An Inequality for the Sum of Divisors Function

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In this paper we will consider one inequality on the sum of divisors function. This inequality is closely related with the Robin's inequality.

As in the theorem 1 of the paper [1], we suppose that $n = q_1^{\lambda_1} \cdot q_2^{\lambda_2} \cdots q_m^{\lambda_m}$ is the prime factorization of n, where $q_1, q_2, \cdots q_m$ are distinct primes and $\lambda_1, \lambda_2, \cdots \lambda_m$ are non-negative integers. We assume $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \geq 1$ here, too. Let $p_1 = 2,\ p_2 = 3,\ p_3 = 5, \cdots, p_n, \cdots$ be the consecutive primes. We will choose $p_m \geq 5$ arbitrarily and fix it. We put $r_0(n) = p_1^{\lambda_1} \cdot p_2^{\lambda_2} \cdots p_m^{\lambda_m}$. Then by the theorem 1 of the paper [2], there exist the optimum points $\overline{\lambda_0} = \left(\lambda_1^0, \lambda_2^0, \cdots, \lambda_m^0\right) \in R^m$ in m-dimensional real space R^m of the function

$$H\left(\bar{\lambda}\right)=H\left(\lambda_1,\,\lambda_2,\cdots,\lambda_m\right)=\frac{\exp\left(\exp\left(e^{-\gamma}\cdot F\left(\bar{\lambda}\right)\right)\right)}{p_1^{\lambda_1}\cdot p_2^{\lambda_2}\cdots p_m^{\lambda_m}},$$

where

$$F\left(\bar{\lambda}\right)=F\left(\lambda_1,\,\lambda_2,\cdots,\lambda_m\right)=\prod_{i=1}^{m}\frac{1-p_i^{-\lambda_i-1}}{1-p_i^{-1}},$$

and γ=0.577$\cdots$ is Euler's constant ([4,5]).

We have

Theorem. There exists a constant $1\le c_0<+\infty$ such that for any $n\ge 2$ we have

$$\sigma\left(n\right)\le e^{\gamma}\cdot n\cdot\log\log\left(c_0\cdot n\cdot\exp\left(\sqrt{\log n}\cdot\exp\left(\sqrt{\log\log(n+1)}\right)\right)\right).$$

Proof. We put

$$G\left(n\right)=\frac{\left(\exp\left(\exp\left(e^{-\gamma}\cdot\sigma\left(n\right)/n\right)\right)\right)/n}{\exp\left(\sqrt{\log n}\cdot\exp\left(\sqrt{\log\log(n+1)}\right)\right)}.$$

There are two steps for the proof of the theorem.

① The function $G\left(n\right)$ has the following properties.

First, For any $n\in S\left(\bar{\lambda},\,m\right)$ ([1]) it holds that $G\left(n\right)\le G\left(r_0(n)\right)$.

In fact, it is clear by the theorem 1 and the theorem 2 of the paper [1].

Second, for $n=p_1^{\lambda_1}\cdot p_2^{\lambda_2}\cdots p_m^{\lambda_m}$ we put $G\left(n\right)=G\left(\bar{\lambda}\right)=G\left(\lambda_1,\,\lambda_2,\cdots,\lambda_m\right)$. Then there exist $\bar{\alpha}_0=\left(\alpha_1^0,\,\alpha_2^0,\cdots\alpha_m^0\right)\in R^m$ such that for any $\left(\lambda_1,\,\lambda_2,\cdots,\lambda_m\right)\in R^m$ we have $G\left(\bar{\lambda}\right)\le G\left(\bar{\alpha}_0\right)$. This is also clear by the theorem 1 of the paper [2]. And for the optimum points $\bar{\alpha}_0=\left(\alpha_1^0,\,\alpha_2^0,\cdots\alpha_m^0\right)\in R^m$ of the function $G\left(\bar{\lambda}\right)$, such the results as in the theorem 2 and the theorem 3 of the paper [2] hold.

Also for any $n \geq 2$ we have $G(n) \leq H(n) = \left(\exp\left(\exp\left(e^{-\gamma} \cdot \sigma(n)/n\right)\right)\right)/n$.

Finally, The every member $\alpha_i^0 \ (i=1,m)$ of the optimum points $\{\alpha_1^0, \alpha_2^0, \cdots \alpha_m^0\}$ of the function $G(\bar{\lambda})$ is not larger than $\lambda_i^0 \ (i=1,m)$ of one of the function $H(\bar{\lambda})$, namely, for any $i \ (1 \leq i \leq m)$ it holds that $\alpha_i^0 \leq \lambda_i^0$.

In fact, by the theorem 2 of [2], for the function $H(\bar{\lambda})$ it holds that

$$p_1^{\lambda_1^0+1} = p_2^{\lambda_2^0+1} = \cdots = p_k^{\lambda_k^0+1} =$$
$$= \left(e^{-\gamma} F(\bar{\lambda}_0)\right) \cdot \exp\left(e^{-\gamma} F(\bar{\lambda}_0)\right) + 1 \quad (1 \leq i \leq k).$$

Similarly, for the function $G(\bar{\lambda})$ it holds that

$$p_1^{\alpha_1^0+1} = p_2^{\alpha_2^0+1} = \cdots = p_k^{\alpha_k^0+1} =$$
$$= \left(e^{-\gamma} F(\bar{\alpha}_0)\right) \cdot \exp\left(e^{-\gamma} F(\bar{\alpha}_0)\right) \cdot \left(\frac{1}{1+\Psi(n)}\right) + 1 \quad (1 \leq i \leq k),$$

where

$$\Psi(n) = \frac{\exp\left(\sqrt{\log\log(n+1)}\right)}{2 \cdot \sqrt{\log n}} +$$
$$+ \frac{\exp\left(\sqrt{\log\log(n+1)}\right)}{2 \cdot \sqrt{\log\log(n+1)}} \cdot \frac{\sqrt{\log n}}{\log(n+1)} \cdot \left(\frac{n}{n+1}\right) \to 0 \ (n \to \infty).$$

Hence for any $i \ (1 \leq i \leq m)$ we have $\alpha_i^0 \leq \lambda_i^0$ and, in particular, we have

$$F(\bar{\alpha}_0) = \prod_{i=1}^{m} \frac{1 - p_i^{-\alpha_i^0 - 1}}{1 - p_i^{-1}} \leq \prod_{i=1}^{m} \frac{1 - p_i^{-\lambda_i^0 - 1}}{1 - p_i^{-1}} = F(\bar{\lambda}_0).$$

② We put

$$D_m = G\left(\alpha_1^0, \alpha_2^0, \cdots, \alpha_m^0\right)$$

and

$$
\begin{cases}
n_0 = p_1^{\alpha_1^0} p_2^{\alpha_2^0} \cdots p_k^{\alpha_k^0} \cdot p_{k+1}^1 \cdots p_m^1, \quad n_0' = n_0 \cdot p_m^{-1}, \\
\bar{\alpha}_0' = \left(\alpha_1^0, \alpha_2^0, \cdots, \alpha_{m-1}^0\right) \in R^{m-1}, \\
D_{m-1}' = G\left(\bar{\alpha}_0'\right) = G\left(\alpha_1^0, \alpha_2^0, \cdots, \alpha_{m-1}^0\right).
\end{cases}
$$

In this connection, we put

$$
D_{m-1} = \max_{(\lambda_1, \lambda_2, \cdots \lambda_{m-1}) \in R^{m-1}} G\left(\lambda_1, \lambda_2, \cdots \lambda_{m-1}\right).
$$

Then it is clear that $D_{m-1}' \le D_{m-1}$ and

$$
\begin{aligned}
\log \frac{D_m}{D_{m-1}'} &= \left(\exp\left(e^{-\gamma} \cdot F\left(\bar{\alpha}_0\right)\right) - \exp\left(e^{-\gamma} \cdot F\left(\bar{\alpha}_0'\right)\right)\right) - \\
&\quad - \left(\log n_0 + \sqrt{\log n_0} \cdot \exp\left(\sqrt{\log\log(n_0+1)}\right)\right) + \\
&\quad + \left(\log n_0' + \sqrt{\log n_0'} \cdot \exp\left(\sqrt{\log\log(n_0'+1)}\right)\right) = \\
&= \exp\left(e^{-\gamma} \cdot F\left(\bar{\alpha}_0'\right)\right)\left(\exp\left(e^{-\gamma} \cdot F\left(\bar{\alpha}_0'\right) \cdot \frac{1}{p_m}\right) - 1\right) - \left(\log p_m\right) - \\
&\quad - \left(\sqrt{\log n_0} \cdot \exp\left(\sqrt{\log\log(n_0+1)}\right) - \sqrt{\log n_0'} \cdot \exp\left(\sqrt{\log\log(n_0'+1)}\right)\right).
\end{aligned}
$$

By the theorem 4 of the paper [3] we have

$$
\begin{aligned}
&\exp\left(e^{-\gamma} \cdot F\left(\bar{\alpha}_0'\right)\right)\left(\exp\left(e^{-\gamma} \cdot F\left(\bar{\alpha}_0'\right) \cdot \frac{1}{p_m}\right) - 1\right) \le \\
&\le \exp\left(e^{-\gamma} \cdot F\left(\bar{\lambda}_0'\right)\right)\left(\exp\left(e^{-\gamma} \cdot F\left(\bar{\lambda}_0'\right) \cdot \frac{1}{p_m}\right) - 1\right) = \\
&= \log p_m + \Theta_1\left(p_m\right),
\end{aligned}
$$

where $\Theta_1\left(p_m\right) = O\left(\frac{\log^4 p_m}{\sqrt{p_m}}\right)$. So there is a constant a $a > 0$ such that

$$
\Theta_1\left(p_m\right) \le a \cdot \frac{\log^4 p_m}{\sqrt{p_m}}.
$$

On the other hand, we have

$$\log n_0 = \log\left(p_1^{\alpha_1^0} p_2^{\alpha_2^0} \cdots p_k^{\alpha_k^0} \cdot p_{k+1}^1 \cdots p_m^1 \right) = \sum_{i=1}^{m} \alpha_i^0 \cdot \log p_i =$$

$$= \sum_{i=1}^{m} \log p_i + \sum_{i=1}^{k} \left(\alpha_i^0 - 1\right) \cdot \log p_i = \vartheta\left(p_m\right) + \vartheta\left(p_k\right) + R_k$$

where $\vartheta\left(p_m\right) = \sum_{i=1}^{m} \log p_i$ is the Chebyshev's function ([6]) and $R_k = o\left(p_k\right)$.

Hence by the prime number theorem ([4,5,6]), we have

$$\frac{\log n_0}{p_m} = \frac{\vartheta\left(p_m\right)}{p_m} + \frac{\vartheta\left(p_k\right)}{p_m} + \frac{R_k}{p_m} \to 1 \ \left(p_m \to \infty\right).$$

From this we get

$$\log n_0 = p_m \cdot \left(1 + \theta_1\left(p_m\right)\right),$$

where $\theta_1\left(p_m\right) = \mathrm{O}\left(\frac{1}{\log p_m}\right)$. So we also obtain

$$\log n_0' = p_{m-1}\left(1 + \theta_2\left(p_{m-1}\right)\right).$$

where $\theta_2\left(p_{m-1}\right) = \mathrm{O}\left(\frac{1}{\log p_{m-1}}\right)$. And it is easy to see that

$$\left(\sqrt{\log n_0} \cdot \exp\left(\sqrt{\log\log(n_0+1)}\right) - \sqrt{\log n_0'} \cdot \exp\left(\sqrt{\log\log(n_0'+1)}\right)\right) =$$

$$= \left(\sqrt{\log n_0} - \sqrt{\log n_0'}\right) \cdot \exp\left(\sqrt{\log\log(n_0+1)}\right) +$$

$$+ \sqrt{\log n_0'} \cdot \left(\exp\left(\sqrt{\log\log(n_0+1)}\right) - \exp\left(\sqrt{\log\log(n_0'+1)}\right)\right) =$$

$$= \exp\left(\sqrt{\log p_m}\right) \cdot \left(\frac{\log p_m}{2 \cdot \sqrt{p_m}}\right) \cdot \left(1 + \Theta_2\left(p_m\right)\right),$$

where $\Theta_2\left(p_m\right) = \mathrm{O}\left(\frac{1}{\log p_m}\right)$. Hence we have

$$\log D_m - \log D_{m-1}' \le a \cdot \frac{\log^4 p_m}{\sqrt{p_m}} -$$

$$- \exp\left(\sqrt{\log p_m}\right) \cdot \frac{\log p_m}{2 \cdot \sqrt{p_m}} \left(1 + \Theta_2\left(p_m\right)\right).$$

On the other hand, it is clear that

$$\frac{\log^3 p_m}{\exp\left(\sqrt{\log p_m}\right)} \to 0 \ \left(p_m \to \infty\right)$$

This shows that there exists a number m_0 such that for any $m \geq m_0$ we have

$$D_m < D'_{m-1} \leq D_{m-1}.$$

From this we get

$$0 < c_0 = \sup_m D_m < +\infty.$$

This is the proof of the theorem. □

Note. ① We are sure that

$$c_0 = D_1 = \frac{\exp\left(\exp\left(e^{-\gamma} \cdot 3/2\right)\right)/2}{\exp\left(\sqrt{\log 2} \cdot \exp\left(\sqrt{\log\log 3}\right)\right)} = 1.6436 \cdots \leq 2$$

② The process for the proof of the theorem by the papers [1,2,3] is graphically as follows. Here ⇒ shows the increasing direction of the values for the function $H(n)$ and $G(n)$.

$$n = q_1^{\lambda_1} \cdot q_2^{\lambda_2} \cdot q_3^{\lambda_3} \cdots q_{m-1}^{\lambda_{m-1}} \cdot q_m^{\lambda_m}$$

⇩ ← paper [1]

$$r_0(n) = p_1^{\lambda_1} \cdot p_2^{\lambda_2} \cdot p_3^{\lambda_3} \cdots \cdot p_m^{\lambda_m}$$

⇩ ← paper [2]

$$n_0 = p_1^{\lambda_1^0} \cdot p_2^{\lambda_2^0} \cdots p_k^{\lambda_k^0} \cdot p_{k+1}^1 \cdots p_m^1,$$

⇩ ← paper [3]

$$n'_0 = p_1^{\lambda_1^0} \cdot p_2^{\lambda_2^0} \cdots p_k^{\lambda_k^0} \cdot p_{k+1}^1 \cdots p_{m-1}^1$$

⇩ ⇙

$$\boxed{n = 2}$$

As it was indicated in the paper [1], one can say that any natural number has the three-dimensional structure. For $\overline{q} = (q_1, q_2, \cdots q_m)$, $\overline{\lambda} = (\lambda_1, \lambda_2, \cdots \lambda_m)$ and $\omega(n) = m$ of $n = q_1^{\lambda_1} \cdot q_2^{\lambda_2} \cdots q_m^{\lambda_m}$ we put $n = n(\overline{q}, \overline{\lambda}, m)$. Then to prove the theorem we have taken the process reducing the dimensional numbers of

$n = n(\overline{q}, \overline{\lambda}, m)$ in the function $G(n)$. The dimensional numbers of n in the function $G(n)$ were reduced by the paper [1], [2] and [4], respectively. That is so; $n = n(\overline{q}, \overline{\lambda}, m) \to n(\overline{\lambda}, m) \to n(\overline{\lambda}_0, m) \to n(m)$.

③ The below table 1 shows the optimum points $\overline{\lambda}_0 = (\lambda_1^0, \lambda_2^0, \cdots, \lambda_m^0)$ of the function $H(\overline{\lambda})$ and the values of $H(n_0)$ and $G(n_0)$ to $\omega(n) = m$.

Table 1

$\omega(n)$ $= m$	$\overline{\lambda} = (\lambda_1^0, \lambda_2^0, \cdots, \lambda_m^0)$ of $n_0 = 2^{\lambda_1^0} \cdot 3^{\lambda_2^0} \cdot 5^{\lambda_2^0} \cdots p_k^{\lambda_k^0} \cdot p_{k+1}^1 \cdots p_m^1$	$H(n_0)$, $G(n_0)$
1	$\lambda_1^0 = 1$	$5.09518716186\cdots$, $1.643686767536\cdots$
2	$\lambda_1^0 = 1.65\cdots, \lambda_2^0 = 1$	$3.58945411446\cdots$, $0.8250082\times10^{-1}\cdots$
3	$\lambda_1^0 = 2.70\cdots, \lambda_2^0 = 1.33\cdots, \lambda_3^0 = 1$	$1.91192398575\cdots$, $0.7148367\times10^{-5}\cdots$
4	$\lambda_1^0 = 3.36\cdots, \lambda_2^0 = 1.75\cdots,$ $\lambda_3^0 = 1, \lambda_4^0 = 1$	$1.32309514626\cdots$, $0.1065950\times10^{-6}\cdots$
5	$\lambda_1^0=4.22\cdots, \lambda_2^0=2.29\cdots,$ $\lambda_3^0=1.24\cdots, \lambda_4^0=\lambda_5^0 = 1$	$0.57062058635\cdots$, $0.3761569\times10^{-9}\cdots$
6	$\lambda_1^0=4.53\cdots, \lambda_2^0=2.49\cdots,$ $\lambda_3^0=1.38\cdots, \lambda_4^0=\lambda_5^0 = \lambda_6^0 = 1$	$0.40977025702\cdots$, $0.767767\times10^{-10}\cdots$
7	$\lambda_1^0=5.02\cdots, \lambda_2^0=2.80\cdots,$ $\lambda_3^0=1.59\cdots, \lambda_4^0=1.14\cdots,$ $\lambda_5^0 = \lambda_6^0 = \lambda_7^0 = 1$	$0.22782964552\cdots$, $0.575576\times10^{-11}\cdots$
8	$\lambda_1^0=5.22\cdots, \lambda_2^0=2.92\cdots,$ $\lambda_3^0=1.68\cdots, \lambda_4^0=1.21\cdots,$ $\lambda_5^0 = \lambda_6^0 = \lambda_7^0 = \lambda_8^0 = 1$	$0.20507350097\cdots$, $0.164730\times10^{-12}\cdots$
9	$\lambda_1^0=5.57\cdots, \lambda_2^0=3.14\cdots,$ $\lambda_3^0=1.83\cdots, \lambda_4^0=1.34\cdots,$ $\lambda_5^0 = \lambda_6^0 = \lambda_7^0 = \lambda_8^0 = \lambda_9^0 = 1$	$0.16722089980\cdots$, $0.287587\times10^{-14}\cdots$
$\cdots$	$\cdots$ $\cdots$ $\cdots$	$\cdots$ $\cdots$

③ The below table 2 shows the Hardy-Ramanujan's numbers ([1]), which give maximum value of the function $G(n_0)$ to $\omega(n) = m$.

Table 2

$\omega(n)$ $= m$	$\tilde{n}_0 = r_0(\tilde{n}_0) = p_1^{\lambda_1} \cdots p_k^{\lambda_k} \cdot p_{k+1}^1 \cdots p_m^1$	$G(\tilde{n}_0)$
1	2	$1.643686767536\cdots$
2	$2 \cdot 3$	$0.82500822 \times 10^{-1} \cdots$
3	$2^2 \cdot 3 \cdot 5$	$0.71483676 \times 10^{-5} \cdots$
4	$2^3 \cdot 3^2 \cdot 5 \cdot 7$	$0.10659507 \times 10^{-6} \cdots$
5	$2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$	$0.37615690 \times 10^{-9} \cdots$
6	$2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13$	$0.76776726 \times 10^{-10} \cdots$
7	$2^5 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17$	$0.575576185 \times 10^{-11} \cdots$
8	$2^5 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19$	$0.164730227 \times 10^{-12} \cdots$
9	$2^5 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$	$0.287587585 \times 10^{-14} \cdots$
$\cdots$	$\cdots \quad \cdots \quad \cdots$	$\cdots$

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See for [1]:
http://commons.wikimedia.org/wiki/File:The_sum_of__divisors_function__and_the_Hardy-Ramanujan%27s_number.pdf
See for [2]:
http://commons.wikimedia.org/wiki/File:An_Exponential_Function_and_itsOptimization_Problem.pdf
See for [3]:
http://commons.wikimedia.org/wiki/File:An_Estimate_for_the_Error_in_a_Formula_on_Prime_Numbers.pdf

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