PHILOSOPHICAL TRANSACTIONS.

Monday, *April* 13. 1668

The Contents.

The Squaring of the Hyperbola, *by an infinite series of Rational Numbers, together with its Demonstration, by that Eminent Mathematician, the Right Honourable the Lord Viscount* Brouncker.

WHat the Acute Dr. *John Wallis* had intimated, some years since, in the Dedication of his Answer to M. *Meibomius de proportionibus, vid.* That the World one day would learn from the Noble Lord *Brounker*, the *Quadrature* of the *Hyperbole*; the Ingenious Reader may see performed in the subjoyned operation, which its Excellent Author was now pleased to communicate, as followeth in his own words;

My Method for Squaring the Hyperbola *is this:*

LEt AB be one *Asymptote* of the Hyperbola EdC; and let AE and BC be parallel to th'other: Let also AE be to BC as 2 to 1; and let the Parallelogram ABDE equal 1. See *Fig.* 1. And note, that the Letter x every where stands for Multiplication.

Supposing the Reader knows, that EA. αζ. KH. βν. dθ. γκ. δλ. εμ. CB.&c. are in an *Harmonic* series, or a *series reciproca primanorum seu arithmetice proportionalium* (otherwise he is referr'd for satisfaction to the 87, 88, 89, 90, 91, 92, 93, 94, 95, prop. *Arithm. Infinitor. Wallisii*:)

I say
$$ABCdEA = \frac{1}{1 \times 2} + \frac{1}{3 \times 4} + \frac{1}{5 \times 6} + \frac{1}{7 \times 8} + \frac{1}{9 \times 10} \; \&c.$$
$$EdCDE = \frac{1}{2 \times 3} + \frac{1}{4 \times 5} + \frac{1}{6 \times 7} + \frac{1}{8 \times 9} + \frac{1}{10 \times 11} \; \&c.$$
$$EdCyE = \frac{1}{2 \times 3 \times 4} + \frac{1}{4 \times 5 \times 6} + \frac{1}{6 \times 7 \times 8} + \frac{1}{8 \times 9 \times 10} \; \&c.$$
in infinitum.

For (in Fig. 2, & 3) the Parallelog.		*And (in Fig. 4.) the Triangl.*	*Note.*
$CA = \frac{1}{1 \times 2}$		$EdC = \frac{1}{2 \times 3 \times 4} = \frac{\square dD - \square dF}{2}$	
$dD = \frac{1}{2 \times 3}$	$dF = \frac{1}{3 \times 4}$	$Ebd = \frac{1}{4 \times 5 \times 6} = \frac{\square br - \square bn}{2}$	$\frac{1}{2}CA = dD + dF$
$br = \frac{1}{4 \times 5}$	$bn = \frac{1}{5 \times 6}$	$dfC = \frac{1}{6 \times 7 \times 8} = \frac{\square fG - \square fk}{2}$	$\frac{1}{2}dD = br + bn$
$fG = \frac{1}{6 \times 7}$	$fk = \frac{1}{7 \times 8}$	$Eab = \frac{1}{8 \times 9 \times 10} = \frac{\square aq - \square ap}{2}$	$\frac{1}{2}dF = fG + fk$
$aq = \frac{1}{8 \times 9}$	$ap = \frac{1}{9 \times 10}$	$bcd = \frac{1}{10 \times 11 \times 12} = \frac{\square cs - \square cm}{2}$	$\frac{1}{2}br = aq + ap$
$cs = \frac{1}{10 \times 11}$	$cm = \frac{1}{11 \times 12}$	$def = \frac{1}{12 \times 13 \times 14} = \frac{\square et - \square el}{2}$	$\frac{1}{2}bn = cs + cm$
$et = \frac{1}{12 \times 13}$	$el = \frac{1}{13 \times 14}$	$fgC = \frac{1}{14 \times 15 \times 16} = \frac{\square gu - \square gh}{2}$	$\frac{1}{2}fG = et + el$
$gu = \frac{1}{14 \times 15}$	$gh = \frac{1}{15 \times 16}$	*&c.*	$\frac{1}{2}fk = gu + gh$
&c.	*&c.*		*&c.*

And

And that therefore in the first series half the first term is greater than the sum of the two next, and half this sum of the second and third greater than the sum of the four next, and half the sum of those four greater than the sum of the next eight, *&c. in infinitum.* For $\frac{1}{2}\, dD = br + bn$; but $bn > fG$, therefore $\frac{1}{2}\, dD > br + fG$, *&c.* And in the *second* series half the first term is less then the sum of the two next, and half this sum less then the sum of the four next, *&c. in infinitum.*

That the first *series* are the *even* terms, *viz.* the 2[d], 4[th], 6[th], 8[th], 10[th], *&c.* and the *second*, the *odd*, *viz.* the 1[st], 3[d], 5[th], 7[th], 9[th], *&c.* of the following series, *viz.* $\frac{1}{1\times 2}\cdot\frac{1}{2\times 3}\cdot\frac{1}{3\times 4}\cdot\frac{1}{4\times 5}\cdot\frac{1}{5\times 6}\cdot\frac{1}{6\times 7}$. *&c. in infinitum* $= 1$. Whereof *a* being put for the number of terms taken at pleasure, $\frac{1}{aa + a}$ is the last, $\frac{a}{a+1}$ is the sum of all those terms from the beginning, and $\frac{1}{a+1}$ the sum of the rest to the end.

That $\frac{1}{4}$ of the first terme in the *third* series is less than the sum of the two next, and a quarter of this sum, less than the sum of the four next, and one fourth of this last sum less than the next eight, I thus demonstrate.

Let a = the 3[d] or last number of any term of the first Column, *viz.* of Divisors,

$$\frac{1}{a \times \overline{a-1} \times \overline{a-2}} = \frac{1}{a^3 - 3a^2 + 2a} = \frac{16a^3 - 48a^2 + 56a - 24}{16a^6 - 96a^5 + 232a^4 - 288a^3 + 184a^2 - 48a} = A$$

$$\left.\begin{array}{l} \dfrac{1}{2a \times \overline{2a-1} \times \overline{2a-2}} = \dfrac{1}{8a^3 - 12a^2 + 4a} \\[2ex] \dfrac{1}{\overline{2a-2} \times \overline{2a-3} \times \overline{2a-4}} = \dfrac{1}{8a^3 - 36a^2 + 52a - 24} \end{array}\right\} = \frac{16a^3 - 48a^2 + 56a - 24}{64a^6 - 384a^5 + 880a^4 - 960a^3 + 496a^2 - 96a} = B$$

$$\frac{64a^6 - 384a^5 + 928a^4 - 1152a^3 + 736a^2 - 192a}{64a^6 - 384a^5 + 880a^4 - 960a^3 + 496a^2 - 96a} \times \tfrac{1}{4} A < B.$$

And $48a^4 - 192a^3 + 240a^2 - 96a =$ *Excess of the Numerator above Denomin.*

$$\left.\begin{array}{llcl} \text{But ---} & \text{The affirm.} & > & \text{the Negat.} \\ \text{That is,} & 48a^4 + 240a^2 & > & 192a^3 + 96a \\ \text{Because} & a^4 + 5a^2 & > & 4a^3 + 2a \\ & a^3 + 5a & > & 4a^2 + 2 \end{array}\right\} \text{if } a > 2.$$

Therefore $B > \frac{1}{4} A$.

Therefore $\frac{1}{4}$ of any number of A: or Terms, is less than their so many respective B. that is, than twice so many of the next Terms. *Quod, &c.*

By

By any one of which three Series, it is not hard to calculate, as near as you please, these and the like *Hyperbolic* spaces, whatever be the *Rational* Proportion of *A E* to *B C*. As for Example, when *A E* is to *B C*, as 5 to 4. (whereof the Calculation follows after that where the Proportion is, as 2 to 1. and both by the third Series.)

First then when (in Fig. 1.) A E. B C :: 2. 1.

2 x 3 x 4)	1.	(0.0416666666	0.0416666666
4 x 5 x 6)	1.	(0.0083333333	0.0113095237
6 x 7 x 8)	1.	(0.0029761904	
8 x 9x10)	1.	(0.0013888888	0.0029019589
10x11x12)	1.	(0.0007575757	
12x13x14)	1.	(0.0004578754	
14x15x16)	1.	(0.0002976190	
16x17x18)	1.	(0.0002042484	0.0007306482
18x19x20)	1.	(0.0001461988	
20x21x22)	1.	(0.0001082251	
22x23x24)	1.	(0.0000823452	
24x25x26)	1.	(0.0000641026	
26x27x28)	1.	(0.0000508751	
28x29x30)	1.	(0.0000410509	
30x31x32)	1.	(0.0000336021	
32x33x34)	1.	(0.0000278520	0.0001829939
34x35x36)	1.	(0.0000233426	
36x37x38)	1.	(0.0000197566	
38x39x40)	1.	(0.0000168691	
40x41x42)	1.	(0.0000145180	
42x43x44)	1.	(0.0000025843	
44x45x46)	1.	(0.0000109793	
46x47x48)	1.	(0.0000096361	
48x49x50)	1.	(0.0000085034	
50x51x52)	1.	(0.0000075415	
52x53x54)	1.	(0.0000067193	
54x55x56)	1.	(0.0000060125	
56x57x58)	1.	(0.0000054014	
58x59x60)	1.	(0.0000048704	
60x61x62)	1.	(0.0000044068	
62x63x64)	1.	(0.0000040002	

0.0416666666
0.0113095237
0.0029019589
0.0007306482
3)0.0001829939(0.0000609980

0.05679179
+ 0.00006100
0.05685279 < E d C y

But 0.0007306482
0.0001829939
0.0000458315 ∺

Therefore 0.05679179
+ 0.00004583
+ 0.00001528
0.05685290 > E d C y.

For, it has been demonstrated that ¼ of any terme in the last Column is less than the terme next after it; and therefore that ⅓ of the last terme, at which you stop

stop, is less than the remaining terms, and that the total of these is less than $\frac{1}{3}$ of a third proportional to the two last.

And therefore ABCyE being $=$ 0.75 ————— 0.75
and EdCy $>$ 0.05685279 —— and $<$ 0.05685290

And ABCdE is $<$ 0.69314720 —— and $>$ 0.69314709

But when AE . BC:: 5. 4. or as EA. to KH. then will the space ABCE. or now, the space AHKE (AH $= \frac{1}{4}$AB.) be found as follows.

8x9x10) 1 (0.0013888888
16x17x18) 1 (0.0002042484 } 0.0003504472
18x19x20) 1 (0.0001461988 }
32x33x34) 1 (0.0000278520 }
34x35x36) 1 (0.0000233426 } 0.0000878204
36x37x38) 1 (0.0000197566 }
38x39x40) 1 (0.0000168691 }

0.00 3888888
0.0003504472
3)0.0000878204(0.0000292735
0.0018271564
+ 0.0000292735
0.0018564299 $<$ Eab

But 0.0003504472 }
0.0000878204 } $\div$
0.00002200737 }

Therefore 0.0018271564
+ 0.0000220074
+ 0.0000073358
0.0018564996 $>$ Eab

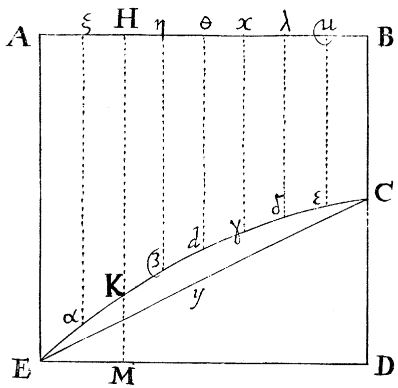
Therefore EMb. (Fig 4.)
being $=$ 0.025 ———— 0.025
Eab $>$ 0.0018564299 —— & $<$ 0.0018564996

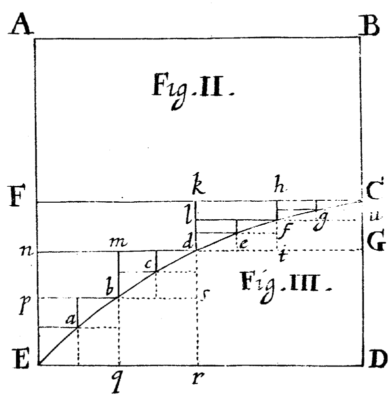
EMba (Fig.4.) or EKM (Fig.1.) $>$ 0.02685643 —— $<$ 0.02685650
AHKM $<$ 0.22314356 —— $>$ 0.22314349

Therefore 3 ABCdE $=$ 2.07944154
and AHKE $=$ 0.2231435 ——
ABCdE (when AE.BC:: 10.1.) $=$ 2. 025850 ——

Therefore the Logar. of 10 is to the Log. of 2, as 2.302585 to 0.693147

Fig. I.





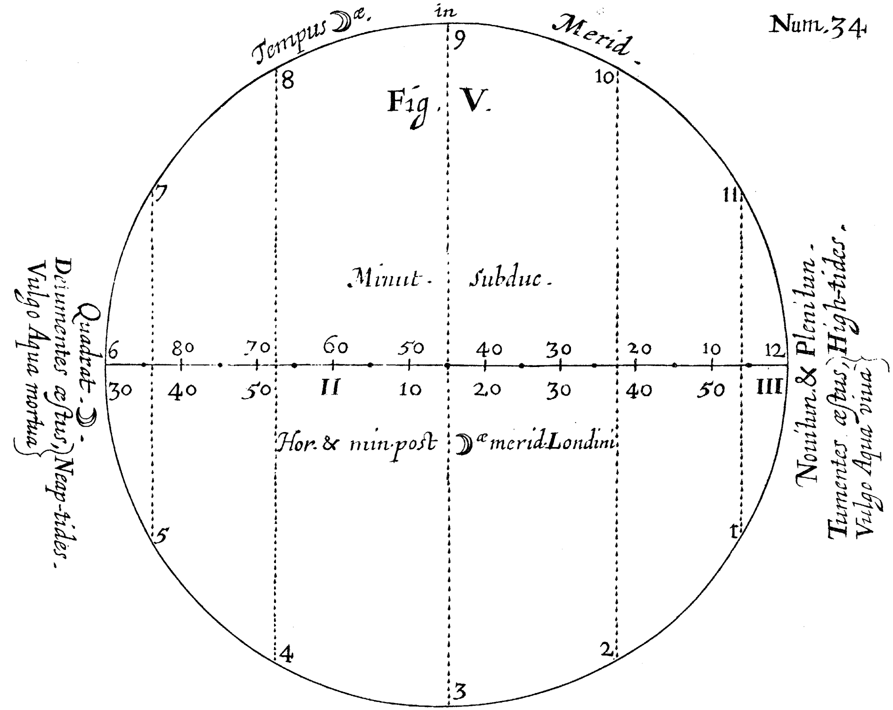


Fig. IIII.