

rate facile perspecturus foret; jam vero, quoniam egregium illud Rei Medicæ Lumen amisimus, eadem aliis Eruditis perpendenda simul proponimus & dijudicanda. Tibi præsertim, Vir Doctissime, cujus auctoritatem & ille plurimi fecit, & nos præcipuam habemus, Judici simul integerrimo & maxime idoneo, totam istam disputationem lubentissime subjicimus.

III. *Methodus Differentialis Newtoniana Illustrata.*
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A Rithmeticæ pars præcipua consistit in inveniendâ in numeris quantitate quâcunque determinatâ; cum vero quantitatum & numerorum natura non patiatur ut omnes quantitates exhibeantur in numeris accurate, necesse habemus ad Approximationes confugere. Hoc est, ubi quantitatum valores mathematice accurati nequeunt obtineri, quærendi sunt ii qui ab accuratis distant minus datâ quâvis differentiâ.

Quicquid hæc de re à Veteribus ad nos pervenit, vel est particulare, ut Methodus eorum reducendi *Æquationes Quadraticas*; vel saltem usibus generalibus male destinatum, ut Methodus Exhaustionum. *Vieta* quidem primus erat qui aliquid generale in hæc arte assequutus est: quippe invenit methodum reducendi *Æquationes Rationales*, quæ solæ tunc in usu erant. In hæc acquievêre omnes Geometræ ex ejus temporibus usque ad ea *Newtoni*. Hic ex Interpolationibus primo pervenit ad Series: quas postea ad reductionem *Æquationum* omnium omnino generum universaliter applicuit. Hæc autem methodus procedit per quantitatum nascentium & evanescentium rationes primas & ultimas, seu si ita loqui liceat, per quantitatum coincidentium.

eidentium differentias infinite parvas. Sed & ulterius promovit *Newtonus* hanc methodum; docuitque quæ ratione approximandum sit ad quantitates quæ determinantur per regularem seriem terminorum, non per *Æquationem* ut vulgo fit. Atque sic posuit fundamenta calculi hujus Differentialis, qui procedit per quantitatuum differentias cujuscunque magnitudinis: ideoque est methodo Serierum universalior. Per hæc artes *Newtonianas*, universa doctrina Approximationum reducitur ad solutionem Problematis, *Invenire Lineam Geometricam quæ per data quotcunque puncta transibit*. Ex hujus inquam solutione inveniuntur radices *Æquationum* quarumcunque, & etiam quantitates quarum relationes ad alias datas per nullas *Æquationes* hæcenus notas possunt exprimi. Existimo igitur *Newtonum* perduxisse methodum Approximandi ad summum perfectionis fastigium; dum ex unico simplicissimo principio totam hanc doctrinam longe lateque patentem deducit. Quapropter credendum est animum *Newtoni* non satis perspectum fuisse iis, qui ejus methodos appellant particulares, & alias tanquam suas & solas genuinas atque generales venditant, quæ aliæ non erant quam Corollaria facillima à *Newtonianis*.

Author noster, in Epistola ad *Oldenburgum*, *Octob. 24. 1676.* data, mentionem fecit de methodo expeditâ ducendi Lineam Parabolicam per data quotcunque puncta; qua dixit se usum fuisse ubi Series simplices non sunt satis tractabiles. Et hanc methodum primo publicavit in Lemmate quinto Libri tertii *Principiorum*. Atque in Lectionibus publicis, circa idem tempus quo dicta Epistola scripta est, *Cantabrigiæ* habitis, exposuit modum generalem determinandi Curvas cujuscunque generis quæ transibunt per totidem data puncta quot earum natura patitur. Hæ Lectiones sub titulo *Arithmetica Universalis* anno 1707. publicatæ sunt, ubi habetur

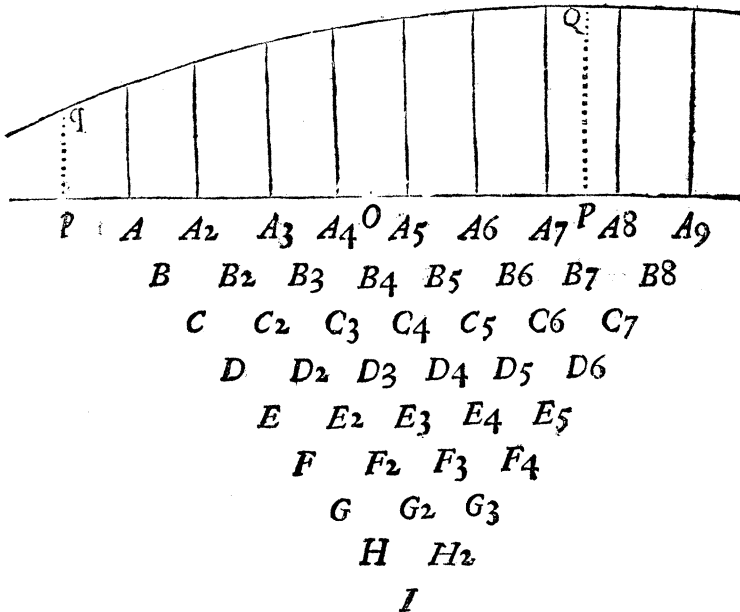
betur methodus exemplis illustrata in sectionibus Conicis. Anno vero 1711. tandem prodiit, inter alios ejusdem Authoris tractatus, ipsa Methodus Differentialis plenius quam ante exposita, cum fundamento ejus demonstrato

Archimedes in methodo Exhaustionum, *Cavallerius* in methodo Indivisibilium, & *Wallisus* noster in Arithmetica Infinitorum, posuerunt fundamenta doctrinæ de determinanda quantitate quæsitâ per locum quem obtinet inter terminos in data Serie: at qua ratione approximandum esset ad valores quantitatum sic determinatarum. horum nemo docuit; Hoc primus & solus perfecit *Newtonus*: atque exinde haud parum ampliata est universa Analysis. Nam sicut ante hoc inventum, ea Problemata Arithmetica sola pro solutis habebantur, ubi relatio quantitatis quæsitæ ad alias datas definiebatur *Æquatione*. jam pro solutis habenda sunt non minus ea, in quibus quantitas quæsitâ locum datum fortitur inter terminos datæ Seriei; siquidem numeri desiderati non minus accurate obtinentur per Methodum Differentialem, quam per extractionem Radicum: hisce vero habitis, parum interest quomodo ad eos devenitum est. Et experientia multiplex docuit, quod plurima Problemata ad *Æquationes* ægre deducuntur, dum ad methodum Differentialem facillime. Qualis est ex multis aliis toties decantata Circuli Quadratura; quam tam perfectam, mea opinione, *Wallisus* in Arithmetica Infinitorum exhibuit quam *Archimedes* illam Parabolæ.

Propositio

Propositio.

Invenire Lineam Parabolicam que transibit per extrema Ordinatorum quocumque æquidistantium.



Casus Primus.

Designent $A, A_2, A_3, A_4, A_5, A_6, A_7, A_8, A_9,$
&c. Ordinatas æquidistantes insistentes Abscissæ in
 dato angulo. Collige earum differentias $B, B_2, B_3,$
 $B_4, B_5, B_6, B_7, B_8,$ *&c.* harumque differentias $C,$
 $C_2, C_3, C_4, C_5, C_6, C_7,$ *&c.* harumque differentias
 $D, D_2, D_3, D_4, D_5, D_6,$ *&c.* harumque differentias
 $E, E_2, E_3, E_4, E_5,$ *&c.* harumque $F, F_2, F_3, F_4,$
&c. Et sic porro. Differentiæ autem colligi debent au-
 ferendo

ferendo priores semper de posterioribus. Hoc est ponendo $B = A_2 - A$, $B_2 = A_3 - A_2$, $B_3 = A_4 - A_3$, $B_4 = A_5 - A_4$, $B_5 = A_6 - A_5$, &c. Tum $C = B_2 - B$, $C_2 = B_3 - B_2$, $C_3 = B_4 - B_3$, $C_4 = B_5 - B_4$, &c. deinde $D = C_2 - C$, $D_2 = C_3 - C_2$, $D_3 = C_4 - C_3$, &c. Et similiter sunt omnes differentiae sequentes colligendae. Vel sint $\alpha, \beta, \gamma, \delta, \epsilon, \zeta, \eta$, &c. aequales $A, A_2, A_3, A_4, A_5, A_6, A_7$, &c. Eritque $A = \alpha$, $B = \beta - \alpha$, $C = \gamma - 2\beta + \alpha$, $D = \delta - 3\gamma + 3\beta - \alpha$, $E = \epsilon - 4\delta + 6\gamma - 4\beta + \alpha$, $F = \zeta - 5\epsilon + 10\delta - 10\gamma + 5\beta - \alpha$, $G = \eta - 6\zeta + 15\epsilon - 20\delta + 15\gamma - 6\beta + \alpha$, &c. In hisce valoribus numerales Coefficientes ipsorum $\alpha, \beta, \gamma, \delta, \epsilon$ &c. generantur ut in dignitatibus integris Binomii $1 - z$, $1 - z^2$, $1 - z^3$, $1 - z^4$, &c. Scribendo numeros 1, 2, 3, 4, 5, &c. in Serie $1 \times \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} \times \dots$ successive pro n . Sit jam P quaelibet Ordinata reliquis intermedia, & AP ejus distantia ab Ordinata prima A appelletur z , tum erit

$$P \cdot Q = A +$$

$$B \times \frac{z}{1} +$$

$$C \times \frac{z}{1} \times \frac{z-1}{2} +$$

$$D \times \frac{z}{1} \times \frac{z-1}{2} \times \frac{z-2}{3} +$$

$$E \times \frac{z}{1} \times \frac{z-1}{2} \times \frac{z-2}{3} \times \frac{z-3}{4} +$$

$$F \times \frac{z}{1} \times \frac{z-1}{2} \times \frac{z-2}{3} \times \frac{z-3}{4} \times \frac{z-4}{5} +$$

$$G \times \frac{z}{1} \times \frac{z-1}{2} \times \frac{z-2}{3} \times \frac{z-3}{4} \times \frac{z-4}{5} \times \frac{z-5}{6} + \&c.$$

Adeoque

Adeoque signum ipsius z mutandum est, quando PQ cadit ad alteras partes Ordinatae primae, ut pq .

Casus Secundus.

Sit jam A_5 Ordinata in medio omnium; pone $A = B_4 + B_5$, $B = D_3 + D_4$, $C = F_2 + F_3$, $D = H + H_2$, &c. & $a = C_4$, $b = E_3$, $c = G_2$, $d = I$, &c. id est, si sint $A_6 = \alpha$, $A_7 = \beta$, $A_8 = \gamma$, $A_9 = \delta$, &c. $A_4 = \kappa$, $A_3 = \lambda$, $A_2 = \mu$, $A = \nu$, &c. Pone $A = \alpha - \kappa$, $B = \beta - 2\alpha + 2\kappa - \lambda$, $C = \gamma - 4\beta + 5\alpha - 5\kappa + 4\lambda - \mu$, $D = \delta - 6\gamma + 14\beta - 14\alpha + 14\kappa - 14\lambda + 6\mu - \nu$, &c. $a = \alpha - 2A_5 + \kappa$, $b = \beta - 4\alpha + 6A_5 - 4\kappa + \lambda$, $c = \gamma - 6\beta + 15\alpha - 20A_5 + 15\kappa - 6\lambda + \mu$, $d = \delta - 8\gamma + 28\beta - 56\alpha + 70A_5 - 56\kappa + 28\lambda - 8\mu + \nu$, &c. Et dicatur A_5P , z , tum erit

$$\begin{aligned}
 PQ = A_5 + & \frac{Az + azz}{1.2} + \\
 & \frac{2Bz + bzz}{1.2} \times \frac{zz-1}{3.4} + \\
 & \frac{3Cz + czz}{1.2} \times \frac{zz-1}{3.4} \times \frac{zz-4}{5.6} + \\
 & \frac{4Dz + dzz}{1.2} \times \frac{zz-1}{3.4} \times \frac{zz-4}{5.6} \times \frac{zz-9}{7.8} + \\
 & \frac{5Ez + ezz}{1.2} \times \frac{zz-1}{3.4} \times \frac{zz-4}{5.6} \times \frac{zz-9}{7.8} \times \frac{zz-16}{9.10} + \\
 & \text{\&c.}
 \end{aligned}$$

Casus Tertius.

Sint jam A_4 , A_5 , Ordinatae duae in medio omnium:
 Pone $A = \frac{A_4 + A_5}{2}$, $B = \frac{C_3 + C_4}{2}$, $C = \frac{E_2 + E_3}{2}$, $D =$
 $10A$ $G + G_2$

$\frac{G + G_2}{2}$, &c. $a = B_4$, $b = D_3$, $c = F_2$, $d = H$, &c.

Vel sint $A_5 = \alpha$, $A_6 = \beta$, $A_7 = \gamma$, $A_8 = \delta$, &c.
 $A_4 = \kappa$, $A_3 = \lambda$, $A_2 = \mu$, $A = \nu$, &c. Deinde crunt
 $2A = \alpha + \kappa$, $2B = \beta - \alpha - \kappa + \lambda$, $2C = \gamma - 3\beta$
 $+ 2\alpha + 2\kappa - 3\lambda + \mu$, $2D = \delta - 5\gamma + 9\beta - 5\alpha -$
 $5\kappa + 9\lambda - 5\mu + \nu$, &c. Et $a = \alpha - \kappa$, $b = \beta - 3\alpha +$
 $3\kappa - \lambda$, $c = \gamma - 5\beta + 10\alpha - 10\kappa + 5\lambda - \mu$, $d =$
 $\delta - 7\gamma + 21\beta - 35\alpha + 35\kappa - 21\lambda + 7\mu - \nu$, &c.
 Et sit O punctum medium inter A_4 , A_5 , atque appel-
 letur OP , z ; eritque Ordinata

$$PQ = \frac{A + az}{4^0} +$$

$$\frac{3B + bz}{4^1} \times \frac{4z-1}{2 \cdot 3} +$$

$$\frac{5C + cz}{4^2} \times \frac{4z-1}{2 \cdot 3} \times \frac{4z-9}{4 \cdot 5} +$$

$$\frac{7D + dz}{4^3} \times \frac{4z-1}{2 \cdot 3} \times \frac{4z-9}{4 \cdot 5} \times \frac{4z-25}{6 \cdot 7} +$$

$$\frac{9E + ez}{4^4} \times \frac{4z-1}{2 \cdot 3} \times \frac{4z-9}{4 \cdot 5} \times \frac{4z-25}{6 \cdot 7} \times \frac{4z-49}{8 \cdot 9} + \&c.$$

In hisce duobus etiam casibus z est negativa, quando Ordinata PQ cadit ad alteras partes initii Abcissæ. Et in omnibus tribus casibus distantia communis Ordinatarum ponitur unitas.

Omnes tres casus demonstrantur facillime per calculum. In casu primo pro PQ scribo successive $\alpha, \beta, \gamma, \delta, \varepsilon$, &c. & pro z interea $0, 1, 2, 3, 4$, &c. quæ sunt longitudines Abcissæ ordine sequentes; & provenient æquationes

$$\alpha = A, \beta = A + B, \gamma = A + 2B + C, \delta = A + 3B + 3C + D,$$

$$\varepsilon = A + 4B + 6C + 4D + E, \&c.$$

$$\beta - \alpha = B, \quad \gamma - \beta = B + C, \quad \delta - \gamma = B + 2C + D,$$

$$\varepsilon - \delta = B + 3C + 3D + E, \quad \&c.$$

$$\gamma - 2\beta + \alpha = C, \quad \delta - 2\gamma + \beta = C + D, \quad \varepsilon - 2\delta + \gamma = C + 2D + E, \quad \&c.$$

$$\delta - 3\gamma + 3\beta - \alpha = D, \quad \varepsilon - 3\delta + 3\gamma - \beta = D + E, \quad \&c.$$

$$\varepsilon - 4\delta + 6\gamma - 4\beta + \alpha = E, \quad \&c.$$

Hæ *Æquationes*, capiendo earum differentias, nullo labore resolvuntur, uti videre est. Et dant eisdem ipsorum *A, B, C, D, &c.* valores, qui antea positi sunt in solutione. Et ad eundem modum demonstrantur casus duo reliqui.

Harum trium serierum unaquæque converget ad valorem *Ordinatæ P Q*, ubi *Ordinarum* datarum differentia sunt iustæ magnitudinis. At ubi non convergunt, aliæ artes adhibendæ sunt. Sed impræsentiarum de hujus Propositionis usu pauca adjiciamus.

Designent $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta, \theta, \kappa, \lambda, \&c.$ terminos quoscunque æquidistantes, quorum differentia sunt perexiguæ; & relationes quas inter se obtinent definientur quamproxime per *Æquationes* sequentes, quæ oriuntur capiendo differentias & differentias differentiarum continuè, & ponendo eas æquales nihilo.

$$\alpha - \beta = 0$$

$$\alpha - 2\beta + \gamma = 0$$

$$\alpha - 3\beta + 3\gamma - \delta = 0$$

$$\alpha - 4\beta + 6\gamma - 4\delta + \varepsilon = 0$$

$$\alpha - 5\beta + 10\gamma - 10\delta + 5\varepsilon - \zeta = 0$$

$$\alpha - 6\beta + 15\gamma - 20\delta + 15\varepsilon - 6\zeta + \eta = 0$$

$$\alpha - 7\beta + 21\gamma - 35\delta + 35\varepsilon - 21\zeta + 7\eta - \theta = 0$$

$$\alpha - 8\beta + 28\gamma - 56\delta + 70\varepsilon - 56\zeta + 28\eta - 8\theta + \kappa = 0$$

$$\alpha - 9\beta + 36\gamma - 84\delta + 126\varepsilon - 126\zeta + 84\eta - 36\theta + 9\kappa - \lambda = 0.$$

&c.

Hæc

Hæc Tabula in usum reservanda est, ut consulatur quoties opus sit. Quod autem hæc Æquationes vel obtinent accurate, vel ad verum approximant, ubi differentię terminorum sunt parvæ, patet ex demonstratio-
ne casus primi Propositionis.

Assumatur quælibet Series $\frac{1}{101}, \frac{1}{102}, \frac{1}{103}, \frac{1}{104}, \frac{1}{105}, \frac{1}{106}, \&c.$
Et quærat^r terminus qui stat proximus ante $\frac{1}{100}$: patet quod ille est $\frac{1}{100}$; videamus ergo qualem hæc methodus exhibebit eundem. Repræsentet α terminum quæsitum, eritque

$\frac{1}{x_{91}} = \beta = 0099,0099,0099,0,$	Æquatio,	$\left. \begin{array}{l} 1ma \\ 2da \\ 3tia \\ 4ta \\ 5ta \\ 6ta \end{array} \right\}$	dat α	$\left\{ \begin{array}{l} 0099,0099,0099,0, \\ 0099,9805,8629,3, \\ 0099,9994,3455,0, \\ 0099,9999,7824,8, \\ 0099,9999,9895,8, \\ 0099,9999,9993,1. \end{array} \right.$
$\frac{1}{x_{92}} = \gamma = 0098,0392,1568,7,$				
$\frac{1}{x_{93}} = \delta = 0097,0873,7864,1,$				
$\frac{1}{x_{94}} = \epsilon = 0096,1538,4615,4,$				
$\frac{1}{x_{95}} = \zeta = 0095,2380,9523,8,$				
$\frac{1}{x_{96}} = \eta = 0094,3396,2264,2.$				

Patet ergo quod hæc methodus continue approxi-
mat. Si terminorum differentię fuissent minores, va-
lores accessissent citius ad verum, & contra tardius
quando differentię sunt majores. Hinc si in Tabulis
numericis desit terminus, potest is per hanc methodum
inferi.

Hoc modo etiam prodeunt ipsissimæ Series Speciosæ, quæ
per alias methodos prodire solent. Proponatur $1 + zz^{-1}$
Ordinata Curvæ quadrandæ: Ea est prima in serie regu-
lari $1 + zz^{-1}, 1 + zz^0, 1 + zz^1, 1 + zz^2, 1 + zz^3, \&c.$
Ordinatarum, quæ omnes præter primam dant suas
arcas $z, z + \frac{1}{3}z^3, z + \frac{2}{3}z^3 + \frac{1}{5}z^5, z + \frac{2}{3}z^3 + \frac{2}{5}z^5 + \frac{1}{7}z^7,$
&c. constituentes novam seriem cujus primus terminus
erit Area quæsitæ: quæ ideo invenietur ponendo pro
eâ α , & pro reliquis in suo Ordine $\beta, \gamma, \delta, \epsilon, \&c.$ Pri-
ma Æquatio dat $\alpha = z$, secunda $\alpha = z - \frac{1}{3}z^3$, tertia
 $\alpha = z - \frac{1}{3}z^3 + \frac{1}{5}z^5$, quarta $\alpha = z - \frac{1}{3}z^3 + \frac{1}{5}z^5 - \frac{1}{7}z^7,$
&c.

&c. Est ergo univcrsim area quæfita $z - \frac{1}{3}z^3 + \frac{1}{5}z^5 - \frac{1}{7}z^7 + \frac{1}{9}z^9 - \frac{1}{11}z^{11}$ &c. Estque hæc Series arcus ad Tangentem z , in circulo radium habente unitati æqualem. Eam invenit *Jacobus Gregorius* noster, & cum *Collinio* communicavit initio anni 1671. à quo, mediante *Ol denburgo* ad *Leibnitium* delata est.

Sit jam &c, $e, d, c, b, a, P, \alpha, \beta, \gamma, \delta, \varepsilon$, &c. Series utrinque excurrrens in infinitum, ubi dantur omnes termini præter P in medio omnium. Sit $A = \alpha + a$, $B = \beta + b$, $C = \gamma + c$, $D = \delta + d$, $E = \varepsilon + e$, &c. atque erit

$$\begin{aligned}
 P &= \frac{A}{2} + \\
 &\quad \frac{A-B}{6} + \\
 &\quad \frac{5A-8B+3C}{60} + \\
 &\quad \frac{7A-14B+9C-2D}{140} + \\
 &\quad \frac{42A-96B+81C-32D+5E}{1260} + \\
 &\quad \frac{66A-165B+165C-88D+25E-3F}{2772} + \\
 &\quad \frac{429A-1144B+1287C-832D+325E-72F+7G}{24024} + \&c.
 \end{aligned}$$

Investigatur hæc Series ex Æquationibus, excerpendo alternas in quibus numerus terminorum est impar. Nam earum differentix relinquent terminos in hac Serie; quæ itaque ad libitum produci potest.

Sit $1+z|^{-1}$ Ordinata Hyperbolæ, & quærat Area ejus quæ jacet supra Abfciffam z , quando ea evadit unitas. Hæc Ordinata est media in Serie Ordinatarum

rum

rum, &c. $\frac{1}{1+z}^{-5}, \frac{1}{1+z}^{-4}, \frac{1}{1+z}^{-3}, \frac{1}{1+z}^{-2}, \frac{1}{1+z}^{-1},$
 $\frac{1}{1+z}^0, \frac{1}{1+z}^1, \frac{1}{1+z}^2, \frac{1}{1+z}^3$ &c. æquidistantium,
 hinc inde excurrente in infinitum. Adeoque Areae ab
 hisce Ordinariis genitæ constituent seriem consimilem,
 cujus medius terminus erit Area quaesita; quæ proinde
 obtinebitur per Seriem modo expolitam. Quando z est
 unitas, ut in casu præsentè, areae curvarum evadunt
 &c. $\frac{15}{64}, \frac{7}{24}, \frac{3}{8}, \frac{1}{4},$ & $1, \frac{3}{2}, \frac{7}{3}, \frac{15}{4},$ &c. Hinc est $A = 1 +$
 $\frac{1}{2} = \frac{3}{2}, B = \frac{3}{2} + \frac{3}{8} = \frac{15}{8}, C = \frac{7}{3} + \frac{7}{24} = \frac{31}{8}, D = \frac{7}{4} + \frac{15}{64} = \frac{255}{64},$
 &c. Hisce in Serie substitutis, prodit P , id est, area
 Hyperbolæ, $\frac{3}{4} - \frac{3}{48} + \frac{3}{480} - \frac{3}{4480} +$ &c. id est, $\frac{3}{4} -$
 $\frac{A}{4 \cdot 3} - \frac{2B}{4 \cdot 5} - \frac{3C}{4 \cdot 7} - \frac{4D}{4 \cdot 9} - \frac{5E}{4 \cdot 11} -$ &c. Ubi jam $A,$
 $B, C D,$ &c. more *Newtoniano*, designant terminos in suo
 ordine ab initio. Calculum appono.

TERMINI.

<i>Affirmativi</i>	<i>Negativi.</i>
7500,0000,0000,0000,0	0625,0000,0000,0000,0
62,5000,0000,0000,0	6,6964,2857,1428,5
7440,4761,9047,6	845,5086,5800,8
97,5586,9130,8	11,3818,4731,9
1,3390,4086,1	1585,7062,8
188,7745,5	22,5708,7
2,7085,0	3260,2
393,4	47,5
5,7	7
+7563,2539,3930,7494,1	-0631,7821,3370,8041,1

Summam negativam subducens ab affirmativâ, habeo
 pro Area, id est, pro Logarithmo Hyperbolico Binarii,
 numerum 6931,4718,0559,9453.

Pro

Pro constructione Tabularum quarumvis numerica-
rum percommoda est Series quæ sequitur. Designent
&c. $e, d, c, b, a, \alpha, \ell, \gamma, \delta, \varepsilon$, &c. terminos alternos in
Serie utrinque serpente in infinitum; Pone $A = \alpha + a$,
 $B = \ell + b$, $C = \gamma + c$, $D = \delta + d$, $E = \varepsilon + e$, &c.
Et terminus inter α & a erit

$$\begin{aligned} & \frac{A}{2} + \\ & \frac{1}{1} \times \frac{A-B}{2^4} + \\ & \frac{1 \cdot 3}{1 \cdot 2} \times \frac{2A-3B+C}{2^7} + \\ & \frac{1 \cdot 3 \cdot 5}{1 \cdot 2 \cdot 3} \times \frac{5A-9B+5C-D}{2^{10}} + \\ & \frac{1 \cdot 3 \cdot 5 \cdot 7}{1 \cdot 2 \cdot 3 \cdot 4} \times \frac{14A-28B+20C-7D+E}{2^{13}} + \\ & \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} \times \frac{42A-90B+75C-35D+9E-F}{2^{16}} + \\ & \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9 \cdot 11}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} \times \frac{132A-297B+275C-154D+54E-11F+G}{2^{19}} + \\ & \text{\&c.} \end{aligned}$$

Hæc Series sequitur ex casu tertio Propositionis,
ponendo $z=0$. Coefficientes numerales literarum sic
producuntur; exempli gratiâ, in quarto termino coef-
ficiens literæ penultimæ C est 5; pone $5 + 1 = n$, &
numeri qui proveniunt ex multiplicatione terminorum
 $1 \times \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} \times \text{\&c.}$ erunt 1, 6, 15, 20,
&c. Horum differentiæ 5, 9, 5, sunt numeri quæsitæ.
Atque adeo Series ad libitum producti potest.

Datis Logarithmis numerorum 46, 48, 50, 52, 54,
56, 58 & 60; invenire Logarithmum numeri 53, qui
consistit in medio omnium. Pone $l, 52 + l, 54 = A =$
 $3,4483,9710,34$, $l, 50 + l, 56 = B = 3,4471,5803,13$.

$l, 48 + l, 58 = C = 3,4446.6923.08$, $l, 46 + l, 60 = D = 3,4409,0908,19$. Hisce valoribus in Serie scriptis, primi quatuor termini dabunt $1,7242,2586,96$ pro Logarithmo numeri 53. Et eadem ratione invenire licet quemvis alium intermedium.

In Constructione ergo Tabularum sufficit primo quæ-
rere aliquos terminos in debitis distantis. nam reli-
qui possunt hoc modo insereri. Etenim continuo
sunt intercalandi termini primo inventi usque dum
perventum fuerit ad ultimos qui desiderantur. Hoc
modo habebitur tota Tabula ex datis paucis terminis
sub initio pro fundamento operationis. Sed non con-
venit ut termini quos primo quærimus, sint omnes per
totam Tabulam æquidistantes; nam si omittimus al-
ternos ubi eorum differentia est maxima, possumus ali-
bi per saltum omittere duos, tres, viginti aut forte
plures terminos. Numerus autem terminorum inter
duos datos consistentium, qui omittuntur, debet sem-
per esse aliquis sequentium 1, 3, 7, 15, 31, 63, &c.
dummodo volumus inserere eos per hanc Seriem; hoc
vero neutiquam incommodabit opus.

Possunt autem pro Praxi termini in unam summam
colligi, ut factum vides in hac Tabella. Prima expressio
est primus terminus; secunda est summa primi & secun-
di; tertia est summa primi, secundi & tertii; & sic porro.

$$\begin{array}{r}
 2 \left| \begin{array}{l} A \\ 2 \end{array} \right. \\
 4 \left| \begin{array}{l} 9A - B \\ 16 \end{array} \right. \\
 6 \left| \begin{array}{l} 150A - 25B + 3C \\ 256 \end{array} \right. \\
 8 \left| \begin{array}{l} 1225A - 245B + 49C - 5D \\ 2048 \end{array} \right. \\
 10 \left| \begin{array}{l} 39690A - 8820B + 2268C - 305D + 35E \\ \hline 65536 \end{array} \right.
 \end{array}$$

Sic

Sic datis aliquibus terminis alternis, intermedii confestim dabuntur per hæc expressiones, nullâ ratione habitâ naturæ Tabulæ particularis. Nam hæ regulæ sunt eadem in omnibus. Areæ curvarum sunt proxime æquales areis Parabolicæ figuræ quæ transit per extrema Ordinatarum suarum. Sed quoniam laboriosum nimis esset semper recurrere ad Parabolam, computavi Tabulam sequentem, quâ Areæ directè exhibentur ex datis Ordinatis.

$$\begin{array}{r}
 1 \left| \frac{A}{1} R \\
 3 \left| \frac{A+4B}{6} R \\
 5 \left| \frac{7A+32B+12C}{90} R \\
 7 \left| \frac{41A+216B+27C+272D}{840} R \\
 9 \left| \frac{989A+5888B-928C+10496D-4540E}{28350} R \\
 11 \left| \frac{16067A+106300B-48525C+272400D-260550E+427368F}{598752} R.
 \end{array}$$

Hic numerus Ordinatarum est impar, A est summa primæ & ultimæ, B secundæ & penultimæ, C tertiæ & antepenultimæ; & sic porro, usque dum deventum sit ad eam in medio omnium, quæ per ultimam literam in quâque expressione repræsentatur. R est basis seu pars Abscissæ inter primam & ultimam Ordinatum interceptæ. Expressiones sunt Areæ contentæ inter Curvam, basin & Ordinatam hinc inde extremas. Tabulam pro pare numero Ordinatarum non apposui, quoniam Area cæteris paribus ex impare earum numero accuratius definitur.

Quærat area quæ generatur ab Ordinatâ $\sqrt{1+zz}$ & jacet supra Abscissam z quando ea evadit unitas. In

10 B

1 + zz

$\overline{1+z}^{-1}$, pro z scribe $\frac{0}{10}, \frac{1}{10}, \frac{2}{10}, \frac{3}{10}, \frac{4}{10}, \frac{5}{10}, \frac{6}{10}, \frac{7}{10}, \frac{8}{10}, \frac{9}{10}, \frac{10}{10}$; & prodibunt undecim Ordinatæ $A, \frac{100}{101}, \frac{25}{101}, \frac{100}{109}, \frac{25}{109}, \frac{4}{5}, \frac{25}{34}, \frac{100}{149}, \frac{25}{41}, \frac{100}{381}, \frac{1}{2}$. Hinc est $A = 1 + \frac{1}{2} = \frac{3}{2}$, $B = \frac{100}{101} + \frac{100}{181} = \frac{28100}{18281}$, $C = \frac{25}{29} + \frac{25}{41} = \frac{1675}{1069}$, $D = \frac{100}{109} + \frac{100}{149} = \frac{25800}{16241}$, $E = \frac{25}{39} + \frac{25}{34} = \frac{1575}{986}$, $F = \frac{4}{5}$. Hisce valoribus substitutis in ultimâ expressione, & unitate pro R , invenies aream esse 785398187. Justus est hic numerus in septimâ figurâ, in octavâ verum superans Binario.

Si undecim Ordinatæ non dent aream fatis exactam, erige plures; & concipe aream divisam esse in plures partes, quarum quamque seorsum quærens habebis pro lubitu justam.

Valor ipsius $\overline{1+Q}^n$ exprimi potest per quamcunque trium serierum sequentium.

$$\begin{aligned} \overline{1+Q}^n &= 1 + \\ & Q \times \frac{n}{1} + \\ & Q^2 \times \frac{n}{1} \times \frac{n-1}{2} + \\ & Q^3 \times \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} + \\ & Q^4 \times \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} + \\ & Q^5 \times \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} + \&c. \end{aligned}$$

$$\begin{aligned} \text{Vel } \overline{1+Q}^n &= 1 + \\ & R \times \frac{n}{1} + \\ & R^2 \times \frac{n}{1} \times \frac{n+1}{2} + \\ & R^3 \times \frac{n}{1} \times \frac{n+1}{2} \times \frac{n+2}{3} + \end{aligned}$$

$R^4 \times$

(1065)

$$R^4 \times \frac{n}{1} \times \frac{n+1}{2} \times \frac{n+2}{3} \times \frac{n+3}{4} +$$

$$R^5 \times \frac{n}{1} \times \frac{n+1}{2} \times \frac{n+2}{3} \times \frac{n+3}{4} \times \frac{n+4}{5} + \&c.$$

posito scilicet $R = \frac{1+Q}{Q}$. Vel

$$\overline{1+Q}^n = 1 +$$

$$\frac{2+n+1 \times Q}{1+Q} \times Q \times \frac{n}{1.2} +$$

$$\frac{4+n+2 \times Q}{1+Q} \times Q^2 \times \frac{n}{1.2} \times \frac{m-1}{3.4} +$$

$$\frac{6+n+3 \times Q}{1+Q} \times Q^3 \times \frac{n}{1.2} \times \frac{m-1}{3.4} \times \frac{m-4}{5.6} +$$

$$\frac{8+n+4 \times Q}{1+Q} \times Q^4 \times \frac{n}{1.2} \times \frac{m-1}{3.4} \times \frac{m-4}{5.6} \times \frac{m-9}{7.8} +$$

$$\frac{10+n+5 \times Q}{1+Q} \times Q^5 \times \frac{n}{1.2} \times \frac{m-1}{3.4} \times \frac{m-4}{5.6} \times \frac{m-9}{7.8} \times \frac{m-16}{9.10}$$

+ &c.

Primæ duæ Series demonstrantur per Casum primum Propositionis Nam si $\overline{1+Q}^0$, $\overline{1+Q}^1$, $\overline{1+Q}^2$, $\overline{1+Q}^3$, $\overline{1+Q}^4$, &c. designent Ordinatas rotidem æquidistantes in Parabolicâ figurâ. erit $\overline{1+Q}^n$ ejusdem Ordinata, cujus distantia à $\overline{1+Q}^0$ est n . Et sic prodit Series prima. At si in alia Parabola $\overline{1+Q}^0$, $\overline{1+Q}^{-1}$, $\overline{1+Q}^{-2}$, $\overline{1+Q}^{-3}$, $\overline{1+Q}^{-4}$, &c. sint æquidistantes Ordinatæ, erit $\overline{1+Q}^n$ Ordinata in eadem, cujus distantia à $\overline{1+Q}^0$ est $-n$; sic proveniet Series secunda Sit jam in tertia Parabola &c. $\overline{1+Q}^{-4}$, $\overline{1+Q}^{-3}$, $\overline{1+Q}^{-2}$, $\overline{1+Q}^{-1}$, $\overline{1+Q}^0$, $\overline{1+Q}^1$, $\overline{1+Q}^2$, $\overline{1+Q}^3$, $\overline{1+Q}^4$, &c. Series Ordinarum

æquidistantium hinc inde progrediens in infinitum, eritque in eadem $\sqrt[n]{1 + 2^n}$ Ordinata, distantia n à termino medio $\sqrt[n]{1 + 2^0}$ remota. Et sic provenit Series tertia per Casum Secundum Propositionis. Prima abruptit quando est n integer & affirmativus, secunda quando est n integer & negativus, & tertia in casu utroque abruptit. Per harum quamque radices numerales commode evolvuntur in Series. Tertia reliquis multo citius convergit: ejus terminus secundus adhiberi potest pro correctione, ubi fit extractio per repetitionem calculi.

Halleius in sua methodo construendi Logarithmos, ex prima harum serierum demonstrat Seriem *Mercatoris* pro Quadratura Hyperbolæ. Sit ejus Ordinata $\sqrt[n]{1 + z^{n-1}}$, vel $\sqrt[n]{1 + z^{n-1}}$, existente n numero infinite parvo; unde per methodos Quadrandi, area quæ jacet supra Abscissam z , id est, Logarithmus numeri $1 + z$, erit $\frac{1 - z^{1-n}}{n}$. Est vero per primam Seriem $\sqrt[n]{1 + z} = 1 + \frac{n}{1}z + \frac{n}{1} \times \frac{n-1}{2} z^2 + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} z^3 + \&c.$ adeoque in casu præsentè, ubi est n infinite parvus, est $\sqrt[n]{1 + z} = 1 + \frac{n}{1}z - \frac{n}{2}z^2 + \frac{n}{3}z^3 - \frac{n}{4}z^4 + \&c.$ quo substituto in valore areæ, ea prodit $z - \frac{1}{2}z^2 + \frac{1}{3}z^3 - \frac{1}{4}z^4 + \frac{1}{5}z^5 - \&c.$ quæ est Series *Mercatoris*.

Similiter per Seriem secundam prodit hæc regula; Sit datus numerus $1 + z$, pone $R = \frac{z}{1+z}$, eritque ejus Logarithmus $R + \frac{1}{2}R^2 + \frac{1}{3}R^3 + \frac{1}{4}R^4 + \frac{1}{5}R^5 + \&c.$

Per Seriem tertiam provenit sequens regula. Sit quilibet numerus R , pone $z = \frac{R-1}{2R}$, eritque ejus Logarithmus

garithmus $\frac{RR-1}{2R} = \frac{1}{3}Az - \frac{2}{5}Bz - \frac{3}{7}Cz - \frac{4}{9}Dz - \frac{5}{11}Ez$
 — &c. Ubi A, B, C, D, E , &c. more *Newtoniano* designant terminos Seriei sicut ab initio. Hæc Series, ut ea ex qua deducitur, reliquis duabus multis vicibus celeberrime approximatur: estque eadem generalius expressa quam, ex fundamento haud absimili, pro inventione Logarithmi Binarii prius dedimus.

Methodus inveniendi valores Serierum Arithmeticarum utcumque tarde convergentium.

In aliquibus Seriebus summa terminorum haberi nequit nisi ad paucissima figurarum loca, dummodo præter simplicem eorum additionem aliæ artes non adhibeantur. Proponatur jam Series quælibet cujus termini omnes iisdem signis afficiuntur, & quorum proximi continue tendunt esse inter se æquales; quales sunt sequentes $\frac{1}{1 \cdot 2} + \frac{1}{3 \cdot 4} + \frac{1}{5 \cdot 6} + \frac{1}{7 \cdot 8} + \&c.$ $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \&c.$ Collige summam aliquot terminorum sub initio, ii proxime addendi sint $\alpha, \beta, \gamma, \delta, \epsilon, \zeta$, &c. In numeris proximis sit $r = \frac{\alpha\gamma - \beta\delta}{\alpha\beta - 2\alpha\gamma + \beta\gamma}$, & quantitatum $\alpha \times \frac{\alpha + r\beta}{\alpha - \beta}$, $\alpha + \beta \times \frac{\beta + r\gamma}{\beta - \gamma}$, $\alpha + \beta + \gamma \times \frac{\gamma + r\delta}{\gamma - \delta}$, $\alpha + \beta + \gamma + \delta \times \frac{\delta + r\epsilon}{\delta - \epsilon}$, &c. differentiarum sint a, b, c, d, e , &c. Deinde in numeris proximis sit $s = \frac{ac - bb}{ab - 2ac + bc}$, & ipsorum $a \times \frac{a + sb}{a - b}$, $a + b \times \frac{b + sc}{b - c}$, $a + b + c \times \frac{c + sd}{c - d}$, $a + b + c + d \times \frac{d + se}{d - e}$, &c. differentiarum sint A, B, C, D , &c. & sit $t = \frac{AC - BB}{AB - 2AC + BC}$: atque sic procede

cede quoad libuerit. Tum crit $\alpha + \beta + \gamma + \delta + \epsilon$
 $+ \&c. = \alpha \times \frac{a+r\beta}{a-\beta} + a \times \frac{a+sb}{a-b} + A \times \frac{A+rB}{A-B} + \&c.$ at-
 que ultra duos terminos hujus novæ Seriei ra-
 ro opus erit progredi.

Ut si desideretur valor Seriei $\frac{1}{1.2} + \frac{1}{3.4} + \frac{1}{5.6} + \frac{1}{7.8}$
 $+ \&c.$ collige primos 21 terminos, quorum summam
 reperio fore 6813,8410,1885. Termini proxime ad-
 dendi sunt $\alpha = ,0005,2854,1226$, $\beta = ,0004,8309,1787$,
 $\gamma = ,0004,4326,2411$, $\delta = ,0004,0816,3265$, &c. Hinc
 fit $r = 1$ proxime, & $\alpha \times \frac{a+r\beta}{a-\beta} = ,0117,6449,6282$,
 $a = -,0000,0017,5096$, $b = -,0000,0014,7410$,
 $c = -,0000,0012,4986$, &c. Unde $s = \frac{1}{2}$ prope, &
 $a \times \frac{a+sb}{a-b} = -,0000,0141,8111$, quem propter signum
 negativum subduco ab $\alpha \times \frac{a+r\beta}{a-\beta}$, & remanet, 0117,6307,
 8171: hic additus summæ primo inventæ 6813,8410,
 1885, dat pro summa totius Seriei numerum 6931,
 4718,0056, qui justus est in nonâ decimali; at ante
 duas hæc correctiones summa erat justa in primâ fi-
 gura solâ. Si animus sit propius scopum attingere,
 pergendum erit ad approximationes sequentes. Si ter-
 mini Seriei diversa habeant signa, conjungendi sunt, ut
 omnes eadem tandem habeant, ut in Serie $1 - \frac{1}{2} + \frac{1}{3}$
 $- \frac{1}{4} + \frac{1}{5} - \&c.$ conjunctis terminis ea evadit $\frac{2}{1.3} +$
 $\frac{2}{5.7} + \frac{2}{9.11} + \frac{2}{13.15} + \&c.$ Sed hic notandum est
 quod differentix a, b, c, d, e , &c. ut & A, B, C, D , &c.
 colligi debent subducendo quantitates antecedentes de
 subsequentiis. Et in omnibus hujusmodi Seriebus si
 p, q, r , repræsentent tres terminos ordine sequentes, p pri-
 mum,

num, q secundum, r tertium, & rectangulum $\frac{p+r}{2} \times q$ non sit majus pr , valor Seriei erit infinite magnus: at magnitudinis semper finitæ ubi accidit contrarium. Potest hæc regula nonnunquam fallere, ubi termini p , q , r parum distant ab initio Seriei, at si consistent inter eos ab initio aliquantum remotos, evadet regula certissima.

Ad alia Serierum genera debent aliæ regulæ adhiberi. Sit Series regularium Polygonorum Circulo Inscriptorum, existente Radio unitate.

$$\begin{aligned} H &= 2,0000,0000,0000,000 \mid 4 \\ G &= 2,8284,2712,4746,190 \mid 8 \\ F &= 3,0614,6745,8920,718 \mid 16 \\ E &= 3,1214,4515,2258,051 \mid 32 \\ D &= 3,1365,4849,0545,938 \mid 64 \\ C &= 3,1403,3115,6954,752 \mid 128 \\ B &= 3,1412,7725,0932,772 \mid 256 \\ A &= 3,1415,1380,1144,299 \mid 512 \end{aligned}$$

Dicatur jam ultimum Polygonum A , penultimum B , antepenultimum C , & reliqua in suo ordine retrosum D , E , F , &c. atque area Circuli quæsitæ erit $A + \frac{A-B}{3}$

$$\begin{aligned} &+ \frac{4A-5B+C}{3 \cdot 15} + \frac{64A-84B+21C-D}{3 \cdot 15 \cdot 63} + \\ &\frac{4096A-5440B+1428C-85D+E}{3 \cdot 15 \cdot 63 \cdot 255} + \&c. \end{aligned}$$

Ubi si pro A , B , C , D , E , &c. scribantur proprii valores, primi quatuor termini dabunt 3,1415,9265,3589,790 pro area Circuli. Hæc autem Series est generalis, ex natura Circuli nequam dependens: applicabilis est quotiescunque numerorum approximantium differentiæ priores sunt posteriorum quasi quadruplæ. Factores in Denominatoribus sunt dignitates integræ numeri 4 unitatibus minutæ:

auræ: quibus datis, coefficientes literarum in diversis terminis formantur ex multiplicatione continua numerorum 1. $\frac{n}{3}$, $\frac{n-3}{15}$, $\frac{n-15}{63}$, $\frac{n-63}{255}$, &c. Ubi pro n substituendus est ultimus Factorum in Denominatore.

Ultima quantitarum $x - 1$, $2^{\sqrt[2]{x}} - 2$, $4^{\sqrt[3]{x}} - 4$, $8^{\sqrt[4]{x}} - 8$, $16^{\sqrt[5]{x}} - 16$, &c. æqualis est Logarithmo numeri x . Pro x scribe 2, & per repetitam extractionem radicis quadratæ exhibunt numeri

$$\begin{aligned} M &= 1,0000,0000,0000,0000. \\ L &= 8284,2712,4746,1901. \\ I &= 7568,2864,0010,8843. \\ H &= 7240,6186,1322,0613. \\ G &= 7083,8051,8838,6214. \\ F &= 7007,0875,6931,7337. \\ E &= 6969,1430,7308,8294. \\ D &= 6950,2734,2438,7611. \\ C &= 6940,8641,2851,8363. \\ B &= 6936,1658,4759,4014. \\ A &= 6933,8182,9699,9493. \end{aligned}$$

Dicatur ultimus numerorum A , penultimus B , & sic retro, atque Logarithmus quæsitus erit $A + \frac{A-B}{1} + \frac{2A-3B+C}{1 \cdot 3} + \frac{8A-14B+7C-D}{1 \cdot 3 \cdot 7} + \frac{64A-120B+70C-15D+E}{1 \cdot 3 \cdot 7 \cdot 15} + \&c.$ Primi quinque termini dant 6931,4718,0559,9457 pro Logarithmo Hyperbolico Binarii. Et quomodo hæc Series procedit in infinitum facile colligitur ex eo quod de priore diximus: estque etiam universalis, proprietates Hyperbolæ minime respiciens.

Extenditur quoque Methodus hæcce Differentialis ad Resolutionem Æquationum & alia quamplurima quorum hic non fit mentio. Continetque fundamenta Serierum generalissima; ut in Reductione Æquationum Irrationalium & Fluxionalium brevi forsan monstrabo.