar{\lambda}_0 = \left(\lambda_1^0, \lambda_2^0, \cdots \lambda_m^0\right) \in R^m$ of the function $H\left(\bar{\lambda}\right)$.

Let's estimate the size of the constant $a > 1$.

In general, since $\lambda_1^0 \ge \lambda_2^0 \ge \cdots \ge \lambda_m^0 \ge 1$, it is sufficient to take a constant $a > 1$ such that $1 < \lambda_1^0 \le a$. On the other hand, since

$$p_1^{\lambda_1^0+1} = \left(e^{-\gamma} F\left(\overline{\lambda}_0\right)\right) \cdot \exp\left(e^{-\gamma} F\left(\overline{\lambda}_0\right)\right) + 1 = \\ = p_m \cdot \log p_m \cdot \left(1 + \varepsilon\left(p_m\right)\right), \tag{35}$$

we get

$$\lambda_1^0 = \frac{\log\left(p_m \cdot \log p_m \cdot \varepsilon\left(p_m\right)\right)}{\log p_1} - 1.$$

Hence we can take the constant $a > 1$ as

$$a = \frac{\log p_m + \log\log p_m}{\log p_1} + 1. \tag{36}$$

4.2.3. The estimate of $F\left(\overline{\lambda}_0\right)$

By the theorem 4.2.1 and the theorem 4.2.2, for the optimum points $\overline{\lambda}_0 = \left(\lambda_1^0,\, \lambda_2^0, \cdots, \lambda_m^0\right) \in R^m$ in m-dimensional real space R^m of the function $H\left(\overline{\lambda}\right)$, the function value $H\left(\lambda_1^0,\, \lambda_2^0, \cdots, \lambda_m^0\right)$ is dependent only on p_m. So we can put

$$C_m = H\left(\lambda_1^0,\, \lambda_2^0, \cdots, \lambda_m^0\right) = \frac{\exp\left(\exp\left(e^{-\gamma} \cdot F\left(\overline{\lambda}_0\right)\right)\right)}{p_1^{\lambda_1^0} \cdot p_2^{\lambda_2^0} \cdots p_m^{\lambda_m^0}}. \tag{37}$$

In this connection, we will put

$$\begin{cases} n_0 = p_1^{\lambda_1^0} p_2^{\lambda_2^0} \cdots p_k^{\lambda_k^0} \cdot p_{k+1}^1 \cdots p_m^1, \quad n_0' = n_0 \cdot p_m^{-1}, \\ \overline{\lambda}_0' = \left(\lambda_1^0,\, \lambda_2^0, \cdots, \lambda_{m-1}^0\right) \in R^{m-1}, \\ C_{m-1}' = H\left(\overline{\lambda}_0'\right) = H\left(\lambda_1^0,\, \lambda_2^0, \cdots, \lambda_{m-1}^0\right) \end{cases} \tag{38}$$

and

$$C_{m-1} = \max_{(\lambda_1,\, \lambda_2, \cdots, \lambda_{m-1}) \in R^m} H\left(\lambda_1,\, \lambda_2, \cdots, \lambda_{m-1}\right). \tag{39}$$

Then it is clear that $C_{m-1}' \le C_{m-1}$.

Let $\overline{\lambda}' = \left(\lambda_1' , \lambda_2' , \cdots, \lambda_{m-1}'\right)$ be the optimum points of the function $H\left(\lambda_1 , \lambda_2 , \cdots, \lambda_{m-1}\right)$ with $(m-1)$-variable in the space R^{m-1}. In general, then we have

$$\lambda_1' > \lambda_2' > \cdots > \lambda_{k-1}' > \lambda_k' = \cdots = \lambda_{m-1}' = 1. \tag{40}$$

Rarely, the last bigger number than 1 in $\left\{\lambda_1' , \lambda_2' , \cdots, \lambda_{m-1}'\right\}$ could be k. But it is not essential. It is important that for any $i \left(1 \le i \le k-1\right)$

$$p_1^{\lambda_1'+1} = p_2^{\lambda_2'++1} = \cdots = p_{k-1}^{\lambda_{k-1}'+1} = \left(e^{-\gamma} \cdot F\left(\overline{\lambda}'\right)\right) \cdot \exp\left(e^{-\gamma} \cdot F\left(\overline{\lambda}'\right)\right) + 1 \tag{41}$$

holds. We note that it doesn't exceed one in $\left\{\lambda_1^0, \lambda_2^0, \cdots, \lambda_m^0\right\}$.

We also put

$$\begin{cases} n' = p_1^{\lambda_1'} \, p_2^{\lambda_2'} \cdots p_{k-1}^{\lambda_k'} \cdot p_k^1 \cdots p_{m-1}^1, \\ n_+' = p_1^{\lambda_1'} \, p_2^{\lambda_2'} \cdots p_{k-1}^{\lambda_k'} \cdot p_k^1 \cdots p_{m-1}^1 \cdot p_m^1 = n' \cdot p_m^1, \\ \overline{\lambda}_+' = \left(\lambda_1' , \lambda_2' , \cdots, \lambda_{m-1}', 1\right), \; C_m' = H\left(\overline{\lambda}_+'\right). \end{cases} \tag{42}$$

The aim of this section is to estimate the size of $\left(\log C_{m-1} - \log C_{m-1}'\right)$.

On the other hand, it is well known that

$$\sum_{p \le p_m} \frac{1}{p} = \log\log p_m + b_0 + E_0\left(p_m\right), \tag{43}$$

where

$$b_0 = \gamma + \sum_p \left(\log\left(1 - \frac{1}{p}\right) + \frac{1}{p} \right) = 0.241\cdots \tag{44}$$

([4,7,8]). And there exists a constant $a > 0$ such that

$$E_0\left(p_m\right) = \mathrm{O}\left(\exp\left(-a\sqrt{\log p_m}\right)\right). \tag{45}$$

In this section we will estimate the value $F\left(\overline{\lambda}_0\right)$ for the optimum points $\overline{\lambda}_0 = \left(\lambda_1^0, \lambda_2^0, \cdots \lambda_m^0\right) \in R^m$ of the function $H\left(\lambda_1, \lambda_2, \cdots, \lambda_m\right)$.

We have

Theorem 4.2.4 For the optimum points $\overline{\lambda}_0 = \left(\lambda_1^0, \lambda_2^0, \cdots \lambda_m^0\right) \in R^m$ of the function $H\left(\lambda_1, \lambda_2, \cdots, \lambda_m\right)$ we have

$$F\left(\overline{\lambda}_0\right) = e^{\gamma} \cdot \log p_m \cdot \left(1 + E_0\left(p_m\right) - \frac{4}{\sqrt{p_m} \cdot \log^{3/2} p_m} + \varepsilon\left(p_m\right)\right), \quad (46)$$

where $\varepsilon\left(p_m\right) = \mathrm{O}\left(E_0^2\left(p_m\right)\right)$. Hence we also have

$$\left(e^{-\gamma} \cdot F\left(\overline{\lambda}_0\right)\right) \cdot \exp\left(e^{-\gamma} \cdot F\left(\overline{\lambda}_0\right)\right) = \quad (47)$$

$$= p_m \cdot \log p_m \cdot \left(1 + \left(\log p_m + 1\right) \cdot E_0\left(p_m\right) - \frac{4 \cdot \left(\log p_m + 1\right)}{\sqrt{p_m} \cdot \log^{3/2} p_m} + \tilde{\varepsilon}\left(p_m\right)\right),$$

where $\tilde{\varepsilon}\left(p_m\right) = \mathrm{O}\left(\log^2 p_m \cdot E_0^2\left(p_m\right)\right)$.

Proof. This could be found in the paper [11].

4.2.4. The estimate of $\left(\log C_{m-1} - \log C'_{m-1}\right)$

In this section we will estimate $\left(\log C_{m-1} - \log C'_{m-1}\right)$**.** This consideration is for next section. We get

Theorem 4.2.5 There exists a number m_0 such that for any $m \geq m_0$ we have

$$\log C_{m-1} - \log C'_{m-1} = \frac{p_m - p_{m-1}}{\sqrt{p_{m-1} \cdot \log p_{m-1}}} \cdot \left(\frac{p_m - p_{m-1}}{p_{m-1}}\right) \cdot \left(1 + \alpha\left(p_m\right)\right), \quad (48)$$

where $\alpha\left(p_m\right) = \mathrm{O}\left(\frac{1}{\log p_m}\right)$. (49)

Proof. This could be also found in the paper [11].

4.2.5. The estimate of $(p_m - p_{m-1})$

In this section we will estimate the size of $(p_{m+1} - p_m)$. Here obtained result on $(p_{m+1} - p_m)$ is a new result for the distribution of the prime number.

We have

Theorem 4.2.6. There exist a number m_0 such that for any $m \geq m_0$ we have

$$(p_m - p_{m-1}) = \mathrm{O}\left(\sqrt{p_{m-1}} \cdot \log^{5/2} p_{m-1}\right). \tag{50}$$

Proof. This could be also found in the paper [11].

4.2.6. The estimate of $E_0(p_m)$

In this section we will estimate the size of the error iterm $E_0(p_m)$ given in the formular (101).

We get

Theorem 4.2.7 There exists a number m_0 such that for any $m \geq m_0$ we have

$$E_0(p_m) = \mathrm{O}\left(\frac{\log^2 p_m}{\sqrt{p_m}}\right). \tag{51}$$

Proof. This could be also found in the paper [11].

4.3. The sum of divisors function and a related inequality (The proof of the theorem 2)

In this section we will consider one inequality on the sum of divisors function. This inequality, in deed, is the proof of the theorem 2.

We put

$$G(n)=\frac{\left(\exp\left(\exp\left(e^{-\gamma}\cdot\sigma(n)/n\right)\right)\right)/n}{\exp\left(\sqrt{\log n}\cdot\exp\left(\sqrt{\log\log(n+1)}\right)\right)}. \tag{52}$$

Proof of the theorem 2. There are two steps for the proof of the theorem 2.

① The function $G(n)$ has the following properties.

First, For any $n\in S(\bar{\lambda},\, m)$ it holds that $G(n)\le G(r_0(n))$.

In fact, it is clear by the theorem 4.1.1 and the theorem 4.1.2.

Second, for $n=p_1^{\lambda_1}\cdot p_2^{\lambda_2}\cdots p_m^{\lambda_m}$ we put $G(n)=G(\bar{\lambda})=G(\lambda_1,\,\lambda_2,\cdots,\lambda_m)$. Then there exist $\bar{\alpha}_0=\left(\alpha_1^0,\,\alpha_2^0,\cdots\alpha_m^0\right)\in R^m$ such that for any $(\lambda_1,\,\lambda_2,\cdots,\lambda_m)\in R^m$ we have $G(\bar{\lambda})\le G(\bar{\alpha}_0)$. This is also clear by the theorem 4.2.1. And, for the optimum points $\bar{\alpha}_0=\left(\alpha_1^0,\,\alpha_2^0,\cdots\alpha_m^0\right)\in R^m$ of the function $G(\bar{\lambda})$, such the results as in the theorem 4.2.2 and the theorem 4.2.3 are valid. Also for any $n\ge 2$ we have

$$G(n)\le H(n)=\left(\exp\left(\exp\left(e^{-\gamma}\cdot\sigma(n)/n\right)\right)\right)/n. \tag{53}$$

Finally, The every member α_i^0 $(i=1,m)$ of the optimum points $\left\{\alpha_1^0,\,\alpha_2^0,\cdots\alpha_m^0\right\}$ of the function $G(\bar{\lambda})$ is not larger than λ_i^0 $(i=1,m)$ of one of the function $H(\bar{\lambda})$, namely, for any i $(1\le i\le m)$ it holds that $\alpha_i^0\le\lambda_i^0$.

In fact, by the theorem 4.2.2, for the function $H(\bar{\lambda})$ it holds that

$$\begin{aligned} p_1^{\lambda_1^0+1}&=p_2^{\lambda_2^0+1}=\cdots=p_k^{\lambda_k^0+1}=\\ &=\left(e^{-\gamma}F(\bar{\lambda}_0)\right)\cdot\exp\left(e^{-\gamma}F(\bar{\lambda}_0)\right)+1\quad(1\le i\le k). \end{aligned} \tag{54}$$

Similarly, for the function $G(\bar{\lambda})$ it holds that

$$\begin{aligned} p_1^{\alpha_1^0+1}&=p_2^{\alpha_2^0+1}=\cdots=p_k^{\alpha_k^0+1}=\\ &=\left(e^{-\gamma}F(\bar{\alpha}_0)\right)\cdot\exp\left(e^{-\gamma}F(\bar{\alpha}_0)\right)\cdot\left(\frac{1}{1+\Psi(n)}\right)+1\quad(1\le i\le k), \end{aligned} \tag{55}$$

where

$$\Psi(n)=\frac{\exp\left(\sqrt{\log\log(n+1)}\right)}{2\cdot\sqrt{\log n}}+ \\ +\frac{\exp\left(\sqrt{\log\log(n+1)}\right)}{2\cdot\sqrt{\log\log(n+1)}}\cdot\frac{\sqrt{\log n}}{\log(n+1)}\cdot\left(\frac{n}{n+1}\right)\to 0\ (n\to\infty). \tag{56}$$

Hence for any $i\ (1\le i\le m)$ we have $\alpha_i^0\le\lambda_i^0$ and, in particular, we have

$$F(\bar{\alpha}_0)=\prod_{i=1}^{m}\frac{1-p_i^{-\alpha_i^0-1}}{1-p_i^{-1}}\le\prod_{i=1}^{m}\frac{1-p_i^{-\lambda_i^0-1}}{1-p_i^{-1}}=F(\bar{\lambda}_0). \tag{57}$$

② We put

$$D_m=G\left(\alpha_1^0,\ \alpha_2^0,\cdots,\alpha_m^0\right) \tag{58}$$

and

$$\begin{cases} n_0=p_1^{\alpha_1^0}p_2^{\alpha_2^0}\cdots p_k^{\alpha_k^0}\cdot p_{k+1}^1\cdots p_m^1, \quad n_0'=n_0\cdot p_m^{-1}, \\ \bar{\alpha}_0'=\left(\alpha_1^0,\ \alpha_2^0,\cdots,\alpha_{m-1}^0\right)\in R^{m-1}, \\ D_{m-1}'=G(\bar{\alpha}_0')=G\left(\alpha_1^0,\ \alpha_2^0,\cdots,\alpha_{m-1}^0\right). \end{cases} \tag{59}$$

In this connection, we put

$$D_{m-1}=\max_{(\lambda_1,\ \lambda_2,\cdots\lambda_{m-1})\in R^{m-1}}G\left(\lambda_1\ ,\ \lambda_2,\cdots\lambda_{m-1}\right). \tag{60}$$

Then it is clear that $D_{m-1}'\le D_{m-1}$ and

$$\begin{aligned} \log\frac{D_m}{D_{m-1}'}&=\left(\exp\left(e^{-\gamma}\cdot F(\bar{\alpha}_0)\right)-\exp\left(e^{-\gamma}\cdot F(\bar{\alpha}_0')\right)\right)- \\ &-\left(\log n_0+\sqrt{\log n_0}\cdot\exp\left(\sqrt{\log\log(n_0+1)}\right)\right)+ \\ &+\left(\log n_0'+\sqrt{\log n_0'}\cdot\exp\left(\sqrt{\log\log(n_0'+1)}\right)\right)= \\ &=\exp\left(e^{-\gamma}\cdot F(\bar{\alpha}_0')\right)\left(\exp\left(e^{-\gamma}\cdot F(\bar{\alpha}_0')\cdot\frac{1}{p_m}\right)-1\right)-(\log p_m)- \\ &-\left(\sqrt{\log n_0}\cdot\exp\left(\sqrt{\log\log(n_0+1)}\right)-\sqrt{\log n_0'}\cdot\exp\left(\sqrt{\log\log(n_0'+1)}\right)\right). \end{aligned} \tag{61}$$

By the theorem 4.2.7, we have

$$\begin{aligned}
&\exp\left(e^{-\gamma}\cdot F\left(\bar{\alpha}_0'\right)\right)\left(\exp\left(e^{-\gamma}\cdot F\left(\bar{\alpha}_0'\right)\cdot\frac{1}{p_m}\right)-1\right)\le \\
&\le \exp\left(e^{-\gamma}\cdot F\left(\bar{\lambda}_0'\right)\right)\left(\exp\left(e^{-\gamma}\cdot F\left(\bar{\lambda}_0'\right)\cdot\frac{1}{p_m}\right)-1\right)= \qquad (62)\\
&=\log p_m+\Theta_1\left(p_m\right),
\end{aligned}$$

where $\Theta_1\left(p_m\right)=\mathrm{O}\left(\frac{\log^4 p_m}{\sqrt{p_m}}\right)$. So there is a constant a $a>0$ such that

$$\Theta_1\left(p_m\right)\le a\cdot\frac{\log^4 p_m}{\sqrt{p_m}}. \qquad (63)$$

On the other hand, we have

$$\begin{aligned}
&\log n_0=\log\left(p_1^{\alpha_1^0}p_2^{\alpha_2^0}\cdots p_k^{\alpha_k^0}\cdot p_{k+1}^1\cdots p_m^1\right)=\sum_{i=1}^{m}\alpha_i^0\cdot\log p_i= \\
&=\sum_{i=1}^{m}\log p_i+\sum_{i=1}^{k}\left(\alpha_i^0-1\right)\cdot\log p_i=\vartheta\left(p_m\right)+\vartheta\left(p_k\right)+R_k
\end{aligned} \qquad (64)$$

where $\vartheta\left(p_m\right)=\sum_{i=1}^{m}\log p_i$ is the Chebyshev's function ([4,8]) and $R_k=o\left(p_k\right)$.

Hence by the prime number theorem ([3,4,8]), we have

$$\frac{\log n_0}{p_m}=\frac{\vartheta\left(p_m\right)}{p_m}+\frac{\vartheta\left(p_k\right)}{p_m}+\frac{R_k}{p_m}\to 1\ \left(p_m\to\infty\right). \qquad (65)$$

From this we get

$$\log n_0=p_m\cdot\left(1+\theta_1\left(p_m\right)\right), \qquad (66)$$

where $\theta_1\left(p_m\right)=\mathrm{O}\left(\frac{1}{\log p_m}\right)$. So we also obtain

$$\log n_0'=p_{m-1}\left(1+\theta_2\left(p_{m-1}\right)\right). \qquad (67)$$

where $\theta_2\left(p_{m-1}\right)=\mathrm{O}\left(\frac{1}{\log p_{m-1}}\right)$. And it is easy to see that

$$\left(\sqrt{\log n_0}\cdot\exp\left(\sqrt{\log\log(n_0+1)}\right)-\sqrt{\log n_0'}\cdot\exp\left(\sqrt{\log\log(n_0'+1)}\right)\right)=$$
$$=\left(\sqrt{\log n_0}-\sqrt{\log n_0'}\right)\cdot\exp\left(\sqrt{\log\log(n_0+1)}\right)+$$
$$+\sqrt{\log n_0'}\cdot\left(\exp\left(\sqrt{\log\log(n_0+1)}\right)-\exp\left(\sqrt{\log\log(n_0'+1)}\right)\right)= \tag{68}$$
$$=\exp\left(\sqrt{\log p_m}\right)\cdot\left(\frac{\log p_m}{2\cdot\sqrt{p_m}}\right)\cdot\left(1+\Theta_2\left(p_m\right)\right),$$

where $\Theta_2\left(p_m\right)=\mathrm{O}\left(\frac{1}{\log p_m}\right)$. Hence we have

$$\log D_m-\log D'_{m-1}\le a\cdot\frac{\log^4 p_m}{\sqrt{p_m}}-$$
$$-\exp\left(\sqrt{\log p_m}\right)\cdot\frac{\log p_m}{2\cdot\sqrt{p_m}}\left(1+\Theta_2\left(p_m\right)\right). \tag{69}$$

On the other hand, it is clear that

$$\frac{\log^3 p_m}{\exp\left(\sqrt{\log p_m}\right)}\to 0\ \left(p_m\to\infty\right) \tag{70}$$

This shows that there exists a number m_0 such that for any $m\ge m_0$ we have

$$D_m< D'_{m-1}\le D_{m-1}. \tag{71}$$

From this we get

$$0<c_0=\sup_m D_m<+\infty. \tag{72}$$

This is the proof of the theorem 2. □

See also more in the paper [12].

Note. ① We are able to see that

$$c_0=D_1=\frac{\exp\left(\exp\left(e^{-\gamma}\cdot 3/2\right)\right)/2}{\exp\left(\sqrt{\log 2}\cdot\exp\left(\sqrt{\log\log 3}\right)\right)}=1.6436\cdots\le 2 \tag{73}$$

② The process for the proof of the theorem 2 is graphically as follows. Here ⇒ shows the increasing direction of the values for the function $H(n)$ and $G(n)$.

$$n = q_1^{\lambda_1} \cdot q_2^{\lambda_2} \cdot q_3^{\lambda_3} \cdots q_{m-1}^{\lambda_{m-1}} \cdot q_m^{\lambda_m}$$

$$\Downarrow \qquad \leftarrow \text{theorem 4.1.2}$$

$$r_0(n) = p_1^{\lambda_1} \cdot p_2^{\lambda_2} \cdot p_3^{\lambda_3} \cdots p_m^{\lambda_m}$$

$$\Downarrow \qquad \leftarrow \text{theorem 4.2.2}$$

$$n_0 = p_1^{\lambda_1^0} \cdot p_2^{\lambda_2^0} \cdots p_k^{\lambda_k^0} \cdot p_{k+1}^1 \cdots p_m^1,$$

$$\Downarrow \qquad \leftarrow \text{theorem 2}$$

$$n_0' = p_1^{\lambda_1^0} \cdot p_2^{\lambda_2^0} \cdots p_k^{\lambda_k^0} \cdot p_{k+1}^1 \cdots p_{m-1}^1$$

$$\Downarrow \quad \swarrow$$

$$\boxed{n = 2}$$

As it was indicated in the paper [1], one can say that any natural number has the three-dimensional structure. For $\overline{q}(n) = (q_1, q_2, \cdots q_m)$, $\overline{\lambda}(n) = (\lambda_1, \lambda_2, \cdots \lambda_m)$ and $\omega(n) = m$ of $n = q_1^{\lambda_1} \cdot q_2^{\lambda_2} \cdots q_m^{\lambda_m}$ we put $n = n(\overline{q}(n),\ \overline{\lambda}(n),\ \omega(n))$. Then to prove the theorem we have taken the process reducing the dimensional numbers of $n = n(\overline{q}(n),\ \overline{\lambda}(n),\ \omega(n))$ in the function $G(n)$. The dimensional numbers of n in the function $G(n)$ were reduced by the theorem 4.1.2 and the theorem 4.2.2, respectively. That is so; $n = n(\overline{q}(n), \overline{\lambda}(n), \omega(n)) \to n(\overline{\lambda}(n), \omega(n)) \to n(\overline{\lambda}_0, \omega(n)) \to n(m)$.

③ The below table 1 shows the optimum points $\overline{\lambda}_0 = (\lambda_1^0, \lambda_2^0, \cdots, \lambda_m^0)$ of the function $H(\overline{\lambda})$ and the values of $H(n_0)$ and $G(n_0)$ to $\omega(n) = m$.

Table 1

$\omega(n)$ $=m$	$\overline{\lambda}=\left(\lambda_1^0,\lambda_2^0,\cdots,\lambda_m^0\right)$ of $n_0=2^{\lambda_1^0}\cdot 3^{\lambda_2^0}\cdot 5^{\lambda_2^0}\cdots p_k^{\lambda_k^0}\cdot p_{k+1}^1\cdots p_m^1$	$H\left(n_0\right)$, $G(n_0)$
1	$\lambda_1^0=1$	$5.09518716186\cdots$, $1.643686767536\cdots$
2	$\lambda_1^0=1.65\cdots$, $\lambda_2^0=1$	$3.58945411446\cdots$, $0.8250082\times 10^{-1}\cdots$
3	$\lambda_1^0=2.70\cdots,\lambda_2^0=1.33\cdots,\lambda_3^0=1$	$1.91192398575\cdots$, $0.7148367\times 10^{-5}\cdots$
4	$\lambda_1^0=3.36\cdots$, $\lambda_2^0=1.75\cdots$, $\lambda_3^0=1$, $\lambda_4^0=1$	$1.32309514626\cdots$, $0.1065950\times 10^{-6}\cdots$
5	λ_1^0=4.22$\cdots$, λ_2^0=2.29$\cdots$, λ_3^0=1.24$\cdots$, λ_4^0=$\lambda_5^0=1$	$0.57062058635\cdots$, $0.3761569\times 10^{-9}\cdots$
6	λ_1^0=4.53$\cdots$, λ_2^0=2.49$\cdots$, λ_3^0=1.38$\cdots$, λ_4^0=$\lambda_5^0=\lambda_6^0=1$	$0.40977025702\cdots$, $0.767767\times 10^{-10}\cdots$
7	λ_1^0=5.02$\cdots$, λ_2^0=2.80$\cdots$, λ_3^0=1.59$\cdots$, λ_4^0=1.14$\cdots$, $\lambda_5^0=\lambda_6^0=\lambda_7^0=1$	$0.22782964552\cdots$, $0.575576\times 10^{-11}\cdots$
8	λ_1^0=5.22$\cdots$, λ_2^0=2.92$\cdots$, λ_3^0=1.68$\cdots$, λ_4^0=1.21$\cdots$, $\lambda_5^0=\lambda_6^0=\lambda_7^0=\lambda_8^0=1$	$0.20507350097\cdots$, $0.164730\times 10^{-12}\cdots$
9	λ_1^0=5.57$\cdots$, λ_2^0=3.14$\cdots$, λ_3^0=1.83$\cdots$, λ_4^0=1.34$\cdots$, $\lambda_5^0=\lambda_6^0=\lambda_7^0=\lambda_8^0=\lambda_9^0=1$	$0.16722089980\cdots$, $0.287587\times 10^{-14}\cdots$
$\cdots$	$\cdots\ \cdots\ \cdots$	$\cdots\ \cdots$

③ The below table 2 shows the Hardy-Ramanujan’s numbers, which give maximum value of the function $G(n_0)$ to $\omega(n)=m$.

Table 2

$\omega(n) = m$	$\tilde{n}_0 = r_0(\tilde{n}_0) = p_1^{\lambda_1} \cdots p_k^{\lambda_k} \cdot p_{k+1}^1 \cdots p_m^1$	$G(\tilde{n}_0)$
1	2	$1.643686767536\cdots$
2	$2 \cdot 3$	$0.82500822 \times 10^{-1} \cdots$
3	$2^2 \cdot 3 \cdot 5$	$0.71483676 \times 10^{-5} \cdots$
4	$2^3 \cdot 3^2 \cdot 5 \cdot 7$	$0.10659507 \times 10^{-6} \cdots$
5	$2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$	$0.37615690 \times 10^{-9} \cdots$
6	$2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13$	$0.76776726 \times 10^{-10} \cdots$
7	$2^5 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17$	$0.575576185 \times 10^{-11} \cdots$
8	$2^5 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19$	$0.164730227 \times 10^{-12} \cdots$
9	$2^5 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$	$0.287587585 \times 10^{-14} \cdots$
$\cdots$	$\cdots \quad \cdots \quad \cdots$	$\cdots$

===&&&===

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