A UNIFIED THEORY OF OPTION PRICING ASSUMING STOCHASTIC VOLATILITY

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Abstract. Stochastic volatility models have grown in popularity in the past decade or two. However, for many stochastic volatility models, the functional form of volatility along with the description of the diffusion process for volatility have been posed with analytic convenience in mind. Here, we consider that analytic tractability may degenerate as realistic modelling improves and that a more general specification for the stock price and volatility processes may be necessary. This leads to an approximating polynomial for European option prices which is benchmarked to two popular stochastic volatility models, the Stein and Stein, and Heston models, before examining a more general specification which is compared to the corresponding Black Scholes price. Stochastic volatility and European option approximation and Heston and Stein and Stein and Black Scholes 35A05 and 35C05 and 35D05 and 35G30 and 47F05 and 65L10
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1. Introduction

Recent decades have given rise to a number of sometimes complex and sophisticated methodologies that attempt to resolve empirical issues related to volatility modelling, with a view, in particular, to pricing options. Models which assume volatility is stochastic have enjoyed particular success in attempting to explain observed volatility smiles with the Heston [3] model in particular receiving much attention in empirical studies.

One of the major reasons that stochastic volatility models have become popular is their ability to model stylised features of volatility more succinctly in their bid to better explain volatility smiles. The models of Scott [6], Stein and Stein [7], and Heston [3] in particular have been useful in their ability to model volatility characteristics such as shocks and persistence by assuming a mean reverting process for volatility or a function of volatility. The Heston [3] model also assumed correlation between the Brownian motions which drive the stock and volatility processes in order to capture another important feature; skewness in the distribution of the asset returns process. That is, zero correlation can only lead to increased kurtosis in the distribution of asset returns and, while this is important in explaining volatility smiles better, empirical evidence supports a skewed distribution for returns, something which cannot be captured properly without correlation.

Given the successes enjoyed by authors including Hull and White [4], Scott [6], Stein and Stein [7], and Heston [3] in modelling volatility stochastically, therefore, we turn to this framework and look to extend it to more general functional forms of volatility under the consideration that there is trade-off between realism and tractability which is generally realised. Specifically, the functional specifications of volatility in models such as those in Stein and Stein [7] and Heston [3] have been predominantly made in order to provide analytic convenience, the cost of which can be realistic option prices.

To that end we recall that, even in the case of zero correlation, analytic solutions to more general stochastic volatility specifications are difficult to obtain. The extra dimension provided by the state variable governing volatility will usually mean that an analytic solution is not attainable (except in specific cases), ultimately requiring some form of numerical approximation. Among those considered along with the approaches of Stein and Stein [7] and Heston [3] is that in Fouque et al [2]. We have commented on the method of Stein and Stein [7] to the extent that they consider only zero correlation and as a result are able to recover the stock price density, though an extension to include correlation is not straightforward. In addition, the approach becomes more complicated under more general volatility specifications arising from the choice of $f(\cdot)$, where this represents the functional form of volatilty.

As alluded to already, for the method in Heston [3], the probabilities of the stock price finishing in a specified region along with its expectation restricted to a specified region are found. These probabilities can be considered as estimates of the pricing density and while the form provided must be calculated by use of a numerical procedure, it can be considered closed-form due to its ease of implementation with modern computer software. However, the method used relies heavily on the functional form specification, namely the square root process. The use of the volatility diffusion power parameter, denoted here by γ, will mean that we cannot generally find an analogous solution here, leading to a consideration of other methods.

The main idea of the approach of Fouque et al [2] is to incorporate a flexible functional form by considering an asymptotic expansion of the option price where it follows that the first term in the expansion is the Black Scholes call option price with volatility based on $f(y)$. Here y is instantaneous volatility and the first-order correction term is based on an expansion of the option price in coefficients of the inverse square-root of mean reversion.

The validation of the asymptotic approach is the consideration of high speed mean reversion and that this is, roughly, equivalent to long-horizon maturities. To see this, we note that the variance of the Ornstien-Ühlenbeck process posed in Fouque et al [2] is

$$var(Y_t) \;=\; \frac{\beta^2}{2\kappa}\left(1 - e^{-2\kappa t}\right), \tag{1.1}$$

which is approximately equal to $\beta^2/(2\kappa)$ for κ large where the link between this approximating value and time is made by the exponential term.

While the methodology is flexible in its ability to handle any suitable functional form of volatility, $f(\cdot)$, there are several issues. To begin with, the Ornstein-Ühlenbeck process is Gaussian and there is only a first-order correction term. This means that the price differences found between options priced in the stochastic volatility setting and those found by standard Black Scholes will be symmetric in correlation about the strike price. Given that the OU process is Gaussian, there is always some probability that it can take on negative values. This can be overcome by the choice of functional form of volatility, thus leading to a skewed distribution, though this will not be captured by the first-order correction term.

In addition, it must be assumed that the speed of mean reversion is numerically high. Even if this is the case, a high enough speed of mean reversion, when volatility of volatility is small, should effectively lead volatility to be approximately constant. This should then result in price differences not too far from a standard Black Scholes option price with effective volatility based on $E[f(Y_t)]$ where Y_t denotes the volatility process. Given that a high speed of mean reversion is connected to long-run behaviour, it follows that the price differences must be compared to those on a much longer maturity, even in the case of an option which has only two weeks (say) until expiry. Finally, as the correlation between the two Brownian motions tends to zero the price differences disappear completely.

To overcome some of these shortcomings, Rasmussen and Wilmott [5] extend the idea presented in Fouque et al [2] to include higher-order terms, a total of five. In this case, the zeroth-order term is the Black-Scholes price based on effective volatility while the other four terms contribute to option prices which fit volatility smiles better than those where only an additional first-order term is considered. However, as is the case in Fouque et al [2], the method relies on asymptotic properties and is thus based on the behaviour of long-run volatility as opposed to volatility at any instant. In particular, the long-run distribution of the volatility process must be known. Naturally, this in turn restricts the choices for which

the stochastic process describing volatility can take.
With these restrictions in mind, the objective of this paper is to approximate option prices which target the *true* option price as if the probability density function of the volatility process is known, even in the event that it isn't. In order to facilitate such an approximation, certain restrictive assumptions are relaxed. Firstly, the functional form of volatility can be arbitrary, so long as it is suitable (in a sense to be made clear). Also, by targeting the true option prices, the approximation is *not* based on long-run densities, but, on the converse, is based on the actual density of the volatility process. In other words, asymptotic properties are not required. Additionally, the functional form of volatility is relaxed to the extent that its probability density function may not be known. In particular, the only assumption made on the underlying volatility process is that it is mean-reverting. That is, the terms comprising its diffusion coefficient can be somewhat arbitrary.
While the methodology will be outlined for general choices of functional form of volatility, $f(Y_t)$, in order to overcome the already mentioned shortcomings. The expansion method centres around the volatility of volatility parameter, β, up to second-order, essentially meaning that any error is of order β^3. For practical purposes, it is usually found that $\beta \in [0, 1)$ and more usual situations are where β is less than 0.4 so that the method is quite accurate to two decimal places at worst. For the sake of exposition, the model is benchmarked to two popular stochastic volatility models, those presented in Stein and Stein [7], and Heston [3], before examining the impact of a more general functional form.
Specifically, the polynomial not only includes correlation, but can also be used when correlation is zero. This is achieved by assuming the zeroth-order term, $u_0(\tau, x, y)$, followed by the retrieval of a first-order term, $u_1(\tau, x, y)$, which is analogous to than found in Fouque et al [2]. The distinction between the first-order term found here and that of Fouque et al [2] is that the latter, as has already been stated, is based on expectations with respect to the density of the long-run volatility process. We continue the expansion further with a second-order term, $u_2(\tau, x, y)$, which is comprised of terms which have zero and non-zero correlation, thus providing flexibility along with increased accuracy. In addition, the algorithm required to implement the approximation is relatively straightforward and quick with modern computer software. Finally, the boundary conditions on the higher-order terms (i.e., the first and second) are automatically satisfied.

1.1. **Plan of the paper.** The rest of this paper is organised as follows: In Section 2 we outline the general form for the stock price and volatility processes before deriving the polynomial which will approximate European call option prices. Section 3 then benchmarks the approach to the Stein and Stein [7] and Heston [3] models before presenting results obtained by posing a more general functional form for volatility. Section 4 then concludes and discusses the results obtained.

2. Mathematical Results

We begin this section by introducing a stochastic system where the asset price and volatility are the state variables. In particular, let us assume that the stock price process evolves according to

$$dS_t = S_t(rdt + \sigma_t dW_t), \tag{2.2}$$

where r is the risk-free force of interest, W_t is a standard one-dimensional Brownian motion, and where we have the volatility specification $\sigma_t = f(Y_t)$, where $f(\cdot)$ is some differentiable function. The volatility specification leads to the introduction of the intrinsic volatility process

$$dY_t = \kappa(\theta - Y_t)dt + \beta Y_t^{\gamma}(\rho dW_t + \sqrt{1-\rho^2}dB_t), \tag{2.3}$$

to which we make the following notes: The long-run mean of the process, Y_t, is θ and κ measures the rate at which the process Y_t reaches the long-run mean. The volatility of volatility parameter is governed by β and the power parameter, γ, controls the path-dependency in the diffusion term of intrinsic volatility. It has also been assumed that the intrinsic volatility process is driven by a Brownian motion which is correlated, as measured by ρ, with the Brownian motion which drives the stock price process. That is, $d\langle W, B\rangle_t = 0$.
We note that the system provided by (2.2) and (2.3) models some quite flexible modelling considerations which are rich in flavour. Firstly, the specification of the volatility function as measured by $f(\cdot)$ can be arbitrary so long as it is suitable. Secondly, the intrinsic volatility process is mean reverting which agrees with empirical studies. Thirdly, the diffusion coefficient is subject to two parameters, β and γ. Lastly, the introduction of the correlation coefficient, ρ, as in Heston [3] allows for skewness in the returns process distribution.
Equations (2.2) and (2.3) give rise to the following partial differential equation (*pde*) operator provided by

$$\begin{aligned}\mathcal{L} &= -\frac{\partial}{\partial\tau} + rx\frac{\partial}{\partial x} + \frac{1}{2}f^2(y)x^2\frac{\partial^2}{\partial x^2} - rx + \kappa(\theta - y)\frac{\partial}{\partial y} \\ &\quad + \frac{1}{2}\beta^2 y^{2\gamma}\frac{\partial^2}{\partial y^2} + \beta f(y)xy^{\gamma}\rho\frac{\partial^2}{\partial x\partial y}, \end{aligned} \tag{2.4}$$

where $\tau = T - t$ for some time $T \geq t \geq 0$. Given the form of (2.4), we can rewrite it in the equivalent form

$$\mathcal{L} = \mathcal{L}_2 + \beta\mathcal{L}_1 + \beta^2\mathcal{L}_0, \tag{2.5}$$

where notation has been lightened by use of the time-deterministic volatility Black Scholes operator

$$\mathcal{L}_2 = -\frac{\partial}{\partial\tau} + rx\frac{\partial}{\partial x} + \frac{1}{2}f^2(y)x^2\frac{\partial^2}{\partial x^2} + \kappa(\theta - y)\frac{\partial}{\partial y} - rx. \tag{2.6}$$

Also, we have the cross-term operator which picks up the correlation between the Brownian motions driving the stock price and volatility processes, provided by

$$\mathcal{L}_1 = f(y)xy^{\gamma}\rho\frac{\partial^2}{\partial x\partial y}. \tag{2.7}$$

Finally, we have the operator which accounts for the diffusion coefficient of the intrinsic volatility process provided by

$$\mathcal{L}_0 = \frac{1}{2}y^{2\gamma}\frac{\partial^2}{\partial y^2}. \tag{2.8}$$

In order to price a European option, $u(\tau, x, y)$, subject to the boundary condition $u(0, x, y) = \{x - K\}^+$, for some strike price, K, we are wishing to solve the equation $\mathcal{L}u(\tau, x, y) = 0$. In order to facilitate solutions to equations such as these, we consider that

$$u(\tau, x, y) = \sum_{n=0}^{\infty}\beta^n u_n(\tau, x, y). \tag{2.9}$$

It then follows that

$$\begin{aligned}\mathcal{L}u(\tau, x, y) &= \sum_{n=0}^{\infty}(\mathcal{L}_2 + \beta\mathcal{L}_1 + \beta^2\mathcal{L}_0)\beta^n u_n(\tau, x, y), \\ &= \sum_{n=0}^{\infty}\beta^n\mathcal{L}_2 u_n(\tau, x, y) + \sum_{n=1}^{\infty}\beta^n\mathcal{L}_1 u_{n-1}(\tau, x, y) + \sum_{n=2}^{\infty}\beta^n\mathcal{L}_0 u_{n-2}(\tau, x, y).\end{aligned}$$

Setting the above to coefficients of β^n, for $n \geq 0$, it follows that

$$\begin{aligned} 0 &= \mathcal{L}_2 u_0(\tau,x,y) + \beta\left(\mathcal{L}_2 u_1(\tau,x,y) + \mathcal{L}_1 u_0(\tau,x,y)\right) \\ &\quad + \beta^2\left(\mathcal{L}_2 u_2(\tau,x,y) + \mathcal{L}_1 u_1(\tau,x,y) + \mathcal{L}_0 u_0(\tau,x,y)\right) \\ &\quad + \sum_{n=3}^{\infty} \beta^n\left(\mathcal{L}_2 u_n(\tau,x,y) + \mathcal{L}_1 u_{n-1}(\tau,x,y) + \mathcal{L}_0 u_{n-2}(\tau,x,y)\right). \end{aligned}$$

We next set each coefficient of β^n, for $n \geq 0$, equal to zero so that we wish to solve the system of equations

$$\mathcal{L}_2 u_0(\tau,x,y) = 0, \tag{2.10}$$

$$\mathcal{L}_2 u_1(\tau,x,y) + \mathcal{L}_1 u_0(\tau,x,y) = 0, \tag{2.11}$$

$$\mathcal{L}_2 u_2(\tau,x,y) + \mathcal{L}_1 u_1(\tau,x,y) + \mathcal{L}_0 u_0(\tau,x,y) = 0. \tag{2.12}$$

In the sequel, the terms arising from the coefficients of β^n for $n > 2$ will be dispensed with. To justify such exclusions we note that in practice the volatility of volatility parameter, β, is usually less than 0.3 and that any error arising will be of order β^3. The reason such higher order terms are dispensed with, as shall be seen, is the increased complexity involved in finding each of the terms $u_n(\tau,x,y)$.

2.1. The Zeroth-Order Term.

Having set the scene by providing an outline to a method of solution, we turn to the first equation, (10). Given that this term corresponds to the time-deterministic Black Scholes operator, we have the Consider a European contingent claim, $u_0(\tau,x,y)$, such that $u_0(0,x,y) = \{x-K\}^+$ and $\mathcal{L}_2 u_0(\tau,x,y) = 0$. In such a setting, we have

$$u_0(\tau,x,y) = x\mathcal{N}(d_1) - Ke^{-r\tau}\mathcal{N}(d_2), \tag{2.13}$$

where $\mathcal{N}(\cdot)$ corresponds to the cumulative Gaussian distribution, and subject to

$$d_1 = \frac{1}{g(\tau,y)}\left[\ln\frac{x}{K} + r\tau\right] + \frac{1}{2}g(\tau,y) \qquad ; \qquad d_2 = d_1 - g(\tau,y).$$

Here

$$g(\tau,y) = \left\{\int_0^{\tau} f^2(v(u))du\right\}^{1/2},$$

where $v(u) = \theta + (y-\theta)e^{\kappa(u-\tau)}$.

Proof. Consider, for brevity, $r = 0$. We have the partial derivatives

$$\frac{\partial}{\partial x}u_0(\tau,x,y) = \mathcal{N}(d_1),$$

along with

$$\frac{\partial^2}{\partial x^2}u_0(\tau,x,y) = \mathcal{N}'(d_1)\frac{\partial}{\partial x}d_1,$$

as well as

$$\frac{\partial}{\partial \tau}u_0(\tau,x,y) = x\mathcal{N}'(d_1)\frac{\partial}{\partial \tau}g(\tau,y),$$

and

$$\frac{\partial}{\partial y}u_0(\tau,x,y) = x\mathcal{N}'(d_1)\frac{\partial}{\partial y}g(\tau,y).$$

It then follows that

$$\begin{aligned}
\mathcal{L}_2 u_0(\tau,x,y) &= -x\mathcal{N}'(d_1)\frac{\partial}{\partial\tau}g(\tau,y)+\frac{1}{2}f^2(y)x^2\mathcal{N}'(d_1)\frac{\partial}{\partial x}d_1\\
&\quad +\kappa(\theta-y)x\mathcal{N}'(d_1)\frac{\partial}{\partial y}g(\tau,y),\\
&= -x\mathcal{N}'(d_1)2g(\tau,y)\frac{\partial}{\partial\tau}g(\tau,y)+f^2(y)x\mathcal{N}'(d_1)\\
&\quad +\kappa(\theta-y)x\mathcal{N}'(d_1)2g(\tau,y)\frac{\partial}{\partial y}g(\tau,y),\\
&= -x\mathcal{N}'(d_1)\frac{\partial}{\partial\tau}g^2(\tau,y)+f^2(y)x\mathcal{N}'(d_1)\\
&\quad +\kappa(\theta-y)x\mathcal{N}'(d_1)\frac{\partial}{\partial y}g^2(\tau,y).
\end{aligned}\tag{2.14}$$

On the other hand,

$$\frac{\partial}{\partial\tau}g^2(\tau,y) = f^2(y)-\kappa(y-\theta)\int_0^\tau e^{\kappa(u-\tau)}f^2_{v(u)}du, \tag{2.15}$$

and

$$\frac{\partial}{\partial y}g^2(\tau,y) = \int_0^\tau e^{\kappa(u-\tau)}f^2_{v(u)}du. \tag{2.16}$$

Substitution now of (2.15) and (2.16) into (2.14) yields the desired result. □ □

We note an important component of the Black Scholes price with deterministic volatility. Specifically, $\lim_{\kappa\to\infty} v(u)=\theta$, thus meaning that high speed mean reversion leads to the Black Scholes price with constant volatility, $f(\theta)$, i.e., volatility is at its long-run value. Conversely, the next result arises as a consideration of zero mean reversion. Assume that the function $f(\cdot)$ is continuous and differentiable. In the case of zero mean reversion, the Black Scholes price of the zero-order term, $u_0(\tau,x,y)$, is the standard Black Scholes price with effective volatility $f(y)$.

Proof. It is straightforward to see that

$$\lim_{\kappa\to 0} g^2(\tau,y) = f^2(y)\tau,$$

as required. □ □

In the sequel we will continue to consider the case of zero mean reversion as a special case for the following reason: Under certain specifications of the functional form it may make sense to use a low speed mean reversion parameter. Indeed, if mean reversion is too low, packages such as Matlab may not recognise these limits and subsequent functions may appear to lead to option prices which explode when this should not be the case.

2.2. **The First-Order Correction Term.** We now turn to the second equation (11). That is,

$$\mathcal{L}_2 u_1(\tau,x,y)+\mathcal{L}_1 u_0(\tau,x,y)=0.$$

This leads to the following The first-order correction term satisfies

$$u_1(\tau,x,y) = \rho H_1(\tau,y)\left(x^2u_{xx}+\frac{1}{2}x^3u_{xxx}\right), \tag{2.17}$$

where the expression uses the second and third derivatives of the Black Scholes call option price, $u_0(\tau, x, y)$, with respect to the underlying spot asset price, x. In addition, the coefficient in τ, y satisfies

$$H_1(\tau, y) = \int_0^\tau \left[y^\gamma f(y) \frac{\partial}{\partial y} g^2(\tau, y) \right] \Big|_{y=v(u),\tau=u} du, \tag{2.18}$$

where $v(u) = \theta + (y - \theta)e^{\kappa(u-\tau)}$.

Proof. See appendix. □

Yet again, we can find the limiting case of zero mean reversion. The result is contained in the Assume that the function $f(\cdot)$ is continuous. In the case of zero mean reversion, we have that

$$\lim_{\kappa\to 0} H_1(\tau, y) = \frac{1}{2} y^\gamma f(y) \frac{\partial}{\partial y} f^2(y)\tau^2.$$

Proof. Straightforward and note that

$$\lim_{\kappa\to 0} v(u) = y.$$

□ □

2.3. **The Second-Order Correction Term.** We now turn to the third and final equation, (12). That is,

$$\mathcal{L}_2 u_2(\tau, x, y) + \mathcal{L}_1 u_1(\tau, x, y) + \mathcal{L}_0 u_0(\tau, x, y) = 0.$$

The result is entailed in the following The second-order correction term satisfies

$$u_2(\tau, x, y) = u_{(2,1)}(\tau, x, y) + u_{(2,2)}(\tau, x, y), \tag{2.19}$$

where notation has been simplified with the use of

$$u_{(2,1)}(\tau, x, y) = H_2(\tau, y) \left(2x^2 u_{xx} + 4x^3 u_{xxx} + x^4 u_{xxxx}\right) + H_{2b}(\tau, y) x^2 u_{xx}, \tag{2.20}$$

and

$$\begin{aligned} u_{(2,2)}(\tau, x, y) = {} & \rho^2 H_3(\tau, y) \Big(4x^2 u_{xx} + 23x^3 u_{xxx} + 23x^4 u_{xxxx} \\ & + \frac{13}{2} x^5 u_{xxxxx} + \frac{1}{2} x^6 u_{xxxxxx}\Big) \\ & + \rho^2 H_{3b}(\tau, y) \left(x^2 u_{xx} + \frac{5}{4} x^3 u_{xxx} + \frac{1}{4} x^4 u_{xxxx}\right). \end{aligned} \tag{2.21}$$

In this case it is true that

$$H_2(\tau, y) = \frac{1}{8} \int_0^\tau \left[y^{2\gamma} \left(\frac{\partial}{\partial y} g^2(\tau, y) \right)^2 \right] \Big|_{y=v(u),\tau=u} du, \tag{2.22}$$

while

$$H_{2b}(\tau, y) = \frac{1}{4} \int_0^\tau \left[y^{2\gamma} \frac{\partial^2}{\partial y^2} g^2(\tau, y) \right] \Big|_{y=v(u),\tau=u} du. \tag{2.23}$$

In addition, we have that

$$H_3(\tau, y) = \frac{1}{2} \int_0^\tau \left[y^\gamma f(y) \frac{\partial}{\partial y} g^2(\tau, y) H_1(\tau, y) \right] \Big|_{y=v(u),\tau=u} du, \tag{2.24}$$

and

$$H_{3b}(\tau, y) = 2\int_0^\tau \left[y^\gamma f(y) \frac{\partial}{\partial y} H_1(\tau, y) \right] \Big|_{y=v(u),\tau=u} du. \tag{2.25}$$

Proof. See appendix. □

The solution to the second-order term highlights some important properties which cannot be explained by the first-order correction alone. In particular, it is noted that the expression for $u_{(2,1)}(\tau, x, y)$ takes no coefficient in correlation. Specifically, when there is no correlation between the Brownian motions in the system, this term alone accounts for differences in option prices over their Black Scholes counterparts. Additionally, the expression for $u_{(2,2)}(\tau, x, y)$ takes a coefficient of correlation squared, that is, ρ^2. In particular, the sign of the first-order correction term, $u_1(\tau, x, y)$, depends on the sign of correlation whereas the former will always have the same sign regardless of the sign of correlation.
The previous result shows that the addition of a second-order correction term has increased the complexity of solution and, as such, no further terms are explored. In many cases it will be easier to evaluate the terms $H_j(\tau, y)$ for $j = 1, 2, 2b, 3, 3b$ numerically in order to overcome cumbersome expressions. It will be seen, in the next Section, that some closed-form solutions do exist. In addition, it is noted that the expressions of the second-order term involve partial derivatives of the zeroth-order term, $u_0(\tau, x, y)$, up to sixth order. Before looking to simplify such expression, we turn to limiting values as mean reversion tends to zero. Assume that the function $f(\cdot)$ is continuous and differentiable. In the case of zero mean reversion, we have that

$$\lim_{\kappa\to 0} H_2(\tau, y) = \frac{1}{24} y^{2\gamma} \left(\frac{\partial}{\partial y} f^2(y) \right)^2 \tau^3.$$

While

$$\lim_{\kappa\to 0} H_{2b}(\tau, y) = \frac{1}{8} y^{2\gamma} \frac{\partial^2}{\partial y^2} f^2(y) \tau^2.$$

In addition

$$\lim_{\kappa\to 0} H_3(\tau, y) = \frac{1}{6} y^{2\gamma} f^2(y) \left(\frac{\partial}{\partial y} f^2(y) \right)^2 \tau^3,$$

and

$$\lim_{\kappa\to 0} H_{3b}(\tau, y) = y^\gamma f(y) \frac{\partial}{\partial y} \left[y^\gamma f(y) \frac{\partial}{\partial y} f^2(y) \right] \tau^2.$$

Having now derived the approximating polynomial for the solution of European-style options, we now look to simplify the resulting expression in order to overcome the tedium of calculating the partial derivatives,

$$\frac{\partial^n}{\partial x^n} u_0(\tau, x, y),$$

for $n \geq 2$. The useful result is contained in the Consider the vector $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) \in \mathbb{R}^5$. Then the following equality holds for linear combinations and partial derivatives of the Black Scholes option price, $u_0(\tau, x, y)$,

$$\begin{aligned} & \alpha_1 x^2 u_{xx} + \alpha_2 x^3 u_{xxx} + \alpha_3 x^4 u_{xxxx} + \alpha_4 x^5 u_{xxxxx} + \alpha_5 x^6 u_{xxxxxx} \\ = \quad & \Bigg\{ \alpha_1 + (3\alpha_4 - \alpha_3 - 11\alpha_5)\frac{1}{g^2(\tau, y)} + 3\frac{\alpha_5}{g^4(\tau, y)} \\ & -\left(\alpha_2 - \alpha_3 + 2\alpha_4 - 6\alpha_5 + (18\alpha_5 - 3\alpha_4)\frac{1}{g^2(\tau, y)}\right)\left(1 + \frac{d_1}{g(\tau, y)}\right) \\ & \left(\alpha_3 - 3\alpha_4 + 11\alpha_5 - 6\frac{\alpha_5}{g^2(\tau, y)}\right)\left(1 + \frac{d_1}{g(\tau, y)}\right)^2 \\ & -(\alpha_4 - 6\alpha_5)\left(1 + \frac{d_1}{g(\tau, y)}\right)^3 + \alpha_5\left(1 + \frac{d_1}{g(\tau, y)}\right)^4 \Bigg\} x^2 u_{xx}, \end{aligned} \tag{2.26}$$

where d_1 is as provided in theorem 2.1.

Proof. It is easy to see that

$$u_{xxx} \;=\; -\frac{1}{x} u_{xx}\left(1 + \frac{d_1}{g(\tau, y)}\right), \tag{2.27}$$

while

$$u_{xxxx} \;=\; \frac{u_{xxx}^2}{u_{xx}} - \frac{1}{x} u_{xxx} - \frac{1}{x^2 g^2(\tau, y)} u_{xx}.$$

Similarly,

$$u_{xxxxx} \;=\; \frac{u_{xxx}^3}{u_{xx}^2} - \frac{3}{x}\frac{u_{xxx}^2}{u_{xx}} + \frac{2}{x^2} u_{xxx}\left(1 - \frac{3}{2g^2(\tau, y)}\right) + \frac{3}{x^3 g^2(\tau, y)} u_{xx},$$

and

$$\begin{aligned} u_{xxxxxx} \;=\; & \frac{u_{xxx}^4}{u_{xx}^3} - \frac{6}{x}\frac{u_{xxx}^3}{u_{xx}^2} - \frac{1}{x^2}\frac{u_{xxx}^2}{u_{xx}}\left(\frac{6}{g^2(\tau, y)} - 11\right) + \frac{1}{x^3} u_{xxx}\left(\frac{18}{g^2(\tau, y)} - 6\right) \\ & + \frac{1}{x^4}\left(\frac{3}{g^4(\tau, y)} - \frac{11}{g^2(\tau, y)}\right) u_{xx}. \end{aligned}$$

Taking linear combinations of these along with the identity

$$\frac{u_{xxx}^n}{u_{xx}^{n-1}} \;=\; (-1)^n x^{-n} u_{xx}\left(1 + \frac{d_1}{g(\tau, y)}\right)^n,$$

as provided by (2.27) yields the desired result. □ □

In other words, in order to evaluate the polynomial approximation, all that is equired is the time-deterministic volatility, $g(\tau, y)$, the limit of integration, d_1, and the Black Scholes option gamma, u_{xx}. The recursion provided in the previous lemma may look cumbersome, though it is easily implemented as illustrated in the following

Example: Consider the second component of the second-order correction term which takes the linear combination of the partial derivatives

$$2x^2 u_{xx} + 4x^3 u_{xxx} + x^4 u_{xxxx}.$$

Direct use of (2.26) then says that this is equal to

$$\left\{2 - \frac{1}{g^2(\tau, y)} - 3\left(1 + \frac{d_1}{g(\tau, y)}\right) + \left(1 + \frac{d_1}{g(\tau, y)}\right)^2\right\} x^2 u_{xx},$$

which simplifies to

$$\left\{\frac{d_1^2}{g^2(\tau,y)} - \frac{d_1}{g(\tau,y)} - \frac{1}{g^2(\tau,y)}\right\} \frac{x}{g(\tau,y)} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_1^2}.$$

Having simplified the terms comprising the approximating polynomial, there is still one consideration to make before analysing various models. In particular, while it has been shown that the zeroth-order term (i.e., the Black Scholes price) satisfies the terminal condition, $u_0(0,x,y) = 0$, we have the implication that the first- and second-order correction terms must also satisfy the appropriate boundary conditions, namely, $u_1(0,x,y) = u_2(0,x,y) = 0$. Fortunately, the argument required is not difficult. In particular, it is clear that

$$\lim_{\tau\to 0} H_j(\tau,y) \;=\; 0.$$

for all $j = 1, 2, 2b, 3, 3b$. In light of Lemma 2.3, we additionally see that

$$\lim_{\tau\to 0} e^{-\frac{1}{2}d_1^2} \;=\; 0$$

It is true that this term converges to zero faster than the absolute value of other terms comprising the linear combination in Lemma 2.3 tend to infinity. In other words, the first- and second-order correction terms actually converge to zero as the time to maturity reaches zero and the boundary conditions are satisfied as required.

A natural consequence of the results derived so far is that they extend easily to other European derivatives. The use of put-call parity reveals that the differences in option prices, as obtained by the first- and second-order polynomial terms, in comparison to the Black Scholes price must be the same for European put options as they are for European call options. From an arbitrage-theoretic view point this progression is necessary; if there are differences in prices from Black Scholes on a European call option, then these differences must also translate to their put counterparts in order to preclude a riskless trading strategy which results in certain profit. Specifically, we recall that put-call parity says

$$w(\tau,x,y) \;=\; u(\tau,x,y) - x + Ke^{-r\tau},$$

where $w(\tau,x,y)$ is equal to the value of the put and where $u(\tau,x,y)$ is equal to the value of the call found thus far. It follows that

$$\frac{\partial^n}{\partial x^n} w_0(\tau,x,y) \;=\; \frac{\partial^n}{\partial x^n} u_0(\tau,x,y),$$

for all $n > 1$ from which the approximation to puts follows.

Having derived an approximating polynomial for the purpose of evaluating European-style options in markets where volatility is stochastic, we now turn to some specific examples. This is the subject of the next Section.

3. Specific Examples

The structure for pricing European options in markets where volatility is stochastic described thus far, we recall, is quite flexible in its ability to allow for various modelling choices. Indeed, in cases where the functional form of volatility is complicated from an analytic standpoint, it is possible to evaluate the time-deterministic volatility, $g(\tau,y)$, as well as the polynomial coefficients, $H_j(\tau,y)$, for $j = 1, 2, 2b, 3, 3b$, by use of numerical integration procedures such as those found in Matlab.

The model of Hull and White [4] follows when we have $\kappa < 0, \theta = 0, \gamma = 1$, and the use of $f(Y_t) = \sqrt{Y_t}$. For the model of Scott [6] we have $\kappa, \theta > 0, \gamma = 0$, and $f(Y_t) = \exp(Y_t)$. We look at the models presented in Stein and Stein [7] and Heston [3] in more detail in the proceeding subsections.

3.1. **The Stein and Stein Model.** In this case, the stock price evolves according to

$$dS_t = S_t(rdt + Y_t dW_t),$$

while the volatility process evolves according to

$$dY_t = \kappa(\theta - Y_t)dt + \beta dB_t.$$

The above system is equivalent to that provided in (2.2) and (2.3) in the event that $f(Y_t) = Y_t$ along with $\gamma = \rho = 0$. Because correlation between the Brownian motions driving each process is zero, we need only find $H_2(\tau, y)$ and $H_{2b}(\tau, y)$. It is not difficult to show that the time-deterministic Black Scholes variance is

$$g^2(\tau, y) = \theta^2\tau + \frac{1}{2\kappa}(y-\theta)^2(1 - e^{-2\kappa\tau}) + \frac{2}{\kappa}\theta(y-\theta)(1-e^{-\kappa\tau}),$$

while

$$\begin{aligned} H_2(\tau, y) = & -\frac{1}{16\kappa^3}(y-\theta)^2(4\kappa\tau e^{-2\kappa\tau} - 1 + e^{-4\kappa\tau}) \\ & -\frac{1}{4\kappa^3}\theta^2(e^{-2\kappa\tau} - 4e^{-\kappa\tau} - 2\kappa\tau + 3) \\ & -\frac{1}{4\kappa^3}\theta(y-\theta)(2\kappa\tau e^{-\kappa\tau} + e^{-3\kappa\tau} - 2 - 2e^{-2\kappa\tau} + 3e^{-\kappa\tau}), \end{aligned}$$

and

$$H_{2b}(\tau, y) = \frac{1}{8\kappa}\left(2\tau - \frac{1}{\kappa}(1 - e^{-2\kappa\tau})\right).$$

It is noted that it is possible to include correlation and find the functions $H_j(\tau, y)$ for $j = 1, 3, 3b$. To examine the usefulness of the approximation in practice, we compare the option prices it produces to a selected sample taken from Stein and Stein [7]. This output can be found in Table 1. The sample selected from Stein and Stein [7] allows one to see the accuracy under a relatively wide range of conditions for varying strike prices relative to a fixed asset price of \$100.

The first four panels of Table 1, that is, panels a)-d) illustrate various parameterisations in the case where instantaneous volatility, y, is equal to its long run mean. In such cases, we have the Black Scholes volatility $g(\tau, y) = \theta\sqrt{\tau}$. In the case of Table 1, panel a), volatility of volatility is at its lowest (β=0.1) while mean reversion is $\kappa = 4$. In this case it is clear that the option prices arising from the approximation are exactly the same as those in Stein and Stein (1991) to two decimal places.

For Table 1, panel b), volatility of volatility increases threefold while the speed of mean reversion increases fourfold. The effect of such a large jump in mean reversion offsets the loss of accuracy arising from a larger volatility of volatility parameter. Specifically, as mean reversion tends to infinity, option prices will approach Black Scholes, that is, increasing the rate at which Y_t is pulled back to its long-run mean θ leads to constant volatility. As a result, all four approximated prices agree with Stein and Stein [7] to two decimal places. Alternatively, panel c) illustrates this where mean reversion is not as fast (only half as much) while volatility of volatility remains the same. On this occasion, the lower speed of mean reversion relative to volatility of volatility has led to a one cent price difference when the strike price is K = \$110 with all other prices agreeing to two decimal places.

The importance of the mean reversion parameter relative to the volatility of volatility parameter is explained further by panel d) in Table 1. In this case mean reversion is low while volatility of volatility is high. Because of the low speed of mean reversion on this occasion, the size of error in the approximation increases as witnessed by the option prices: The approximation is not accurate to two decimal places for any of the four strike prices, however, in this extreme case, the loss of accuracy is not large.

Finally, panel e) in Table 1 illustrates the impact of typical prices arising when instantaneous volatility, y, starts above its long-run mean, θ. In this case, the speed of mean reversion is quite large at $\kappa = 16$ while volatility of volatility remains at its highest value in the table. Specifically, the high speed of

Table 1. **Sample of option price comparisons for the Stein and Stein model and polynomial approximation:** Parameters are as provided along with $r = 0.0953$, $\tau = 0.5$, $x = \$100$

	initial vol y	long-run mean θ	reversion κ	volvol β	strike K	Stein and Stein	Approximation
a)	0.20	0.20	4.0	0.1	90	15.16	15.16
	0.20	0.20	4.0	0.1	100	8.18	8.18
	0.20	0.20	4.0	0.1	110	3.69	3.69
	0.20	0.20	4.0	0.1	120	1.42	1.42
b)	0.20	0.20	16.0	0.3	90	15.22	15.22
	0.20	0.20	16.0	0.3	100	8.28	8.28
	0.20	0.20	16.0	0.3	110	3.80	3.80
	0.20	0.20	16.0	0.3	120	1.50	1.50
c)	0.25	0.25	8.0	0.3	90	16.09	16.09
	0.25	0.25	8.0	0.3	100	9.63	9.63
	0.25	0.25	8.0	0.3	110	5.22	5.21
	0.25	0.25	8.0	0.3	120	2.61	2.61
d)	0.35	0.35	4.0	0.4	90	18.25	18.24
	0.35	0.35	4.0	0.4	100	12.43	12.42
	0.35	0.35	4.0	0.4	110	8.16	8.14
	0.35	0.35	4.0	0.4	120	5.23	5.22
e)	0.35	0.25	16.0	0.4	90	16.33	16.33
	0.35	0.25	16.0	0.4	100	10.00	10.00
	0.35	0.25	16.0	0.4	110	5.61	5.61
	0.35	0.25	16.0	0.4	120	2.91	2.92

mean reversion results in the volatility process being pulled quickly to its long-run rate and this high level of mean reversion again offsets any possible loss of accuracy attributable to the large volatility of volatility parameter. In all, the option prices agree to two decimal places with the exception of a one cent discrepancy for the deep out of the money option, that is, where the strike price is $K = \$120$.
The examples examined here have shown that the polynomial approximates solutions to the Stein and Stein [7] model quite well under reasonable parameterisations. In the next section we examine a model which incorporates correlation between the Brownian motions driving the stock price and volatility processes.

3.2. **The Heston Model.** In this case, the stock price evolves according to

$$dS_t = S_t\left(rdt + \sqrt{Y_t}dW_t\right),$$

while the variance process evolves according to

$$dY_t = \kappa(\theta - Y_t)dt + \beta\sqrt{Y_t}(\rho dW_t + \sqrt{1-\rho^2}dB_t).$$

The above system is equivalent to that provided in (2.2) and (2.3) in the event that $f(Y_t) = \sqrt{Y_t}$ along with $\gamma = 0.5$, $\rho \neq 0$. Because correlation between the Brownian motions driving each process is non-zero in this case, we need to find $H_j(\tau, y)$, $j = 1, 2, 2b, 3, 3b$. It is not difficult to show that the time-deterministic Black Scholes variance is

$$g^2(\tau, y) = \theta\tau + \frac{1}{\kappa}(y - \theta)(1 - e^{-\kappa\tau}).$$

In addition, we have the following polynomial coefficient functions,

$$H_1(\tau, y) = -\frac{1}{\kappa}(y-\theta)\left(\tau e^{-\kappa\tau} - \frac{1}{\kappa}(1-e^{-\kappa\tau})\right) + \frac{1}{\kappa}\theta\left(\tau - \frac{1}{\kappa}(1-e^{-\kappa\tau})\right),$$

along with

$$\begin{aligned} H_2(\tau, y) &= -\frac{1}{4\kappa^2}(y-\theta)\left(\tau e^{-\kappa\tau} - \frac{1}{2\kappa}(1-e^{-2\kappa\tau})\right) \\ &\quad + \frac{1}{4\kappa^2}\theta\left(\frac{\tau}{2} - \frac{1}{4\kappa}(3 - 4e^{-\kappa\tau} + e^{-2\kappa\tau})\right), \end{aligned}$$

while $H_{2b}(\tau, y) = 0$ and

$$\begin{aligned} H_3(\tau, y) &= \frac{1}{4\kappa^2}(y-\theta)^2\left(\kappa^2\tau^2 e^{-2\kappa\tau} - 2\kappa\tau(e^{-\kappa\tau} - e^{-2\kappa\tau}) + 1 - 2e^{-\kappa\tau} + e^{-2\kappa\tau}\right) \\ &\quad + \frac{1}{2\kappa^4}\theta(y-\theta)\left((2-\kappa^2\tau^2)e^{-\kappa\tau} - (1+\kappa\tau)e^{-2\kappa\tau} - 1 + \kappa\tau\right) \\ &\quad + \frac{1}{4\kappa^4}\theta^2\left(\kappa^2\tau^2 - 2\kappa\tau + 2\kappa\tau e^{-\kappa\tau} + e^{-2\kappa\tau} - 2e^{-\kappa\tau} + 1\right), \end{aligned}$$

as well as

$$\begin{aligned} H_{3b}(\tau, y) &= -\frac{2}{\kappa}(y-\theta)\left(\frac{\tau e^{-\kappa\tau}}{2\kappa}(2+\kappa\tau) - \frac{1}{\kappa^2}(1-e^{-\kappa\tau})\right) \\ &\quad + \frac{2}{\kappa^2}\theta\left(\tau(1+e^{-\kappa\tau}) - \frac{2}{\kappa}(1-e^{-\kappa\tau})\right). \end{aligned}$$

To examine the usefulness of the approximation under differing circumstances we look to replicate option prices under conditions similar to those presented in Table 1. In keeping with the former example, we leave the mean reversion and volatility of volatility parameters fixed along with the stock price and strike prices. The interest rate and time to expiry is the same also. The fundamental differences in the present case are a revision of the instantaneous variance and long-run mean to account for the change in the functional form of volatility. Correlation has also been included and an arbitrary value of $\rho = -0.5$ has been chosen.

The results obtained for comparison are presented in Table 2. On this occasion, only four of the option prices obtained from the approximating polynomial differ to their actual counterparts as found with the Heston [3] model. Otherwise, the patterns found are quite similar to those presented in Table 1. Specifically, panel a) of Table 2 considers a moderate level of mean reversion coupled with low volatility of volatility. The end result is that the approximation matches the actual values to two decimal places in all four cases.

For panel b) in Table 2, the speed of mean reversion is increased fourfold while the volatility of volatility parameter is increased threefold. Yet again, the high speed of mean reversion leads to an exact match in prices for all four examples. In panel c) we see the results of decreasing the speed of mean reversion relative to the volatility of volatility parameter coupled with an increase in instantaneous variance and its long-run mean. Remarkably, the approximated option prices match their actual counterparts to two decimal places in all cases.

Turning to panel d) in Table 2 we see the effect of having a moderate level of mean reversion relative to a large value for volatility of volatility. On this occasion, the accuracy of the polynomial has degenerated somewhat, though this degeneracy is only marginal. In particular, three of the four approximated option prices differ from their actual counterparts at a level of one cent. The option prices presented for a strike price, $K = \$100$, that is, at the money, match to two decimal places.

Finally, we look to panel e) of Table 2 to see the effect of a different value for instantaneous variance to

TABLE 2. **Sample of option price comparisons for the Heston model and polynomial approximation:** Parameters are as provided along with $r = 0.0953, \rho = -0.5, \tau = 0.5, x = \$100$

	initial vol y	long-run mean θ	reversion κ	volvol β	strike K	Heston	Approximation
a)	0.04	0.04	4.0	0.1	90	15.19	15.19
	0.04	0.04	4.0	0.1	100	8.17	8.17
	0.04	0.04	4.0	0.1	110	3.59	3.59
	0.04	0.04	4.0	0.1	120	1.28	1.28
b)	0.04	0.04	16.0	0.3	90	15.20	15.20
	0.04	0.04	16.0	0.3	100	8.17	8.17
	0.04	0.04	16.0	0.3	110	3.58	3.58
	0.04	0.04	16.0	0.3	120	1.26	1.26
c)	0.0625	0.0625	8.0	0.3	90	16.06	16.06
	0.0625	0.0625	8.0	0.3	100	9.47	9.47
	0.0625	0.0625	8.0	0.3	110	4.91	4.91
	0.0625	0.0625	8.0	0.3	120	2.22	2.22
d)	0.1225	0.1225	4.0	0.4	90	18.08	18.09
	0.1225	0.1225	4.0	0.4	100	12.08	12.08
	0.1225	0.1225	4.0	0.4	110	7.58	7.57
	0.1225	0.1225	4.0	0.4	120	4.46	4.45
e)	0.1225	0.0625	16.0	0.4	90	16.30	16.30
	0.1225	0.0625	16.0	0.4	100	9.86	9.86
	0.1225	0.0625	16.0	0.4	110	5.35	5.35
	0.1225	0.0625	16.0	0.4	120	2.61	2.60

its long-run mean. On this occasion, the high speed of mean reversion offsets the high level of volatility of volatility and the end result is an exact match in prices to two decimal places in all four cases.
This example has again shown that the approximating polynomial is highly accurate under varying parameterisations. We conclude the increasing the volatility of volatility will lead to a degeneracy in accuracy and that such loss in accuracy can be countered by increasing the speed of mean reversion, thus leading to volatility which becomes approximately constant. It has also been shown that the accuracy holds well under reasonable parameterisations when the instantaneous variance is not equal to its long-run mean.

3.3. **A More General Model - Impact of Choosing the Functional Form of Volaility.** On this occasion we look to a more arbitrary specification of the functional form of volatility to examine how useful the approximation becomes. The main objective of this investigation arises out of the consideration that the functional forms presented in Stein and Stein [7] and Heston [3] may have been posed for analytic convenience, i.e., to obtain closed-form solutions to option prices. We will consider that the functional form may not behave in a manner which can be considered nice from an analytic standpoint and, as such, look to exploit the power of the approximation procedure. Specifically, let us assume that the stock price process evolves according to

$$dS_t = S_t\left(rdt + \left(0.1 + \frac{1}{9\times 10^8 Y_t^{10}}\right)dW_t\right),$$

as well as the intrinsic volatility process

$$dY_t = \kappa(\theta - Y_t)dt + \beta\sqrt{Y_t}(\rho dW_t\sqrt{1-\rho^2}dB_t).$$

It is clear that the above system is equivalent to that provided in (2.2) and (2.3) in the event that $\gamma = 0.5$, $\rho \neq 0$, and in the event that we have the functional form of volatililty

$$f(Y_t) \;=\; 0.1 + \frac{1}{9 \times 10^8 Y_t^{10}}. \tag{3.28}$$

Figure 1 shows a typical realisation of a sample path of $f(Y_t)$ where the initial volatility is equal to its

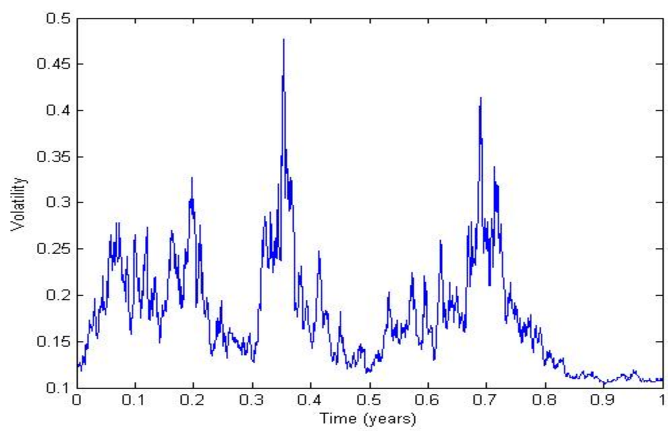

Figure 1. **Simulated volatility path for the functional form (3.28):** Parameters are instantaneous volatility equals long-run mean, $y = \theta = 0.2$, mean reversion, $\kappa = 20$, and volatility of volatility $\beta = 0.3$

long-run mean. The process is also relatively quickly mean reverting with a reasonably high input for volatility of volatility. It is clear from this figure that the path has the properties usually attributed to a sample path of volatility. Specifically, the sample path is volatile with the high speed of mean reversion pulling it back to its long-run mean. In addition, there are several spikes in the path which revert quite quickly to the long-run mean.

On this occasion we have the time-deterministic Black Scholes variance represented by

$$g^2(\tau, y) \;=\; \int_0^{\tau} \left(0.1 + \frac{1}{9 \times 10^8 v(u)^{10}}\right)^2 du,$$

where

$$v(u) \;=\; \theta + (y - \theta)e^{\kappa(u-\tau)}.$$

The above integral is evaluated numerically using the pre-programmed family of integration formulas in Matlab. In addition, it is necessary to evaluate each of the approximating polynomial functional coefficients, $H_j(\tau, y)$, for $j = 1, 2, 2b, 3, 3b$ numerically, again using the pre-programmed family of integration formulas in Matlab.

Because there is no benchmark model to reconcile the results of the approximation with, we instead compare prices to those found with the time-deterministic Black Scholes formula, that is, where the input volatility is $g(\tau, y)$. In addition, all inputs including the instantaneous volatility, its long-run mean, the level of mean reversion, and the volatility of volatility parameters will be the same as those used to illustrate the Stein and Stein [7] model in Table 1. The purpose of this is to provide a two-way comparison. Firstly, comparing the approximated values to the Black Scholes benchmark will enable us to gage how option priced under the stochastic volatility specification differ across moneyness relative to Black Scholes. Secondly, the (approximate) option prices obtained can be compared against their counterparts found by the evaluation of the Stein and Stein [7] model to illustrate that the choice of functional form is an important modelling consideration.

Table 3 contains the approximated option prices and their Black Scholes counterparts. Firstly, we comment to the extent that the option prices obtained assuming stochastic volatility are only approximations and that the accuracy of these can be gaged given the analysis of the accuracy of the approximated prices found using the Stein and Stein [7] model. Specifically, the accuracy (or lack thereof) in that case should translate directly to those found here.

TABLE 3. **Sample of option prices for the polynomial approximation:** Parameters are as provided along with $r = 0.0953, \rho = -0.5, \tau = 0.5, x = \$100$

	initial vol y	long-run mean θ	reversion κ	volvol β	strike K	Black Scholes	Stochastic Volatility	% Change
a)	0.20	0.20	4.0	0.1	90	14.32	14.34	0.14
	0.20	0.20	4.0	0.1	100	6.24	6.41	2.72
	0.20	0.20	4.0	0.1	110	1.67	1.85	10.78
	0.20	0.20	4.0	0.1	120	0.26	0.29	11.54
b)	0.20	0.20	16.0	0.3	90	14.32	14.49	1.19
	0.20	0.20	16.0	0.3	100	6.24	7.29	16.83
	0.20	0.20	16.0	0.3	110	1.67	2.76	65.27
	0.20	0.20	16.0	0.3	120	0.26	0.56	115.38
c)	0.25	0.25	8.0	0.3	90	14.23	14.43	1.41
	0.25	0.25	8.0	0.3	100	5.74	7.43	29.44
	0.25	0.25	8.0	0.3	110	1.14	2.91	155.26
	0.25	0.25	8.0	0.3	120	0.10	0.49	390.00
d)	0.35	0.35	4.0	0.4	90	14.22	14.53	2.18
	0.35	0.35	4.0	0.4	100	5.69	8.39	47.45
	0.35	0.35	4.0	0.4	110	1.08	3.92	262.96
	0.35	0.35	4.0	0.4	120	0.08	0.69	762.50
e)	0.35	0.25	16.0	0.4	90	14.23	14.41	1.26
	0.35	0.25	16.0	0.4	100	5.70	7.27	27.54
	0.35	0.25	16.0	0.4	110	1.10	2.74	149.09
	0.35	0.25	16.0	0.4	120	0.09	0.45	400.00

The first thing to notice is that the prices found with the stochastic volatility model are all higher than the Black Scholes prices in all 20 cases. The extent to which this true is mostly determined by the mean reversion and volatility of volatility parameters. Specifically, the most explanatory power in the amount of price differences over Black Scholes belongs to the volatility of volatility parameter. That is, increasing the input of volatility of volatility is leading to greater price differences. On this occasion, the level of mean reversion is having less of a role of explaining the price changes and the only exception is in panel e) where the parameter has increased fourfold over that provided in panel d).

Indeed, with volatility of volatility remaining at 0.3 as shown in panels b) and c) there is an unexpected result for the two options with a strike price of $K = \$90$. Specifically, the option priced where mean reversion is twice as large, that is, panel b) has a higher price than that found in panel c). Otherwise, panel b) where mean reversion is double that in panel c) leads to lower option prices.

4. CONLCUSION

This paper has investigated a methodology which can be used to unify several well-known stochastic volatility models with a view to option pricing. In particular, a general model of stochastic volatility where the Stein and Stein [7] and Heston [3] models act as special cases has been posed. The method of

solving the generalised system posed here is an approximating polynomial which should carry an error of β^3 where β is our volatility of volatility parameter.
The flexibility and accuracy of the approximating procedure has been investigated from two important directions. The first of these is a direct comparison of its performance against the benchmark models of Stein and Stein and Heston. It has been shown that for both models the accuracy of the approximation is quite high under reasonable parameterisations. For the former model, it was shown that fourteen of the twenty option prices compared matched to two decimal places. For the other ten, the inaccuracies where generally at a one cent level.
Correlation between the Brownian motions comprising the system was then investigated with comparisons of option prices to those found with the Heston [3] model. In this case, the order of accuracy was higher with sixteen of the twenty option prices being compared being accurate to two decimal places. In the remaining four cases, the order of error was one cent.
Having investigated the accuracy of the approximating polynomial, we then turned to the second direction of investigation. This approach was designed to illustrate the flexibility and usefulness of the polynomial in situations where the functional form of volatility may not be nice from and an analytic standpoint. In other words, the restricting need to pose a functional form of volatility as well as a diffusion process for volatility which is analytically convenient to work with is removed and the results indicate that the functional form of volatility is not a trivial consideration.
In light of the results presented in this paper, two useful directions for future research arise, and these can be considered somewhat linked. The first direction is the posing of a functional form of volatility along with an accurate diffusion model for volatility in order to capture the characteristics of a sample path of volatility with as much accuracy as can be obtained. Related to such accuracy is the estimation and/or calibration of the functional form and diffusion process parameters to stock price data. Such estimation was omitted here in favor of an investigation which highlights the usefulness of the generalised methodology.

Appendix A. Proof of Theorem 2.2

The expansion in (11) provides the coefficient of β,

$$\mathcal{L}_2 u_1(t,x,y) + \mathcal{L}_1 u_0(t,x,y) \;=\; 0.$$

However,

$$\begin{aligned}
\mathcal{L}_1 u_0(\tau,x,y) &= \rho x y^{\gamma} f(y) \frac{\partial}{\partial y} \mathcal{N}(d_1), \\
&= -\rho x y^{\gamma} f(y) \frac{1}{g(\tau,y)} d_2 \mathcal{N}'(d_1) \frac{\partial}{\partial y} g(\tau,y), \\
&= -\rho y^{\gamma} f(y) d_2 x^2 u_{xx} \frac{\partial}{\partial y} g(\tau,y).
\end{aligned}$$

Therefore, we look for α_1, α_2, and $h(g(\tau,y))$ such that

$$\alpha_1 x^2 u_{xx} + \alpha_2 x^3 u_{xxx} \;=\; -h(g(\tau,y)) d_2 x^2 u_{xx},$$

whence it becomes clear that we must have $\alpha_1 = 1, \alpha_2 = 1/2, h(g(\tau,y)) = 1/2g(\tau,y)$. That is,

$$\begin{aligned}
\mathcal{L}_2 u_1(\tau,x,y) &= -2\rho y^{\gamma} f(y) g(\tau,y) \left(x^2 u_{xx} + \frac{1}{2} x^3 u_{xxx} \right) \frac{\partial}{\partial y} g(\tau,y), \\
&= -\rho y^{\gamma} f(y) \left(x^2 u_{xx} + \frac{1}{2} x^3 u_{xxx} \right) \frac{\partial}{\partial y} g^2(\tau,y).
\end{aligned}$$

Alternatively,

$$\mathcal{L}_2 x^n \frac{\partial^n}{\partial x^n} u_0(\tau, x, y) \;=\; 0,$$

and, because $\mathcal{L}_2$ is linear in its derivatives in t and y, this means there exists some function, $H_1(\tau, y)$, such that

$$\mathcal{L}_2 u_1(\tau, x, y) \;=\; \rho\left(x^2 u_{xx} + \frac{1}{2}x^3 u_{xxx}\right)\mathcal{L}_2 H_1(\tau, y),$$

where

$$\mathcal{L}_2 H_1(\tau, y) \;=\; -y^{\gamma} f(y)\frac{\partial}{\partial y} g^2(\tau, y).$$

The function $H_1(\tau, y)$ provided in (2.18) satisfies this equality. □

Appendix B. Proof of Theorem 2.3

The expansion in (12) provides the coefficient of β^2,

$$\begin{aligned}\mathcal{L}_2 u_2(\tau, x, y) &= \mathcal{L}_2[u_{2,1}(\tau, x, y) + u_{2,2}(\tau, x, y)],\\ &= -\mathcal{L}_0 u_0(\tau, x, y) - \mathcal{L}_1 u_1(\tau, x, y),\end{aligned}$$

where we will let $u_{2,1}(\tau, x, y)$ correspond to the operator, $\mathcal{L}_0$, and $u_{2,2}(\tau, x, y)$ correspond to $\mathcal{L}_1$. We look first to the right part of the right hand side of this where we find that

$$\begin{aligned}-\mathcal{L}_0 u_0(\tau, x, y) &= -\frac{1}{2}y^{2\gamma}\frac{\partial}{\partial y}\left[x\mathcal{N}'(d_1)g_y\right],\\ &= -\frac{1}{2}y^{2\gamma}\left(gg_{yy} + d_1 d_2 (g_y)^2\right)x^2 u_{xx},\end{aligned}$$

where g_y and g_{yy} denote the usual first and second derivatives of $g(\tau, y)$ with respect to y respectively. Because of the term involving $d_1 d_2$, we look for the constants, $\alpha_1, \alpha_2, \alpha_3$ such that

$$\alpha_1 x^2 u_{xx} + \alpha_2 x^3 u_{xxx} + \alpha_3 x^4 u_{xxxx} \;=\; h(g(\tau, y), d_1 d_2) x^2 u_{xx},$$

for some suitable function, h. It is not difficult to show that the choice of $\alpha_1 = 2, \alpha_2 = 4, \alpha_4 = 1$, leads to

$$\mathcal{L}_0 u_0(\tau, x, y) \;=\; \frac{1}{2}y^{2\gamma}\left(gg_{yy} + (g_y)^2\right)x^2 u_{xx} + \frac{1}{2}y^{2\gamma}(g_y g)^2(2x^2 u_{xx} + 4x^3 u_{xxx} + x^4 u_{xxxx}).$$

However, $gg_{yy} + (g_y)^2 = \frac{1}{2}g^2_{yy}$, leaving us with

$$-\mathcal{L}_0 u_0(\tau, x, y) \;=\; -\frac{1}{2}y^{2\gamma}(g_y g)^2(2x^2 u_{xx} + 4x^3 u_{xxx} + x^4 u_{xxxx}) - \frac{1}{2}y^{2\gamma}g^2_{yy}x^2 u_{xx}.$$

That is,

$$\begin{aligned}\mathcal{L}_2 u_{2,1}(\tau, x, y) &= -\frac{1}{2}y(g_y g)^2(2x^2 u_{xx} + 4x^3 u_{xxx} + x^4 u_{xxxx}) - \frac{1}{2}y^{2\gamma}g^2_{yy}x^2 u_{xx}\\ &= (2x^2 u_{xx} + 4x^3 u_{xxx} + x^4 u_{xxxx})\mathcal{L}_2 H_2(\tau, y) + x^2 u_{xx}\mathcal{L}_2 H_{2b}(\tau, y),\end{aligned}$$

and we find $H_2(\tau, y)$ provided in (2.22) and $H_{2b}(\tau, y)$ provided in (2.23) satisfies this equality.
In order to recover $u_{2,2}(\tau, x, y)$, we now look to the term, $-\mathcal{L}_1 u_1(\tau, x, y)$. To that end, we have

$$\begin{aligned}\rho\frac{\partial}{\partial y}H_1(\tau, y)\left(x^2 u_{xx} + \frac{1}{2}x^3 u_{xxx}\right) &= \rho\left(x^2 u_{xx} + \frac{1}{2}x^3 u_{xxx}\right)\frac{\partial}{\partial y}H_1(\tau, y)\\ &\quad + \rho H_1(\tau, y)\frac{\partial}{\partial y}\left(x^2 u_{xx} + \frac{1}{2}x^3 u_{xxx}\right).\end{aligned} \tag{2.29}$$

The first term on the right hand side is straightforward, for we must then have

$$\begin{aligned} &\rho^2 x y^\gamma f(y) \frac{\partial}{\partial x}\left(x^2 u_{xx} + \frac{1}{2}x^3 u_{xxx}\right)\frac{\partial}{\partial y}H_1(\tau, y) \\ = \quad & 2\rho^2 y^\gamma f(y)\left(x^2 u_{xx} + \frac{5}{4}x^3 u_{xxx} + \frac{1}{4}x^4 u_{xxxx}\right)\frac{\partial}{\partial y}H_1(\tau, y). \end{aligned}$$

That is, there exists the function, $u_{2,2,1}(\tau, x, y)$, such that

$$\mathcal{L}_2 u_{2,2,1}(\tau, x, y) = 2\rho^2\left(x^2 u_{xx} + \frac{5}{4}x^3 u_{xxx} + \frac{1}{4}x^4 u_{xxxx}\right)\mathcal{L}_2 H_{3b}(\tau, y),$$

from which we find that (2.25) satisfies this equality. We now turn to the second term on the right side of (2.29). In light of theorem 2.2, we seek

$$\rho H_1(\tau, y)\frac{\partial}{\partial y}\frac{1}{2g(\tau, y)}d_2 x^2 u_{xx},$$

and find this equal to

$$\rho H_1(\tau, y)\frac{1}{2g^2}g_y\left(d_1 d_2^2 - d_1 - 2d_2\right)x^2 u_{xx}.$$

Because the expression involves $d_1 d_2^2$ we look for the constants, $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, and find that

$$\frac{1}{x^2 u_{xx}}g^3(\tau, y)(4x^2 u_{xx} + 14x^3 u_{xxx} + 8x^2 u_{xxxx} + x^5 u_{xxxxx}) \;=\; d_1 d_2^2 - d_1 - 2d_2.$$

That is, the choice of $\alpha_1 = 4, \alpha_2 = 14, \alpha_3 = 8$, and $\alpha_4 = 1$, leads to

$$\rho H_1(\tau, y)\frac{\partial}{\partial y}\left(x^2 u_{xx} + \frac{1}{2}x^3 u_{xxx}\right) \;=\; \frac{1}{2}\rho H_1(\tau, y) g g_y (4x^2 u_{xx} + 14x^3 u_{xxx} + 8x^2 u_{xxxx} + x^5 u_{xxxxx}).$$

Because we are using the operator, $\mathcal{L}_1$, we need to differentiate with respect to x and multiply by $\rho x y^\gamma f(y)$. Doing so says that

$$\begin{aligned} &\mathcal{L}_2 u_{2,2,2}(\tau, x, y) \\ = \quad & \rho^2 y^\gamma f(y) H_1(\tau, y) g g_y\left(4x^2 u_{xx} + 23x^3 u_{xxx} + 23x^2 u_{xxxx} + \frac{13}{2}x^5 u_{xxxxx} + \frac{1}{2}x^6 u_{xxxxxx}\right) \\ = \quad & \rho^2 \mathcal{L}_2 H_3(\tau, y)\left(4x^2 u_{xx} + 23x^3 u_{xxx} + 23x^2 u_{xxxx} + \frac{13}{2}x^5 u_{xxxxx} + \frac{1}{2}x^6 u_{xxxxxx}\right), \end{aligned}$$

from which we find that (2.24) satisfies this equality. □

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