An Equivalent Condition to the Robin Inequality

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The Robin's inequality (RI) is well known ([7,11]). This is related with many problems of the analytical number theory. And there are some statements equivalent to one ([8,10]). In this paper we would show a condition related with the RI. This condition is generalized rather than the Robin's one.

Recall that it is called the Robin's inequality that for any $n \geq 5041$

$$\sigma(n) \leq e^{\gamma} \cdot n \cdot \log\log n . \qquad (1)$$

where $\sigma(n) = \sum_{d|n} d$ is the sum of divisors function, $\gamma = 0.577\cdots$ is Euler's constant ([9,13]).

We have

Theorem 1. The RI holds if and only if there exists a constant $c_0 \geq 1$ such that for any number $n \geq 2$ we have

$$\sigma(n) \leq e^{\gamma} \cdot n \cdot \log\log(c_0 \cdot n) \tag{2}$$

Proof. Suppose that the RI holds. Then for any $n \geq 5041$ it holds that

$$\sigma(n) \leq e^{\gamma} \cdot n \cdot \log\log n .$$

Now we put

$$H(n) = \frac{\exp\left(\exp\left(e^{-\gamma} \cdot \sigma(n)/n\right)\right)}{n} . \tag{3}$$

Then for any $n \geq 5041$ we have $H(n) \leq 1$. Let $c_0 = \max\limits_{2 \leq n \leq 5040} H(n)$. Then $c_0 \geq 1$ and for any $n \geq 2$ we have (2).

Suppose that (2) holds, but the RI doesn't hold. Then by the Robin's theorem ([10,11]) there exist constants $c > 0$ and $0 < \beta < 1/2$ such that, for infinitely many numbers n, we have

$$e^{\gamma} \cdot n \cdot \log\log n + c \cdot \frac{n \cdot \log\log n}{(\log n)^{\beta}} \leq \sigma(n) . \tag{4}$$

On the other hand, it is clear that

$$\begin{aligned} &\log\log(c_0 \cdot n) = \log(\log n + \log c_0) = \\ &= \log\left(\log n\left(1 + \frac{\log c_0}{\log n}\right)\right) = \log\log n + \log\left(1 + \frac{\log c_0}{\log n}\right) \leq \\ &\leq \log\log n + \frac{\log c_0}{\log n} . \end{aligned} \tag{5}$$

From (4) and (5), for infinitely many numbers n we have

$$\begin{aligned} &e^{\gamma} \cdot n \cdot \log\log n + c \cdot \frac{n \cdot \log\log n}{(\log n)^{\beta}} \leq \sigma(n) \leq \\ &\leq e^{\gamma} \cdot n \cdot \log\log n + e^{\gamma} \cdot n \cdot \frac{\log c_0}{\log n} \end{aligned} \tag{6}$$

and

$$c \cdot \frac{\log\log n}{(\log n)^{\beta}} \le e^{\gamma} \cdot \frac{\log c_0}{\log n}. \tag{7}$$

If $c_0 = 1$ then (7) is impossible. If $c_0 > 1$ then, since $(1/2-\beta) > 0$, we have

$$0 < \frac{c \cdot e^{-\gamma}}{\log c_0} \le \frac{1}{\log\log n} \cdot \frac{1}{(\log n)^{1-\beta}} \to 0 \ (n \to \infty). \tag{8}$$

This is a contradiction. □

Note. There are some statements equivalemt to the RI.

Theorem. The below statements are equivalent to each other.

a) The RI holds

b) For any $n \ge 1$

$$\sigma(n) \le H_n + \exp(H_n) \cdot \log(H_n), \tag{9}$$

holds, where $H_n = \sum_{i=1}^{n} i^{-1}$ is n-th harmonic sum. This is called Lagarias' inequality ([10]).

c) For any real x

$$\pi(x) = \int_2^x \frac{1}{\log t}\,dt + \mathrm{O}\left(\sqrt{x} \cdot \log x\right) \tag{10}$$

holds, where $\pi(x)$ is the number of the prime numbers not exceeding the given x ([8,9]).

By the prime number theorem ([8]), it holds that

$$\pi(x) = \log x \cdot (1 + E(x)), \tag{11}$$

where $E(x) = \mathrm{O}\left(\frac{1}{\log^2 x}\right)$. The best-known result until now is that there exists a constant $a > 0$ such that $E(x) = \mathrm{O}\left(-a \cdot \exp\left(\sqrt{\log x}\right)\right)$ ([8]).

Further more, there are some, too. But here we will show it only with the sum of divisors function.

Similarly, as in the theorem 1, we could obtain

Theorem 2. The below statements are equivalent to each other.

1) The RI holds

2) There exist constants $c_0 \geq 1$, $c_1 \geq 0$ and $c_2 \geq 0$ such that for any $n \geq 2$

$$\sigma(n) \leq e^{\gamma} \cdot n \cdot \log\log\left(c_0 \cdot n \cdot \exp\left(c_1 \cdot \sqrt{\log n} \cdot \exp\left(c_2 \cdot \left(\log\log(n+1)\right)^{\alpha}\right)\right)\right) \tag{12}$$

holds, where $0 < \alpha < 1$.

3) There exists a constant $c_0 \geq 1$ such that for any $n \geq 2$

$$\sigma(n) \leq e^{\gamma} \cdot n \cdot \log\log n + \frac{c_0 \cdot (\log\log n)^{\alpha_1}}{\sqrt{\log n}} \cdot \exp\left(\left(\log\log(n+1)\right)^{\alpha}\right) \tag{13}$$

holds, where $0 < \alpha_1$, $0 < \alpha < 1$.

Proof. We will see only the proof of 2). It is easy to see that

$$\begin{aligned}
&\log\log\left(c_0 \cdot n \cdot \exp\left(c_1 \cdot \sqrt{\log n} \cdot \exp\left(c_2 \cdot \left(\log\log(n+1)\right)^{\alpha}\right)\right)\right) = \\
&= \log\left(\log n + \log c_0 + c_1 \cdot \sqrt{\log n} \cdot \exp\left(c_2 \cdot \left(\log\log(n+1)\right)^{\alpha}\right)\right) = \\
&= \log\left(\log n\left(1 + \frac{\log c_0}{\log n} + \frac{c_1 \cdot \sqrt{\log n} \cdot \exp\left(c_2 \cdot \left(\log\log(n+1)\right)^{\alpha}\right)}{\log n}\right)\right) = \\
&= \log\log n + \log\left(1 + \frac{\log c_0}{\log n} + \frac{c_1 \cdot \exp\left(c_2 \cdot \left(\log\log(n+1)\right)^{\alpha}\right)}{\sqrt{\log n}}\right) \leq \\
&\leq \log\log n + \frac{\log c_0}{\log n} + \frac{c_1 \cdot \exp\left(c_2 \cdot \left(\log\log(n+1)\right)^{\alpha}\right)}{\sqrt{\log n}}.
\end{aligned} \tag{14}$$

On the other hand, if $0 < \beta < 1/2$ then we have

$$\frac{1}{\log\log n}\cdot\left(\frac{\log c_0}{\left(\log n\right)^{1-\beta}}+\frac{c_1\cdot\exp\left(c_2\cdot\left(\log\log(n+1)\right)^{\alpha}\right)}{\left(\log n\right)^{1/2-\beta}}\right)\to 0\ \left(n\to\infty\right). \quad (15)$$

This shows that 1) and 2) are equivalent to each other. □

Of course, from (12) we can get (2). But (2) shows the clear relation with the RI. And we can say that (2) is the more generalized proposition rather than the RI. But, (13) is the most generalized rather than the RI.

In the paper [11], Robin have proved that for any $n \ge 2$

$$\sigma\left(n\right)\le e^{\gamma}\cdot n\cdot\log\log n+\frac{0.26\cdots}{\log\log n} \quad (16)$$

holds unconditionally.

In the paper [10], Lagalias had indicated that his inequality (9) holds for nearly all n without any condition.

In the paper [7], they had shown that the RI holds under all odd numbers.

To prove the RI, it is sufficient to show that

$$\underset{n\ge 5041}{Sup}\frac{\exp\left(\exp\left(e^{-\gamma}\cdot\sigma\left(n\right)/n\right)\right)}{n}\le 1 \quad (17)$$

holds unconditionally.

Similarly, to prove (2) it is sufficient to see that

$$c_0=\underset{n\ge 2}{Sup}\frac{\exp\left(\exp\left(e^{-\gamma}\cdot\sigma\left(n\right)/n\right)\right)}{n}<+\infty\,. \quad (18)$$

And to prove (12) it is sufficient to take such constants $c_1 \ge 0$ and $c_2 \ge 0$ as

$$\underset{n\ge 2}{Sup}\frac{\exp\left(\exp\left(e^{-\gamma}\cdot\sigma\left(n\right)/n\right)\right)/n}{\exp\left(c_1\cdot\sqrt{\log n}\cdot\exp\left(c_2\cdot\left(\log\log(n+1)\right)^{\alpha}\right)\right)}<+\infty\,. \quad (19)$$

Thus one can say that the best way to prove the RI is to prove (18) or (19).

In connection with the proof of (18) or (19), we recommend the paper [1,2,3,4,5,6]. In these papers we presented a new idea to prove (18) and (19). For the function $H(n)$ from (3), we would like to call it an σ-index of the natural number $n \in N$. And for the prime factorization $n = q_1^{\lambda_1} \cdot q_2^{\lambda_2} \cdots q_m^{\lambda_m}$ of n with $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_m \ge 1$, we put $\bar{q}(n) = (q_1, q_2, \cdots q_m)$, $\bar{\lambda}(n) = (\lambda_1, \lambda_2, \cdots \lambda_m)$ and $\omega(n) = m$ ([1]). Here for $\bar{q}(n)$, $\bar{\lambda}(n)$ and $\omega(n)$, we would like to call it the prime factor pattern, the exponential pattern, the exponential length, respectively. And for the optimum points $\bar{\lambda}_0 = (\lambda_1^0, \lambda_2^0, \cdots, \lambda_m^0)$ of the function $H(\lambda_1, \lambda_2, \cdots \lambda_m)$ ([2]), we would like to call $n_0 = p_1^{\lambda_1^0} \cdot p_2^{\lambda_2^0} \cdots p_m^{\lambda_m^0}$ a special Hardy-Ramanujan's number. Then the plan [6] to prove (18) or (19) could be understood clearly. In other words, the scheme to understand the papers [1~6] and [*] is as follows, where [*] is the present paper;

$$[1] \to [2] \to [3] \to [4] \to [5] \to [6] \to [*],$$

or

$$[5] \to [6] \to [*] \to [1] \to [2] \to [3] \to [4].$$

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One can find [1~6] in;
http://commons.wikimedia.org/wiki/File:**title of paper**.pdf

For example; address for [1]
http://commons.wikimedia.org/wiki/File:The_sum_of_divisors_function_and_the_Hardy-Ramanujan%27s_number.pdf

And address for [4]
http://commons.wikimedia.org/wiki/File:An_Inequality_for_the_Sum_of_Divisors_Function.pdf

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