Category Theory

Pertaining to Dynamical Systems

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May 16, 2009

Abstract

After presenting a definition of categories, various inverses and endomorphisms are explored. A large category of dynamical systems is discussed, in the context of which it is proved that a chaotic observable implies a chaotic dynamic system.

1 Categories

Category Theory started forming in 1945 and now appears in many branches of mathematics. Discussing dynamical systems through categories can be considered useful or insightful.

Definition A *category* is made up of *objects* and *arrows* with the following four operations. [2, p.7]

- The *domain* assigns to each arrow f, an object $A = \text{dom } f$.
- The *codomain* assigns to each arrow f, an object $B = \text{cod } f$.
- The *composition* assigns to each pair {g,f} of arrows with dom g = cod f an arrow $g \circ f$ called their composition with $g \circ f : \text{dom } f \to \text{cod } g$.
- The *identity* operation assigns to each object A an arrow $id_A = 1_A : A \to A$.

The operations of categories must abide by the following axioms: [2, p.7-8]

Associativity: For given objects and arrows in the configuration $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{k} D$, one always gets the equality

$$k \circ (g \circ f) = (k \circ g) \circ f \tag{1}$$

Unit Law: For all arrows $f : A \xrightarrow{f} B$ and $g : B \xrightarrow{g} C$, composition with the identity arrow 1_B gives

$$1_B \circ f = f \tag{2}$$

$$g \circ 1_B = g \tag{3}$$

Categories of dynamical systems are the principal example in this paper, so as to not bias the scope of categories, provided below is a short list of categories. [2, p.12]

Category	Objects	Arrows
Monoid	one object	any
Group	one object	every arrow has a (two sided) inverse
Rings	all rings	unit preserving maps between rings
Directed Graph	verticies	edges
Sets	all sets; X, Y, ...	$f : X \to Y$
Topological	Topological spaces	continuous maps
Homotopy	Topological spaces	homotopy classes of maps

2 Morphisms

The name for arrows in categories is interchangeable with morphisms, functions, maps, and operators. Left and right inverses are pertinent to many fields of mathematics, and category theory is no exception.

2.1 Inverses

Definition If $A \xrightarrow{f} B$,

- a *retraction* for f is a map $B \xrightarrow{r} A$ for which $r \circ f = 1_A$.
- a *section* for f is a map $B \xrightarrow{s} A$ for which $f \circ s = 1_B$. [1, p.49]

Retractions are also known as left inverses, where as sections are right inverses. If there exists a retraction for a function, then that function is often described as injective, monomorphic or one-one. When there exists a section for a function, said function is called surjective, epimorphic and onto.

Proposition 2.1 *If a function $f : A \to B$ has both a section and a retraction, then they are the same map.*

Proof

$$\begin{aligned} r &= r \circ 1_B \\ &= r \circ (f \circ s) \\ &= (r \circ f) \circ s \\ &= 1_A \circ s \\ &= s \end{aligned}$$

Definition A map $f : A \to B$ is called an *isomorphism* if there exists another map $f^{-1} : B \to A$, which is both a retraction and section for f:

$$\begin{aligned} f \circ f^{-1} &= 1_B \\ f^{-1} \circ f &= 1_A \end{aligned}$$

Such a map f^{-1} is called the *inverse map* for f. [1, p.54]

2.2 Endomorphisms

Often in dynamical systems when there is a set X and a function $f : X \to X$, that system is notated as (X, f).

Definition An *endomorphism* is a map where the domain and the codomain are the same object. [1, p.15]

All dynamical system maps are endomorphic, some having more structure than others.

Definition An *automorphism* is a function which is both an endomorphism and an isomorphism. [1, p.55,155]

For a set Y and the bi-infinite shift space $Y^{\mathbb{Z}}$, the shift operator is an automorphism.

Definition An *involution* is a function such that f composed with itself is the identity operator. $f^2 = f \circ f = 1$ [1, p.139]

All points in an object are period-2 points under involutions. Some involutions include $f(x) = -x$ as well as the shift space on period-2 sequences.

Definition An *idempotent* is a function such that $f \circ f = f$. [1, p.54,155]

For a dynamic system with many points attracted to fixed points (such as Newton's method applied to $f(x) = x(x^2-4)$ as in Module 2), a function which takes points to their attractor (and unattracted points to one particular unattracted point), is an idempotent map.

Proposition 2.2 *If a map is both an idempotent and involution, than this is the identity map.*

Proof

$$f = f \circ f = 1$$

3 Dynamical Systems

3.1 A Category of Dynamical Systems

We can construct a category of dynamical systems. This category of dynamical systems has objects which are dynamical systems; all objects are sets with an attached endomorphism. The arrows between objects in this category are required to commute with the endomaps, i.e. if the dynamical systems (X, α) and (Y, β) are in our category, then $f : X \to Y$ is allowed iff $\alpha \circ f = f \circ \beta$. [1, p.152,164] Requiring f to be isomorphic and continuous produces a category of homeomorphic dynamical systems.

Each dynamical system is itself a category of one object and one endomorphic arrow. A category of dynamical systems is a category of categories where the maps between objects are functors.

Definition A *functor* is a structure preserving morphism of categories, taking objects to objects and taking maps to maps while preserving domain and codomain. [1, p.167]

3.2 Chaos

In the context of dynamical systems, we have defined a concept of chaos:

Definition For a dynamical system (X, α), the function α is *chaotic* if it has sensitivity to initial conditions, transitivity, and a dense set of periodic points.

This definition of chaos requires that the set X is more varied a metric space than the discrete metric. A chaotic observable, however, does not require this structure on our given set.

Definition Let (X, α) be a dynamical system and f be a function $f : X \to Y$. Define the function $\bar{f} : X \to Y^{\mathbb{N}}$ as the f-image of an α-itinerary of a point in X. The function $\bar{f}$ is said to be a *chaotic observable* if it is onto for states (i.e. there exists a section, $Y^{\mathbb{N}} \xrightarrow{s} X$ such that $f \circ s = 1_{Y^{\mathbb{N}}}$). [1, p.317]

If $\bar{f} : X \rightarrow Y^{\mathbb{N}}$ is onto for states, then the image of X under $\bar{f}$ is the full shift on Y. By nessessity of further discussion, we are defining a metric on $Y^{\mathbb{N}}$ such that for two points a,b $\in Y^{\mathbb{N}}$

$$a = .y_0y_1y_2y_3 \dots z_k \dots \quad (4)$$

$$b = .y_0y_1y_2y_3 \dots y_k \dots \quad (5)$$

where $z_k \neq y_k$, $\rho(a,b) = 2^{-k}$. Note that we have a dynamical systems category of two objects, $(X, \alpha) \xrightarrow{\bar{f}} (Y^{\mathbb{N}}, \sigma)$ where σ is the shift operator on sequences in $Y^{\mathbb{N}}$.

Lemma 3.1 *If $\bar{f} : X \rightarrow Y^{\mathbb{N}}$ is a chaotic observable then $(\bar{f} \circ X, \sigma)$ has a dense set of periodic points.*

Proof Let $\epsilon > 0$ and choose $k \in \mathbb{N}$ such that $2^{-k} < \epsilon$. For any $x \in Y^{\mathbb{N}}$, x can be written as below and y can be chosen as such:

$$x = .x_0x_1x_2x_3 \dots x_k \dots \quad (6)$$

$$y = .x_0x_1x_2x_3 \dots x_kx_0x_1x_2x_3 \dots x_kx_0 \dots \quad (7)$$

As we can see, $\rho(x,y) < \epsilon$ and $\sigma^{k+1}(y) = y$.

Lemma 3.2 *If $\bar{f} : X \rightarrow Y^{\mathbb{N}}$ is a chaotic observable, then $(\bar{f} \circ X, \sigma)$ is transitive.*

Proof Choose $u, w \in Y^{\mathbb{N}}$ and $k < j$ as

$$u = .u_0u_1u_2u_3 \dots u_k \dots \quad (8)$$

$$w = .w_0w_1w_2w_3 \dots \dots w_j \dots \quad (9)$$

Define opens sets $U, W \subset Y^{\mathbb{N}}$ as all points $y \in Y^{\mathbb{N}}$ such that $\rho(y,u) < 2^{-k}$ and $\rho(y,w) < 2^{-j}$ respectively. Now choose a point $x \in U$ as

$$x = .u_0u_1u_2u_3 \dots u_kw_0w_1w_2w_3 \dots w_j \dots \quad (10)$$

and an open set $X \subset U$ defined by the points $y \in Y^{\mathbb{N}}$ such that $\rho(y, x) < 2^{-(j+k+1)}$. Using the shift operation

$$\sigma^{k+1}(x) = .w_0 w_1 w_2 w_3 \ldots w_j \ldots \tag{11}$$

it is shown that the image of X under $k+1$ operations is a subset of W.

Lemma 3.3 *If $\bar{f} : X \to Y^{\mathbb{N}}$ is a chaotic observable, then $(\bar{f} \circ X, \sigma)$ has sensitive dependence on initial conditions.*

Proof Choose $1 > D > 0$ and $\epsilon > 0$, then choose $n \in \mathbb{N}$ such that $2^{-n} < \epsilon$. For any point $x \in X$, choose a point $y \in X$ such that $\rho(x, y) < \epsilon$ and defined as below,

$$x = .x_0 x_1 x_2 x_3 \ldots x_n x_{n+1} \ldots \tag{12}$$

$$y = .x_0 x_1 x_2 x_3 \ldots y_n y_{n+1} \ldots \tag{13}$$

where $y_n \neq x_n$ for all $i \geq n$. Under shift operators

$$\sigma^n(x) = .x_n x_{n+1} x_{n+2} \tag{14}$$

$$\sigma^n(y) = .y_n y_{n+1} y_{n+2} \tag{15}$$

it follows that $\rho(\sigma^n \circ x, \sigma^n \circ y) = 2^{-0} > D$.

Proposition 3.4 *If $\bar{f} : X \to Y^{\mathbb{N}}$ is a chaotic observable, then the dynamical system $(\bar{f} \circ X, \sigma)$ is a chaotic dynamical systems.*

Proof This follows from the previous lemmas that if $\bar{f}$ is chaotic observable then the image of X under $\bar{f}$ has sensitive dependence on initial conditions, transitivity, and a dense set of periodic points in $Y^{\mathbb{N}}$ with the shift operator.

Example Let Γ be the middle third Cantor set and define $\gamma : \Gamma \to \Gamma$ as

$$\gamma(x) = \begin{cases} 3x & \text{if } x < 1/2 \\ 3(1-x) & \text{if } x > 1/2 \end{cases} \tag{16}$$

Let Y_1 be the interval $(0,1)$, $f_1(x) = x$ and $\bar{f}_1 : \Gamma \to Y_1^{\mathbb{N}}$ be the itinerary of a point in Γ under γ. This observable is not onto for states as there are sequences in $Y_1^{\mathbb{N}}$ for which no point in Γ will produce such an itinerary (such as $.\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}\ldots$) However, consider the set $Y_2 = \{0,2\}$, $f_2(x) =$ terterary expansion of x and $\bar{f}_2 : \Gamma \to Y_2^{\mathbb{N}}$ defined as the itinerary function in Module 7. The image of Γ under $\bar{f}_2$ is the full shift on Y_2 and thus $\bar{f}_2$ is a chaotic observable. As you can see, there can exist both chaotic and non-chaotic observables for the same dynamical system.

4 Conclusion

While the definition presented here is for a category of dynamical systems with objects which are sets, a generalization can be made to drop this requirement. It should be noted that category theory only provides another language to discuss dynamical systems, or any other branch of mathematics. In the same way that one could argue that Java is better than C or Python, category theory provides another way to discuss the mathematics of dynamical systems.

References

[1] F. William Lawvere and Stephen H. Schanuel, *Conceptual mathematics a first introduction to categories*, Cambridge University Press, Cambridge, UK, 1997.

[2] Saunders Mac Lane, *Categories for the working mathematician*, Springer Verlag, New York Heidelberg Berlin, 1971.