Reasonable Basic Algebra

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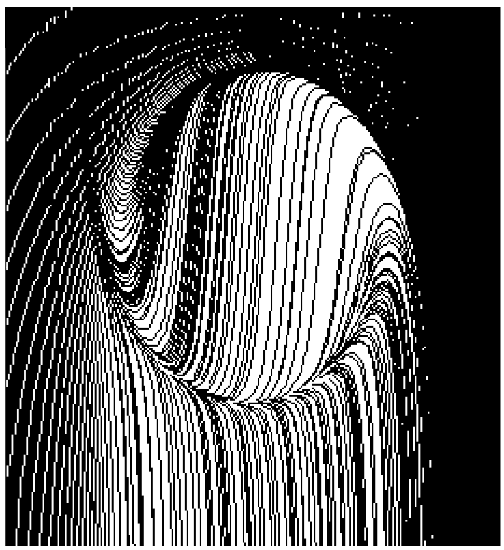
(Verbose Edition)



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[1]Educologists who deem the footnotes “inflammatory” need only uncomment “renewcommand-footnote” in the ADJUSTMENTS TO DOCUMENT to turn them off.

To Françoise, Bruno and Serge.

Contents

Preface

The prospect facing students still in need of Basic Algebra as they enter two-year colleges[2] is a discouraging one inasmuch as it usually takes at the very least two semesters before they can arrive at the course(s) that they are interested in—or required to take, not to dwell on the fact that their chances of *overall* success tend to be extremely low[3].

Reasonable Basic Algebra (RBA) is a standalone version of part of From Arithmetic To Differential Calculus (A2DC), a *course of study* developed to allow a significantly higher percentage of students to complete Differential Calculus in three semesters. As it is intended for a one-semester course, though, RBA may serve in a similar manner students with different goals.

The general intention is to get the students to change from being "answer oriented", the inevitable result of "show and tell, drill and test", to being "question oriented[4]" and thus, rather than try to "remember" things, be able to "reconstruct" them as needed. The specific means by which RBA hopes to accomplish this goal are presented at some length below but, briefly, they include:

- An *expositional approach*, based on what is known in mathematics as model theory, which carefully distinguishes "real-world" situations from their "paper-world" representations[5]. A bit more precisely, we start with processes involving "real-world" collections that yield either a relationship between these collections or some new collection and the students then have to develop a paper procedure that will yield the sentence representing the relationship or the number-phrase representing the new collection.

[2]Otherwise known, these days, as "developmental" students.

[3]For instance, students who wish eventually to learn Differential Calculus, the "mathematics of change", face five or six semesters with chances of overall success of no more than one percent.

[4]See John Holt's classic *How Chidren Fail*, Delacorte Press,1982.

[5]See Zoltan P. Dienes, for instance *Building Up Mathematics*.

EXAMPLE 1. Given that, in the real-world, when we attach to a collection of three apples to a collection of two apples we get a collection of five apples, the question for the students is to develop a paper procedure that, from 3 **Apples** and 2 **Apples**, the number-phrases representing on paper these real-world collections, will yield the number-phrase 5 **Apples**.

In other words, the students are meant to abstract the necessary concepts from a familiar "real-world" since, indeed, "We are usually more easily convinced by reasons we have found ourselves than by those which have occurred to others." (Blaise Pascal).

- A very carefully structured *contents architecture*—in total contrast to the usual more or less haphazard string of "topics"—to create systematic reinforcement and foster an exponential learning curve based on a Coherent View of Mathematics and thus help students acquire a Profound Understanding of Fundamental Mathematics[6].
- A systematic attention to *linguistic issues* that often prevent students from being able to focus on the *mathematical concepts* themselves.
- An insistence on *convincing* the students that the reason things mathematical are the way they are is not because "experts say so" but because *common sense* says they *cannot* be otherwise.

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The *contents architecture* was designed in terms of three major requirements.

1. From the *students*' viewpoint, each and every mathematical issue should:

- flow "naturally" from what just precedes it,
- be developed only as far as *needed* for what will follow "naturally",
- be dealt with in sufficient "natural" *generality* to support further developments without having first to be recast.

EXAMPLE 2. After counting dollars sitting on a counter, it is "natural" to count dollars changing hands over the counter and thus to develop signed numbers. In contrast, multiplication, division or fractions all involve a complete change of venue.

2. Only a very few very simple but very powerful *ideas* should be used to underpin all the presentations and discussions even if this may be at the cost of some additional length. After they have *familiarized* themselves with such an idea, in its simplest possible embodiment, later, in more complicated situations, the students can then focus on the *technical* aspects of getting the idea to *work* in the situation at hand. In this manner, the students eventually get to feel that they can *cope* with "anything".

[6]See Liping Ma's *Knowing and Teaching Elementary School Mathematics.*

EXAMPLE 3. The concept of *combination-phrase* is introduced with 3 **Quarters** + 7 **Dimes** in which **Quarters** and **Dimes** are *denominators* and where + does *not* denote addition as it does in 3 **Quarters** + 7 **Quarters** but stands for "and". (In fact, for a while, we write 3 **Quarters** & 7 **Dimes**.) The concept then comes up again and again: with 3 HUNDREDS + 7 TENS, with $\frac{3}{4} + \frac{7}{10}$, with $3x^2 + 7x^5$, with $3x + 7y$, etc, culminating, if much later, with $3\vec{i} + 7\vec{j}$.

EXAMPLE 4. *If* we can *change*, say, 1 **Quarter** for 5 **Nickels** and 1 **Dime** for 2 **Nickels**, we can then change the *combination-phrase* 3 **Quarters** + 7 **Dimes** for $3\ \cancel{\textbf{Quarters}} \times \frac{5\ \textbf{Nickels}}{\cancel{\textbf{Quarter}}} + 7\ \cancel{\textbf{Dimes}} \times \frac{2\ \textbf{Nickels}}{\cancel{\textbf{Dime}}}$ that is for the *specifying-phrase* 15 **Nickels** + 14 **Nickels** which we *identify* as 29 **Nickels**. (Note by the way that here × is a very particular type of multiplication, as also found in $3\ \cancel{\textbf{Dollars}} \times \frac{7\ \textbf{Cents}}{\cancel{\textbf{Dollar}}} = 21$ **Cents**.) Later, when having to "add" $\frac{3}{4} + \frac{7}{10}$, the students will then need only to concentrate on the *technical* issue of developing a procedure to find the denominators that **Fourth** and **Tenth** can *both* be changed for, e.g. **Twentieths**, **Hundredths**, etc.

3. The issue of "undoing" whatever has been done should always be, if not always resolved, at least always discussed.

EXAMPLE 5. Counting *backward* is introduced by the need to undo counting *forward* and both *subtracting* and *signed* numbers are introduced by the need to *undo* adding, that is by the need to solve the *equation* $a + x = b$.

∴

As a result of these requirements, the *contents* had to be stripped of the various "kitchen sinks" to be found in current BASIC ALGEBRA courses and the two essential themes RBA focuses on are *affine inequations & equations* and *Laurent polynomials*. This focus *empowers* the students in that, once they have mastered these subjects, they will be able both: **i.** to investigate the CALCULUS OF FUNCTIONS as in A2DC and **ii.** to acquire in a similar manner whatever other algebraic tools they may need for other purposes.

However, a problem arose in that the background necessary for a treatment that would make solid sense to the students was not likely to have been acquired in any course the students might have taken previously while, for lack of time, a full treatment of ARITHMETIC, such as can be found in A2DC, was out of the question here.

Following is the "three PARTS compromise" that was eventually reached. PART I consists of a treatment of ARITHMETIC, taken from A2DC but minimal in two respects: i. It is limited to what is strictly necessary to make sense of *inequations* & *equations* in Part II and *Laurent polynomials* in Part III, that is to the ways in which number-phrases are *compared* and *operated with*. ii. It is developed only in the case of *counting* number-phrases with the extension to *decimal* number-phrases to be taken for granted even though the latter are really of primary importance—and fully dealt with in A2DC.

- Chapter 1 introduces and discusses the general model theoretic concepts that are at the very core of RBA: *real-world collections* versus *paper-world number-phrases*, *combinations*, *graphic representations*.
- Chapter 2 discusses *comparisons*, with real-world collections compared *cardinally*, that is by way of one-to-one matching, while paper-world number-phrases are compared *ordinally*, that is by way of counting. The six *verbs*, $<$, $>$, $\leqq$, $\geqq$, $=$, $\neq$, together with their interrelationships, are carefully discussed in the context of *sentences*, namely *inequalities* and *equalities* that can be TRUE or FALSE.
- Chapter 3 discusses the *effect* of an *action* on a *state* and introduces *addition* as a *unary* operator representing the real-world *action* of attaching a collection to a collection.
- Chapter 4 introduces *subtraction* as a *unary* operator meant to "undo" addition, that is as representing the real-world *action* of detaching a collection from a collection.
- Chapter 5 considers collections of "two-way" items which we represent by *signed number-phrases*.

  EXAMPLE 6. Collections of steps forward versus collections of steps backward, Collections of steps up versus collections of steps down, Collections of dollars gained versus collections of dollars lost, etc

  In order to deal with *signed* number-phrases, the *verbs*, $<$, $>$, etc, are extended to $\circledless$, $\circledgtr$, etc and the *operators* $+$ and $-$ to $\oplus$ and $\ominus$.
- Chapter 6 introduces *co-multiplication* between number-phrases and *unit-value* number-phrases as a way to find the *value* that represents the *worth* of a collection.

  EXAMPLE 7. 3 Apples $\times\, 2\frac{\text{Cents}}{\text{Apple}} = 6$ Cents as well as 3 Dollars $\times\, 7\frac{\text{Cents}}{\text{Dollar}} = 21$ Cents

  We continue to distinguish between *plain* number-phrases and *signed* number-phrases with $\times$ and $\otimes$.

PART II then deals with number-phrases *specified* as solution of *problems*.

- Chapter 7 introduces the idea of real-world collections selected from a set of selectable collections by a requirement and, in the paper-world, of nouns specified from a data set by a form. Letting the data set then consist of *counting* numerators, we discuss *locating* and *representing* the solution subset (of the data set) specified by a *basic* formula, i.e. of type $x = x_0$, $x < x_0$, etc where x_0 is a given *gauge*.
- Chapter 8 extends the previous ideas to the case of *decimal* numerators by introducing a general procedure, to be systematically used henceforth, in which we locate separately the *boundary* and the *interior* of the solution subset. Particular attention is given to the representation of the solution subset, both by *graph* and by *name*.

- Chapter 9 begins the focus on the computations necessary to *locate* the boundary in the particular case of "special affine" problems, namely *translation* problems and *dilation* problems, which are solved by *reducing* them to *basic* problems.
- Chapter 10 then solves *affine* problems by *reducing* them to *dilation* problems and hence to *basic* problems. It concludes with the consideration of some *affine-reducible* problems.
- Chapter 11 discusses the *connectors* AND, AND/OR, EITHER/OR, in the context of *double* basic problems, that is problems involving two *basic* inequations/equations (in the same unknown). Here again, particular attention is given to the representation of the solution subset, both by *graph* and by *name*.
- Chapter 12 wraps up the discussion of how to select collections with the investigation of *double* affine problems, that is problems involving two affine inequations/equations (in the same unknown).

PART III investigates *plain polynomials* as a particular case of *Laurent polynomials*.

- Chapter 13 discusses what is involved in *repeated multiplications* and *repeated divisions* of a number-phrase by a *numerator* and introduces the notion of *signed* power.
- Chapter 14 extends this notion to *Laurent monomials*, namely signed powers of x. Multiplication and division or *Laurent monomials* are carefully discussed.
- Chapter 15 extends the fact that *decimal* numerators are *combinations* of signed powers of TEN to the introduction of *Laurent* polynomials as combinations of *signed* powers of x. Addition and subtraction of polynomials are then defined in the obvious manner.
- Chapter 16 continues the investigation of Laurent polynomials with the investigation of multiplication.
- Chapter 17 discusses a particular case of multiplication, namely the successive powers of $x_0 + u$.
- Chapter 18 closes the book with a discussion of the division of polynomials both in descending and ascending powers

$\therefore$

This is probably the place where it should be disclosed that, as the development of this text was coming to an end, the author came across a 1905 text[7] that gave him the impression that, in his many deviations from the

[7] H. B. Fine, *College Algebra*, reprinted by American Mathematical Society Chelsea,

current praxis, he had often reinvented the wheel. While rather reassuring, this was also, if perhaps surprisingly, somewhat disheartening.

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Some of the *linguistic issues* affecting the students's progress are very specific and are directly addressed *as such*. The concept of *duality*, for instance, is a very powerful one and occurs in very many guises.

- When it occurs as "passive voice", *duality* is almost invariably confused with *symmetry*, a more familiar concept[8]. But, in particular, while duality preserves *truth*, symmetry may or may not.

  EXAMPLE 8. "**Jack** is a *child* of **Sue**" is the *dual* of "**Sue** is a *parent* of **Jack**" and, since both refer to the same real-world relationship, they are either both TRUE or both FALSE.
  On the other hand, "**Jack** is a *child* of **Sue**" is the *symmetrical* of "**Sue** is a *child* of **Jack**" and, *here*, the truth of one forces the falsehood of the other. But compare with what would happen with "brother" or "sibling" instead of "child".

- When it occurs as *indirect* definition, *duality* is quite foreign to most students but absolutely indispensable in certain situations.

  EXAMPLE 9. While **Dollar** can be defined *directly* in terms of **Quarters** by saying that 1 **Dollar** is equal to 4 **Quarters**, the definition of **Quarter** in terms of **Dollar** is an *indirect* one in that we must say that a **Quarter** is *that* kind of coin *of which* we need 4 to change for 1 **Dollar** and students first need to be reconciled with this syntactic form. The same stumbling block occurs in dealing with roots since $\sqrt{9}$ is to be understood as "*that* number the square *of which* is 9"[9].

Other linguistic issues, even though more diffuse, are nevertheless systematically taken into account. For instance:

- While mathematicians are used to all sorts of things "going without saying", students feel more comfortable when everything is made *explicit* as, for instance, when & is distinguished from +. Hence, in particular, the *explicit* use in this text of *default rules*.
- The meaning of mathematical symbols usually depends on the context while students generally feel more comfortable with *context-free* terminology, that is in the case of a *one-to-one* correspondence between *terms* and *concepts*.

2005.

[8]The inability to use the "passive voice" is a most important *linguistic* stumbling block for students and one that Educologists have yet to acknowledge.

[9]Educologists will surely agree that, for instance, these particular "reverse" problems would in fact be better dealt with in an *algebraic* context, i.e. as the investigation of $4x = 1$ and $x^2 = 9$. Iincidentally, this is the point of view adopted in A2DC where arithmetic and algebra are systematically "integrated".

- Even small linguistic variations in *parallel* cases disturb the students who take these variations as having to be significant and therefore as implying in fact an unsaid but actual lack of parallelism.

In general, being aware of what *needs* to be said versus what can go without saying is part of what makes one a mathematician and, as such, requires learning and getting used to. Thus, although being pedantic is not the goal here, RBA tries very hard to be as pedestrian as possible and, if only for the purpose of "discussing matters", to make sure that *everything* is *named* and that every term is "explained" even if usually not *formally* defined.

∴

The standard way of establishing truth in *mathematics* is by way of proof but the capacity of being *convinced* by a proof is another part of what makes one a mathematician. And indeed, since the students for whom RBA was written are used only to drill based on "template examples", they tend to behave as in the joke about Socrates' slave who, when led through the proof of the Pythagorean Theorem, answers "Yes" when asked if he agrees with the current step and "No" when asked at the end if he agrees with the truth of the Theorem. So, to try to be *convincing*, we use a mode of *arguing* somewhat like that used by lawyers in front of a *court*[10].

Another reason for using a mode of reasoning more akin to everyday argumentation is that even people unlikely to become prospective mathematicians ought to realize the similarities between having to establish the truth in *mathematics* and having to establish the truth in *real-life*. Yet, as Philip Ross wrote recently, "*American psychologist Edward Thorndike first noted this lack of transference over a century ago, when he showed that* [...] *geometric proofs do not teach the use of logic in daily life.*"[11].

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Finally, it is perhaps worth mentioning that this text came out of the author's conviction that it is not good for a society to have a huge majority of its citizens saying they were "never good in math". To quote Colin McGinn at some length:

"*Democratic States are constitutively committed to ensuring and furthering the intellectual health of the citizens who compose them: indeed, they are only possible at all if people reach a certain cognitive level* [...]. *Democracy*

[10]See Stephen E. Toulmin, *The Uses of Argument* Cambridge University Press, 1958

[11]Philip E. Ross, *The Expert Mind.* Scientific American, August 2006.

and education (in the widest sense) are thus as conceptually inseparable as individual rational action and knowledge of the world. [. . .] *Plainly,* [education] *involves the transmission of knowledge from teacher to taught. But* [knowledge] *is true justified belief that has been arrived at by rational means.* [. . .] *Thus the norms governing political action incorporate or embed norms appropriate to rational belief formation.* [. . .]"

"*A basic requirement is to cultivate in the populace a respect for intellectual values, an intolerance of intellectual vices or shortcomings.* [. . .] *The forces of cretinisation are, and have always been, the biggest threat to the success of democracy as a way of allocating political power: this is the fundamental conceptual truth, as well as a lamentable fact of history.*"

"[However] *people do not really like the truth; they feel coerced by reason, bullied by fact. In a certain sense, this is not irrational, since a commitment to believe only what is true implies a willingness to detach your beliefs from your desires.* [. . .] *Truth limits your freedom, in a way, because it reduces your belief-options; it is quite capable of forcing your mind to go against its natural inclination.* [. . .] *One of the central aims of education, as a preparation for political democracy, should be to enable people to get on better terms with reason—to learn to live with the truth.*" [12]

[12] Colin McGinn, *Homage to Education*, London Review of Books, August 16, 1990

Part I

Elements of Arithmetic

What is important is the real world, that is physics, but it can be explained only in mathematical terms.

Dennis Serre[1]

procedure
process

Chapter 1

Counting Number-Phrases

This chapter takes a brief look back at ARITHMETIC to present it in a way that will be a better basis for looking at ALGEBRA because we will then be able to look at ALGEBRA as just a continuation of ARITHMETIC.

1.1 What Arithmetic and Algebra are About

To put it as briefly as possible, ARITHMETIC and ALGEBRA are both about developing **procedures** to figure out on paper the *result* of real-world **processes** without having to go through the real-world processes themselves. To make this a bit clearer, here are two examples from ARITHMETIC the ALGEBRA counterpart of which we will deal with in Part III of this book.

EXAMPLE 1. In the real world, we may want to hand-out six one-dollar bills to each of four people. To find out ahead of time how many one-dollar bills this would amount to, we would put on the table six one-dollar bills for the first person, then six one-dollar bills for the second person, etc. The result of this real-world process is that this amounts to twenty-four one-dollar bills.

But with, say, hundreds of one-dollar bills to each of thousands of people, this *process* would be impractical and what we do instead is to *represent* on paper both the one-dollar bills and the people and then develop the *procedure* called *multiplication*,

[0]Bulletin of the AMS, Vol 47 Number 1 Pages 139-144

mathematical language

that is a *procedure* for figuring-out on paper how many one-dollar bills we will need as a result of the real-world process.

EXAMPLE 2. In the real world, we may want to split fourteen one-dollar bills among three people. To find out ahead of time how many one-dollar each person should get, we would put on the table one one-dollar bill for the first person, one one-dollar bill for the second person, one one-dollar bill for the third person, and then, in a second round, another one-dollar bill for the first person, another one-dollar bill for the second person, and so on until we cannot do a full round. The result of this real-world *process* is that each person would get four one-dollar bills with two one-dollar bills remaining un-split.

But with thousands of one-dollar bills to be split among hundreds of people, this *process* would be impractical and what we do instead is to *represent* on paper both the one-dollar bills and the people and then develop the procedure called *division*, that is a *procedure* for figuring-out on paper how many one-dollar bills to give to each person and how many one-dollar bills will remain un-split as a result of the real-world process.

The difference between these two examples illustrate is not obvious but, as we shall see, it is a significant one which, in fact, is at the root of the distinction between ARITHMETIC and ALGEBRA.

1.2 Specialized Languages

People working in any trade need to use words with a special meaning. Sometimes, these are special words but often they are common words used with a meaning special to the trade. For instance, what *electricians* call a "pancake" is a junction box that is just the thickness of drywall.

In the same manner, in order to develop and discuss the procedures of ARITHMETIC and ALGEBRA, we will have to use a **mathematical language**, that is words that will sometimes be special words but will most of the time be just common words with a meaning special to MATHEMATICS.

EXAMPLE 3. While the words "process" and "procedure" usually mean more or less the same thing, in this book we shall reserve the word "process" for when we talk about what we do in the *real world* and we shall reserve the word "procedure" for when we talk about what we do on *paper*.

In this book, we will encounter a great many such words with special meaning, likely more than usual. The idea, though, is certainly not that the students should *memorize* the special meaning of all these words. These words are used as *focusing devices* to help the students see *exactly* what they are intended to see whenever we discuss an issue. Thus, quite often, these words with special meaning will not reappear once the discussion has been completed as they will have served their purpose.

However, in order to help students find where the special meaning of these words is explained, these words with special meaning will always be:

- **boldfaced** the first time they appear—which is where they are explained,
- printed in the **margin** of the page where they first appear and are explained,
- listed in the **index** at the end of the book with the number of the page where they first appear and are explained.

boldfaced
margin
index
item
represent
picture
denominator
collect
collection

1.3 Real-World

While in the real-world it is often possible to exhibit the **items** that are to be dealt with this is not possible in a book. So, to start with, we need a way to make it clear when we are talking about real-world *items* as opposed to when we are talking about what we will use to **represent** these items on *paper*.

In this book, when we will want to talk about real-world *items*, we will use **pictures** of these *items*.

EXAMPLE 4. When we will be talking about real-world one-dollar bills, we will use the following picture

1.4 Number-Phrases

Our first task in ARITHMETIC is to find a way to *represent* real-world items on paper. The underlying idea is quite simple.

1. Given *real-world* items, in order to represent them on *paper*, we need to convey two pieces of information:

- We must write a **denominator** to say *what kind* of items we are dealing with. Of course, for this to be possible, all the items will have to be of the *same kind* and this will not work when the items are of different kinds as, for instance, when we are dealing with ten-dollar bills together with one-dollar bills. So, for the time being, we will deal only with items that are all of the *same* kind and in this case we will say that we can **collect** the items into a **collection**.

  EXAMPLE 5. Given the following real-world items,

  since they are all of the same kind (they make up a *collection*) we can use as a *denominator* the name of the President whose picture is on them, that is

numerator
string
slash
/
number-phrase
nature (of a collection)
size (of a collection)
number

Washington

- We must write a **numerator** to say *how many* of these items there are in the collection we are dealing with.
  The first approach that comes to mind is just to write a **string** of **slashes**, that is to write a *slash* / for each and every *item* in the real-world collection.

  EXAMPLE 6. Given the following real-world items,

  since they are all of the same kind they make up a *collection* and to get a *numerator* we can just write a / for each and every item in the collection that is

  ///

It is usual first to write the *numerator* and then to write the *denominator* and the result then makes up what we shall call a **number-phrase**.

EXAMPLE 7. Given the following real-world items,

since they are all of the same kind they make up a *collection* which we can represent by the *number-phrase*

/// **Washingtons**

2. Conversely, given a *number-phrase*, to get the *collection* that it represents,

i. The *denominator* tells us the **nature of the collection**, that is what *kind* of items are in the collection,

ii. The *numerator* tells us the **size of the collection**, that is the **number** of items that are in the collection.

EXAMPLE 8. Given the *number-phrase* /// **Washingtons**, to get the *collection* of real-world items that it represents:

i. The *denominator* **Washingtons** tells us that the items in the collection are like ,

ii. The *numerator* /// then tells us that there must be a  in the collection for each slash in the numerator.

iii. Altogether, the *slash number-phrase*

/// **Washingtons**

represents the *collection* of real-world items

 .

3. In other words, compared to a photograph of the collection, a number-phrase causes no loss of information as all we did was just to separate **quantity**—represented by the *numerator*— *from* **quality**—represented by the *denominator*[2]. (Keep in mind, though, that this only works for *collections.*)

quantity
quality
accounting
count
counting number-phrase
digit
shorthand

As a matter of fact, this is most likely how, several thousands of years ago, ARITHMETIC, got started when, one may imagine, Sumerian merchants, faced with the problem of **accounting** for more goods in the warehouse and/or money in the safe than they could handle directly, decided to have both the goods and the money *represented* by various *scratches* on clay tablets so that they could see from these *scratches* the situation their business was in without the inconvenience of having to go to the warehouse and/or to open the safe.

1.5 Representing Large Collections

With *large* collections, a problem arises in that it becomes difficult to see, at a glance, how many items a long string of slashes represents.

EXAMPLE 9. Given the *number-phrase*

//////////////////////////////// **Washingtons**

it is not immediately clear how many items are in the collection that the number-phrase represents.

What we will do is to **count** the collection and we will write what we shall therefore call a **counting number-phrase**. There are three stages to developing the procedure.

1. We must begin by *memorizing* the following **digits** as **shorthands** for the first *nine* strings of slashes:

[2] In spite of which this is precisely the point where, in the name of "abstraction", Educologists cut their students away from *denominators* without noticing, of course, that this is exactly the point where they start losing them.

basic succession
basic collection
basic counting number-phrase
basic counting
count
end-digit

string	digit
/	1
//	2
///	3
////	4
/////	5
//////	6
///////	7
////////	8
/////////	9

Moreover, the various procedures that we shall use will also require that we have already memorized the **basic succession**, that is the *digits* in the *order*:

$$\xrightarrow{\quad 1, \quad 2, \quad 3, \quad 4, \quad 5, \quad 6, \quad 7, \quad 8, \quad 9 \quad}$$

"one, two, three, four, five, six, seven, eight, nine"

NOTE. There is nothing sacred about TEN: it is simply because of how many fingers we have on our two hands—"digit" is just a fancy word for "finger"—and we could have used just about any number of digits instead of TEN.

In fact, deep down, computers use only TWO digits, 0 and 1, because any electronic device is either *off* or *on*. At intermediate levels, computers may use EIGHT $(0, 1, 2, 3, 4, 5, 6, 7)$ or SIXTEEN digits $(0, 1, 2, 3, 4, 5, 6, 7, 8, 9, a, b, c, d, e, f)$.

The Babylonians used SIXTY digits, a historical remnant of which can be seen in the fact that there are SIXTY seconds to a minute and SIXTY minutes to an hour.

The point is that all that we do with TEN digits could easily be done with *any* number of digits[3].

2. We can then represent a **basic collection**, that is a collection with *no more* items than we have digits, that is no more than NINE items, by a **basic counting number-phrase**.

a. Given a *basic collection*, to get the *numerator* of the *counting number phrase* the procedure, called **basic counting**, is:

i. We **count** the collection, that is we point successively at each and every item in the collection while saying the digits in the *basic succession* that we memorized.

ii. The numerator is the **end-digit**, that is the *last* digit we say.

EXAMPLE 10. Given the *collection*

[3] Z. P. Dienes always used to *start* his workshops with second graders, base-THREE ***arithmetic blocks*** and the digits 0, 1, 2.

pick

to get the *basic counting number-phrase* that represents it:

i. We can use for the *denominator* the name of the President whose picture is on them, that is **Washington**.

ii. We *count* the collection to get the *numerator*, that is

We *point at* each and everyone of: while we *say*:

$\xrightarrow{1,\quad 2,\quad 3}$

and the *end-digit* gives us 3 for the numerator.

iii. Altogether, the *collection*

is represented by the *basic counting number-phrase*

3 **Washingtons**

.

b. Conversely, given a *basic counting number-phrase*, to get the *basic collection* that it represents:

i. We **pick** one *item*—of the kind specified by the denominator—each and every time we say a *digit* in the basic succession

ii. We stop after we have picked the item for the numeral in the *numerator*

EXAMPLE 11. Given the *basic counting number-phrase*

5 **Washingtons**

to get the *basic collection* that it represents:

i. The *denominator* **Washingtons** tells us that the items to be picked must be of the same *kind* as 

ii. The *numerator* 5 tell us to pick an item each and every time we say a digit in the succession; we stop after we have picked the item for the *end-digit*:

We *say*: We *pick* each and every one of:

$\xrightarrow{1,\ 2,\ 3,\ 4,\ 5}$

iii. Altogether, the *basic counting number-phrase*

5 **Washingtons**

represents the *basic collection*

extended counting
extended collections
extended succession
numerals
endless

.

3. For **extended counting**, that is for counting **extended collections**, that is for collections with more *items* than we have *digits*, we can continue to proceed essentially as above: we must begin by *memorizing* the **extended succession**, that is the **numerals** that follow the *basic succession* $\xrightarrow{1,\,2,\,3,\,4,\,5,\,6,\,7,\,8,\,9}$, namely

$$\xrightarrow{\quad 10, \quad 11, \quad 12, \quad 13, \ldots}$$

that is:

numerals	we say	meaning	to make us think of:
10	ten		~~/////////~~
11	eleve - n	ten - one	~~/////////~~ /
12	elve-tw	ten - two	~~/////////~~ //
13	thir - teen	ten - three	~~/////////~~ ///
...	...	...	...
19	nine - teen	ten - nine	~~/////////~~ /////////
20	twen - ty	two - tens	~~/////////~~ ~~/////////~~
21	twen - ty - one	two - tens & one	~~/////////~~ ~~/////////~~ /
...	...	...	...

NOTE. The *words* we say for the *numerals* are far from being as systematic as the numerals themselves. This is due in part to the fact that these words slowly evolved over a very long time.

However, and this is possibly the single most important fact about ARITHMETIC, while there are only so many digits in the *basic* succession—NINE in our case, the *extended* succession is **endless**.

a. Given an *extended collection*, to get the *numerator* of the *counting number-phrase* that represents it:

i. We begin by pointing successively at each and every item in the collection while saying the *digits* in the *basic succession* that we memorized,

ii. We continue by pointing successively at each and every item in the collection while saying the *numerals* in the *extended succession* that we memorized.

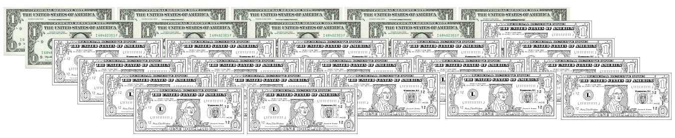
iii. The numerator is the *end-numeral*, that is the last numeral we say.

EXAMPLE 12. Given the *extended collection*,

to get the *counting number-phrase*:

i. We *start* with a *basic* count, that is:

we *point at* each and everyone of: while we *say*:



1, 2, 3, 4, 5, 6, 7, 8, 9 ⟶

ii. We *continue* with an *extended* count, that is:
we *point at* each and everyone of: while we *say*:

10 ⟶

11, 12, 13, 14, . . . ⟶

. . . 29. 30, 31, 32 ⟶

iii. Altogether, the *extended collection*

is represented by the *counting number-phrase*

32 **Washingtons**

b. Given an *extended counting number-phrase*, to get the *collection*:

i. We begin by picking one item each and every time we say a *digit* in the *basic succession*

ii. We continue by picking one item each and every time we say a *digit* in the *extended succession*

iii. We stop after we have picked the item for the *end-numeral*.

EXAMPLE 13. Given the *extended counting number-phrase*,

32 **Washingtons**

to get the *collection* that it represents:

illustrate
graph
graph (to)
ruler
arrowhead
tick-marks
label
graph

i. The *denominator* **Washingtons** tells us that the items to be picked must be of the same *kind* as 
ii. The *numerator* 32 tells us to pick an item each and every time we say a digit in the basic succession and then one each and every time we say a numeral in the extended succession; we stop after we have picked the item for the *numerator*.
iii. Altogether, the *extended counting number-phrase*

32 **Washingtons**

represents the *extended collection*

NOTE. The sticklers among us will have rightfully observed that, strictly speaking, *counting* is neither a paper *procedure* since it involves the real-world items nor a real world *process* since it involves the digits we write on paper. Indeed, *counting* is a bridge from the real-world to the paper-world.

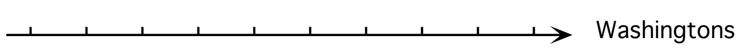
1.6 Graphic Illustrations

As pointed-out at the beginning of this book, it is usually easier to work with *representations* of collections on paper than with the real-world collections themselves. But, once we have *represented* collections with *number-phrases*, we will often also want to **illustrate** the number-phrase with a **graph**. For short, we shall often say that we **graph** the number-phrase.

For that purpose, we will use **rulers** that are straight lines with:

- an **arrowhead** to indicate the way the *succession* goes
- **tick-marks** to be **labeled** with the *numerators*
- a *label* for the *denominator*.

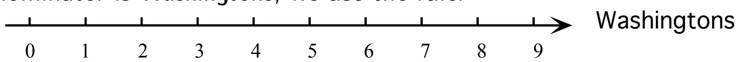
EXAMPLE 14. To *graph* collections represented by *basic* counting number-phrases whose denominator is **Washingtons**, we use *rulers* such as



However, **graphing** collections represented by number-phrases can raise issues of its own.

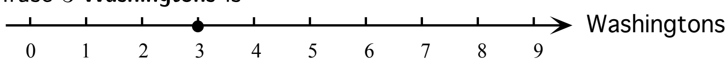
1. In the case of *basic* counting number-phrases, there is no problem and, in fact, as soon as we *label* the tick-marks with *numerators*, the *arrowhead* ceases to be necessary. (But then, there is no point in erasing it either.)

EXAMPLE 15. To *graph* collections represented by *basic* counting number-phrases whose denominator is **Washingtons**, we use the *ruler*



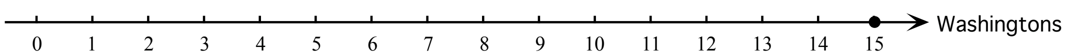
Then, given a *basic* counting number-phrase, one usually places a dot on the corresponding tick-mark.

EXAMPLE 16. The *graph* that represents the collection represented by the counting number-phrase 3 **Washingtons** is



2. In the case of *extended* counting number-phrases, one problem is that we may not be able to draw a long enough ruler.

EXAMPLE 17. We can barely graph 15 **Washingtons** (by extending the ruler into the margin):



but we cannot extend the ruler enough to represent 37 **Washingtons**

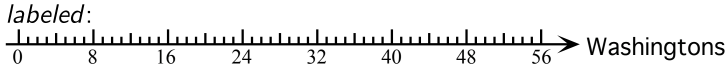
A work-around could be to draw the tick-marks closer together. But then we may not be able to label all the tick-marks.

EXAMPLE 18. On the following ruler

we don't have enough room to write two-digit numerators.

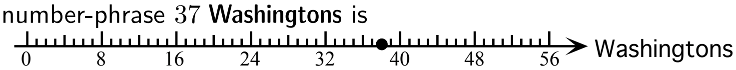
One workaround to *that* is to label the tick-marks only every so often. However it is usually better to do so *regularly*, that is every so many. To make it easier to read the ruler, it is usual in this case to make the tick-marks that are labeled longer and, if these are far apart, to make the middle tick-marks a bit longer too.

EXAMPLE 19. In the following ruler, only every *eighth* tick-mark, that is 8, 16, 24, 32, etc, is *labeled*:



and the middle tick-marks, 4, 12, 20, etc, are made easier to see by being made a bit larger.

EXAMPLE 20. The *graphic* that represents the collection represented by the *extended* counting number-phrase 37 **Washingtons** is

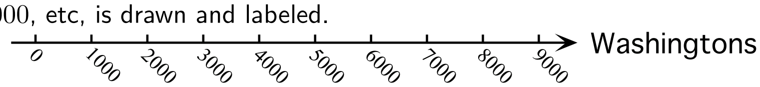


3. But, to *graph* collections represented by really large counting number-phrases, we will not even be able to draw all the tick-marks—and even so

sort
set
combination-phrase
&

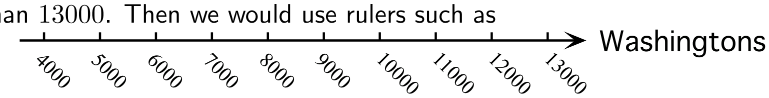
we will often have to write the labels at an degree angle for them to fit.

EXAMPLE 21. In the following ruler, only every thousandth tick-mark, that is 1000, 2000, 3000, etc, is drawn and labeled.



And another workaround may be *not* to start at 0.

EXAMPLE 22. Suppose we are not involved with any numerator less than 4000 and more than 13000. Then we would use rulers such as



1.7 Combinations

When there is more than one *kind* of items, they do not make up a *collection* and we cannot represent them by *number-phrases*.

EXAMPLE 23. Given the following real-world items,

since they are *not* all of the same kind (they do *not* make up a *collection*) there is no one President whose name we can use as a *denominator*.

We then proceed as follows:

1. We **sort** the items by *kind* into *collections* so that we now have a **set** of collections.

EXAMPLE 24. In the above example, we can sort the real-world items into a *set* of collections:

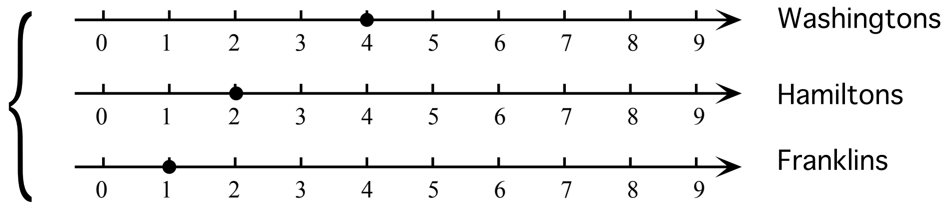
2. We represent the *set* of collections by a **combination-phrase** by writing the number-phrases that represent each one of the collections separated by the symbol **&** to be read as "and".

EXAMPLE 25. In the above example, we can represent the *set* of collections by the *combination-phrase*:

4 **Washington** & 2 **Hamiltons** & 1 **Franklin**

3. The *graphic* representation of a *combination-phrase* requires as many rulers as there are kinds of collections in the set of collections that the combination-phrase represents.

EXAMPLE 26. In the above example, since there are three kinds of bills, we need three rulers:



1.8 About Number-Phrases

We end this chapter with a few remarks about why we are using the term *number-phrases* as opposed to just the term *numbers* as is usual in most current ARITHMETIC textbooks[4].

1. A *numerator* by itself, that is without a *denominator*, represents a *number* which is *not* something in the *real-world* that we can see and touch.

EXAMPLE 27. When asked "Can you show what 3 represents?", we usually respond by showing three real world items, for instance   , but this is what the *number-phrase* 3 **Washingtons** represents and not what 3 by itself represents. In fact, there is no way we can show what 3 *by itself* represents.

In contrast, *number-phrases* represent *collections* which are things in the *real-world* that we can see and touch. This is exactly the reason why we use *number-phrases* even if they make things more cumbersome.

2. Aside from anything else, we should realize that when textbooks use the word *number* they are talking—usually without saying it—about the *concepts* represented by the *numerators* that are actually printed.

EXAMPLE 28. When a textbook says "3 is the number of one-dollar bills on the desk", what is meant is "3 is the *numerator* that represents the *number* of one-dollar bills on the desk". Indeed, 3 is only a mark on paper that tells us how many s

---

[4]Educologists will be glad to measure the progress accomplished since Chrystal's *Textbook of Algebra* infamous opening: "*The student is already familiar with the distinction between abstract and concrete arithmetic. The former is concerned with those laws of, and operations with, numbers that are independent of the things numbered; the latter is taken up with applications of the former to the numeration of various classes of things.*"

there are in the real-world collection that is on the desk.

So, when textbooks use the term *number* instead of the term *numerator*, they are not just using one term instead of another, they are, at best, blurring the distinction between the *real-world* and the *paper-world* we use to discuss the real-world[5].

3. *Number phrases* allow us to be very precise as to what we are dealing with. In particular, the use of number phrases allows us to distinguish:

- matters of *quality*, that is questions about the *kind* of the items under consideration

from

- matters of *quantity*, that is questions about the *number* of the items under consideration.

EXAMPLE 29. Given the collection  sitting on a desk, we can ask three very different questions:

- "*What* is on the desk?" which we answer in ARITHMETIC by writing the *counting-number-phrase*

  5 **Washingtons**

- "*What kind* of items are on the desk?" which we answer in ARITHMETIC by writing the *denominator*

  Washingtons

- "*How many* items are on the desk?" which we answer in ARITHMETIC by writing the *numerator*

  5

4. The distinction we make in ARITHMETIC between *denominators* and number-phrases with the numerator 1 is very similar to the *informal* distinction we make in English between "a" and "one".

EXAMPLE 30. In ARITHMETIC, we distinguish the *denominator* **Washington** from the *number-phrase* 1 **Washington** the same way as in English we distinguish between

- "This looks like *a* five-dollar bill"

which, just like "This looks like *a* ten-dollar bill" or "This looks like *a* twenty-dollar bill" is a *qualitative* statement because they all are statements about what *kind* of bills they look like.

- "This looks like *one* five-dollar bill"

which, just like "This looks like *two* five-dollar bills" or "This looks like *three* five-dollar bills", is a *quantitative* statement because they all are statements about *how many* bills it looks there are.

[5] At worst, one can wonder if educologists are not just *confusing* the two worlds.

Quite often, though and as we will see in many different situations, the numerator 1 "goes without saying".

amount
stuff
number-phrase, decimal
numerator, decimal
denominator

EXAMPLE 31.

$$3 \text{ Washingtons} + \text{Washingtons}$$

is understood to mean

$$3 \text{ Washingtons} + 1 \text{ Washington}$$

and, to take an example from things to come, in the same manner

$$3x + x$$

is understood to mean

$$3x + 1x$$

So, even though *we* shall avoid letting the numerator 1 "go without saying", just in case and to be on the safe side, we set the

DEFAULT RULE # 1. *When there is* no numerator *in front of a denominator and it is otherwise clear that we are dealing with a* counting number-phrase*, it then goes without saying that the numerator is understood to be* 1*.*

NOTE. Unfortunately, this default rule is often abbreviated as "when there is no numerator, the numerator is 1" which is dangerous because when we say that there is *no numerator* it is tempting to think that the numerator is 0!

1.9 Decimal Number-Phrases

We will work not only with collections of items but also with **amount***s* of **stuff** and, just as we use *counting* number-phrases to represent *collections* of items, in order to represent *amounts* of stuff we will use **decimal number-phrases** that consist of a **decimal numerator** and a **denominator**:

Collection of Items Counting Number-phrase	Amount of Stuff Decimal Number-phrase
Kind of items Denominator	Kind of stuff Denominator
Number of items Counting Numerator	Quantity of stuff Decimal Numerator

EXAMPLE 32. We can represent twenty-four apples by the *counting* number-phrase

$$24 \text{ Apples}$$

but in order to represent an *amount* of gold, we need a *decimal* number-phrase such as

31.72 **Grams of gold**

Unfortunately, this being a text on BASIC ALGEBRA, there was space only for the smallest possible investigation of ARITHMETIC, that is one limited to the introduction, illustration and discussion of the concepts strictly necessary to the understanding of BASIC ALGEBRA. So, for lack of space, this was done using only *counting* number-phrases even though, as just noted above, many real-world situations require *decimal* number-phrases instead.

More precisely, even though the investigation of *decimal* number-phrases is intimately related to the representation of *large collections*, in the above section and for lack of space we had to take a short cut, namely use *extended counting* rather than only *basic counting* together with *combinations.* Had we had the space to develop the latter approach for representing large collections, it would then have immediately and effortlessly led to *decimal* number-phrases.

So, here we will have to rely on the reader's own knowledge of *decimal* numbers. However, the interested reader will find a full investigation in SELF-CONTAINED ARITHMETIC as well as in FROM ARITHMETIC TO DIFFERENTIAL CALCULUS.

count
start-digit
end-digit
count from ... to ...
direction

Chapter 2

Comparisons: Equalities and Inequalities

We investigate the *first* of the three fundamental processes involving two collections. We will introduce the procedure in the case of *basic* collections using *basic counting* number-phrases.

2.1 Counting From A Counting Number-Phrase To Another

Before we can develop the procedures for these three fundamental processes, we must make the concept of *counting* more flexible by allowing a **count**

- to start with *any* digit which we will call the **start-digit**. (So, the start-digit doesn't have anymore to be 1 as it always did in Chapter 1.)
- to end with *any* digit which we will call the **end-digit**. (So, the end-digit may be "before" the start digit as well as "after" the start digit.)

Specifically, when we **count from** the start-digit **to** the end-digit:

i. We start (just) *after* the start-digit
ii. We stop (just) *after* the end-digit.

However, given a *start-digit* and a *end-digit*, we may have to count in either one of two possible **directions**:

count-up
count-down
precession

- We may have to **count-up**, that is we may have to use the *succession*

$$\xrightarrow{1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9}$$

which we read along the arrow, that is *from left to right*.

EXAMPLE 1. To count from the start-digit 3 to the end-digit 7:

i. We must count *up*, that is we must use the *succession*

$$\xrightarrow{1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9}$$

ii. We *start* counting *up* in the succession *after* the start-digit 3, so that 4 is the first digit we say,

$$\xrightarrow{4,\ \ldots}$$

iii. We *stop* counting *up* in the succession *after* the end-digit 7 so that 7 is the last digit we say

$$\xrightarrow{\ldots 7}$$

Altogether, the count from the start-digit 3 to the end-digit 7 is

$$\xrightarrow{4,\ 5,\ 6,\ 7}$$

- We may have to **count-down**, that is we may have to use the **precession**

$$\xleftarrow{1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9}$$

which we read along the arrow, that is *from right to left*. **NOTE.** If we prefer to read *from left to right*, we may also write the *precession* as

$$\xrightarrow{9,\ 8,\ 7,\ 6,\ 5,\ 4,\ 3,\ 2,\ 1}$$

which we read along the arrow, that is *from left to right*.

EXAMPLE 2. To count from the start-digit 6 to the end-digit 2:

i. We must count *down*, that is we must use the *precession*

$$\xrightarrow{9,\ 8,\ 7,\ 6,\ 5,\ 4,\ 3,\ 2,\ 1}$$

ii. We *start* counting *down* in the precession *after* the start-digit 6 so that 5 is the *first* digit we say

$$\xrightarrow{5,\ \ldots}$$

iii. We *stop* counting *down* in the precession *after* the end-digit 2 so that 2 is the *last* digit we say.

$$\xrightarrow{\ldots\ 2}$$

Altogether, the count from the start-digit 6 to the end-digit 2 is

$$\xrightarrow{5,4,3,2}$$

NOTE. Memorizing the *precession* $\xrightarrow{9,\,8,\,7,\,6,\,5,\,4,\,3,\,2,\,1}$ just like we memorized the *succession* $\xrightarrow{1,\,2,\,3,\,4,\,5,\,6,\,7,\,8,\,9}$ makes life a lot easier.

length (of a count)
compare
match one-to-one
leftover
relationship
hold (to)

Finally, the **length of a count** from a start-digit to an end-digit is how many digits we say regardless of the direction, that is whether up in the succession or down in the precession.

EXAMPLE 3. When we count from the start-digit 3 to the end-digit 7, the *length* of the count is 4.

EXAMPLE 4. When we count from the start-digit 6 to the end-digit 2, the *length* of the count is 4.

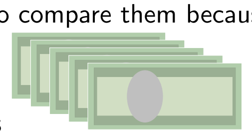
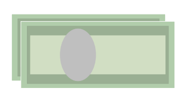
What that does, as in Chapter 1, is again to separate *quality*—represented by the *direction* of the count, up or down, from *quantity*— represented by the *length* of the count, how many digits we count.

NOTE. As already mentioned, we will only use *basic* counting, whether up or down, but *extended* counting would work essentially the same way.

2.2 Comparing Collections

Given two collections, the first thing we usually want to do is to **compare** the first collection to the second collection but an immediate issue is whether the kinds of items in the two collections are the *same* or *different*.

- When the two given collections involve *different* kinds of items, they don't they cannot be compared.

EXAMPLE 5. If Jane's collection is  and Nell's collection is , we don't really want to compare them because that would mean that we are really looking at the items as  and , that is that we would be ignoring some of the details in the *pictures*.

- When the two given collections involve the *same* kind of items, the real-world *process* we will use to compare the two collections will be to **match one-to-one** each item of the first collection with an item of the second collection and to look in which of the two collections the **leftover** items are in.

When the two given collections involve the *same* kind of items, there are *six* several different **relationships** that can **hold** from the first collection to the

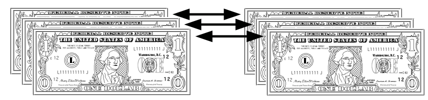
simple
is-the-same-in-size-as
is-different-in-size-from

second collection.

1. Up front, we have two very **simple** relationships:

- When there are *no* leftover objects, we will say that the first collection **is-the-same-in-size-as** the second collection.

Example 6. To compare in the real-world Jack's  with Jill's , we match Jack's collection one-to-one with Jill's collection:



Since there is *no* leftover item in *either* collection, the *relationship* between Jack's collection and Jill's collection is that:

Jack's collection *is-the-same-in-size-as* Jill's collection

- When there *are* leftover objects, regardless of where they are, we will say that the first collection **is-different-in-size-from** the second collection.

Example 7. To compare Jack's  with Jill's  in the *real-world*, we match Jack's collection one-to-one with Jill's collection:

Since there *are* leftover items in *one* of the two collections, the *relationship* between Jack's collection and Jill's collection is that:

Jack's collection *is-different-in-size-from* Jill's collection

Example 8. To compare in the real-world Jack's  with Jill's , we match Jack's collection one-to-one with Jill's collection:

strict
is-smaller-in-size-than
is-larger-in-size-than
mutually exclusive

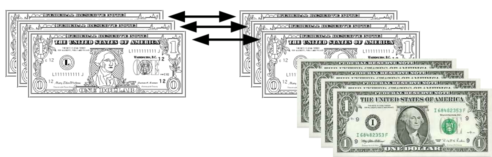
Since there *are* leftover items in *one* of the two collections, the *relationship* between Jack's collection and Jill's collection is that:

Jack's collection *is-different-in-size-from* Jill's collection

2. When two collections are *different-in-size*, then there are two possible **strict** relationships depending on which of the two collections the *leftover* item, if any, are in:

- When the leftover items are in the *second* collection, we will say that the first collection **is-smaller-in-size-than** the second collection.

Example 9. To compare Jack's  with Jill's  in the *real-world*, we match Jack's collection one-to-one with Jill's collection:



Since the leftover items are in Jill's collection, the *relationship* between Jack's collection and Jill's collection is that:

Jack's collection *is-smaller-in-size-than* Jill's collection

- When the leftover objects are in the *first* collection, we will say that the first collection **is-larger-in-size-than** the second collection.

Example 10. To compare in the real-world Jack's  with Jill's , we match Jack's collection one-to-one with Jill's collection:

Since the leftover items are in Jack's collection, the *relationship* between Jack's collection and Jill's collection is that:

Jack's collection *is-larger-in-size-than* Jill's collection

The relationship *is the same as* and the two *strict* relationships, *is-smaller-than* and *is-larger-than*, are **mutually exclusive** in the sense that as soon as we know that one of them holds, we know that neither one of the other two can hold.

is-no-larger-than
is-no-smaller

3. Quite often, though, instead of the above three relationships, we will need to use another two relationships that we shall call **lenient**.

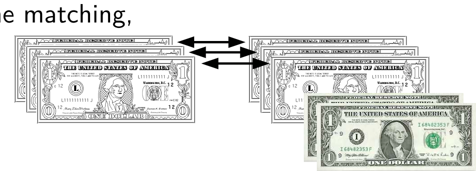
a. Instead of wanting to make sure that a first collection *is-smaller-than* a second collection, we may just want to make sure that the first collection **is-no-larger-than** the second collection, that is we may include collections that *are-the-same-as*.

What this mean is that instead of requiring that, after the one-to-one matching, the leftover items be in the *second* collection, we only require that the leftover items *not* be in the *first* collection and this is of course the case when the leftover items are in the *second* collection as before ... but *also* when there are *no* leftover items in *either* collection and therefore certainly no leftover in the first collection.

Example 11. If Jack's collection is  and Jill's collection is , then we have that:

Jack's collection *is no-larger-in-size-than* Jill's collection

since, after one-to-one matching,

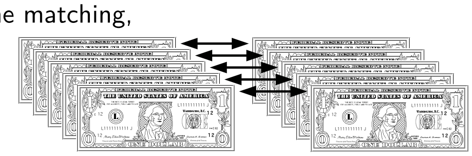


there is *no leftover item* in Jack's collection.

Example 12. If Mike's collection is  and Jill's collection is , it is also the case that:

Mike's collection *is no-larger-in-size-than* Jill's collection

since, after one-to-one matching,



there is no leftover item in *either* collection and therefore certainly no leftover item in Mike's collection.

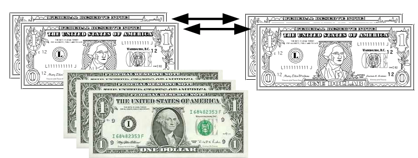
b. Similarly, instead of wanting to make sure that a first collection *is-larger-than* a second collection, we may just want to make sure that the first collection **is-no-smaller** than the second collection, that is we include collections that *are-the-same*.

What this mean in the real-world is that instead of requiring that, after the one-to-one matching, the leftover items be in the *first* collection, we only require that the leftover items *not* be in the *second* collection and this is of course the case when the leftover items are in the *first* collection as before . . . but *also* when there are *no* leftover items in *either* collection and therefore certainly no leftover in the second collection.

EXAMPLE 13. If Dick's collection is  and Jane's collection is , then we have that:

Dick's collection *is no-smaller-in-size-than* Jane's collection

since, after one-to-one matching,

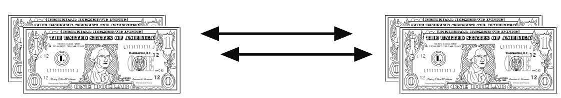


there is *no leftover item* in Jane's collection.

EXAMPLE 14. If Mary's collection is  and Jane's collection is , it is also the case that:

Mary's collection *is no-smaller-in-size-than* Jane's collection

since, after one-to-one matching,



there is no leftover item in *either* collection and therefore certainly no leftover item in Jane's collection.

The two *lenient* relationships are *not* mutually exclusive in the sense that, given two collections, even if we know that one *lenient* relationship *is* holding from the first collection to the second collection, we can*not* be sure that the other *lenient* relationship does *not* hold from the first collection to the second collection because the first collection could be holding because the first collection is-the-same-as the second collection in which case the other *lenient* relationship would be holding too.

On the other hand, if *both* lenient relationships hold from a first collection to a second collection, then we know for sure that the first collection *is-the-same-as* the second collection.

verb
=
is-equal-to
≠
is-not-equal-to
<
is-less-than
>
is-more-than
≦
is less-than-or-equal-to
≧
is more-than-or-equal-to
strict
lenient verbs
sentence (comparison)

2.3 Language For Comparisons

In order to *represent* on paper *relationships* between two collections, we first need to expand our *mathematical* language beyond *number-phrases*.

1. Given a *relationship* between two collections, we need a **verb** to represents this *relationship*. In keeping with our distinguishing between what we do in the *real-world* and what we write on *paper* to represent it, as between a real-world *process* and the paper *procedure* that represents it, we use different words for a real-world *relationships* and for the *verbs* we write on paper to represent it:

- To represent on paper the real-world *simple* relationships:
  - *is-the-same-in-size-as*, we will use the *verb* $=$ which we will read as **is-equal-to**,
  - *is-different-in-size-from*, we will use the *verb* $\neq$ which we will read as **is-not-equal-to**,
- To represent on paper the real-world *strict* relationships:
  - *is-smaller-in-size-than*, we will use the *verb* $<$, which we will read as **is-less-than**.
  - *is-larger-in-size-than*, we will use the *verb* $>$ which we will read as **is-more-than**,
- To represent on paper the real-world *lenient* relationships
  - *is-no-larger-in-size-than*, we will use the verb $\leqq$, which we will read as **is less-than-or-equal-to**.
  - *is-no-smaller-in-size-than*, we will use the verb $\geqq$, which we will read as **is more-than-or-equal-to**.

We will say that

- The verbs $>$ and $<$ are **strict** *verbs* because they represent the *strict* relationships *is-smaller-in-size-than* and *is-larger-in-size-than*.
- The verbs $\geqq$ and $\leqq$ are **lenient verbs** because they represent the *lenient* relationships *is-no-larger-in-size-than* and *is-no-smaller-in-size-than*.

2. Then, to indicate that a relationship *hold***s** from one collection to another, we write a **comparison-sentence** that consists of the *number-phrases* that represent the two collections with the *verb* that represents the *relationship* in-between the two number-phrases.

EXAMPLE 15. Given Jack's  and Jill's , we represent the *relationship*

Jack's collection *is the same as* Jill's collection

by writing the *comparison-sentence*

$$3 \textbf{ Dollars } = 3 \textbf{ Dollars}$$

which we read as

THREE dollars is-equal-to THREE dollars.

Example 16. Given Jack's  and Jill's , we represent the *relationship*

Jack's collection *is different from* Jill's collection

by writing the *comparison-sentence*

$$3 \textbf{ Dollars } \neq 7 \textbf{ Dollars}$$

which we read as

THREE dollars is-not-equal-to SEVEN dollars.

Example 17. Given Jack's  and Jill's , we represent the *relationship*

Jack's collection *is different from* Jill's collection

by writing the *comparison-sentence*

$$5 \textbf{ Dollars } \neq 3 \textbf{ Dollars}$$

which we read as

FIVE dollars is-not-equal-to THREE dollars.

Example 18. Given Jack's  and Jill's , we represent the *relationship*

Jack's collection *is smaller than* Jill's collection

by writing the *comparison-sentence*

$$3 \textbf{ Dollars } < 7 \textbf{ Dollars}$$

which we read as

THREE dollars is less than SEVEN dollars.

Example 19. Given Jack's  and Jill's , we represent the *relationship*

Jack's collection *is larger than* Jill's collection

by writing the *comparison-sentence*

$$5 \textbf{ Dollars } > 3 \textbf{ Dollars}$$

which we read as

FIVE dollars is more than THREE dollars.

equality
inequality (plain)

EXAMPLE 20. Given Jack's  and Jill's , we represent the *relationship*

Jack's collection *is no-larger than* Jill's collection

by writing the *comparison-sentence*

$$3 \textbf{ Dollars} \leqq 5 \textbf{ Dollars},$$

which we read as

THREE dollars is less-than-or-equal-to FIVE dollars.

EXAMPLE 21. Given Mike's  and Jill's , we represent the *relationship*

Mike's collection *is no-larger than* Jill's collection

by writing the *comparison-sentence*

$$5 \textbf{ Dollars} \leqq 5 \textbf{ Dollars},$$

which we read as

FIVE dollars is less-than-or-equal-to FIVE dollars.

EXAMPLE 22. Given Dick's  and Jane's , we represent the *relationship*

Dick's collection *is no-smaller than* Jane's collection

by writing the *comparison-sentence*

$$5 \textbf{ Dollars} \geqq 2 \textbf{ Dollars},$$

which we read as

THREE dollars is more-than-or-equal-to FIVE dollars.

EXAMPLE 23. Given Mary's  and Jane's , we represent the *relationship*

Mary's collection *is no-smaller than* Jane's collection

which we represent by writing the *comparison-sentence*

$$2 \textbf{ Dollars} \geqq 2 \textbf{ Dollars},$$

which we read as

THREE dollars is more-than-or-equal-to FIVE dollars.

3. Finally, comparison-sentences are named according to the *verb* that they involve

- Comparison-sentences involving the *verb* $=$ are called **equalities**.

  EXAMPLE 24.

  $3 \textbf{ Dollars} = 3 \textbf{ Dollars}$ is an equality

- Comparison-sentences involving the *verb* $\neq$ are called **plain inequalities**.

inequality (strict)
inequality (lenient)

EXAMPLE 25.

3 **Dollars** $\neq$ 5 **Dollars** is a *plain* inequality

- Comparison-sentences involving the *verbs* $>$ or $<$ are called **strict inequalities**

  EXAMPLE 26.

  3 **Dollars** $<$ 7 **Dollars** and 8 **Dollars** $>$ 2 **Dollars** are *strict* inequalities

- Comparison-sentences involving the verbs $\leqq$ and $\geqq$ are called **lenient inequalities**.

  EXAMPLE 27.

  3 **Dollars** $\leqq$ 7 **Dollars** and 8 **Dollars** $\geqq$ 2 **Dollars** are *lenient* inequalities

2.4 Procedures For Comparing Number-Phrases

Given two number-phrases, the *procedure* for writing the comparison-sentences that are true will depend on whether the number-phrases are *basic counting* number-phrases or *decimal* number-phrases.

Given two *basic counting* number-phrases, we must see whether we must count-*up* or count-*down* from the first numerator to the second numerator[1].

There are three possibilities depending on the *direction* we have to count when we count from the numerator of the first number-phrase to the numerator of the second number-phrase:

- We may have to count *up*, in which case the comparison-sentence is:

  first *counting* number-phrase $<$ second *counting* number-phrase
  (with $<$ read as "is-less-than")

  EXAMPLE 28. To compare the given *basic counting* number-phrases 3 **Washingtons** and 7 **Washingtons**

  i. We must count from 3 to 7: $\xrightarrow{4,\ 5,\ 6,\ 7}$

  that is we must count *up*.

  ii. So, we write the *strict inequality*:

  3 **Washingtons** $<$ 7 **Washingtons**

- We may have to count *down*, in which case the comparison-sentence is:

  first *counting* number-phrase $>$ second *counting* number-phrase
  (with $>$ read as "is-more-than")

  EXAMPLE 29. To compare the given *basic counting* number-phrases 8 **Washingtons** and 2 **Washingtons**

[1]Educologists will be glad to know that, already in 1905, Fine was using the *cardinal* aspect for comparison processes *in the real world* and the *ordinal* aspect for comparison procedures *on paper*.

true
false

i. We must count from 8 to 2:

$$\xrightarrow{7,\ 6,\ 5,\ 4,\ 3,\ 2}$$

that is, we must count *down*.
ii. So, we write the *strict inequality*:

$$8 \text{ **Washingtons**} > 2 \text{ **Washingtons**}$$

- We may have neither to count *up* nor to count *down*, in which case the comparison-sentence is:

  first *counting* number-phrase = second *counting* number-phrase
  (with = read as "is-equal-to")

  EXAMPLE 30. To compare the given basic sentences 3 **Washingtons** and 3 **Washingtons**.
  i. We must count from 3 to 3, that is we must count neither *up* nor *down*.
  ii. So, we write the *equality*:

  $$3 \text{ **Washingtons**} = 3 \text{ **Washingtons**}$$

2.5 Truth Versus Falsehood

Inasmuch as the *comparison-sentences* that we wrote until now represented relationships between real-world collections, they were **true**.

However, there is nothing to prevent us from writing comparison-sentences regardless of the real-world. In fact, there is nothing to prevent us from writing comparison-sentences that are **false** in the sense that there is no way that anyone could come up with real-world collections for which *one-to-one matching* would result in the relationship represented by these comparison-sentences.

EXAMPLE 31. The sentence

$$5 \text{ **Dollars**} < 3 \text{ **Dollars**}$$

is *false* because there is no way that anyone could come up with real-world collections for which *one-to-one matching* would result in there being leftover items in the *second* collection.

EXAMPLE 32. The sentence

$$5 \text{ **Dollars**} = 3 \text{ **Dollars**},$$

is *false* because there is no way that anyone could come up with real-world collections for which *one-to-one matching* would result in there being *no* leftover item.

EXAMPLE 33. The sentence

$$3 \text{ **Dollars**} \leqq 3 \text{ **Dollars**},$$

negation
NOT[ ]
slash
duality (linguistic)
symmetry (linguistic)

is *true* because we can come up with real-world collections for which *one-to-one matching* would result in there being *no* leftover item.

EXAMPLE 34. The sentence

$$5 \textsf{ Dollars} \leqq 3 \textsf{ Dollars},$$

is *false* because there is no way that anyone could come up with real-world collections for which *one-to-one matching* would result in there being leftover items in the second collection or *no* leftover item.

However, while occasionally useful, it is usually not very convenient to write sentences that are *false* because then we must not forget to write that they are false when we write them and we may miss that it says somewhere that they are false when we read them. So, inasmuch as possible, we shall write only sentences that are *true* and we will use

DEFAULT RULE # 2. *When no indication of truth or falsehood is given, mathematical sentences will be understood to be true and this will go without saying.*

When a sentence is *false*, rather than writing *it* and say that it is *false*, what we shall usually do is to write *its* **negation**—which is *true* and therefore "goes without saying". We can do this either in either one of two manners:

- We can place the *false* sentence within the symbol **NOT**[ ],
- We can just **slash** the *verb* which is what we shall usually do.

EXAMPLE 35. Instead of writing that

$$\text{the sentence } 5 \textsf{ Dollars} = 3 \textsf{ Dollars} \text{ is } \textit{false}$$

we can either write the sentence

$$\textbf{NOT}[5 \textsf{ Dollars} = 3 \textsf{ Dollars}]$$

or the sentence

$$5 \textsf{ Dollars} \neq 3 \textsf{ Dollars}$$

2.6 Duality Versus Symmetry

The **linguistic duality** that exists between $<$ and $>$ must not be confused with **linguistic symmetry**, a concept which we tend to be more familiar with[2].

[2]This confusion is a most important *linguistic* stumbling block for students and one that Educologists utterly fail to take into consideration.

opposite
dual

1. Linguistic *symmetry* involves pairs of sentences—which may be *true* or *false*—that represent **opposite** relationships between the two people/collections because, even though the verbs are *the same*, the two people/collections are mentioned in *opposite* order.

Example 36.

- **Jack** is a child of **Jill** versus **Jill** is a child of **Jack**
- **Jill** beats **Jack** at poker versus **Jack** beats **Jill** at poker
- **Jack** loves **Jill** versus **Jill** loves **Jack**
- 9 **Dimes** $>$ 2 **Dimes** versus 2 **Dimes** $>$ 9 **Dimes**

Observe that just because one of the two sentences is *true* (or *false*) does *not*, by itself, automatically force the other to be either *true* or *false* and that whether or not it does depends on the *nature* of the relationship.

2. Linguistic *duality* involves pairs of sentences—which may be *true* or *false*—that represent the *same* relationship between the two people/collections because, even though the people/collections are *mentioned in opposite order*, the two *verbs* are **dual** of each other which "undoes" the effect of the order so that only the *emphasis* is different.

Example 37.

- **Jack** is a *child* of **Jill** versus **Jill** is a *parent* of **Jack**
- **Jill** *beats* **Jack** at poker versus **Jack** *is beaten by* **Jill** at poker
- **Jack** *loves* **Jill** versus **Jill** *is loved by* **Jack**
- 9 **Dimes** $>$ 2 **Dimes** versus 2 **Dimes** $<$ 9 **Dimes**

Observe that here, as a result, if one of the two sentences is *true*(or *false*) this *automatically* forces the other to be *true* (or *false*) and this regardless of the *nature* of the relationship.

3. When the *verbs* are *the same* and the order does *not* matter for these verbs, the sentences are at the same time (*linguistically*) *symmetric* and (*linguistically*) *dual* .

Example 38.

- **Jack** is a sibling of **Jill** versus **Jill** is a sibling of **Jack**
- 2 **Nickels** $=$ 1 **Dime** versus 1 **Dime** $=$ 2 **Nickels**

Observe that, here again, as soon as one sentence is *true* (or *false*), by itself this *automatically* forces the other to be *true* (or *false*) and that it does not depend on the *nature* of the relationship.

operation
attach
result
resulting collection

Chapter 3

Addition

We investigate the *second* of the three fundamental processes involving two collections. We will introduce the procedure in the case of *basic* collections using *basic counting* number-phrases and we will then extend the procedure to *extended* collections using *decimal-number phrases*.

3.1 Attaching A Collection To Another

Given two collection, the second fundamental issue is the first instance of what is called an **operation**—as opposed to a *relationship*: it is to **attach** the *second* collection to the *first* collection.

To get the collection that is the **result** of the real-world *process*:

i. We set the second collection alongside the first collection

ii. We move the second collection along the first collection

iii. The **resulting collection** is made of all the items in the first collection as well as the moved items. .

EXAMPLE 1. To attach Jill's  to Jack's 

i. We set Jill's collection to the right of Jack's collection:

operator
+
specifying-phrase
bar

ii. We move Jill's collection along Jack's collection

iii. The items in the first collection together with the moved items make up the resulting collection.

3.2 Language For Addition

In order to represent on paper the result of an *operation*, such as attaching a second collection to a first collection, we need to expand again our mathematical language.

1. The first thing we need is a symbol, called **operator**, to represent the *operation*. In the case of *attaching* a second collection to a first collection, we will of course use the *operator* +, read as "plus". **NOTE.** It should be stated right away, though, that this use of the symbol + is only one among very many different uses of the symbol + and that this will create in turn many difficulties. We shall deal with these difficulties one at a time, as we encounter each new use of the symbol +.

2. Given two collections represented by number-phrases, we will represent f attaching the second collection to the first by a **specifying-phrase** that we write as follows:

i. We write the first number phrase:

first number phrase

ii. We write the symbol for adding:

first number phrase $\underline{\ +\qquad\qquad\qquad\qquad\ }$

iii. We write the second number-phrase over the **bar**:

first number phrase $\underline{\ +\ \text{second number phrase}\ }$

Altogether then, the *specifying-phrase* that corresponds to attaching to a first collection a second collection is:

first number phrase $\underline{\ +\ \text{second number phrase}\ }$

EXAMPLE 2. In order to say that we want to add to the first number-phrase 5 **Washingtons** the second number-phrase 3 **Washingtons** we write the *specifying phrase*:

5 **Washingtons** $\underline{\ +\ 3\ \textbf{Washingtons}\ }$

3. This language gives us a lot of flexibility:

identify
identification-sentence
arrowhead
=

- *Before* we count the result of attaching a second collection to a first collection, we can already represent the *result* by using a *specifying-phrase*.
- *After* we have found the result of attaching a second collection to a first collection, we can represent the result by a *number-phrase*.
- Altogether, to summarize the whole *process*, we can **identify** the *specifying phrase* with an **identification-sentence** which we write as follows
  i. We write the specifying phrase
  ii. We lengthen the *bar* with an **arrowhead**
  iii. We write the number-phrase that represents the result.

EXAMPLE 3.

i. *Before* we attach to Jack's  Jill's , we can already represent the *result* by the *specifying-phrase*

$$6\ \textsf{Washingtons}\ \underline{\ +\,3\ \textsf{Washingtons}\ }$$

ii. *After* we have found that the result of attaching to Jack's  Jill's  is  we can represent the result by

$$9\ \textsf{Washingtons}$$

iii. Altogether, to summarize the whole *process* with an identification-sentence we lengthen the bar with an arrowhead and we write the number-phrase that represents the result of the attachment.

$$6\ \textsf{Washingtons} \xrightarrow{+\,3\ \textsf{Washingtons}} 9\ \textsf{Washingtons}$$

4. Usually, though, we will not write things this way and we only did it above to show how the mathematical language represented the reality. As usual, some of it "goes without saying":

- In the *specifying phrase*, the *bar* goes without saying
- In the identification sentence, the arrowhead is replaced by the symbol =

EXAMPLE 4. Instead of writing the specifying phrase

$$6\ \textsf{Washingtons}\ \underline{\ +\,3\ \textsf{Washingtons}\ }$$

we shall write

$$6\ \textsf{Washingtons} +\ 3\ \textsf{Washingtons}$$

and instead of writing the identification sentence

$$6\ \textsf{Washingtons} \xrightarrow{+\,3\ \textsf{Washingtons}} 9\ \textsf{Washingtons}$$

addition

we shall write

$$6 \text{ Washingtons} + 3 \text{ Washingtons} = 9 \text{ Washingtons}$$

3.3 Procedure For Adding A Number-Phrase

Given two collections, the paper procedure that gives (the *numerator* of) the number-phrase that represents the result of attaching the second collection to the first collection is called **addition** and depends on whether the two number-phrases are *basic counting* number-phrases or *decimal* number-phrases.

In order to *add* a second *basic* collection to a first *basic* collection, we count *up* from the numerator of the first collection by a length equal to the numerator of the second collection.

There are then two cases depending on whether, when we count up from the numerator of the first number-phrase by a length equal to the second numerator, we need to end up *past* 9 or not.

- If we do not need to end up *past* 9, the result of the addition is just the end-digit.

  EXAMPLE 5. To add Jill's 5 **Washingtons** to Jack's 3 **Washingtons**, that is, to *identify* the *specifying-phrase*

  $$3 \text{ Washingtons} + 5 \text{ Washingtons}$$

  i. Starting from 3, we count *up* by a length equal to 5:

  $$\xrightarrow{4,\ 5,\ 6,\ 7,\ 8}$$

  ii. The end-digit is 8.

  iii. We write the *identification-sentence*:

  $$3 \text{ Washingtons} + 5 \text{ Washingtons} = 8 \text{ Washingtons}$$

- If we need to end up *past* 9, then we must bundle and change TEN of the items.

  EXAMPLE 6. To add Jill's 8 **Washingtons** to Jack's 5 **Washingtons**, that is to identify the *specifying-phrase*

  $$5 \text{ Washingtons} + 8 \text{ Washingtons}$$

  i. Starting from 5, we count *up* by a length equal to 8 but stop after TEN:

  $$\xrightarrow{4,\ 5,\ 6,\ 7,\ 8,\ 9,\ \text{TEN}}$$

  ii. We bundle TEN **Washingtons** and change for a 1 **DEKAWashingtons** and count the rest

  $$\xrightarrow{1,\ 2,\ 3}$$

iii. We write the *identification-sentence*:

$$5 \textbf{ Washingtons} + 8 \textbf{ Washingtons} = 1 \textbf{ DEKAWashingtons} \,\&\, 3 \textbf{ Washingtons}$$

which of course we could also write

$$5 \textbf{ Washingtons} + 8 \textbf{ Washingtons} = 1.3 \textbf{ DEKAWashingtons}$$

or

$$5 \textbf{ Washingtons} + 8 \textbf{ Washingtons} = 13. \textbf{ Washingtons}$$

or . . .

Actually, we usually do the latter a bit differently, that is, instead of *basic* counting up just past 9, interrupt ourselves to bundle and change, and then start *basic* counting again, it is easier to use some *extended* counting and count all the way and *then* bundle and change what we must and count the rest.

EXAMPLE 7. To add Jill's 8 **Washingtons** to Jack's 5 **Washingtons**, that is to identify the *specifying-phrase*

$$5 \textbf{ Washingtons} + 8 \textbf{ Washingtons}$$

i. We count up from 5 by a length equal to 8 using *extended-counting*:

$$\xrightarrow{\text{4, 5, 6, 7, 8, 9, TEN, ELEVEN, TWELVE, THIRTEEN}}$$

ii. Then we say that we can't *write* THIRTEEN **Washingtons** since we only have digits up to 9 so that we should bundle TEN **Washingtons** and change for a 1 **DEKAWashingtons** with 3 **Washingtons** left

iii. We write the *identification-sentence*:

$$5 \textbf{ Washingtons} + 8 \textbf{ Washingtons} = 1 \textbf{ DEKAWashingtons} \,\&\, 3 \textbf{ Washingtons}$$

that is, using a decimal number-phrase,

$$5 \textbf{ Washingtons} + 8 \textbf{ Washingtons} = 1.3 \textbf{ DEKAWashingtons}$$

or, if we prefer,

$$5 \textbf{ Washingtons} + 8 \textbf{ Washingtons} = 13. \textbf{ Washingtons}$$

or . . .

The difference is of course not a great one. It is only that we said that we would deal with *extended* collections using only *basic* counting and indeed, in the second example, we fudged a bit when, after having counted to THIRTEEN, we said that after bundling and changing we had 3 left: officially, we cannot do so since we have not yet introduced *subtraction*.

However, if the first example illustrates the fact that, when needed, we can indeed do things "cleanly", the second example illustrates the fact that, while we are usually not willing to count *very* far, a bit of (extended) counting beyond 9 makes life easier.

detach
resulting collection

Chapter 4

Subtraction

We investigate the *third* of the three fundamental processes involving two collections. We will introduce the procedure in the case of *basic* collections using *basic counting* number-phrases and we will then extend the procedure to *extended* collections using *decimal-number phrases.*

4.1 Detaching A Collection From Another

Given two collections, the third fundamental issue is to **detach** the *second* collection from the *first* collection. This is the second instance of an *operation.*

The real-world *process* is to mark off the items of the first collection that are also in the second collection and to look at all the unmarked items as making up a single collection that we shall also call the **resulting collection**.

Example 1. To detach Jill's  from Jack's 

i. We set Jill's collection to the right of Jack's collection

ii. We mark off the items in Jack's collection that are also in Jill's collection

–
–
minus
bar

iii. The unmarked items in the first collection make up the *resulting collection*

4.2 Language For Subtraction

In order to represent on paper the result of an *operation*, such as *detaching* a second collection from a first collection, we need to expand again our mathematical language but we will proceed in essentially the same manner as we did with the language for *addition*.

1. The first thing we need is a symbol, called *operator*, to represent the *operation*. In the case of *detaching* a second collection from a first collection, we will of course use the *operator* $-$, read as "minus".
To represent on paper the result of *detaching* a second collection from a first collection, we will of course use the *operator* $-$ read **minus**.
Here again, just as with the symbol $+$, this use of the symbol $-$ is only one among very many different uses of the symbol $-$ and that this will create in turn many difficulties. We shall deal with these difficulties one at a time, as we encounter each new use of the symbol $-$.

Note. It should be stated right away, though, that this use of the symbol $-$ is only one among very many different uses of the symbol $-$ and that this will create in turn many difficulties. We shall deal with these difficulties one at a time, as we encounter each new use of the symbol $-$.

2. Given two collections represented by number-phrases, we will represent *detaching* the second collection from the first by a *specifying-phrase* that we write as follows:

i. We write the first number phrase:

first number phrase

ii. We write the symbol for *subtracting*:

first number phrase $\overline{\;-\;\;\;\;\;\;\;\;\;\;\;\;\;}$

iii. We write the second number-phrase over the **bar**:

first number phrase $\overline{\;-\text{ second number phrase}}$

Altogether then, the specifying phrase that corresponds to *detaching* from a first collection a second collection is:

$$\text{first number phrase}\ \overset{-\ \text{second number phrase}}{\overline{\qquad\qquad\qquad\qquad}}$$

identify
identification-sentence
arrowhead
=

EXAMPLE 2. In order to say that we want to *subtract* from the first number-phrase 5 **Washingtons** the second number-phrase 3 **Washingtons** we write the *specifying phrase*:

$$5\ \textbf{Washingtons}\ \overset{-\ 3\ \textbf{Washingtons}}{\overline{\qquad\qquad\qquad}}$$

3. This language gives us a lot of flexibility:

- *Before* we count the result of attaching a second collection to a first collection, we can already represent the *result* by using a *specifying-phrase*.
- *After* we have found the result of attaching a second collection to a first collection, we can represent the result by a *number-phrase*.
- Altogether, to summarize the whole *process*, we can **identify** the *specifying phrase* with an **identification-sentence** which we write as follows
  i. We write the specifying phrase
  ii. We lengthen the *bar* with an **arrowhead**
  iii. We write the number-phrase that represents the result.

EXAMPLE 3.

i. *Before* we detach from Jack's  Jill's , we can already represent the *result* by the *specifying-phrase*

$$6\ \textbf{Washingtons}\ \overset{-\ 4\ \textbf{Washingtons}}{\overline{\qquad\qquad\qquad}}$$

ii. *After* we have found that the result of detaching from Jack's  Jill's  is  we can represent the result by

4 **Washingtons**

iii. Altogether, to summarize the whole *process* with an identification-sentence we lengthen the bar with an arrowhead and we write the number-phrase that represents the result of the *detachment*.

$$6\ \textbf{Washingtons}\ \xrightarrow{-\ 2\ \textbf{Washingtons}}\ 4\ \textbf{Washingtons}$$

4. Usually, though, we will not write things this way and we only did it above to show how the mathematical language represented the reality. As usual, some of it "goes without saying":

- In the *specifying phrase*, the *bar* goes without saying
- In the identification sentence, the arrowhead is replaced by the symbol =

EXAMPLE 4. Instead of writing the specifying phrase subtraction

$$6 \textbf{ Washingtons} \xrightarrow{-\ 2 \textbf{ Washingtons}}$$

we shall write

$$6 \textbf{ Washingtons} -\ 2 \textbf{ Washingtons}$$

and instead of writing the identification sentence

$$6 \textbf{ Washingtons} \xrightarrow{-\ 2 \textbf{ Washingtons}} 4 \textbf{ Washingtons}$$

we shall write

$$6 \textbf{ Washingtons} \ -\ 2 \textbf{ Washingtons} \ =\ 4 \textbf{ Washingtons}$$

4.3 Procedure For Subtracting A Number-Phrase

Given two collections, the paper procedure that gives (the *numerator* of) the number-phrase that represents the result of detaching the second collection from the first collection is called **subtraction** and depends on whether the two number-phrases are *basic counting* number-phrases or *decimal* number-phrases.

In order to *subtract* a second *basic* collection from a first *basic* collection, we count *down* from the numerator of the first collection by a length equal to the numerator of the second collection.

There are then two cases depending on whether, when we count *down* from the numerator of the first number-phrase by a length equal to the second numerator, we can complete the count or not.

- If we can complete the count, then the result of the subtraction is just the end-digit.

  Example 5. To subtract Jill's 3 **Washingtons** from Jack's 7 **Washingtons**, that is to *identify* the *specifying-phrase*

  $$7 \textbf{ Washingtons} - 3 \textbf{ Washingtons}$$

  i. Starting from 7, we count *down* by a length equal to 3:

  $$\xrightarrow{6,\ 5,\ 4}$$

  ii. We can complete the count and the end-digit is 4

  iii. We write the *identification-sentence*:

  $$7 \textbf{ Washingtons} - 3 \textbf{ Washingtons} = 4 \textbf{ Washingtons}$$

- In particular, the end-digit can be 0.

  Example 6. To subtract Jill's 5 **Washingtons** from Jack's 5 **Washingtons**, that is to *identify* the *specifying-phrase*

  $$5 \textbf{ Washingtons} - 5 \textbf{ Washingtons}$$

  i. Starting from 5, we count *down* by a length equal to 5:

  $$\xrightarrow{4,\ 3,\ 2,\ 1,\ 0}$$

outcast
incorrect
strike out

ii. We can complete the count and the end-digit is 0
iii. We write the *identification-sentence*:

$$5 \text{ Washingtons} - 5 \text{ Washingtons} = 0 \text{ Washingtons}$$

- If we cannot complete the count, then the subtraction just cannot be done. (At least in this type of situation. We shall see in the next Chapter other situations in which we can end down *past* 0.)

  EXAMPLE 7. To subtract Jill's 5 **Washingtons** from Jack's 3 **Washingtons**, that is to *identify* the *specifying-phrase*

  $$3 \text{ Washingtons} - 5 \text{ Washingtons}$$

  But, to *identify* the specifying-phrase, we would have to start from 3 and count *down* by a length of 5 but, by the time we got to 0, we would have counted only by a length of 3 and so we cannot complete the count which is as it should be.

4.4 Subtraction As Correction

Subtraction often comes up after we have done a long string of additions and realized that there is an **outcast**, that is a number-phrase that we shouldn't have added (for whatever reason), so that, as a consequence, the *total* is **incorrect**.

EXAMPLE 8. Suppose we had an ice-cream stand and that we had added *sales* as the day went which gave us the following specifying-phrase:

6 **Washingtons** + 3 **Washingtons** + 7 **Washingtons** + 9 **Washingtons**

and that at the end of day we identified the specifying-phrase which gave us

$$25 \text{ Washingtons}$$

but that we then realized that 3 **Washingtons** was an *outcast* (it was not a *sale* but money given for some other purpose) with the consequence that 25 **Washingtons** is *incorrect* in that it is not the sum total of the *sales* for the day.

To get the correct total, we have the following two choices for the procedure:

- **Procedure A** would be to **strike out** the *outcast* and redo the entire addition:

  EXAMPLE 9. In the above example, we *strike out* the *outcast* 3 **Washingtons**

  6 **Washingtons** + ~~3 **Washingtons**~~ + 7 **Washingtons** + 9 **Washingtons**

  which gives us

  $$22 \text{ Washingtons}$$

  Of course, since Procedure A is going to involve a lot of unnecessary work redoing all that had been done correctly, it is very inefficient.

cancel out
adjustment

- **Procedure B** would be to **cancel out** the *effect* of the outcast in the incorrect total by *subtracting* the outcast from the incorrect total. (Accountants call this "entering an **adjustment**".)

  EXAMPLE 10. In the above example, we *subtract* 3 **Washingtons** (the outcast) from 25 **Washingtons** (the incorrect total):

  25 **Washingtons** − 3 **Washingtons**

  which gives us:

  22 **Washingtons**

We now want to *see* that the two procedures *must* give us the same result either way. For that, we place the specifying-phrases in the two procedures side by side and we see that that the remaining number-phrases are the same either way.

EXAMPLE 11. In the above example, we have:

6 **Washingtons** + ~~3 **Washingtons**~~ + 7 **Washingtons** + 9 **Washingtons**

and

6 **Washingtons** + ~~3 **Washingtons**~~ + 7 **Washingtons** + 9 **Washingtons** − ~~3 **Washingtons**~~

We see that, either way, the remaining number-phrases are:

6 **Washingtons** + 7 **Washingtons** + 9 **Washingtons**

Chapter 5

Signed Number-Phrases

We have seen in Chapter 1 that we can use **plain number-phrases**, that is either *counting* number-phrases or *decimal* number-phrases, only in situations where the items are all of the same *one* kind. We shall now introduce and discuss a *new type* of number-phrase that we shall use in a type of situations that occurs frequently in which the items are all of *either one of two* kinds.

Just as we did for *plain* number-phrases in Chapters 2, 3, and 4, we will have to *define* for this *new type* of number-phrase what we mean by:

i. To "compare" two number-phrases,

ii. To "add" a second number-phrase to a first number-phrase,

iii. To "subtract" a second number-phrase from a first number-phrase.

and in particular to develop the corresponding *procedures*.

What will complicate matters a little bit, though, is that the procedures for the *new type* of number-phrases will involve the procedure that we developed for *plain* number-phrases. So, until we feel completely comfortable with the distinction, we shall use new symbols for "comparison", "addition" and "subtraction" for the new kind of number-phrases [1].

[1] One can only wonder as to how Educologists can let their students use, without warning, the same symbols in these rather different situations.

cancel
two-way collections
action
step
state
degree
benchmark

5.1 Actions and States

Quite often we don't deal with items that are all of the same kind but with items of two different kinds and a special case of this is when two items of different kinds cannot be together as they somehow **cancel** each other. As a result, we will now consider what we shall call **two-way collections**, that is collections of items that are all of one kind or all of another kind with items of different kinds canceling each other.

1. In the real-world, *two-way collections* come up very frequently and in many different types of situations but they generally fall in either one of two types:

- In one type of two-way collections, called **actions**, the items are **steps** in either *this-direction* or *that-direction*.

  Example 1. In fact, we already encountered in the previous chapter this kind of items: counting *up* and counting *down*. Of course, the situation there was not symmetrical: we could always count steps *up* but we could not always count steps *down*. But there would have been no point counting at the same time three steps up and five steps down since steps up would cancel out steps down and this would have just amounted to counting two steps down.

  Example 2.
  - Actions that a businesswoman may take on a bank account are to *deposit* three thousand dollars, *withdraw* two thousand dollars, etc
  - Actions that a gambler may take are to *win* fifty-eight dollars, *lose* sixty-two dollars, etc
  - Actions that a mark may take on a horizontal line include moving two feet *leftward*, five feet *rightward*, etc.
  - Actions that a mark may take on a vertical line include moving five inches *upward*, five inches *downward*, etc.

- In the other type of two-way collections, called **states**, the items are **degrees** of one kind or another but they have to be either on *this-side* or *that-side* of some **benchmark**.

  Example 3.
  - States that a business may be in include being three thousand dollars *in the red*, being seven thousand dollars *in the black*, etc.
  - States that a gambler may be in include being sixty-two dollars *ahead of the game*, being thirty-seven dollars *in the hole*, etc.
  - States that a mark may be in on a horizontal line with some benchmark include being two feet *to the left* of the benchmark, being nine feet *to the right* of the benchmark, etc.
  - States that a mark may be in on a vertical line with some benchmark include being five inches *above* the benchmark, being three inches *below* the benchmark, etc.

nature (of an action)
nature (of a state)
extent (of an action)
size (of a state)
direction (of an action)
side (of a state)
signed number-phrase

2. Since all the items in a given two-way collection are of the same kind, a two-way collection is essentially a collection with a twist. So, just as we said that, in the real world,

- the *nature of a collection* is the *kind* of items in the collection,
- the *extent of a collection* is the *number* of items in the collection,

we shall now say that:

- the **nature of an action** is the *kind* of steps in the action and the **nature of a state** is the *kind* of degrees in which the state can be
- the **extent of an action** is the *number* of steps in the action and the **size of a state** is the *number* of degrees of the state.
- the **direction of an action** is the *direction* of the steps in the action and the **side of a state** is the *side* of the degrees in the state.

  EXAMPLE 4. When a person climbs up and down a ladder, an *action* may be climbing up seven rungs. Then,
  - the *nature* of the action is *climbing rungs*
  - the *size* of the action is *seven*
  - the *direction* is *up*

5.2 Signed Number-Phrases

Plain number-phrases are not sufficient to represent on paper either *actions* or *states* because they do not indicate the *direction* of the action or the *side* of the state.

EXAMPLE 5.

- 3000 **Dollars** does not say if the businesswoman made a deposit or a withdrawal or if the business is in the red or in the black.
- 62 **Dollars** does not say if the gambler is ahead of the game or in the hole.
- 2 **Feet** does not say if the mark is to the left or to the right of the benchmark.
- 5 **Inches** does not say if the mark is moving up or down.

1. Since a two-way collection is just a collection with a *direction* or a *side*, we will represent on paper a two-way collection by a **signed number-phrase** that will consist of:

- a *denominator* to represent on paper the *nature* of the action (that is the *kind* of the steps in the action) or of the state (that is the *kind* of the degrees in the state).
- a *numerator* to represent on paper the *extent* of the action (that is the *number* of steps in the action) or the *extent* of the state (that is the *number* of degrees in the state),
- a *sign* to represent on paper the *direction* of the action (that is the *direction* of the steps in the action) or the *side* of the state (that is the

record
standard direction
opposite direction
standard side
opposite side
sign
+
positive
−
negative
context
signed-numerator
positive numerators
negative numerators

side of the benchmark that the degrees of the state are on.)

2. However, in order to say what direction the action or what side the state, we must always begin by **recording** for future reference:

- which direction is to be the **standard direction** and which direction is therefore to be the **opposite direction**,
- which side of the benchmark is going to be the **standard side** and which side is therefore to be the **opposite side**,

NOTE. Historically, it has long gone without saying that *standard* was what was "good" and *opposite* what was "bad".

EXAMPLE 6.

– To *deposit* money is usually considered to be "good" as it goes with *saving* while to *withdraw* money is usually considered to be "bad" as it goes with *spending*.
– To *win* is usually considered to be "good" while to *lose* is considered to be "bad".
– To go *up* is usually considered to be "good" while to go *down* is usually considered to be "bad".

3. Once we have recorded what is *standard* and therefore what is *opposite*, we can use a **sign** to represent on paper the *direction* of the action (that is the direction of the steps in the action) or the *side* of the state (that is the side of the benchmark that the degrees of the state are on):

- we will use the sign +, read here as **positive**, to represent on paper whatever is *standard*, whether an action or a state.
- we will use the sign −, read here as **negative**, to represent on paper whatever is *opposite*, whether an action or a state.

NOTE. This use of the symbols + and − is entirely different from their use in Chapter 1 where they denoted *addition* and *subtraction*. This complicates *reading* the symbol as we need to rely on the **context**, that is the text that is around the symbol, to decide what the symbol stands for.

4. However, because this will make developing and using *procedures* a lot easier, we will lump the *sign* together with the *numerator* and call the result a **signed-numerator**. Signed-numerator with a + are said to be **positive numerators** and signed-numerators with a − are said to be **negative numerators**. **NOTE.** Historically, just as with standard and opposite and perhaps as a result, *positive* has been identified with "good" and *negative* with "bad".

So, altogether, a *signed* number-phrase will consist of:

- a signed-numerator
- a denominator

EXAMPLE 7. Say that we have put on record that the *standard* direction is to *win* money so that to *lose* money is the *opposite* direction. Then,

When a *real-world* gambler:	We write on *paper*:
• *wins* forty-seven dollars	$+47$ **Dollars**
• *loses* sixty-two dollars	-62 **Dollars**

sign, of the numerator
size

EXAMPLE 8. Say we have put on record that the *standard* side is *in-the-black* so that *in-the-red* is the *opposite* side. Then,

When a *real-world* business is:	We write on *paper*:
• three thousand dollars *in-the-black*	$+3000$ **Dollars**
• seven hundred dollars *in-the-red*	-700 **Dollars**

5. We are using the same symbol, 0, both for

- the counting numerator that is left of the succession of counting numerators $1, 2, 3, 4, \ldots$
- the signed numerator which is inbetween the succession of positive numerators $+1, +2, +3, +4, \ldots$ and the recession of negative numerators $-1, -2, -3, -4, \ldots$.

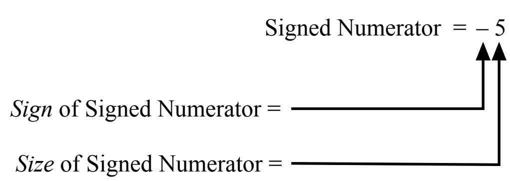
In this case, we shall have to live with the ambiguiity and decide each time, according to the context, which one the numerator 0 really is.

5.3 Size And Sign

On the other hand, given a *signed numerator*, we shall say that:

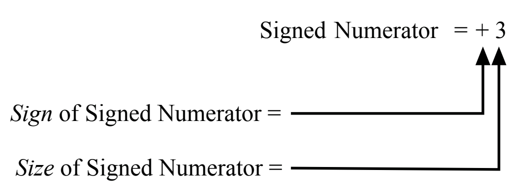
- the **sign of the numerator** is the sign which was put in front of the plain numerator to make the signed numerator
- the **size** of the numerator is the plain numerator from which the signed numerator was made.

EXAMPLE 9.



In other words, -5 is a signed-numerator whose *size* is 5 and whose *sign* is $-$.

EXAMPLE 10.



signed ruler
minus infinity
$-\infty$
plus infinity
$+\infty$

In other words, $+3$ is a signed-numerator whose *size* is 3 and whose *sign* is $+$.

Indeed, signed number-phrases can contain more information than is necessary for a particular purpose and then all we need is either the *sign* or the *size* of the signed number-phrase.

1. In many circumstances, what matters is only the *size* of the signed number-phrases and not the *sign*.

Example 11. Say we are told that

- Jill's balance is $+70,000,000$ **Dollars**
- Jack's balance is $-70,000,000$ **Dollars**.

We can safely conclude that neither Jack nor Jill belongs to "the rest of us".

Example 12. If we are stopped on the turnpike doing $+100$ $\frac{\textbf{Miles}}{\textbf{Hour}}$, that is while driving from Philadelphia to New York, or doing -100 $\frac{\textbf{Miles}}{\textbf{Hour}}$ that is while driving back from New York to Philadelphia , it does not matter which way we were going: regardless of the *direction*, we are going to get into big trouble.

So, in such cases, it is the *size* of the given *signed numerator* that matters.

Example 13. The *size* of Jill's $+70,000,000$ **Dollars** is $70,000.000$ and the *size* of Jack's $-70,000,000$ **Dollars** is also $70,000,000$ **Dollars**.
So, what makes Jack and Jill different from "the rest of us" is the *size* of their balance and not its *sign*.

Example 14. The *size* of our speed when we are going $+100$ $\frac{\textbf{Miles}}{\textbf{Hour}}$ (that is from Philadelphia to New York) is 100 $\frac{\textbf{Miles}}{\textbf{Hour}}$ and the *size* of our speed when we are going -100 $\frac{\textbf{Miles}}{\textbf{Hour}}$ (that is from New York to Philadelphia) is also 100 $\frac{\textbf{Miles}}{\textbf{Hour}}$.
So, what gets us into trouble is the *size* of our speed.

2. In many other circumstances, what matters is only the *sign* of the signed number-phrase and not the *numerator*.

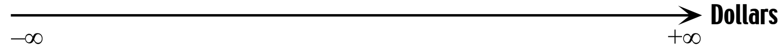
Example 15. Usually, banks do not accept *negative* balances, regardless of their *size*. In other words, all bank care about is the *sign* of the balance.

Example 16. If we are stopped going the wrong way on a one way street, it won't matter if we were well under the speed limit. In other words, what gets us into trouble is the *sign* of our speed and not its *size*.

5.4 Graphic Illustrations

To *graph* a *two-way collection* represented on paper by a *signed number-phrase*, we proceed essentially just as with counting number-phrases and/or decimal number-phrases. The only differences are that on a **signed ruler**:

- we shall have the symbol for **minus infinity**, $-\infty$, and the symbol for **plus infinity**, $+\infty$, at the corresponding ends of the ruler

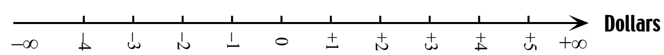


algebraic viewpoint
$<$ (signed)
$>$ (signed)
$\leqq$ (signed)
$\geqq$ (signed)

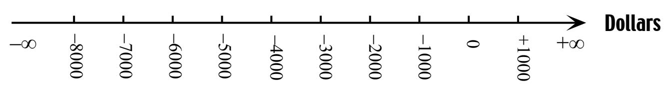
- the tick-marks, if any, are labeled with *signed number-phrases*.

As with all rulers and depending on the circumstances, 0 may or may not appear.

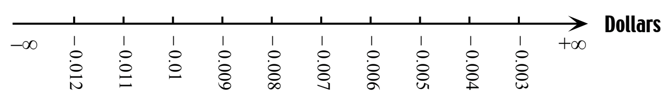
EXAMPLE 17.



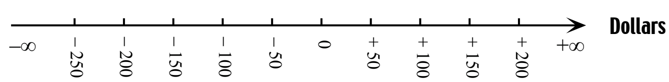
EXAMPLE 18.



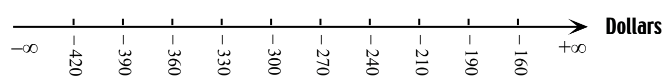
EXAMPLE 19.



EXAMPLE 20.



EXAMPLE 21.



5.5 Comparing Signed Number-Phrases

We investigate the *first* fundamental process involving *actions* and *states*: Given two *actions* or two *states* we would like to be able to *compare* the signed number-phrases that represent them.

However, there are actually *two* viewpoints from which to compare signed number-phrases.

1. From what we shall call the **algebraic viewpoint**, the comparison depends both on the *sign* and the *size* of the two signed number-phrases.
In the real-world, the comparison corresponds to the relationship is-smaller-than understood as is-poorer-than extended to the case when being in debt is allowed.
It is traditional to use the same *verbs* as with counting number-phrases and decimal number-phrases, that is: $<$, $>$, $=$, and $\leqq$, $\geqq$.

algebra-compare
algebra-more-than
algebra-less-than
is-left-of
is-right-of

a. There are two cases depending on the signs of the two signed number-phrases:

- When the signs of the two signed number-phrases are *the same*
  - any two positive number-phrases **algebra-compare** the same way as their *sizes* compare

    EXAMPLE 22.

    $$+365.75 \text{ Dollars} > +219.28 \text{ Dollars}$$

    because $365.75 > 219.28$.
  - any two negative number-phrases *algebra-compare* the way opposite to the way their *sizes* compare

    EXAMPLE 23.

    $$-432.69 \text{ Dollars} < -184.41 \text{ Dollars}$$

    because $432.69 > 184.41$.
- When the sign of the two signed number-phrases are *opposite*, we can say *either* that
  - any positive number-phrase is **algebra-more-than** any negative number-phrase

  or, *dually*, that
  - any negative number-phrase is **algebra-less-than** any positive number-phrase

    EXAMPLE 24.

    $$-2386.77 \text{ Dollars} < +17.871 \text{ Dollars}$$

    because any negative number-phrase is less-than any positive number-phrase.

b. In other words, when we *picture* on a ruler the signed number-phrases involved in an *algebraic comparison*, an algebraic comparison is about the relative positions of the two signed number-phrases relative to each other:

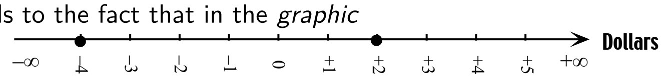
- *is-algebra-less-than* is pictured as **is-left-of**
- *is-algebra-more-than* is pictured as **is-right-of**

EXAMPLE 25.

- The algebra-comparison sentence

  $$-4 \text{ Dollars} < +2 \text{ Dollars}$$

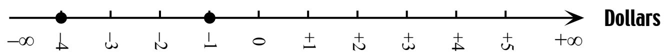
  corresponds to the fact that in the *graphic*

  the mark that represents -4 *is-left-of* the mark that represents $+2$
- The algebra-comparison sentence

  $$-1 \text{ Dollars} > -4 \text{ Dollars}$$

  corresponds to the fact that in the graphics



the mark that represents -1 *is-right-of* the mark that represents -4

size viewpoint
is-larger-in-size-than
is-smaller-in-size-than
is-farther-away-from-the-center

This illustrates the reason that we can reuse the same verbs with signed number-phrases as we did with counting number-phrases and decimal number-phrases.

2. From what we shall call the **size viewpoint**, the comparison depends only on the *size* of the two signed number-phrases and *not* on the *sign*.

a. It is quite usual in the real-world to say that a hundred dollar debt is larger than a fifty dollar debt even though someone owing a hundred dollars is-poorer-than a person owing fifty dollars.
So, we will say that:

- A first signed number-phrase **is-larger-in-size-than** a second signed number-phrase when the size of the first signed number-phrase is larger than the size of the second signed number-phrase.

or, dually, we can say

- A first signed number-phrase **is-smaller-in-size-than** a second signed number-phrase when the size of the first signed number-phrase is smaller than the size of the second signed number-phrase.

We shall not use *symbols* and we shall just write the words.

EXAMPLE 26. We have of course that

$$+365.75 \text{ Dollars is-larger-in-size-than } +219.28 \text{ Dollars}$$

which corresponds to the fact that 365.75, the size of the first signed number-phrase, is larger than 219.28, the size of the second signed number-phrase.
We also have that

$$-365.75 \text{ Dollars is-larger-in-size-than } -219.28 \text{ Dollars}$$

which corresponds to the fact that 365.75, the size of the first signed number-phrase, is larger than 219.28, the size of the second signed number-phrase.
And we also have that

$$-365.75 \text{ Dollars is-larger-in-size-than } +219.28 \text{ Dollars}$$

which corresponds to the fact that 365.75, the size of the first signed number-phrase, is larger than 219.28, the size of the second signed number-phrase.

None of this has anything to do with the fact that, from the *algebra viewpoint*,

$$+365.75 \text{ Dollars} > +219.28 \text{ Dollars}$$
$$-365.75 \text{ Dollars} < -219.28 \text{ Dollars}$$
$$-365.75 \text{ Dollars} < +219.28 \text{ Dollars}$$

b. In other words, when we *illustrate* on a ruler the signed number-phrases involved in a *size comparison*, the comparison is about which numerator **is-farther-away-from-the-center**.

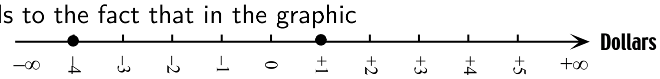
EXAMPLE 27.

follow up
merge

- The size-comparison sentence

  -4 **Dollars** is-larger-in-size-than $+1$ **Dollars**

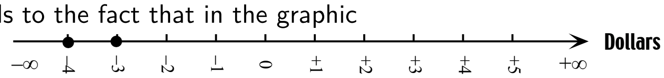
  corresponds to the fact that in the graphic



  the mark that represents -4 **Dollars** is *farther-away-from-the-center-than* the mark that represents $+1$ **Dollars**.
- The size-comparison sentence

  -4 **Dollars** is-larger-in-size-than -3 **Dollars**

  corresponds to the fact that in the graphic



  the mark that represents -4 *farther-away-from-the-center-than* the mark that represents -3

5.6 Adding a Signed Number-Phrase

We investigate the *second* fundamental process involving *actions* and *states*.

1. Just as in in the case of *collections* we could *attach* a *second* collection to a *first* collection, here we can

- **follow up** a *first* action with a *second* action.

  EXAMPLE 28.
  - a gambler may *win* forty-five dollars and then follow up with *winning* sixty-two dollars.
  - a gambler may *win* thirty-one dollars and then follow up with *losing* forty-four dollars.
  - a gambler may *lose* twenty-one dollars and then follow up with *winning* fifty-seven dollars.
  - a gambler may *lose* seventy-eight dollars and then follow up with *losing* thirty-four dollars.
- **merge** a *first* state with a *second* state

  EXAMPLE 29.
  - a business that is three thousand dollars *in the black* may merge with a business that is six hundred dollars *in the black*.
  - a business that is three hundred dollars *in the black* may merge with a business that is five hundred dollars *in the red*.
  - a business that is two thousand dollars *in the red* may merge with a business that is seven hundred dollars *in the black*.
  - a business that is seven hundred dollars *in the red* may merge with a business that is two hundred dollars *in the red*.

NOTE. English forces us to use a different word order here: while we attached a *second* collection *to* a *first* collection, here we must say that we follow up a *first* action *with* a *second* action. In order to be consistent, and although it is not necessary, we will also say that we merge a *first* state *with* a *second* state. adding $\oplus$

2. Then, just like *adding a counting-number-phrases* was the paper procedure to get the result of *attaching* a collection, **adding** a *signed number-phrase* will be the paper procedure to get the *result* of *following up* an action and/or *merging* a state.

In order to distinguish adding *signed* number-phrases from adding *counting* number-phrases as we develop the procedure, we shall use for a while the symbol $\oplus$. Later, we will just use $+$ and learn to rely on the *context*.

3. Just like, in Chapter 1, we introduced *counting* number-phrases with *slashes*, /, to discuss addition of *signed* number-phrases, we will use temporarily *arrows* of two kinds, $\leftarrow$ and $\rightarrow$.

EXAMPLE 30. We will use temporarily

$\rightarrow \rightarrow \rightarrow \rightarrow \rightarrow$ **Dollars** instead of $+5$ **Dollars**

and

$\leftarrow \leftarrow \leftarrow \leftarrow \leftarrow$ **Dollars** instead of -5 **Dollars**.

When adding a signed number-phrase, we must distinguish two cases.

a. The second signed number-phrase has the *same* sign as the first signed number-phrase. Then, all the items are of the *same kind* and so *following up* is the same as *attaching*. So, in that case, to get the *size* of the result, we add the *sizes* of the two signed number-phrases.

EXAMPLE 31.

In the *real-world*, when we:	We write on *paper*:
deposit five dollars	$\rightarrow \rightarrow \rightarrow \rightarrow \rightarrow$ **Dollars**
and then	$\oplus$
deposit three dollars,	$\rightarrow \rightarrow \rightarrow$ **Dollars**
altogether	$=$
this	$[\rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \oplus \rightarrow \rightarrow \rightarrow]$ **Dollars**
is the same as	$=$
when	$\rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow$ **Dollars**
we	$=$
deposit eight dollars	$+8$ **Dollars**

or

EXAMPLE 32.

In the *real-world*, when we	We write on *paper*:
withdraw five dollars	← ← ← ← ← **Dollars**
and then	⊕
withdraw three dollars,	← ← ← **Dollars**
altogether	=
this	[← ← ← ← ← ⊕ ← ← ←] **Dollars**
is the same as	=
when	← ← ← ← ← ← ← ← **Dollars**
we	=
withdraw eight dollars	−8 **Dollars**

b. The second signed number-phrase has the *opposite* sign from the first signed number-phrase. Then, the items are of the *same kind* and so *following up* is the same as *attaching*. So, in that case, to get the *size* of the result, we add the *sizes* of the two signed number-phrases.

EXAMPLE 33.

In the *real-world*, when we	We write on *paper*:
deposit three dollars	→ → → **Dollars**
and then	⊕
withdraw five dollars,	← ← ← ← ← **Dollars**
altogether	=
this	[→ → → ⊕ ← ← ← ← ←] **Dollars**
is	[→ → → ← ← ← ← ←] **Dollars**
the same	[→ → ⫻ ⫻ ← ← ← ←] **Dollars**
as	[→ ⫻ ⫻ ← ← ←] **Dollars**
when	[⫻ ⫻ ← ←] **Dollars**
we just	← ← **Dollars**
withdraw two dollars	−2 **Dollars**

or

EXAMPLE 34.

In the *real-world*, when we	We write on *paper*:
deposit three dollars	→ → → **Dollars**
and then	⊕
withdraw five dollars,	← ← ← ← ← **Dollars**
altogether	=
this	[→ → → ⊕ ← ← ← ← ←] **Dollars**
is	[→ → → ← ← ← ← ←] **Dollars**
the same	[→ → ⫻ ⫻ ← ← ← ←] **Dollars**
as	[→ ⫻ ⫻ ← ← ←] **Dollars**
when	[⫻ ⫻ ← ←] **Dollars**
we just	← ← **Dollars**
withdraw two dollars	−2 **Dollars**

THEOREM 1. *To add signed-numerators:*

- *When the two signed number-phrases have the* same *sign,*
  - *We get the* sign *of the result by taking the common sign*
  - *We get the* size *of the result by* adding *the two sizes.*
- *When the two signed number-phrase have* opposite *signs, we must first* compare *the* sizes *of the two signed number-phrases and then*
  - *We get the* sign *of the result by taking the sign of the signed number-phrase whose* size *is* larger*,*
  - *We get the* size *of the result by* subtracting *the smaller size from the larger size.*

EXAMPLE 35. To identify the specifying-phrase $(+3) \oplus (+5)$ and since $(+3)$ and $(+5)$ have the *same* sign, we *proceed* as follows:

- We get the *sign* of the result by taking the common sign which gives us $+$
- We get the *size* of the result by *adding* the sizes 3 and 8 which gives us 8

In symbols,

$$\begin{aligned}(+3) \oplus (+5) &= (+[3+5]) \\ &= (+8)\end{aligned}$$

EXAMPLE 36. To identify the specifying-phrase $(+3) \oplus (-5)$ and since $(+3)$ and (-5) have *opposite* signs, we must compare the *sizes*. Since $3 < 5$,

- We get the *sign* of the result by taking the sign of the number-phrase with the larger *size* which gives us $-$
- We get the *size* of the result by subtracting the smaller size, 3, from the larger size, 5 which gives us 2

In symbols,

$$\begin{aligned}(+3) \oplus (-5) &= (-[5-3]) \\ &= (-2)\end{aligned}$$

5.7 Subtracting a Signed Number-Phrase

We investigate the *third* fundamental process involving *actions* and *states*.

While, in the case of *collections*, *detaching* a collection made immediate sense as "un-attaching", in the case of actions "un-following up" and in the case of states "un-merging" do not make immediate sense. So, instead, we shall look at subtraction from the point of view of *correction* after we have done a long string of signed-additions and realized that there is an *incorrect*

$\ominus$
add the opposite

entry, that is a signed number-phrase that we shouldn't have added (for whatever reason), so that the *total* is *incorrect*.

1. Up front, things would seem to work out exactly as in the case of un-signed number-phrases.

EXAMPLE 37. Suppose that we work in a bank and that we had added transactions as the day went which gave us the following specifying phrase

-2 **Dollars** $\oplus$ -7 **Dollars** $\oplus$ $+5$ **Dollars** $\oplus$ ... $\oplus$ $+3$ **Dollars** and that at the end of day we identified the specifying-phrase which gave us

-132 **Dollars**

but that we then realized that -7 **Dollars** was an *outcast* (it was not for a *transaction* but for money involved in some other matter) with the consequence that -132 **Dollars** is *incorrect* in that it is not the sum total of the *transaction* for the day.

2. To get the correct total, we have the following two choices for the procedure:

- **Procedure A** would be to *strike out* the incorrect signed number-phrase and *redo* the entire addition:

  EXAMPLE 38. In the above example, we would *strike out* the *incorrect entry* -7 **Dollars**

  -2 **Dollars** $\oplus$ ~~-7 **Dollars**~~ $\oplus$ $+5$ **Dollars** $\oplus$ ... $\oplus$ $+3$ **Dollars**

  Of course, since Procedure A is going to involve a lot of unnecessary work redoing all that had been done correctly, it is very inefficient.

- **Procedure B** would be to *cancel out* the *effect* of the incorrect entry on the incorrect total by *subtracting* the incorrect entry from the incorrect total.

  EXAMPLE 39. In the above example, we would *subtract* the incorrect entry -7 **Dollars** from the incorrect total -132 **Dollars**

  -132 **Dollars** $\ominus$ -7 **Dollars**

  except that, at this point, we have no *procedure* for $\ominus$! Indeed, at this point, the only procedure we have for subtracting is for subtracting *unsigned* number-phrases.

  On the other hand, the obvious way to *cancel out* the *effect* of the incorrect entry on the incorrect total and that it is by **adding the opposite** of the incorrect entry to the incorrect total. (Accountants call this "entering an *adjustment*".)

  EXAMPLE 40. In the above example, we would *add the opposite* of the incorrect entry -7 **Dollars**, that is we would add -7 **Dollars** to the incorrect total -132 **Dollars**

  -132 **Dollars** $\oplus$ $+7$ **Dollars**

3. We now want to *see* that the two procedures *must* give us the same result either way. For that, we place the specifying-phrases in the two procedures side by side and we see that that the remaining number-phrases are the same either way. subtract

EXAMPLE 41. In the above example, we place the specifying-phrases in the two procedures side by side:

- The specifying-phrase in **Procedure A** is:

  -2 Dollars $\oplus$ ~~-7 Dollars~~ $\oplus$ $+5$ Dollars $\oplus$ ... $\oplus$ $+3$ Dollars

- The specifying-phrase in **Procedure B** is:

  -2 Dollars $\oplus$ ~~-7 Dollars~~ $\oplus$ $+5$ Dollars $\oplus$ ... $\oplus$ $+3$ Dollars $\oplus$ ~~$+7$ Dollars~~

We see that, either way, the remaining number-phrases are:

-2 Dollars $\oplus$ $+5$ Dollars $\oplus$ ... $\oplus$ $+3$ Dollars

4. Altogether then:

- *Adding the opposite* of the incorrect entry (**Procedure B**):

$$-132 \text{ Dollars} \quad \oplus \quad +7 \text{ Dollars}$$

necessarily amounts to exactly the same as

- *Striking out* the incorrect entry (**Procedure A**):

$$-132 \text{ Dollars} \quad \ominus \quad -7 \text{ Dollars}$$

Since **Procedure B** is much faster than **Procedure A**, we say that the procedure for **subtracting** a signed number-phrase will be to *add its opposite*.

EXAMPLE 42. In order to identify the specifying-phrase $(+3) \ominus (+5)$,

i. we identify instead the specifying-phrase $(+3) \oplus (-5)$

ii. we do the addition which gives us -2

EXAMPLE 43. In order to identify the specifying-phrase $(-3) \ominus (-5)$,

i. we identify instead the specifying-phrase $(-3) \oplus (+5)$

ii. we do the addition which gives us $+2$

EXAMPLE 44. In order to identify the specifying-phrase $(-3) \ominus (+5)$,

i. we identify instead the specifying-phrase $(-3) \oplus (-5)$

ii. we do the addition which gives us -8

EXAMPLE 45. In order to identify the specifying-phrase $(+3) \ominus (-5)$,

i. we identify instead the specifying-phrase $(+3) \oplus (+5)$

ii. we do the addition which gives us $+8$

5.8 Effect Of An Action On A State

We now look at the connection between *states* and *actions*.

initial state
final state
change
gain
loss

1. A *state* does not exist in isolation but is always one of many.

EXAMPLE 46. The *state* of an account is usually different on different days.

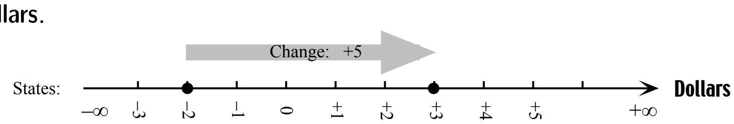
Given two states, we shall refer to the first one as the **initial state** and to the second one as the **final state**. The **change** from the *initial* state to the *final* state can be *up* in which case we shall call the change a **gain** or can be *down* in which case we shall call the change a **loss**.

On paper, we shall use $+$ for a *gain* and we shall use $-$ for a *loss*.

EXAMPLE 47.

- At the *beginning* of a month, Jill's account was two dollars in-the-red
- At the *end* of the month, Jill's account was three dollars in-the-black

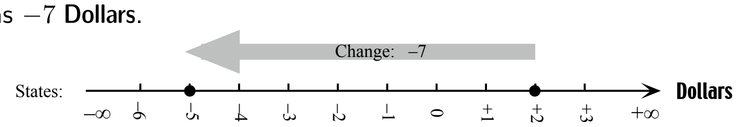
So, during that month Jill's account went *up* by five dollars and we shall write the *gain* as $+5$ **Dollars**.



EXAMPLE 48.

- At the *beginning* of a month, Jack's account was two dollars in-the-black
- At the *end* of the month, Jack's account was five dollars in-the-red

So, during that month Jack's account went *down* by seven dollars and we shall write the *loss* as -7 **Dollars**.



THEOREM 2. *Regardless of what the sign of the initial state and the sign of the final state are, we have that*

$$\textit{change} = \textit{final state} \ominus \textit{initial state}$$

EXAMPLE 49.

- At the *beginning* of a month, Jill's account was two dollars in-the-red
- At the *end* of the month, Jill's account was three dollars in-the-black

$$\begin{aligned}\text{change} &= +3 \textbf{ Dollars} \ominus -2 \textbf{ Dollars}\\ &= +3 \textbf{ Dollars} \oplus +2 \textbf{ Dollars}\\ &= +5 \textbf{ Dollars}\end{aligned}$$

EXAMPLE 50.

- At the beginning of a month, Jack's account was two dollars in-the-black
- At the end of the month, Jack's account was five dollars in-the-red

$$\begin{aligned}\text{change} &= -5 \textbf{ Dollars } \ominus +2 \textbf{ Dollars}\\ &= -5 \textbf{ Dollars } \oplus -2 \textbf{ Dollars}\\ &= -7 \textbf{ Dollars}\end{aligned}$$

2. A *change* always happens as the result of an *action*.

EXAMPLE 51. On an account,

- A *deposit* results in a *gain*,
- A *withdrawal* results in a *loss*.

In fact, we have exactly

$$\text{action} = \text{change}$$

so that, as a consequence of the previous THEOREM, *actions* and *states* are related as follows:

THEOREM 3 (Conservation Theorem)**.**

$$\textit{action} = \textit{final state} \ominus \textit{initial state}$$

EXAMPLE 52.

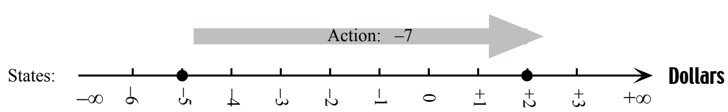
- On Monday, Jill's account was five dollars *in-the-red*,
- On Tuesday, Jill *deposits* seven dollars.

So, we have:

i.

$$\textbf{Action} = +7 \textbf{ Dollars}$$

ii.



So, on Wednesday, Jill's account is two dollars *in-the-black*

iii. Then we compute the change:

$$\begin{aligned}\textbf{Change} &= \textbf{Final State} \ominus \textbf{Initial State}\\ &= +2 \textbf{ Dollars} \ominus -5 \textbf{ Dollars}\\ &= +2 \textbf{ Dollars} \oplus +5 \textbf{ Dollars}\\ &= +7 \textbf{ Dollars}\end{aligned}$$

And we have indeed that

$$\text{action} = \text{final state} - \text{initial state}$$

What happened is that each state is the result of *all prior* actions. So, by subtracting the *initial* state from the final state, we eliminate the effect of all the actions that resulted in the *initial* state, that is the effect of all the actions except the effect of the last one, namely the seven dollars deposit.

5.9 From Plain To Positive

We now have two kinds of number-phrases: *plain* number-phrases and *signed* number-phrases. The two, though, overlap and we want to analyze the connections between the two and what is gained when we go from using *plain* number-phrases to using *signed* number-phrases.

1. We developed

- *plain* number-phrases in order to deal with collections of items that are all of *one* kind,
- *signed* number-phrases in order to deal with collections of items that are all of one kind or all of another kind—with items of different kinds canceling each other.

But then, given collections of items that are all of *one* kind, it often happens that we can eventually think of another kind of items that cancel the first kind of items.

EXAMPLE 53. We may start *counting* steps to find out *how much we walked*. But eventually, we may want to know *how far we progressed*, being that there are steps *backward* as well as step *forward* and, if it doesn't matter what kind of steps they are when it comes to *how much we walked*, it does matter very much when it comes to *how far we progressed* and so we need to keep track of the direction of the steps.

2. But then, we can represent the original collection of items in two ways:

- With a *plain* number-phrase
- With a *positive* number-phrase

EXAMPLE 54. Given a collection of seven steps (necessarily all in the same direction since all items in a collection have to be the same), we can represent the collection by:

- the plain number-phrase

$$7 \text{ Steps}$$

- or we can adopt that direction as *standard direction* and then represent the collection by the *positive* number-phrase

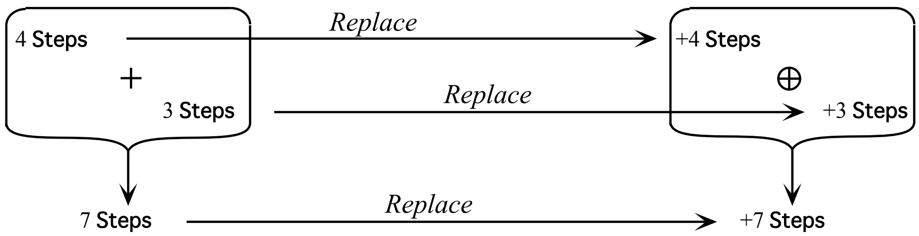
$$+7 \text{ Steps}$$

3. We now check that, when we do an addition, we can go either one of two routes:

- We can first *replace* the two *plain* number-phrases by *positive* number-phrases and then oplus the two *positive* number-phrases,
- We can add the two *plain* number-phrases and then *replace* the result of the addition by a *positive* number-phrase.

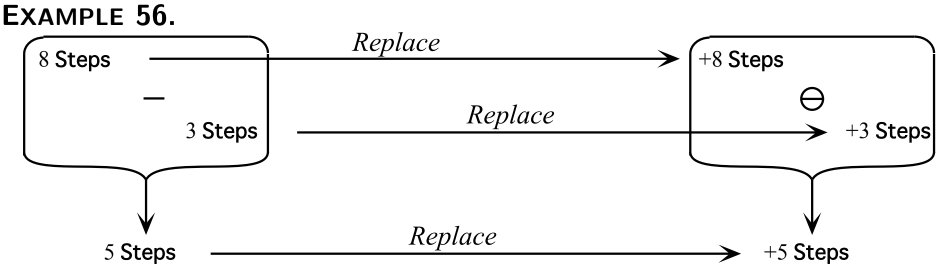
Both routes get us to the same result.

EXAMPLE 55.



This works also with *subtraction*.

EXAMPLE 56.



NOTE. The reader should check on her/his own that if, instead of *replacing plain* number-phrases by *positive* number-phrases, we were to *replace plain* number-phrases by *negative* number-phrases, then things would not always work in the sense that the two routes would not always result with the same number-phrase.

worth
unit-worth
value
unit-value

Chapter 6

Co-Multiplication and Values

.

We seldom deal with a collection without wanting to know what the (money?) **worth** of the collection is, that is how much money the collection could be exchanged for.

6.1 Co-Multiplication

Since all the items in a *collection* are the same, to find the *worth* of that collection, we need only know the **unit-worth** of the items, that is the amount of money that any one of these items can be exchanged for.

EXAMPLE 1. Given a collection of five apples, and given that the *unit-worth* of apples is seven cents, the real-world *process* for finding the *worth* of the collection is to exchange each apple for seven cents. Altogether, we end up exchanging the whole collection for thirty-five cents which is therefore the *worth* of the collection.

We now want to develop a paper *procedure* to get the number-phrase that represents the *worth* of the given collection, which we will call **value**, in terms of the number-phrase that represents the *unit-worth* of the items in the collection, which we will call **unit-value**.

1. We know how to write the number-phrase that represents the given *collection* and how to write its *value*, that is the number-phrase that represents its *worth*, but what is not obvious is how we should write the *unit-value* that is the number-phrase that represents the *unit-worth*.

EXAMPLE 2. In EXAMPLE 1, we represent the collection of five apples by writing the

co-denominator
co-multiplication

number-phrase 5 **Apples** and we represent its worth by writing its *value*, that is the number-phrase 35 **Cents**.

What is not obvious is how to write the *unit-value* of the **Apples**, that is the number-phrase that represents the *unit-worth* of the apples, that is the fact that "each apple is worth seven cents".

More specifically, we know what the *numerator* of the unit-value should be but what we don't know is how to write the *denominator* of the unit-value which we will call **co-denominator**.

Looking at the real-world shows that the *procedure* for finding the *value* must involve *multiplication* so that the *specifying-phrase* must look like:

$$\textit{Number-phrase} \text{ for collection} \times \textit{Unit-value} = \text{Number-phrase for money}$$

EXAMPLE 3. In EXAMPLE 2, the number-phrase that represents the *collection* is 5 **Apples** and the numerator of the unit-phrase that represents the unit-value of the items is 7 so the *specifying-phrase* must look like

$$5 \textbf{ Apples} \times 7\ ???$$

where ??? stands for the *co-denominator*.

2. The *co-denominator* should be such that the procedure for going from the specifying phrase to the result should prevent the denominator of the number-phrase for the *collection* from appearing in the *result* and, at the same time, be such as to force the denominator of the number-phrase for the *value* to appear in the *result*.

EXAMPLE 4. In EXAMPLE 3, since we must have

$$5 \textbf{ Apples} \times 7\ ??? = 35 \textbf{ Cents}$$

the *procedure* to go from the specifying phrase on the left, that is 5 **Apples** × 7 ???, to the result on the right, that is 35 **Cents**, must

- prevent **Apples** from appearing on the right
- but force **Cents** to appear on the right.

3. What we will do is to write the *co-denominator* just like a *fraction* with:

- the denominator of the *value above* the *bar*
- the denominator of the *items below* the *bar*.

EXAMPLE 5. In EXAMPLE 4, we write $\frac{\textbf{Cents}}{\textbf{Apple}}$ in place of ??? so that the specifying-phrase becomes

$$5 \textbf{ Apples} \times 7\ \frac{\textbf{Cents}}{\textbf{Apple}}$$

That way, the procedure for identifying such a specifying phrase, called **co-multiplication**, is quite simply stated:

i. multiply the *numerators*

ii. multiply the *denominators* with cancellation.

EXAMPLE 6. When we carry out the procedure on the specifying phrase in EXAMPLE

co-number-phrase
evaluate
percentage

5, we get

$$5 \textsf{ Apples} \times 7 \frac{\textsf{Cents}}{\textsf{Apple}} = (5 \times 7) \left(\cancel{\textsf{Apples}} \times \frac{\textsf{Cents}}{\cancel{\textsf{Apple}}} \right)$$
$$= 35 \textsf{ Cents}$$

which is what we needed to represent the real-world situation in EXAMPLE 1.

4. From now on, in order to remind ourselves that the reason why *unit-values* are written this way is to make it easy to *co-multiply*, we shall call them **co-number-phrases**[1].

Also, just as we often say "To *count* a collection" as a short for "To find the numerator of the number-phrase that represents a collection", we shall say "To **evaluate** a collection" as a short for "To find the numerator of the number-phrase that represents the *value* of a collection".

NOTE. Co-multiplication is at the heart of a part of mathematics called DIMENSIONAL ANALYSIS that is much used in sciences such as PHYSICS, MECHANICS, CHEMISTRY and ENGINEERING where people have to "cancel" denominators all the time.

EXAMPLE 7.

$$5 \textsf{ Hours} \times 7 \frac{\textsf{Miles}}{\textsf{Hour}} = (5 \times 7) \left(\cancel{\textsf{Hours}} \times \frac{\textsf{Miles}}{\cancel{\textsf{Hour}}} \right) = 35 \textsf{ Miles}$$

EXAMPLE 8.

$$5 \textsf{ Square-Inches} \times 7 \frac{\textsf{Pound}}{\textsf{Square-Inch}} = (5 \times 7) \left(\cancel{\textsf{Square-Inches}} \times \frac{\textsf{Pound}}{\cancel{\textsf{Square-Inch}}} \right) = 35 \textsf{ Pounds}$$

Co-multiplication is also central to a part of mathematics called LINEAR ALGEBRA that is in turn of major importance both in many other parts of mathematics and for all sort of applications in sciences such as ECONOMICS.

EXAMPLE 9.

$$5 \textsf{ Hours} \times 7 \frac{\textsf{Dollars}}{\textsf{Hour}} = (5 \times 7) \left(\cancel{\textsf{Hours}} \times \frac{\textsf{Dollar}}{\cancel{\textsf{Hour}}} \right) = 35 \textsf{ Dollars}$$

More modestly, *co-multiplication* also arises in **percentage** problems:

EXAMPLE 10.

$$5 \textsf{ Dollars} \times 7 \frac{\textsf{Cents}}{\textsf{Dollar}} = (5 \times 7) \left(\cancel{\textsf{Dollars}} \times \frac{\textsf{Cents}}{\cancel{\textsf{Dollar}}} \right) = 35 \textsf{ Cents}$$

[1]Educologists will of course have recognized number-phrases and co-number-phrases for the *vectors* and *co-vectors* that they are—albeit one-dimensional ones.

extend
signed co-number-phrase

6.2 Effect of Transactions on States: Signed Co-Multiplication

We now want to **extend** the concept of *co-multiplication* to *signed*-number-phrases in order to deal with *actions* and *states*.

1. We begin by looking at the real-world. As before, we want to investigate the *change* in a given state, *gain* or *loss*, that results from a given transaction, "in" or "out" as before but with *two-way collections* of "good" items or "bad" items.

Example 11. Consider a store where, for whatever reason best left to the reader's imagination, collections of apples can either get in or out of the store. Moreover, the collections are really two-way collections in that the apples can be either *good*—inasmuch as they will generate a sales profit—or *bad*—inasmuch as they will have to be disposed of at a cost.

2. We now look at the way we will represent things on paper.

a. To represent collections that can get *in* or *out*, we use *signed number-phrases* and we use a + sign for collections that get *in* and a − sign for *collections* that get *out*.

So, we will represent

- *collections* getting "in" by *positive* number-phrases,
- *collections* getting "out" by *negative* number-phrases,

Example 12. In the above example, we would represent

- a collection of three apples getting *in* the store by the number-phrase $+3$ **Apples**
- a collection of three apples getting *out* of the store by the number-phrase -3 **Apples**

b. To represent unit-values that can be *gains* or *losses*, we use **signed co-number-phrase** and we use a + sign to represent *gains* and a − sign to represent *losses*.

So, we will represent

- the unit-value of *"good"* items by *positive* co-number-phrases,
- the unit-value of *"bad"* items by *negative* co-number-phrases,

Example 13. In the above example, we would represent

- the unit-value of apples that will generate a sales *profit* of seven cents per apples by the co-number-phrase $+7\ \frac{\textsf{Cents}}{\textsf{Apple}}$
- the unit-value of apples that will generate a disposal *cost* of seven cents per apple by the co-number-phrase $-7\ \frac{\textsf{Cents}}{\textsf{Apple}}$

3. Looking at the *effect* that *transactions* (of two-way collections) can have on (money) *states*, that is at the fact that:

- A two-way collection of "good" items getting "in" makes for a "good" change.

- A two-way collection of "good" items getting "out" makes for a "bad" change. signed co-multiplication
- A two-way collection of "bad" items getting "in" makes for a "bad" change.
- A two-way collection of "bad" items getting "out" makes for a "good" change.

we can now write the procedure for **signed co-multiplication** for which we will use the symbol $\otimes$:

i. multiply the *denominators* (with cancellation).

ii. multiply the *numerators* according to the way gains and losses occur:

- $(+) \otimes (+)$ gives $(+)$

EXAMPLE 14.

Three apples get *in* the store. $+3$ **Apples**

The apples have a unit-value of seven cents-per-apple *gain*. $+7 \, \frac{\textbf{Cents}}{\textbf{Apple}}$

The specifying phrase is $[+3 \textbf{ Apples}] \otimes \left[+7 \, \frac{\textbf{Cents}}{\textbf{Apple}}\right]$

We co-multiply $[(+3) \otimes (+7)] \left[\cancel{\textbf{Apples}} \times \frac{\textbf{Cents}}{\cancel{\textbf{Apple}}}\right]$

We get a twenty-one cent *gain*. $= +21$ **Cents**

- $(+) \otimes (-)$ gives $(+)$

EXAMPLE 15.

Three apples get *in* the store. $+3$ **Apples**

The apples have a unit-value of seven cents-per-apple *loss*. $-7 \, \frac{\textbf{Cents}}{\textbf{Apple}}$

The specifying phrase is $[+3 \textbf{ Apples}] \otimes \left[-7 \, \frac{\textbf{Cents}}{\textbf{Apple}}\right]$

We co-multiply $[(+3) \otimes (-7)] \left[\cancel{\textbf{Apples}} \times \frac{\textbf{Cents}}{\cancel{\textbf{Apple}}}\right]$

We get a twenty-one cent *loss*. $= -21$ **Cents**

- $(-) \otimes (+)$ gives $(+)$

EXAMPLE 16.

Three apples get *out* of the store. -3 **Apples**

The apples have a unit-value of seven cents-per-apple *gain*. $+7 \, \frac{\textbf{Cents}}{\textbf{Apple}}$

The specifying phrase is $[-3 \textbf{ Apples}] \otimes \left[+7 \, \frac{\textbf{Cents}}{\textbf{Apple}}\right]$

We co-multiply $[(-3) \otimes (+7)] \left[\cancel{\textbf{Apples}} \times \frac{\textbf{Cents}}{\cancel{\textbf{Apple}}}\right]$

We get a twenty-one cent *loss*. $= -21$ **Cents**

- $(-) \otimes (-)$ gives $(+)$

EXAMPLE 17.

Three apples get *out* of the store. -3 **Apples**

The apples have a unit-value of seven cents-per-apple *loss*. $-7\ \frac{\text{Cents}}{\text{Apple}}$

The specifying phrase is $[-3\ \textbf{Apples}] \otimes \left[-7\ \frac{\text{Cents}}{\text{Apple}}\right]$

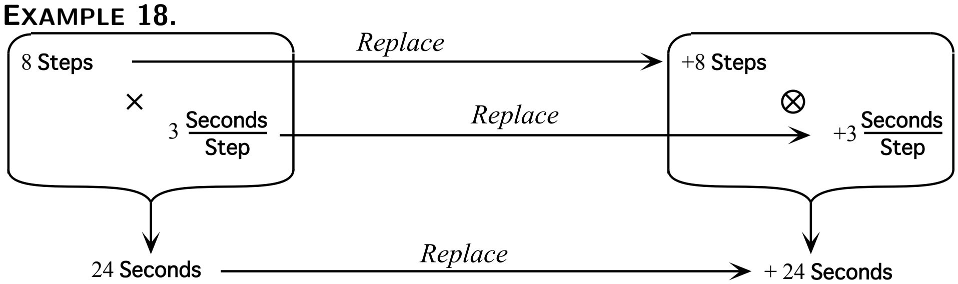
We co-multiply $[(-3) \otimes (-7)]\ \left[\cancel{\textbf{Apples}} \times \frac{\text{Cents}}{\cancel{\text{Apple}}}\right]$

We get a twenty-one cent *gain*. $= +21$ **Cents**

NOTE. The choice of symbols, $+$ to represent *good* and $-$ to represent *bad*, was not an arbitrary choice because of the way they interact with the symbols for *in* and *out*. We leave it as an exercise for the reader to investigate what happens when other choices are made.

4. Just as with *addition* and *subtraction*, in the case of *co-multiplication* too, we can replace *plain* number-phrases by *positive* number-phrases .

EXAMPLE 18.



Part II

Inequations & Equations Problems

select
requirement
meet
enter
noun
blank
form
instruction
nonsense

Chapter 7

Basic Problems 1:
(Counting Numerators)

In the *real world*, we often **select** collections on the basis of **requirements** that these collections must **meet**. After introducing some more mathematical language and discussing real-word situations, we will develop a *paper world* approach and introduce what will be our general procedure when dealing with such problems.

7.1 Forms, Data Sets And Solution Subsets

We begin by looking at the way we deal in ENGLISH with the selection of collections in the real world.

1. Essentially, what we use are "incomplete sentences" like those we encounter on certain exams or when we have to **enter** a **noun** in the **blanks** of a **form**.

EXAMPLE 1. The following

| | is a past President of the United States.

is a *form* in which the box is the *blank* in which we are supposed to *enter* a *noun*.

2. The **instruction** to enter some given *noun* in the *blank* of a *form* may result in:

- **nonsense**, that is words that say *nothing* about the real world.

sentence
TRUE
FALSE
data set
curly brackets
{ }
problem

EXAMPLE 2. The instruction to *enter* the *data*,

Mathematics

in the blank of the *form*

[ ] is a past President of the United States.

results in

[Mathematics] is a past President of the United States.

which is *nonsense*.

- a **sentence**, that is words that say *something* about the real world but that, like something we may write on a exam, can be **true** or **false**.

  EXAMPLE 3. Given the form

  [ ] is a past President of the United States.

  - The instruction to *enter* the *noun*,

    Jennifer Lopez

    in the blank of the *form* results in

    [Jennifer Lopez] is a past President of the United States.

    which is a sentence that (unfortunately) happens to be FALSE.
  - The instruction to enter the *noun*

    Bill Clinton

    in the blank of the *form* results in

    [Bill Clinton] is a past President of the United States.

    which is a sentence that happens to be TRUE.

3. In order to avoid having to deal with *nonsense*, that is in order to make sure that when we enter a *noun* we always get a *sentence*, regardless of whether that sentence turns out to be TRUE or FALSE, we will always have a **data set** from which to take the *nouns*.
We shall write the *data set* by writing the data within a pair of **curly brackets** { }

EXAMPLE 4. Given the form

[ ] is a past President of the United States.

the following could be a data set
{Bill Clinton, Ronald Reagan, Jennifer Lopez, John Kennedy, Henry Ford}
but the following could *not* be a *data set*
{Bill Clinton, Ronald Reagan, Jennifer Lopez, Mathematics, Henry Ford}

4. A **problem** will consist of a *form* together with a *data set*.

solution
non-solution
solution subset
select
set of *selectable* collections
require
gauge collection

EXAMPLE 5. The form

________ is a past President of the United States.

and the *data set*

{Bill Clinton, Ronald Reagan, Jennifer Lopez, John Kennedy, Henry Ford}

make up a *problem*.

a. Given a *problem*, that is given a *data set* and a *form*,

- a **solution** (of the given problem) is a *noun* such that, when we enter this noun into the blank of the form, the result in a sentence that is TRUE
- a **non-solution** (of the given problem) is a *noun* such that, when we enter this noun into the blank of the form, the result in a sentence that is FALSE

EXAMPLE 6. Given the *problem* consisting of

the form

________ is a past President of the United States.

and the *data set*

{Bill Clinton, Ronald Reagan, Jennifer Lopez, John Kennedy, Henry Ford}

- The *solutions* of the problem are

  Bill Clinton, Ronald Reagan, John Kennedy
- The *non-solutions* of the problem are

  Jennifer Lopez, Henry Ford

b. Given a *problem*, that is given a *data set* and a *form*, the **solution subset** for the *problem* consists of all the *solutions*.

We write a *solution subset* the same way as we write s *data set*, that is we write the *solutions* between *brackets* { }.

EXAMPLE 7. Given the *problem* consisting of

the form

________ is a past President of the United States.

and the *data set*

{Bill Clinton, Ronald Reagan, Jennifer Lopez, John Kennedy, Henry Ford} the *solution subset* of that problem is

{Bill Clinton, Ronald Reagan, John Kennedy}

7.2 Collections Meeting A Requirement

The simplest way to **select** collections from a given **set of *selectable* collections** is to **require** them to *compare* in a given way to a given **gauge collection** which we do by matching the collections one-to-one with the select subset

gauge collection. (See Chapter 2.) The result is what we will call the **select subset**.

Example 8. Jack has the following collection of one-dollar bills

So the bids that he can at all make in an auction (set of *selectable* collections) are:

If the *starting* bid (gauge collection) for a particular object is three dollars (a *selectable* collection), the bids that Jack could make (*select subset*) would then be:

1. The *gauge* collection may or may not be a *selectable* collection.

Example 9. Jack has the following collection of one-dollar bills

So the bids that he can at all make in an auction (set of *selectable* collections) are:

- If the *starting* bid for a particular object is three dollars (a *selectable* collection), then the bids that he could make (*select subset*) would be:

- If the *starting* bid for a particular object is three dollars and forty cents (*not* a *selectable* collection), then the bids that he could make (*select subset*) would be:

2. The way the selectable collections are required to compare with the gauge collection can be to be:

- *larger-in-size* than the *gauge collection*,

or

- *smaller-in-size* than the *gauge collection*,

or

- *same-in-size* as the *gauge collection*.

or

- *different-in-size* from the *gauge collection*,

empty
full

or

- *no-larger-in-size* than the *gauge collection*,

or

- *no-smaller-in-size* than the *gauge collection*,

Example 10. Jack has the following collection of one-dollar bills

So the bids that he can at all make in an auction correspond to the collections of one-dollar bills that he can use (set of *selectable* collections):

- If it is the *starting* bid for a particular object that is three dollars, then the bids that Jack could make (*select subset*) would be:

- If it is the *current* bid for a particular object that is three dollars, then the bids that Jack could make would be:

3. Occasionally, the subset of selected collections can be **empty** meaning that *none* of the selectable collections meets the given requirement.

Example 11. Jack has the following collection of one-dollar bills

So the bids that he can at all make in an auction (set of *selectable* collections) are:

If the *starting* bid (gauge collection) for a particular object is seven dollars, then Jack cannot make any bid so that the *select subset* is *empty*.

4. Occasionally, the subset of selected collections can be **full** meaning that *all* of the selectable collections meet the given requirement.

Example 12. Jack has the following collection of one-dollar bills

So the bids that he can at all make in an auction (set of *selectable* collections) are:

If the *starting* bid (gauge collection) for a particular object is one dollars, then the *select subset* is *full*.

unspecified-numerator
x
specifying-formula
formula
equation

There is of course nothing difficult with the one-to-one matching *process* involved in checking whether selectable collections compare or do not compare in a given way with a given gauge collection, but, as with most*real-world processes*, all this one-to-one matching of items is certainly going to get very quickly very tedious.

7.3 Basic Formulas

In order to *represent on paper* real-world the various situations involving the selection on the basis of a *requirement* of a *subset of selected collection* from among a *set of selectable collections* we will use:

- *Number-phrases* to represent the collections,
- The six *verbs* that were introduced in Chapter 2 to compare collections
$$>, <, =, \neq, \leqq, \geqq$$
- A special kind of *form* to represent the requirement.

1. The main difficulty with *forms* as we discussed them in Section 7.1 above is with the *blanks*. So we begin by introducing a kind of form that will be appropriate for "computations".

a. Instead of *blanks*, we will use an **unspecified-numerator** such as, for instance, the letter x.

EXAMPLE 13. Instead of writing

$$\boxed{} < 5$$

we will write

$$x < 5$$

b. A **specifying-formula** —we will often say **formula** for short—is a kind of forms in which:

- the verb can be any one of:
$$>, <, =, \neq, \leqq, \geqq$$
- the nouns are *numerators*
- the common denominator is *factored out*.

EXAMPLE 14. The following are *specifying-formulas*

$$x \leqq 8$$
$$x + 3 \geqq 8$$
$$3 \times x < 12$$
$$+3 \otimes x \oplus -7 = -12$$

We will distinguish between:

- **Equations**, that is specifying-formulas that involve the verb
$$=$$

- **Inequations**[1], that is specifying-formulas that involve any one of the other five verbs:

$$>, <, \neq, \leqq, \geqq$$

inequation
instruction
replace
code
vertical bar

c. Then, instead of giving the **instruction**

enter the *given* numerator in the *blank*.

we will give the *instruction*

replace the *unspecified* numerator x by the *given* numerator.

Example 15. Instead of giving the *instruction*

Enter 7 in the *blank* of the *form*:

$$\boxed{} < 5.$$

we will give the *instruction*

Replace x by 7 in the *formula*:

$$x < 5$$

d. While a *formula* is *not* a *sentence* because it does not say anything about the real world (how could it since all that x stands for is a *blank*!), once we have replaced in a formula the unspecified numerator x by a given numerator, we have of course a *sentence*. (That this sentence is going to be either TRUE or FALSE depending on the given numerator is beside the point here.)

Example 16. The *specifying-formula*

$$x < 5$$

is not a sentence because it does not sat anything about the real world since x does not stand for a given numerator.
The instruction to replace x by 7 in the *specifying-formula*

$$x < 5$$

results in

$$7 < 5$$

which is a *sentence*. (That it happens to be FALSE is beside the point here.)

e. What will complicate matters a bit is that we will often **code** the *instruction* to replace the unspecified numerator x by some given numerator into the specifying-formula itself. For that, we will

i. draw, to the right of the specifying formula a **vertical bar** extending a bit *below* the line, which we read as "where"

ii. write to the bottom right of the *vertical bar*:

– the unspecified numerator x

[1] Although supposedly exceedingly concerned with the relevance of mathematics to the "ordinary life" of their students—as opposed to their "school life" one can only suppose, but judging by the textbooks they produce in vast numbers, Educologists are strangely indifferent to the fact that, in the real world, *inequations* are vastly more prevalent than *equations*.

followed by
– the symbol :=, to be read as "is to be replaced by",
followed by
– the given *numerator*

Example 17. Instead of using the instruction

Replace x by 7 in the *specifying-formula*:

$$x < 5$$

we shall *write* the *instruction* right into the specifying formula as follows:

$$x < 5|_{x:=7}$$

and the result is to be read as:

$x < 5$ where x is to be replaced by 7.

The reason this complicates matters is that while

$$x < 5$$

is a *specifying-formula*,

$x < 5$ where x is to be replaced by 7.

is a *sentence* since it is the same as the sentence

$$7 < 5$$

f. In particular, we have that:

- replacing the unspecified numerator by a given numerator in an *inequation* results in an *inequality*.

Example 18.

	Using a *form*, we would write:	Using a *formula*, we will write:
"Before": Inequation	$\square > 3.14$ (neither TRUE nor FALSE)	$x > 3.14$ (neither TRUE nor FALSE)
"Action":	7.82 → $\square > 3.14$ Enter 7.82 in the blank	$x > 3.14|_{x:=7.82}$ Replace x by 7.82
"After": Inequality	$\boxed{7.82} > 3.14$ is TRUE	$7.82 > 3.14$ is TRUE

- replacing the unspecified numerator by a given numerator in an *equation* results in an *equality*.

Example 19.

formula, associated
associated equation
associated strict inequation

	Using a *form*, we would write:	Using a *formula*, we will write:
"Before": Equation	$\square = +5$ (neither TRUE nor FALSE)	$x = +5$ (neither TRUE nor FALSE)
"Action":	–3 → $\square = +5$ Enter -3 in the blank	$x = +5\|_{x:=-3}$ Replace x by -3
"After": Equality	$\boxed{-3} = +5$ is FALSE	$-3 = +5$ is FALSE

2. Given a formula, the **associated formulas** for that formula are the formulas that differ from the given formula only by the *verb*.

Crucial for the general procedure that we will develop in the next chapter and given an inequation regardless of whether this given inequation is *strict* or *lenient*, are:

- The **associated equation**, that is the *equation* we obtain by replacing the verb in the given inequation by the verb =.

  EXAMPLE 20. The equation associated with the *lenient* inequation
  $$-3 \otimes x \geqq +90.43$$
  is the equation
  $$-3 \otimes x = +90.43$$

  EXAMPLE 21. The equation associated with the *strict* inequation
  $$x \oplus -14.08 < +53.71$$
  is the equation
  $$x \oplus -14.08 = +53.71$$

- The **associated strict inequation**, that is the *inequation* we obtain by replacing the verb in the given inequation by the corresponding *strict* verb.

  EXAMPLE 22. Given the *lenient* inequation
  $$x + 6.08 \geqq 17.82$$
  the *associated strict inequation* is
  $$x + 6.08 > 17.82$$

  So, the *strict* inequation associated to a *strict* inequation is the *strict* inequation itself.

  EXAMPLE 23. Given the *strict* inequation
  $$x \ominus -6.08 < -44.78$$
  the *associated strict inequation* is
  $$x \ominus -6.08 < -44.78$$

  While certainly surprising, this will help us developing a general procedure in the next chapter.

basic formulas
unspecified numerator
gauge numerator
x_0
equation, basic
inequation, basic simple

In particular, we can say that a *lenient* inequation gives the choice between the associated *strict inequation* and the associated *equation*.

EXAMPLE 24. The *lenient inequation*

$$x \leqq +53.71$$

gives the choice between the associated *strict inequation*:

$$x < +53.71$$

and the associated *equation*

$$x = +53.71$$

For instance,

-61.05 is a solution of $x \leqq +53.71$ because -61.05 is a solution of $x < +53.71$

and

$+53.71$ is a solution of $x \leqq +53.71$ because $+53.71$ is a solution of $x = +53.71$

3. The simplest kind of specifying formula, which we will call **basic formulas**, are formulas involving two *nouns* related by a *verb* in the following manner:

i. The first *noun* is the **unspecified numerator** x,

ii. The *verb* is any of the verbs introduced in Chapter 2 to *compare* collections:

iii. The second *noun* is a given **gauge numerator**

EXAMPLE 25. The following specifying-phrases are basic formulas:

$$x < 5$$
$$x \geqq -3$$
$$x \neq -52.19$$

but the following specifying phrases are not basic formulas:

$$x + 3 \geqq 8$$
$$3 \times x < 12$$
$$3 \otimes x \oplus -7 = -12$$

In order to talk in general about basic formulas, we will use the symbol x_0 to stand for the *gauge numerator*.

4. We will sort *basic formulas* according to the kind of verb that is involved and we will distinguish four types of *basic formula* corresponding to the four types of *comparison sentences* that we encountered in Chapter 2.

- **Basic equations** are basic formulas of the type:

  $$x = x_0$$

  EXAMPLE 26. The formula

  $$x = 31.19$$

  is a *basic equation*

- **Basic simple inequations** are basic formulas of type:

  $$x \neq x_0$$

  EXAMPLE 27. The formula

  $$x \neq 742.05$$

inequation, basic strict
inequation, basic lenient
basic problem

is a *basic simple inequation*

- **Basic strict inequations** are basic formulas of type:

$$x > x_0 \qquad \text{or} \qquad x < x_0$$

EXAMPLE 28. The formulas

$$x > 132.17$$

and

$$x < -283.41$$

are both *basic strict inequations*

- **Basic lenient inequations** are basic formulas of type:

$$x \leqq x_0 \qquad \text{or} \qquad x \geqq x_0$$

EXAMPLE 29. The formulas

$$x \geqq 132.17$$

and

$$x \leqq +283.41$$

are both *basic lenient inequations*

5. A **basic problem** with thus be a *problem* in which

- the *data set* consists of number-phrases
- the *formula* is a basic formula
- the *common denominator* has been factored out and declared up-front.

EXAMPLE 30. Given the basic problem in **Dollars** where

- The *data set* is:

$$\{2, 3, 4, 5, 6, 7, 8\}$$

- The *formula* is:

$$x > 5$$

(where x is an unspecified numerator and 5 is the *gauge numerator*)
the *solution subset* is

$$\{6, 7, 8\}$$

7.4 Basic Problems

Given a *basic problem* involving *counting* number-phrases,

i. We *determine* the solution subset by replacing the unspecified numerator successively by each and every numerator in the data set. We then have *comparison sentences* that are TRUE or FALSE depending on

- which one of the six verbs is the *verb* in the formula.
- which way, *up* or *down* or *not at all*, we have to count from the numerator replacing the unspecified numerator to the given gauge numerator

(See Chapter 2.)

ii. We *represent* the solution subset:

graph
dot, solid
dot, hollow
name

- To **graph** the solution subset, we will use:
  - a **solid dot** to represent a *solution*: ●
  - a **hollow dot** to represent a *non-solution*: ○
- To **name** the solution subset, we will use, just as for *data sets*, two *curly brackets*, $\{\quad\}$, and write the solutions in-between the curly brackets.

1. Usually, a problem has both *non-solutions* and *solutions*.

EXAMPLE 31.

I. In the real world, Jack has the following collection of one-dollar bills

So the bids that he can at all make in an auction (set of *selectable* collections) are:

If the *starting* bid for a particular object is three dollars (a *selectable* collection), then the bids that he could make (*select subset*) would be:

II. On paper, we represent this by the following problem:

- We represent the *set of selectable collections* by the *data set*:

$$\{1, 2, 3, 4, 5\} \text{ Dollars}$$

- We represent the *requirement* that the bid must be no less than three dollars by the *formula*

$$x \geqq 3$$

III. To determine the solution subset we check each and every numerator in the data set. The verb $\geqq$ requires that, from the numerator that replaces the unspecified numerator to the gauge numerator, we must count *down* or must *not* count.

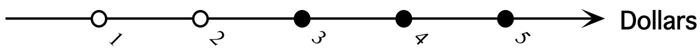
$x \geqq 3|_{x:=1}$ is FALSE because, from 1 to 3, we must count *up*
$x \geqq 3|_{x:=2}$ is FALSE because, from 2 to 3, we must count *up*
$x \geqq 3|_{x:=3}$ is TRUE because, from 3 to 3, we must *not* count
$x \geqq 3|_{x:=4}$ is TRUE because, from 4 to 3, we must count *down*
$x \geqq 3|_{x:=5}$ is TRUE because, from 5 to 3, we must count *down*

So:

1 is a non-solution
2 is a non-solution
3 is a solution
4 is a solution
5 is a solution

IV. We represent the solution subset

- The *graph* of the solution subset is: empty



- The *name* of the solution subset is:

$$\{3, 4, 5\} \text{ Dollars}$$

2. Occasionally, it can happen that there is no *solution* in which case we say that the solution subset is **empty**.

EXAMPLE 32.

I. In the real world, Jack has the following collection of one-dollar bills

So the bids that he can at all make in an auction (set of *selectable* collections) are:

If the *starting* bid for a particular object is seven dollars (a *selectable* collection), then he would not be able to make any bid (the *select subset* is empty):

II. On paper, we represent this by the following problem:

- We represent the *set of selectable collections* by the *data set*:

$$\{1, 2, 3, 4, 5\} \text{ Dollars}$$

- We represent the *requirement* that the bid must be no less than three dollars by the *formula*

$$x \geqq 7$$

III. To determine the solution subset we check each and every numerator in the data set. The verb $\geqq$ requires that, from the numerator that replaces the unspecified numerator to the gauge numerator, we must count *down* or must *not* count.

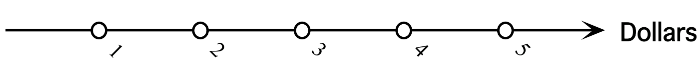
$x \geqq 7|_{x:=1}$ is FALSE because, from 1 to 7, we must count *up*
$x \geqq 7|_{x:=2}$ is FALSE because, from 2 to 7, we must count *up*
$x \geqq 7|_{x:=3}$ is FALSE because, from 3 to 7, we must count *up*
$x \geqq 7|_{x:=4}$ is FALSE because, from 4 to 7, we must count *up*
$x \geqq 7|_{x:=5}$ is FALSE because, from 5 to 7, we must count *up*

So:

1 is a non-solution
2 is a non-solution
3 is a non-solution
4 is a non-solution
5 is a non-solution

IV. We represent the solution subset

- The *graph* of the solution subset is:



full

- The *name* of the solution subset is:

$$\{\quad\}\text{ Dollars}$$

3. Occasionally, it can happen that there is no *non-solution* in which case we say that the solution subset is **full**.

EXAMPLE 33.

I. In the real world, Jack has the following collection of one-dollar bills

So the bids that he can at all make in an auction (set of *selectable* collections) are:

If the *starting* bid for a particular object is one dollars (a *selectable* collection), then he can any bid any selectable collection (the *select subset* is full):

II. On paper, we represent this by the following problem:

- We represent the *set of selectable collections* by the *data set*:

$$\{1, 2, 3, 4, 5\}\text{ Dollars}$$

- We represent the *requirement* that the bid must be no less than three dollars by the *formula*

$$x \geqq 1$$

III. To determine the solution subset we check each and every numerator in the data set. The verb $\geqq$ requires that, from the numerator that replaces the unspecified numerator to the gauge numerator, we must count *down* or must *not* count.

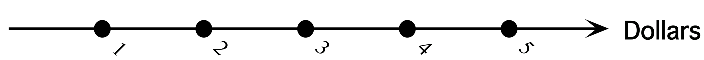
$x \geqq 1|_{x:=1}$ is TRUE because, from 1 to 1, we must *not* count
$x \geqq 1|_{x:=2}$ is TRUE because, from 2 to 1, we must count *down*
$x \geqq 1|_{x:=3}$ is TRUE because, from 3 to 1, we must count *down*
$x \geqq 1|_{x:=4}$ is TRUE because, from 4 to 1, we must count *down*
$x \geqq 1|_{x:=5}$ is TRUE because, from 5 to 1, we must count *down*

So:

1 is a solution
2 is a solution
3 is a solution
4 is a solution
5 is a solution

IV. We represent the solution subset

- The *graph* of the solution subset is:



- The *name* of the solution subset is:

$$\{1, 2, 3, 4, 5\}\text{ Dollars}$$

4. When the *data set* is **infinite**, we cannot check every numerator in the data set and we must make the case that beyond a certain numerator, the numerators are all solutions or all non-solutions.

infinite

$\cdots$

EXAMPLE 34.

I. On paper, we represent such a situation by the following problem:

- We represent the *set of selectable collections* by the *data set*:

  $$\{1, 2, 3, 4, 5, \ \dots\} \text{ Dollars}$$

  where $\dots$ is read "and so on".
- We represent the *requirement* that the bid must be no less than three dollars by the *formula*

  $$x \geqq 3$$

II. To determine the solution subset we are supposed to check each and every numerator in the data set. The verb $\geqq$ requires that, from the numerator that replaces the unspecified numerator to the gauge numerator, we must count *down* or must *not* count.

i. We start by checking each and every numerator in the data set until we pass the gauge numerator 3:

$x \geqq 3|_{x:=1}$ is FALSE because, from 1 to 3, we must count *up*

$x \geqq 3|_{x:=2}$ is FALSE because, from 2 to 3, we must count *up*

$x \geqq 3|_{x:=3}$ is TRUE because, from 3 to 3, we must *not* count

$x \geqq 3|_{x:=4}$ is TRUE because, from 4 to 3, we must count *down*

So:

1 is a non-solution

2 is a non-solution

3 is a solution

4 is a solution

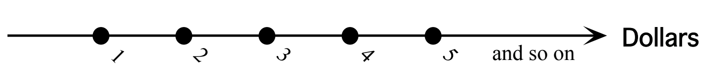
ii. We now make the case that any numerator beyond 4, that is $5, 6, 7, \ \dots$, is a solution:

- Since, from any numerator beyond 4, that is $5, 6, 7, \ \dots$, to 4, we must count *down*,
- And since, from 4 to the gauge 3, we must count *down*,
- It follows that from any numerator beyond 4, that is $5, 6, 7, \ \dots$, to the gauge 3, we must count *down*.

So, any numerator beyond 4, that is $5, 6, 7, \ \dots$ is also going to be a *solution*.

III. We represent the solution subset

- The *graph* of the solution subset is:



  where we actually write "and so on" because ... would run the risk of not being seen.
- The *name* of the solution subset is:

  $$\{1, 2, 3, 4, 5, \ \dots\} \text{ Dollars}$$

where we use . . . to mean “and so on”.

5. When the *data set* involves *signed* numerators, we proceed essentially in the same manner as with *plain* numerators.

EXAMPLE 35.

I. On paper, we represent such a situation by the following problem:

- We represent the *set of selectable collections* by the *data set*:

  $$\{-5, -4, -3, -2, -1, 0, +1, +2, +3, +4, +5, \dots\}$$ **Dollars**

  where . . . is read “and so on”.
- We represent the *requirement* that the balance must be more than a three dollar debt by the *formula*

  $$x > -3$$

II. i. We start by checking each and every numerator in the data set until we pass the gauge numerator 3:

$x \geqq -3|_{x:=-5}$ is FALSE because, from -5 to -3, we must count *up*

$x \geqq -3|_{x:=-4}$ is FALSE because, from -4 to -3, we must count *up*

$x \geqq -3|_{x:=-3}$ is TRUE because, from -3 to -3, we must *not* count

$x \geqq -3|_{x:=-2}$ is TRUE because, from -2 to -3, we must count *down*

So:

-5 is a non-solution

-4 is a non-solution

-3 is a solution

-2 is a solution

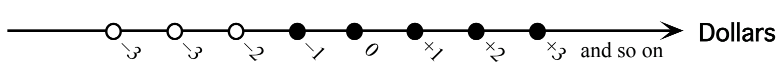
ii. We now make the case that any numerator beyond -2, that is $-1, 0, +1, +2, \dots$ is a solution:

- Since, from any numerator beyond -2, that is $-1, 0, +1, +2, \dots$, to -2, we must count *down*,
- And since, from -2 to the gauge -3, we must count *down*,
- It follows that from any numerator beyond -2, that is $-1, 0, +1, +2, \dots$ to the gauge -3, we must count *down*.

So, any numerator beyond -2, that is $-1, 0, +1, +2, \dots$, is also going to be a *solution*.

III. We represent the solution subset

- The *graph* of the solution subset is:



  where we actually write “and so on” because . . . would run the risk of not being seen.
- The *name* of the solution subset is:

  $$\{-3, -2, -1, 0, +1, +2, +3, \dots\}$$ **Dollars**

  where we use . . . to mean “and so on”.

Chapter 8

Basic Problems 2:

Decimal Numerators

We continue our investigation of BASIC PROBLEMS in the case when the numerators are *decimal* numerators rather than *counting* numerators as was the case in the previous chapter.

The reason we are investigating the case of *decimal* numerators separately is that we cannot compare *decimal* numerators just by *counting up* or *counting down* as we did in the previous chapter where the numerators were *counting* numerators. While there is of course a procedure for comparing *decimal* numerators, we will not use it here for two reasons:

- We have not discussed in this book the comparison procedures for *decimal* numerators since, for reasons of space and time, we have had to take *decimal* numerators for granted,
- As it happens, we will not need to use any *comparison procedure* because we will introduce a *general procedure* that is extremely powerful in that it will allow us to investigate not only BASIC PROBLEMS in the case when the numerators are *decimal* but also many other types of problems.

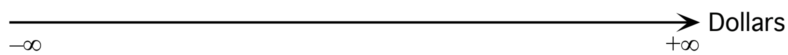
So, this chapter is turned towards the chapters to follow for which it is in fact a direct preparation as well as a foundation.

Finally, we shall use

DEFAULT RULE # 3. *When no data set is* declared*, it will go without saying that the data set consists of* all *signed decimal numerators.*

But, of course, in order to make sense in terms of the real world, we will still have to *declare* the *denominator*.

Also, to graph solution subsets, we will use rulers that have no *tick-mark* other than the ones directly relevant to the problem at hand but will have the symbol for *minus infinity*, $-\infty$, and the symbol for *plus infinity*, $+\infty$, at the corresponding ends of the ruler:



8.1 Basic Equation Problems

When a problem involves an *equation* with *decimal* number-phrases, things remain pretty much the same as with *counting* number-phrases because *equations* usually do not have many solutions.

In the present case of a *basic* equation,

i. We *determine* the solution subset from the fact that the one and only one solution is the *gauge* numerator.

ii. We *represent* the *solution subset* just as in the case of counting numerators, namely:

- To *graph* the solution subset, we will use:
  - a *solid dot* to represent a *solution*: ●
  - Since, here, there is no reason to consider any numerator aside from the *gauge*, there is no *non-solution* and so no need for *hollow* dots.
- To *name* the solution subset, just as for *data sets*, we will use two *curly brackets*,{ }, and write the solution in-between the curly brackets.

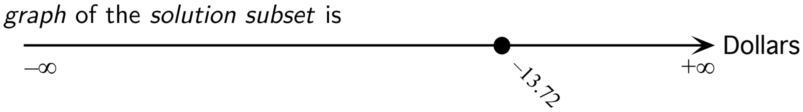
EXAMPLE 1. Given the problem in **Dollars** in which

- the *data set* consists of all signed decimal numerators
- the formula is the *basic equation*

$$x = -13.72$$

we proceed as follows:

- The only solution is -13.72
- The *graph* of the *solution subset* is



- The *name* of the *solution subset* is

$$\{-13.72\}\ \textbf{Dollars}$$

8.2 Basic Inequation Problems

boundary
interior
boundary point

In the case of an *inequation*, though, things are very different with *decimal* numerators from what they were with *counting* numerators because *inequations* can have too many solutions for us to handle them individually and we will develop and use a general procedure which we will call PASCH PROCEDURE.

1. Roughly, to determine the solution subset of a given inequation problem with *decimal* numerators, we will proceed in two stages:

I. We will locate the **boundary** of its solution subset, that is the solution subset of the associated *equation* problem.

II. We will locate the **interior** of its solution subset, that is the solution subset of the associated *strict inequation* problem.

EXAMPLE 2. Given the problem in **Dollars** in which

- the *data set* consists of all signed decimal numerators
- the *formula* is the *lenient inequation*

$$x \geqq -13.72$$

we will locate separately:

i. the *boundary* of the solution subset, that is the solution subset of the associated *equation*

$$x = -13.72$$

ii. the *interior* of the solution subset, that is the solution subset of the associated *strict inequation*

$$x > -13.72$$

As already noted in the previous chapter, when the problem involves a *strict* inequation in the first place, this would appear rather senseless but, in fact, it is precisely by distinguishing the *boundary* from the *interior* that we will be able us to develop a general procedure.

EXAMPLE 3. Given the problem in **Dollars** in which

- the *data set* consists of all signed decimal numerators
- the *formula* is the *basic inequation*

$$x < -55.06$$

we will locate separately:

i. the *boundary* of the solution subset, that is the solution subset of the associated *equation*

$$x = -55.06$$

ii. the *interior* of the solution subset, that is the solution subset of the associated *strict inequation*

$$x < -55.06$$

2. More precisely, in the case of a given *basic* inequation problem,

I. We locate the *boundary* as follows:

i. There is only one **boundary point** namely the *gauge*.

section
half-line
ray, solid

ii. The *boundary point*, though, may be a solution or a non-solution of the given *inequation* problem and we must check which it is:

- If the basic inequation is *strict*, then the boundary point is a *non-solution* and is therefore *non-included* in the solution set.
- If the basic inequation is *lenient*, then the boundary point is a *solution* and is therefore *included* in the solution set.

II. We locate the *interior* as follows:

i. The *boundary point* separates the *data set* in two **sections**, Section A and Section B.

ii. We pick a **test numerator** in each of Section A and Section B and we check if the test point is a solution or a non-solution of the given inequation.

iii. We conclude with the help of

THEOREM 4 (PASCH THEOREM)**.**

- *If the* test numerator *in a section is a* solution*, then* all *numerators in that same section are* included *in the solution subset.*
- *If the* test numerator *in a section is a* non-solution*, then* all *numerators in that same section are* non-included *in the solution subset.*

NOTE. Why the PASCH THEOREM should be the case requires of course an explanation as, *up front*, there is no obvious reason why this should be so. However, while the explanation is certainly not difficult and in fact rather interesting, it has been relegated to the supplementary text for the sake of saving time.

3. The solution subset of a basic inequation problem with decimal numerators is called a **half-line**. In order to represent a *half-line*,

i. We *graph* the *half-line* as follows:

i. We graph the *boundary* of the *half-line* exactly the same way as we graphed *counting* number-phrases that is we use

- a *solid dot* to graph a boundary point that is a *solution* (is *included* in the *half-line*): ●
- a *hollow dot* to graph a boundary point that is a *non-solution* (is *non-included* in the *half-line*): ○

ii. • We graph the sections of the data set that make up *interior* of the *half-line* with a **solid ray**

because this is what we would get if we were to draw a whole lot of *solid dots* right next to each other to graph all the *decimal* numerators that are solutions:

ray, hollow
square bracket
round parenthesis
infinity

- We graph the sections of the data set that are not in the *interior* of the *half-line* with a **hollow ray**

because this is what we would get if we were to draw a whole lot of *hollow dots* right next to each other to graph all the *decimal* numerators that are non-solutions:

NOTE. Once done investigating a problem, though, it is customary only to indicate the solution subset. In other words, it is customary to use the following

DEFAULT RULE # 4. *It goes without saying that those parts of the data set that are* not marked as being included *in the solution subset are in fact* non-included *in the solution subset.*

ii. To *name* the *half-line*

i. We *name* the *boundary* by writing on the side of the boundary point:

- a **square bracket** when the inequation is *lenient* (when the verb *does* involve the symbol $=$, that is when the verb is either $\leqq$ or $\geqq$).
- a **round parenthesis** when the inequation is *strict* (when the verb does *not* involve the symbol $=$, that is when the verb is either $<$ or $>$).

and by writing a *round parenthesis* on the other side.

ii. We *name* the *interior* by separating the boundary point x_0 by a comma from a symbol for **infinity** depending on the *verb*:

- $x_0, +\infty$ when the inequation is $x > x_0$ or $x \geqq x_0$
- $-\infty, x_0$ when the inequation is $x < x_0$ or $x \leqq x_0$

iii. Altogether then, the name of a half-line will be one of the following:

- When the inequation is *strict*:

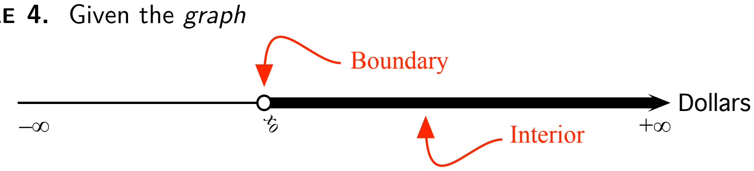
$$(x_0, +\infty) \quad \text{or} \quad (-\infty, x_0)$$

- When the inequation is *lenient*:

$$[x_0, +\infty) \quad \text{or} \quad (-\infty, x_0]$$

NOTE. One advantage of marking only those parts of the data set that are *included* in the solution subset is that the *graph* and the *name* then correspond exactly.

EXAMPLE 4. Given the *graph*



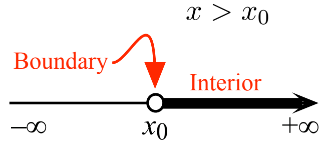
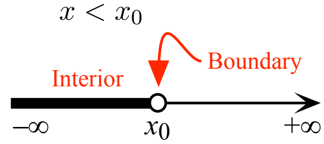
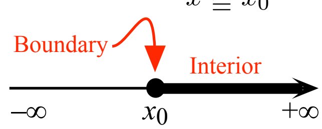
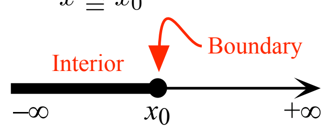
the corresponding *name* is

$$(x_0, +\infty)$$

kind (of half-line)

8.3 The Four Basic Inequation Problems

There are four **kinds** of basic inequation problems and they correspond to four different *kinds* of half-line:

verb ...	... involves >	... involves <
... is *strict*, (does *not* involve =)	$x > x_0$  $(x_0, +\infty)$	$x < x_0$  $(-\infty, x_0)$
... is *lenient* (involves =)	$x \geqq x_0$  $[x_0, +\infty)$	$x \leqq x_0$  $(-\infty, x_0]$

Note. Recall that it goes without saying that those parts of the data set that are *not marked as being included* in the solution subset are in fact *non-included* in the solution subset.

We now look at an example of each one of these four *kinds* of basic inequation problems.

1. Basic *strict* inequations of the kind $x > x_0$

Example 5. Given the basic inequation problem in **Dollars** in which

- the data set consists of all possible signed decimal numbers of **Dollars**.
- the formula is

$$x > +37.42$$

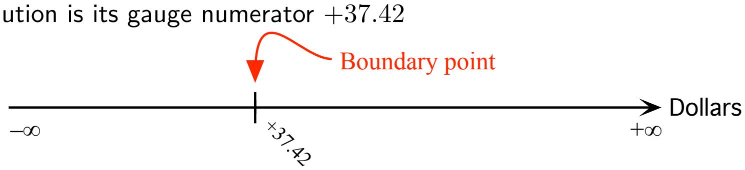
we proceed as follows:

I. We determine the *boundary* of the solution subset:

i. To *locate* the boundary point we use the associated equation

$$x = +37.42$$

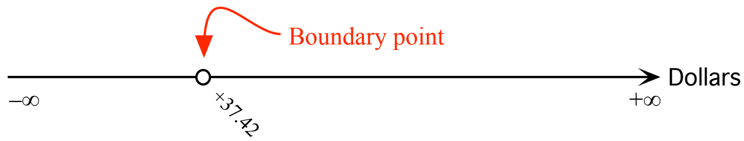
whose solution is its gauge numerator $+37.42$



ii. We check whether the boundary point $+37.42$ is *included* or *non-included* in the solution subset. Since the inequation is *strict*,

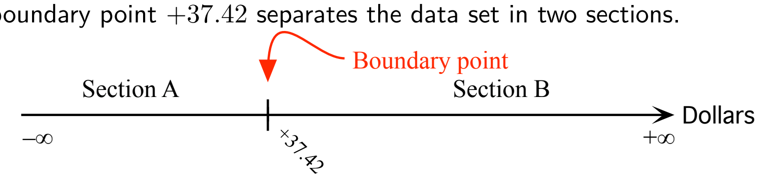
$$x > +37.42|_{x:=+37.42} \text{ is FALSE}$$

we get that the boundary point $+37.42$ is non-included in the solution subset and we graph it with a *hollow* dot:



II. We determine the *interior* of the solution subset:

i. The boundary point $+37.42$ separates the data set in two sections.



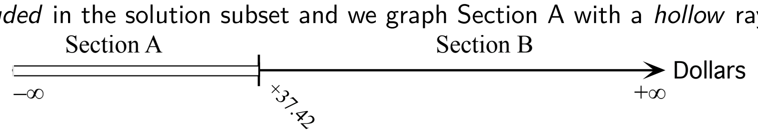
ii. We test Section A with, for instance, -100, and since

$$x > +37.42|_{x:=-100} \text{ is FALSE}$$

we get that -100 is a non-solution of the inequation

$$x > +37.42$$

and the PASCH THEOREM then tells us that all the numerators in Section A are *non-included* in the solution subset and we graph Section A with a *hollow* ray:



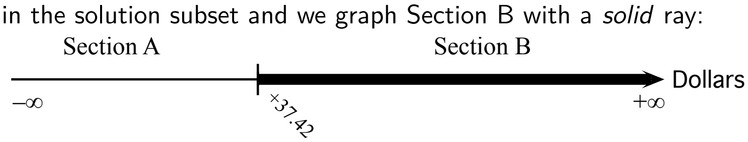
iii. We test Section B with, for instance, $+100$, and since

$$x > +37.42|_{x:=+100} \text{ is TRUE}$$

we get that $+100$ is a solution of the inequation

$$x > +37.42$$

and the PASCH THEOREM then tells us that all the numerators in Section B are *included* in the solution subset and we graph Section B with a *solid* ray:

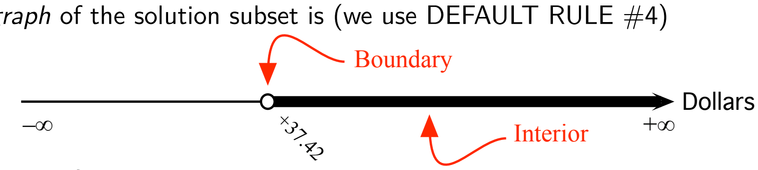


III. Altogether, we represent the solution subset of the inequation problem in **Dollars**

$$x > +37.42$$

as follows:

- The *graph* of the solution subset is (we use DEFAULT RULE #4)



- The *name* of the solution subset is

$$(+37.42, +\infty) \text{ Dollars}$$

2. Basic *strict* inequations of the kind $x < x_0$

EXAMPLE 6. Given the basic inequation problem in **Dollars** in which

- the data set consists of all possible signed decimal numbers of **Dollars**.
- the formula is

$$x < -153.86$$

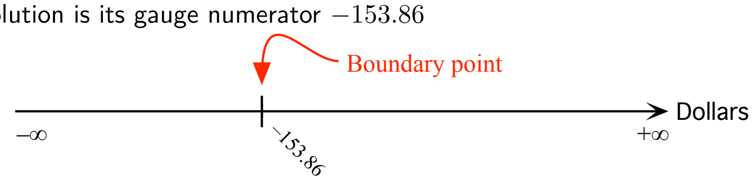
we proceed as follows:

I. We determine the *boundary* of the solution subset:

i. To *locate* the boundary point we use the associated equation

$$x = -153.86$$

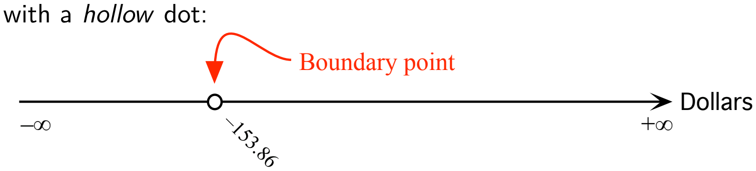
whose solution is its gauge numerator -153.86



ii. We check whether the boundary point -153.86 is *included* or *non-included* in the solution subset. Since the inequation is *strict*,

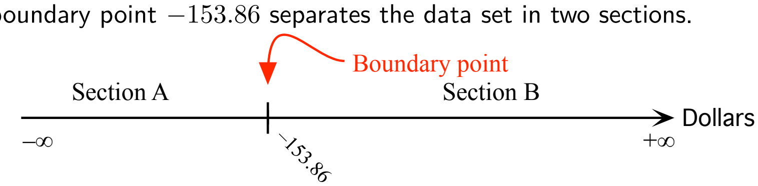
$$x < -153.86|_{x:=-153.86} \text{ is FALSE}$$

we get that the boundary point -153.86 is non-included in the solution subset and we graph it with a *hollow* dot:



II. We determine the *interior* of the solution subset:

i. The boundary point -153.86 separates the data set in two sections.



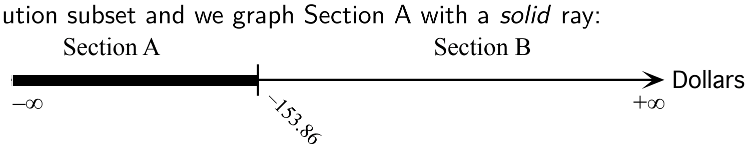
ii. We test Section A with, for instance, -1000, and since

$$x < -153.86|_{x:=-1000} \text{ is TRUE}$$

we get that -1000 is a solution of the inequation

$$x < -153.86$$

and the PASCH THEOREM then tells us that all numerators in Section A are *included* in the solution subset and we graph Section A with a *solid* ray:



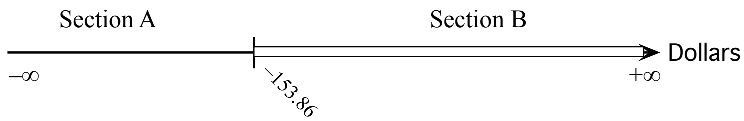
iii. We test Section B with, for instance, $+1000$, and since

$$x < -153.86|_{x:=+1000} \text{ is FALSE}$$

we get that $+1000$ is a non-solution of the inequation

$$x < -153.86$$

and the PASCH THEOREM then tells us that all numerators in Section B are *non-included* in the solution subset and we graph Section B with a *hollow* ray:

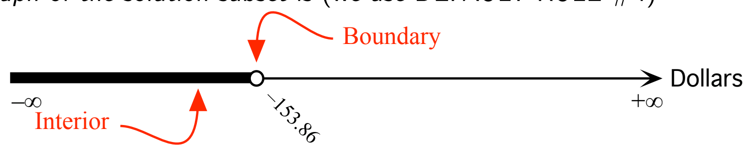


III. Altogether, we represent the solution subset of the inequation problem in **Dollars**

$$x < -153.86$$

as follows:

- The *graph* of the solution subset is (we use DEFAULT RULE #4)



- The *name* of the solution subset is

$$(-\infty, -153.86)\ \textbf{Dollars}$$

3. Basic *lenient* inequations of the kind $x \geqq x_0$

EXAMPLE 7. Given the basic inequation problem in **Dollars** in which

- the data set consists of all possible signed decimal numbers of **Dollars**.
- the formula is

$$x \geqq -93.78$$

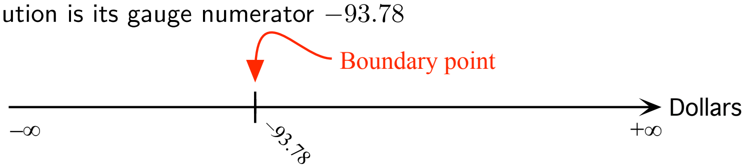
we proceed as follows:

I. We determine the *boundary* of the solution subset:

i. To *locate* the boundary point we use the associated equation

$$x = -93.78$$

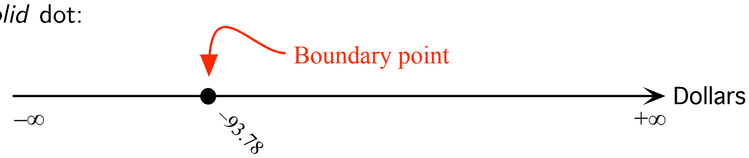
whose solution is its gauge numerator -93.78



ii. We check whether the boundary point -93.78 is *included* or *non-included* in the solution subset. Since the inequation is *lenient*,

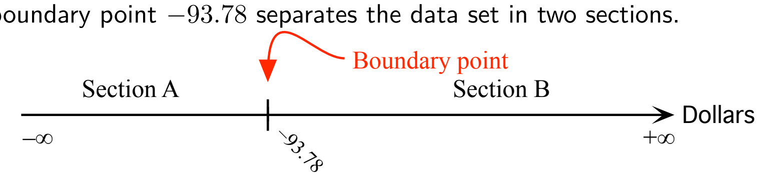
$$x \geqq -93.78|_{x:=-93.78} \text{ is TRUE}$$

so that the boundary point -93.78 is *included* in the solution subset and we graph it with a *solid* dot:



II. We determine the *interior* of the solution subset:

i. The boundary point -93.78 separates the data set in two sections.



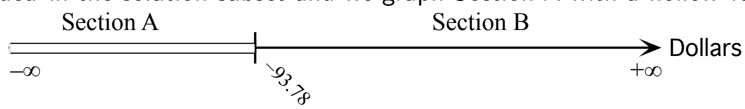
ii. We test Section A with, for instance, -1000, and since

$$x \geqq -93.78|_{x:=-1000} \text{ is FALSE}$$

we get that -1000 is a non-solution of the inequation

$$x \geqq -93.78$$

and the PASCH THEOREM then tells us that all the numerators in Section A are *non-included* in the solution subset and we graph Section A with a *hollow* ray:



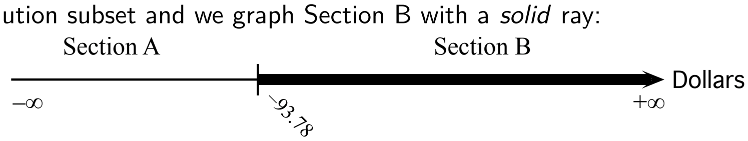
iii. We test Section B with, for instance, $+1000$, and since

$$x \geqq -93.78|_{x:=+1000} \text{ is TRUE}$$

we get that $+1000$ is a solution of the inequation

$$x \geqq -93.78$$

and the PASCH THEOREM then tells us that all numerators in Section B are *included* in the solution subset and we graph Section B with a *solid* ray:

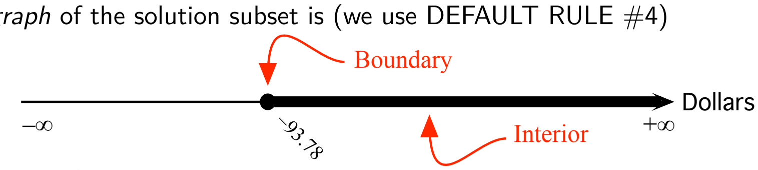


III. Altogether, we represent the solution subset of the inequation problem in **Dollars**

$$x \geqq -93.78$$

as follows:

- The *graph* of the solution subset is (we use DEFAULT RULE #4)



- The *name* of the solution subset is

$$[-93.78, +\infty) \text{ Dollars}$$

4. Basic *lenient* inequations of the kind $x \leqq x_0$

EXAMPLE 8. Given the basic inequation problem in **Dollars** in which

- the data set consists of all possible signed decimal numbers of **Dollars**.
- the formula is

$$x \leqq -358.13$$

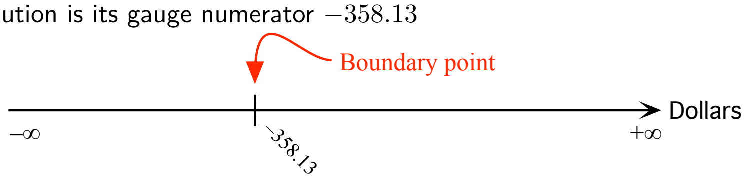
we proceed as follows:

I. We determine the *boundary* of the solution subset:

i. To *locate* the boundary point we use the associated equation

$$x = -358.13$$

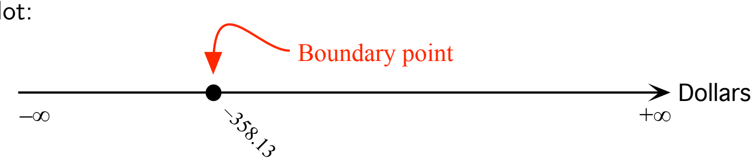
whose solution is its gauge numerator -358.13



ii. We check whether the boundary point -358.13 is *included* or *non-included* in the solution subset. Since the inequation is *strict*,

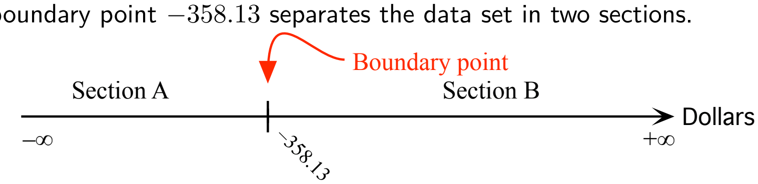
$$x \leqq -358.13|_{x:=-358.13} \text{ is FALSE}$$

and the boundary point -358.13 is included in the solution subset and we graph it with a *solid* dot:



II. We determine the *interior* of the solution subset:

i. The boundary point -358.13 separates the data set in two sections.



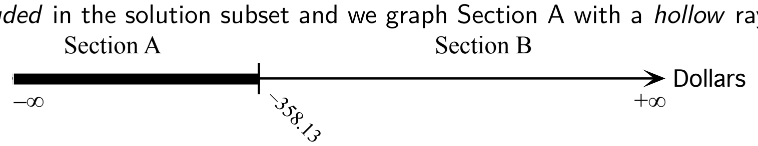
ii. We test Section A with, for instance, -100, and since

$$x \leqq -358.13|_{x:=-100} \text{ is FALSE}$$

we get that -100 is a non-solution of the inequation

$$x \leqq -358.13$$

and the PASCH THEOREM then tells us that all the numerators in Section A are *non-included* in the solution subset and we graph Section A with a *hollow* ray:



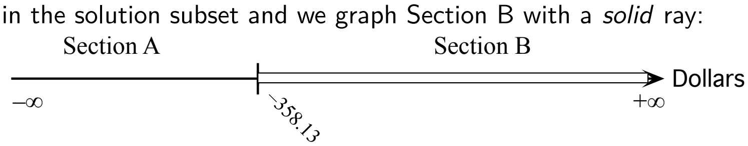
iii. We test Section B with, for instance, $+100$, and since

$$x \leqq -358.13|_{x:=+100} \text{ is TRUE}$$

we get that $+100$ is a solution of the inequation

$$x \leqq -358.13$$

and the PASCH THEOREM then tells us that all the numerators in Section B are *included* in the solution subset and we graph Section B with a *solid* ray:

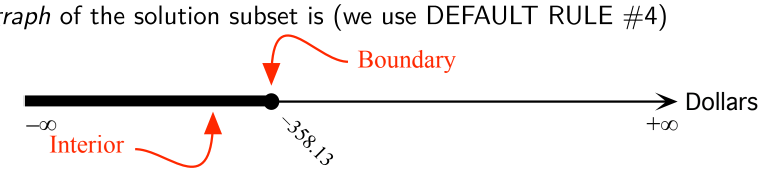


III. Altogether, we represent the solution subset of the inequation problem in **Dollars**

$$x \leqq -358.13$$

as follows:

- The *graph* of the solution subset is (we use DEFAULT RULE #4)



- The *name* of the solution subset is

$$(-\infty, -358.13] \text{ Dollars}$$

reduce
equation, original
equation, reduced
equivalent
invoke

Chapter 9

Translation & Dilation Problems

A large part of ALGEBRA is concerned with the investigation of the solution subset of problems. In this chapter, we begin with problems barely more complicated than *basic* problems.

We will continue to use the PASCH PROCEDURE so that we will be able to focus on solving the *associated equation* to locate the *boundary point* of the solution subset.

The approach that we will follow, which we will call the REDUCTION APPROACH, will be to **reduce** the **original equation** to an equation of a kind we have already investigated and which we can therefore solve and we will call that equation the **reduced equation**. Of course, the *reduced equation* will have to be **equivalent** to the *original equation* in the sense that the *reduced equation* will have to have the same solution subset as the *original equation*. This will be automatically ensured as long as we can **invoke** the

THEOREM 5 (Fairness). *Given any* equation*, as long as, whatever we do onto one side of the verb =, we do exactly the same onto the other side of the verb =, we get an* equivalent *equation.*

NOTE. While the **Fairness Theorem** seems obviously true, making the case that it is true is not that easy because what is not obvious is on what evidence to base the case. We will thus leave this issue for when the reader takes a course in MATHEMATICAL LOGIC.

After we have located the *boundary point*, we will find the *interior* of the solution subset just by following the GENERAL PROCEDURE we introduced in the case of *basic* problems.

The only—small—difficulty will be that, although similar in nature, different problems may involve numerators of different kinds:

- *plain counting* numerators to represent *numbers* of items,
- *signed counting* numerators to represent *two-way numbers* of items,
- *plain decimal* numerators to represent *quantities* of stuff,
- *signed decimal* numerators to represent *two-way quantities* of stuff.

9.1 Translation Problems

The simplest kind of real-world situations is where, given a collection, we *attach* another collection and we then want the result to compare in a given way with a given *gauge* collection.

1. More precisely, in order for the result to compare in a given way with the given gauge collection, we have two possibilities depending on what we are *given*:

- When we are given the *initial* collection, we will have to find what collection(s) can be *attached*.

  EXAMPLE 1. Jill already has two and half tons of sand in her dump-truck and she wants to know how much more sand she can load given that her dump-truck is licensed for carrying seven and a quarter tons.

- When we are given what collection is *to be attached*, we will have to find out what *initial* collections are possible.

  EXAMPLE 2. Jack knows his aunt will give him three apples as he visits her on the way to school but he wants to have more than seven apples for his friends at school. How many apples could he take with him as he sets out?

2. In order to represent these kinds of real-world situations, we just need one denominator to represent the kind of items in the collections.

EXAMPLE 3. We represent Jill's real-world situation in EXAMPLE 1 by the inequation

$$2.5 \textbf{ Tons of sand} + x \textbf{ Tons of sand} \leqq 7.25 \textbf{ Tons of sand}$$

where 2.5 **Tons of sand** represents what Jane has already loaded, 7.25 **Tons of sand** represents the gauge and x **Tons of sand** represent what she can load on the way.

EXAMPLE 4. We represent Jack's real-world situation in EXAMPLE 2 by the inequation

$$x \textbf{ Apples} + 3 \textbf{ Apples} > 7 \textbf{ Apples}$$

where x **Apples** represents Jack's initial collection of apples, 3 **Apples** represents the collection his aunt will give him and where 7 **Apples** represents the gauge.

Since we have a *common denominator*, we can *factor out* this common denominator. We then see that, from the investigation viewpoint, the kind of formula we get in both types of situations is essentially the same so that we won't have to deal with them separately. We will call this kind of problem a **translation problem**.

problem, translation
formula, translation
equation, translation
inequation, translation

EXAMPLE 5. We can factor out the common denominator **Tons of sand** in the inequation in EXAMPLE 3

$$2.5 \textbf{ Tons of sand} + x \textbf{ Tons of sand} \leqq 7.25 \textbf{ Tons of sand}$$

which gives us the *translation problem* in **Tons of sand**

$$2.5 + x \leqq 7.25$$

EXAMPLE 6. We can factor out the common denominator **Apples** in the inequation in EXAMPLE 4

$$x \textbf{ Apples} + 3 \textbf{ Apples} > 7 \textbf{ Apples}$$

which gives us the *translation problem* in **Apples**

$$x + 3 > 7$$

3. So far, for the sake of simplicity, we have been dealing only with *simple* collections but we will also have to deal with *two-way* collections and it will indeed matter whether the real-world situations involve *simple* collections or *two-way* collections because *plain* numerators cannot always be subtracted from while *signed* numerators can always be subtracted from.
So, we will have to deal separately with problems involving *plain* numerators and problems involving *signed* numerators.

EXAMPLE 7. Given that his starting balance is three dollars and twenty cents in the red, Mike wants to know how many dollars he can gain or lose given that his ending balance has to be higher than seven dollars and seventy five cents in the red.
We represent this real-world situation by the *translation* problem in **Dollars**

$$-3.25 \oplus x > -7.75$$

where x stands for a *signed* numerator.

4. Depending on how we want the resulting collection to compare with the given gauge, the *formula*, called **translation formula**, may involve any one of the following verbs: $\neq$, $>$, $<$, $\geqq$, $\leqq$ and $=$, and we will also use the terms **translation equation** and **translation inequation**.

EXAMPLE 8. Given an initial collection with three apples and a gauge collection with seven apples, the problem can involve any of the following translation inequations:

$$3 + x \neq 7$$
$$3 + x < 7$$
$$3 + x > 7$$
$$3 + x \leqq 7$$
$$3 + x \geqq 7$$

as well as with the translation equation

$$3 + x = 7$$

5. *Translation* problems are the simplest problems after *basic* problems and, in fact, *basic* problems are a special case of *translation* problems: If the number of items in the given collection in a *translation* problem is 0, then the *translation* problem is really just a *basic* problem.

EXAMPLE 9. If, in EXAMPLE 1, Jill had *no* apple instead of *three*, then the *translation* problem in **Apples** would be

$$0 + x > 7$$

which boils down to the *basic* inequation in **Apples**

$$x > 7$$

9.2 Solving Translation Problems

We now turn to the investigation of the solution subset of translation problems which we will do in accordance with the GENERAL PROCEDURE.

1. We locate the *boundary point* of the solution subset. This involves the following steps:

i. We write the *associated equation* for the given problem.

EXAMPLE 10. Given the inequation in **Apples**

$$3 + x > 7$$

the *associated equation* in **Apples** is

$$3 + x = 7$$

ii. We try to solve the associated equation by way of the REDUCTION APPROACH, that is we try to *reduce* the given *translation* problem to a *basic* problem by subtracting from *both* sides the numerator that is being added to x. The **Fairness Theorem** will then ensure that the resulting *basic* equation is equivalent to the original given *translation* equation.

This, though, is where it matters if the equation involves *plain* numerators or *signed* numerators and we look at the two cases separately.

- If the numerators involved in the equation are *plain* numerators, we may or may not be able to *subtract* depending on whether the numerator of the gauge is larger or smaller than the numerator being added to x.

  EXAMPLE 11. Given the *plain* equation in **Apples**

  $$3 + x = 7$$

  we subtract 3 from both sides

  $$3 + x - 3 = 7 - 3$$

  which boils down to the *basic* equation in **Apples**

  $$x = 4$$

  which the **Fairness Theorem** ensures to be *equivalent* to the translation equation in **Apples**

  $$3 + x = 7$$

which therefore has the solution of the basic equation, 4, as its own solution.

EXAMPLE 12. Given the *plain* equation in **Apples**

$$7 + x = 4$$

we cannot subtract 7 from the right side so we cannot subtract 7 from *both* sides as required by the **Fairness Theorem**.

So, the original translation equation

$$7 + x = 4$$

cannot be reduced to a basic equation and therefore has no solution.

- If the numerators involved in the equation are *signed* numerators, we can always *subtract* since "ominussing" means "oplussing the opposite".

EXAMPLE 13. Given the *signed* equation in **Apples**

$$+7 \oplus x = +3$$

we "ominus" $+7$ from both sides, that is we "oplus" both sides with the *opposite* of $+7$

$$+7 \oplus x \oplus -7 = +3 \oplus -7$$

which boils down to the *basic* equation in **Apples**

$$x = -4$$

which the **Fairness Theorem** ensures to be *equivalent* to the original *signed* translation equation in **Apples**

$$+7 \oplus x = +3$$

which therefore has the solution of the basic equation, $+4$, as its own solution.

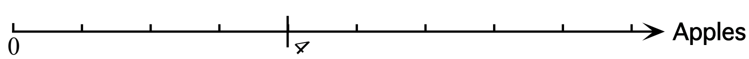
2. We locate the *interior* of the solution subset according to the GENERAL PROCEDURE. (For the sake of showing *complete* investigations, we will mention in each EXAMPLE the step where we locate the *boundary point*.)

EXAMPLE 14. Given the *translation* problem in **Apples**:

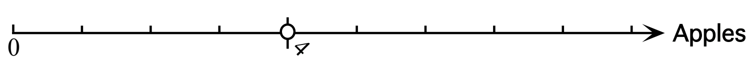
$$3 + x > 7$$

i. To locate the *boundary* of the solution subset:

i. We solve the *associated equation* using the REDUCTION APPROACH: 4

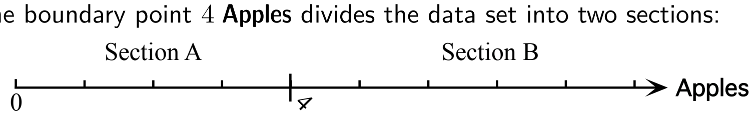


ii. Since the inequation is *strict*, the boundary point 4 **Apples** is *non-included* in the solution subset and so we graph it with a *hollow* dot.



ii. To locate the *interior* of the solution subset:

i. The boundary point 4 **Apples** divides the data set into two sections:



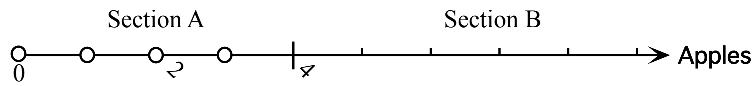
ii. We test Section A, for instance with 2. and, since

$$3 + x > 7|_{x:=2} \text{ is FALSE}$$

we get that 2 is a *non-solution* of the inequation in **Apples**

$$3 + x > 7$$

and **Pasch's Theorem** then tells us that *all* number-phrases in Section A are *non-included* in the solution subset.



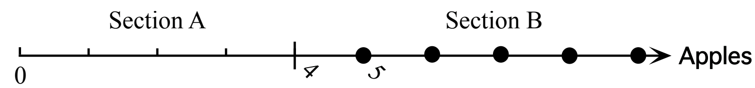
iii. We test Section B, for instance with 5, and, since

$$3 + x > 7|_{x:=5} \text{ is TRUE}$$

we get that 5 is a *solution* of the inequation in **Apples**

$$3 + x > 7$$

and **Pasch's Theorem** then tells us that *all* number-phrases in Section B are *included* in the solution subset.

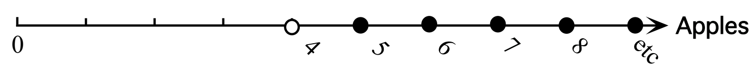


iii. *Altogether*, we represent the solution subset of the inequation in **Apples**

$$3 + x > 7$$

as follows:

- The *graph* of the solution subset is



- The *name* of the solution subset is

$$\{5, 6, 7, 8, 9, \text{etc}\} \textbf{ Apples}$$

EXAMPLE 15. Given the *plain* translation problem in **Apples**:

$$8 + x < 2$$

i. To locate the *boundary* of the solution subset:

i. The REDUCTION APPROACH does not work so that the *associated equation* has no solution.

ii. As a result, the solution subset has no boundary point.

ii. To locate the *interior* of the solution subset:

i. Since there is no boundary point, the interior of the solution subset is either the *full* data set (*all* number-phrases are included) or is *empty* (*no* number-phrase is included):

ii. We test with, for instance, 3 and, since

$$8 + x < 2|_{\text{where } x:=3} \text{ is FALSE}$$

we get that 3 is a *non-solution* of the inequation in **Apples**

$$8 + x < 2$$

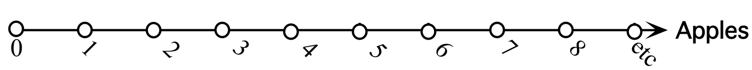
and **Pasch's Theorem** then tells us that *all* number-phrases are *non-included* in the solution subset.

iii. *Altogether*, we represent the solution subset of the inequation in **Apples**

$$8 + x < 2$$

as follows:

- The *graph* of the solution subset is



(While, normally, we do not mark the non-solutions, here we mark them as otherwise we would be leaving the ruler unmarked which would be ambiguous.)

- The *name* of the solution subset is

$$\{\ \}$$ **Apples**

EXAMPLE 16. Given the *plain* translation problem in **Apples**:

$$8 + x > 2$$

i. To locate the *boundary* of the solution subset:

i. The REDUCTION APPROACH does not work so that the *associated equation* has no solution.

ii. As a result, the solution subset has no boundary point.

ii. To locate the *interior* of the solution subset:

i. Since there is no boundary point, the interior of the solution subset is either the *full* data set (*all* number-phrases are included) or is *empty* (*no* number-phrase is included):

ii. We test with, for instance, 3 and, since

$$8 + x > 2|_{\text{where } x:=3} \text{ is TRUE}$$

we get that 3 is a *solution* of the inequation in **Apples**

$$8 + x > 2$$

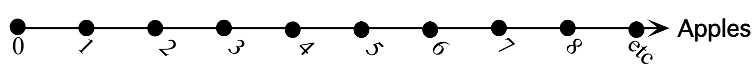
and **Pasch's Theorem** then tells us that *all* number-phrases are *included* in the solution subset.

iii. *Altogether*, we represent the solution subset of the inequation in **Apples**

$$8 + x > 2$$

as follows:

- The *graph* of the solution subset is



- The *name* of the solution subset is

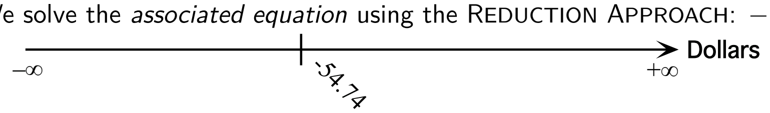
$$\{0, 1, 2, 3, 4, 5, 6, 7, 8, \text{etc}\}$$ **Apples**

EXAMPLE 17. Given the translation problem in **Dollars**:

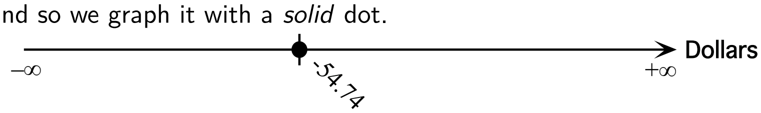
$$-3.08 \oplus x \leqq -57.82$$

i. To locate the *boundary* of the solution subset:

i. We solve the *associated equation* using the REDUCTION APPROACH: -54.74

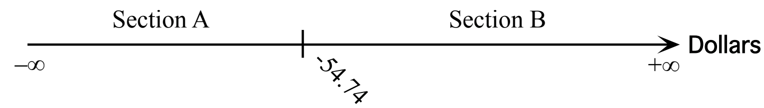


ii. Since the inequation is *lenient*, the boundary point is *included* in the solution subset and so we graph it with a *solid* dot.



ii. To locate the *interior* of the solution subset:

i. The boundary point -54.74 **Dollars** divides the data set into two sections:



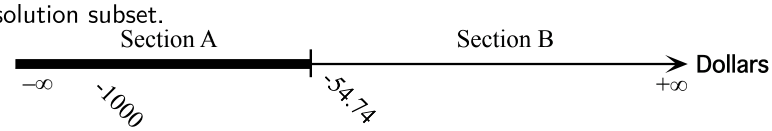
ii. We test Section A with, for instance, -1000 and, since

$$-3.08 \oplus x \leqq -57.82|_{\text{where } x:=-1000} \text{ is TRUE}$$

we get that -1000 is a *solution* of the inequation in **Dollars**

$$-3.08 \oplus x \leqq -57.82$$

and **Pasch's Theorem** then tells us that *all* number-phrases in Section A are *included* in the solution subset.



iii. We test Section B with, for instance, 5 and, since

$$3 + x > 7|_{\text{where } x:=+1000} \text{ is TRUE}$$

we get that $+1000$ is a *none-solution* of the inequation in **Dollars**

$$3 + x > 7$$

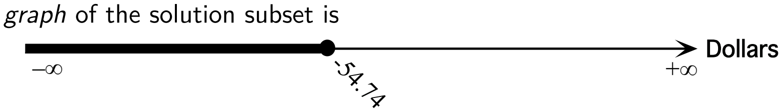
and **Pasch's Theorem** then tells us that *all* number-phrases in Section B are *non-included* in the solution subset.

iii. *Altogether*, the solution subset of the inequation in **Dollars**

$$3 + x > 7$$

is a *ray* which we represent as follows:

- The *graph* of the solution subset is



- The *name* of the solution subset is

$$(-\infty, -54.74] \text{ Dollars}$$

9.3 Dilation Problems

Another kind of real-world situation, almost as simple as those represented by *translation* problems but very different in nature, is where we want to find the situations in which the *money* worth of a collection compares in a given way with a given *money* gauge.

1. More precisely, in order for the worth to compare in a given way with the given money gauge, we have two possibilities depending on what we are *given*.

- When we are given the *number of items* in the collection, we will have to find what *unit-worths* will let the worth of the collection compare with the gauge in the given way.

EXAMPLE 18. The BananaCompany is twelve dollars in the black and just lost three apples. So, whether or not the BananaCompany is still in the black will depend on the going *unit profit/removal worth* of the good/bad apples.

EXAMPLE 19. Dick wants to sell *three and a half pounds* of flour but he needs at least fourteen dollars and seventy cents. So, whether or not he will be able to sell the flour will depend on the going *unit worth* of the flour.

- When we are given the *unit-worth* of the items in the collection, we will have to find what *numbers of items* will let the worth of the collection compare with the gauge in the given way.

EXAMPLE 20. Jane wants to sell flour at four dollars and twenty cents a pound and she needs fourteen dollars and seventy cents. How much flour can she sell?

EXAMPLE 21. The CranberryCompany is seven dollars in the black and cannot be in the red. It needs to get bad cranberries removed. So, how many pounds of cranberries it can get rid of will depend on the going *unit worth* of the cranberry removal.

2. In order to represent these kinds of real-worls situations, we need three *denominators*:

i. A *denominator* to represent the kind of *items*,

ii. A *denominator* to represent the *denomination* (that is "kind of money") in which the money *gauge* is given,

iii. A *co-denominator* to represent the *unit-worth* of the items expressed in that *denomination*.

EXAMPLE 22. We represent the BananaCompany real-world situation in EXAMPLE 18 by the inequation

$$-3 \textbf{ Apples} \times x \ \frac{\textbf{Dollars}}{\textbf{Apple}} \geqq -12 \textbf{ Dollars}$$

where -3 **Apples** represents the three apples that were lost, $x \ \frac{\textbf{Dollars}}{\textbf{Apple}}$ represents the unit profit/removal worth of the apples and -12 **Dollars** represents the money gauge.

EXAMPLE 23. We represent Dick's real-world situation in EXAMPLE 19 by the inequation

$$3.5 \textbf{ Pounds of flour} \times x \ \frac{\textbf{Dollars}}{\textbf{Pound of flour}} \geqq 14.70 \textbf{ Dollars}$$

where the unit value $x \ \frac{\textbf{Dollars}}{\textbf{Apple}}$ represents the unit-worth of the apples and where -12.70 **Dollars** represents the money gauge.

EXAMPLE 24. We represent Jack's real-world situation in EXAMPLE 20 by the inequation

$$x \textbf{ Apples} \times 4 \ \frac{\textbf{Dollars}}{\textbf{Apple}} \leqq 12 \textbf{ Dollars}$$

where the unit value $x \ \frac{\textbf{Dollars}}{\textbf{Apple}}$ represents the unit-worth of the apples and where 12 **Dollars** represents the money gauge.

EXAMPLE 25. We represent Jane's real-world situation in EXAMPLE 21 by the inequation

$$x \textbf{ Pounds of flour} \times 4.20 \ \frac{\textbf{Dollars}}{\textbf{Pound of flour}} \leqq 14.70 \textbf{ Dollars}$$

problem, dilation

where the unit value $4.20 \frac{\textsf{Dollars}}{\textsf{Pound of flour}}$ represents the unit-worth of the flour and where 14.70 **Dollars** represents the money gauge.

However, when we carry out the *co-multiplication*, we get a *common denominator* which is the denominator that represents the *denomination* in which the collection of items—or the amount of stuff—is to be *evaluated*. We can then factor out this common denominator and we can then see that the kind of formula we get in both types of situations is essentially the same and we will call the resulting kind of problem a **dilation problem**.

EXAMPLE 26. When we carry out the co-multiplication in EXAMPLE 22, we get

$$-3 \; \cancel{\textsf{Apples}} \times x \; \frac{\textsf{Dollars}}{\cancel{\textsf{Apple}}} \geqq -12 \; \textsf{Dollars}$$

that is

$$[-3 \times x] \; \textsf{Dollars} \geqq -12 \; \textsf{Dollars}$$

where we can factor out the common denomnator which gives us the *dilation problem* in **Dollars**

$$-3 \times x \; \geqq \; -12$$

EXAMPLE 27. When we carry out the co-multiplication in EXAMPLE 24, we get

$$x \; \cancel{\textsf{Apples}} \times 4 \; \frac{\textsf{Dollars}}{\cancel{\textsf{Apple}}} \leqq 12 \; \textsf{Dollars}$$

that is

$$[x \times 4] \; \textsf{Dollars} \leqq 12 \; \textsf{Dollars}$$

where we can factor out the common denominator which gives us the *dilation problem* in **Dollars**

$$x \times 4 \; \leqq \; 12$$

EXAMPLE 28. When we carry out the co-multiplication in EXAMPLE 23, we get

$$3.5 \; \cancel{\textsf{Pounds of flour}} \times x \; \frac{\textsf{Dollars}}{\cancel{\textsf{Pound of flour}}} \leqq 14.70 \; \textsf{Dollars}$$

that is

$$[3.5 \times x] \; \textsf{Dollars} \leqq 14.70 \; \textsf{Dollars}$$

where we can factor out the common denomnator which gives us the *dilation problem* in **Dollars**

$$3.5 \times x \; \leqq \; 14.70$$

formula, dilation
equation, dilation
inequation, dilation

EXAMPLE 29. When we carry out the co-multiplication in EXAMPLE 25, we get

$$x\ \cancel{\textbf{Pounds of flour}} \times 4.20\ \frac{\textbf{Dollars}}{\cancel{\textbf{Pound of flour}}} \leqq 14.70\ \textbf{Dollars}$$

that is

$$[x \times 4.20]\ \textbf{Dollars} \leqq 14.70\ \textbf{Dollars}$$

where we can factor out the common denomnator which gives us the *dilation problem* in **Dollars**

$$x \times 4.20 \leqq 14.70$$

3. We will see that for the purpose of investigating *dilation* problems, it will not really matter whether the real-world situations that they represent involve *simple* situations or *two-way* situations. What will very much matter is whether the real-world situations involve items that can be divided or items that cannot be divided because *counting* numerators cannot always be divided while *decimal* numerators can always be divided.
So, we will deal separately with problems involving *counting* numerators and problems involving *decimal* numerators.

4. The *formula* in a *dilation problem* may involve any one of the following verbs: $\neq$, $>$, $<$, $\geqq$, $\leqq$ and $=$. It is called a **dilation formula** and we will also use the terms **dilation equation** and **dilation inequation**.

EXAMPLE 30. In EXAMPLE 18, depending on the situation, we could have to solve any of the following *dilation formulas* in **Dollars**:

$$3 \times x \neq 4.95$$
$$3 \times x < 4.95$$
$$3 \times x > 4.95$$
$$3 \times x \leqq 4.95$$
$$3 \times x \geqq 4.95$$

and/or the associated equation in **Dollars**

$$3 \times x = 4.95$$

5. In some ways, *dilation* problems are very similar to *translation* problems. In particular, *basic* problems are also a special case of *dilation* problems: If the number of items in the collection in a *dilation* problem is 1, then the *dilation* problem is really just a *basic* problem.

EXAMPLE 31. If Jill's collection in EXAMPLE 18 included only one apple instead of three apples, then the *dilation* problem would be

$$1\ \textbf{Apples} \times x\ \frac{\textbf{Dollars}}{\textbf{Apple}} \leqq 4.95\ \textbf{Dollars}$$

which boils down to the *basic* inequation in **Dollars**

$$x \leqq 4.95$$

9.4 Solving Dilation Problems

We can now turn to the *investigation* of dilation problems which we will do according to the GENERAL PROCEDURE.

1. We locate the *boundary point* of the solution subset. This involves the following steps:

i. We write the *associated equation* for the given problem.

EXAMPLE 32. Given the dilation problem in **Dollars** in EXAMPLE 32

$$3 \times x \leqq 12$$

the associated equation in **Dollars** is

$$3 \times x = 12$$

EXAMPLE 33. Given the dilation problem in **Dollars** in EXAMPLE 35

$$x \times 4.20 \leqq 14.70$$

the associated equation in **Dollars** is

$$x \times 4.20 = 14.70$$

ii. We try to solve the associated equation by way of the REDUCTION APPROACH, that is we try to *reduce* the given *dilation* problem to a *basic* problem by dividing *both* sides by the numerator that is being multiplied by x. The **Fairness Theorem** will then ensure that the resulting *basic* equation is equivalent to the original given *translation* equation.

This, though, is where it matters if the equation involves *counting* numerators or *decimal* numerators and we look at the two cases separately.

- If the numerators involved in the equation are *counting* numerators, we may or may not be able to divide depending on whether the numerator of the gauge is or is not a multiple of the numerator being multiplied by x.

  EXAMPLE 34. Given the associated equation in **Live Rabbits**

  $$3 \times x = 12$$

  we can divide both sides by 3

  $$3 \times x \div 3 = 12 \div 3$$

  which boils down to the basic equation in **Live Rabbits**

  $$x = 4$$

  which the **Fairness Theorem** ensures to be equivalent to the original dilation problem in **Live Rabbits**

  $$3 \times x = 12$$

  which therefore has the solution of the basic equation, 4, as its own solution.

  EXAMPLE 35. Given the associated equation in **Live Rabbits**

  $$3 \times x = 13$$

  we cannot divide 13 by 3 so we cannot divide both sides by 3 as required by the **Fairness Theorem**.

  So, the original dilation equation,

  $$3 \times x = 13$$

cannot be reduced to a basic equation and therefore has no solution. This of course corresponds to the fact that we cannot have fractions of *live* rabbits.

- If the numerators involved in the equation are *decimal* numerators, we can always *divide*.

EXAMPLE 36. Given the equation in **Grams of Gold**

$$x \times 3.2 = 13.76$$

we can divide both sides by 3.2

$$x \times 3.2 \div 3.2 = 13.76 \div 3.2$$

which boils down to the basic equation in **Grams of Gold**

$$x = 4.3$$

which the **Fairness Theorem** ensures to be equivalent to the original dilation problem in **Grams of Gold**

$$x \times 3.2 = 13.76$$

which therefore has the solution of the basic equation, 4, as its own solution.

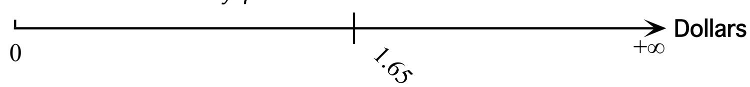
2. We locate the *interior* of the solution subset according to the GENERAL PROCEDURE. (For the sake of showing *complete* investigations, we will mention in each EXAMPLE the step in which we locate the *boundary point*.)

EXAMPLE 37. Given the *dilation* problem in **Dollars**:

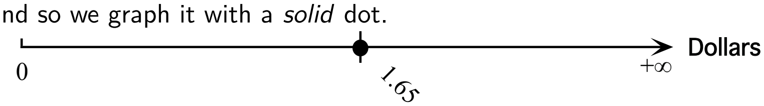
$$3 \times x \leqq 4.95$$

i. To get the *boundary* of the solution subset

i. We *locate* the *boundary point* as above: 1.65

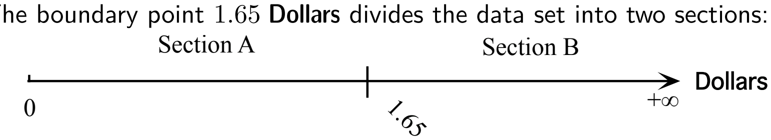


ii. Since the inequation is *lenient*, the *boundary point* is *included* in the solution subset and so we graph it with a *solid* dot.



ii. To get the *interior* of the solution subset

i. The boundary point 1.65 **Dollars** divides the data set into two sections:



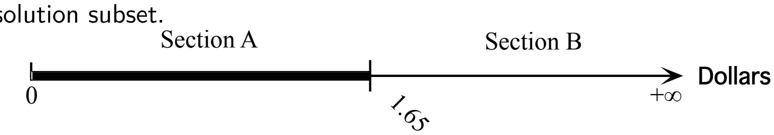
ii. We test Section A with, for instance, 1 and, since

$$3 \times x \leqq 4.95|_{\text{where } x:=1} \text{ is TRUE}$$

we get that 1 is a *solution* of the inequation in **Dollars**

$$3 \times x \leqq 4.95$$

and **Pasch's Theorem** then tells us that *all* number-phrases in Section A are *included* in the solution subset.



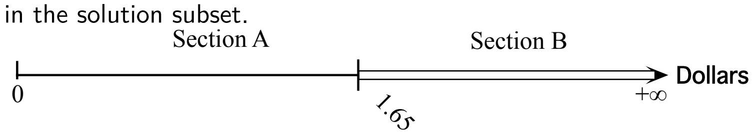
iii. We test Section B with, for instance, 5 and, since

$$3 \times x > 4.95|_{\text{where } x:=5} \text{ is FALSE}$$

we get that 5 is a *non-solution* of the inequation in **Dollars**

$$3 \times x \leqq 4.95$$

and **Pasch's Theorem** then tells us that *all* number-phrases in Section B are *non-included* in the solution subset.

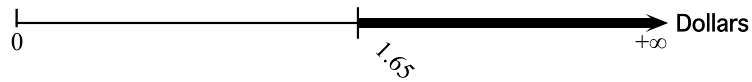


iii. *Altogether*, we represent the solution subset of the inequation in **Dollars**

$$3 \times x \leqq 4.95$$

as follows:

- The *graph* of the solution subset is



- The *name* of the solution subset is

$$(-\infty, 1.65) \text{ Dollars}$$

problem, affine formula, affine equation, affine inequation, affine term, constant

Chapter 10

Affine Problems

10.1 Introduction

The most frequent type of real-world situations is where we want to find the situations in which the money worth of a collection *plus some fixed money amount* compares in a given way with a given gauge.

1. The corresponding problem is called an **affine problem** and we shall also use the terms **affine formula**, **affine equation** and **affine inequation**. The number-phrase that represents the fixed money amount is called the **constant term**.

Example 1. Jane wants to buy three apples but there is a fixed transaction charge of four dollars and fifty cents and the most she wants to spend is twenty-three dollars and thirty-four cents. So, whether or not she will be able to get the three apples will depend on the on the going *unit-worth* of the apples.

The real-world situation is represented by the inequation

$$3\ \textsf{Apples} \times x\ \frac{\textsf{Dollars}}{\textsf{Apple}} + 4.5\ \textsf{Dollars} \leqq 23.34\ \textsf{Dollars}$$

where 4.5 **Dollars** is the *constant term*.

When we carry out the co-multiplication we get the affine inequation

$$3\ \cancel{\textsf{Apples}} \times x\ \frac{\textsf{Dollars}}{\cancel{\textsf{Apple}}} + 4.5\ \textsf{Dollars} \leqq 23.34\ \textsf{Dollars}$$

$$[3 \times x]\ \textsf{Dollars} + 4.5\ \textsf{Dollars} \leqq 23.34\ \textsf{Dollars}$$

When we factor out the common denominator **Dollars**, we get the *affine* problem in **Dollars**

$$3 \times x + 4.5 \leqq 23.34$$

2. *Translation* problems and *dilation* problems as well as *basic* problems turn out to be special cases of *affine* problems which are therefore a more general type of problems:

- If the number of items in an affine problem is 1, then the affine problem is really just a *translation* problem.

  EXAMPLE 2. If the number of items in EXAMPLE 1 were 1 instead of 3, then the inequation would be
  $$1 \textbf{ Apples} \times x \frac{\textbf{Dollars}}{\textbf{Apple}} + 4.5 \textbf{ Dollars} \leqq 23.34 \textbf{ Dollars}$$
  which boils down to the inequation in **Dollars**
  $$x + 4.5 \leqq 23.34$$
  which is a *translation* problem.

- If the *fixed* number-phrase in an affine problem is 0, then that affine problem is really just a *dilation* problem.

  EXAMPLE 3. If the *fixed* number-phrase in EXAMPLE 1 were 0 **Dollars** instead of 4.5 **Dollars**, then the inequation would be
  $$3 \textbf{ Apples} \times x \frac{\textbf{Dollars}}{\textbf{Apple}} + 0 \textbf{ Dollars} \leqq 23.35 \textbf{ Dollars}$$
  which boils down to the inequation in **Dollars**
  $$3 \times x \leqq 23.35$$
  which is a *dilation* problem.

- If, in an affine problem, both the additional number-phrase is 0 and the number of items is 1, then that affine problem is really just a *basic* problem.

  EXAMPLE 4. If, in EXAMPLE 24 the number of items were 1 instead of 3 and the additional number-phrase were 0 **Dollars** instead of 4.5 **Dollars**, then the inequation would be
  $$1 \textbf{ Apples} \times x \frac{\textbf{Dollars}}{\textbf{Apple}} + 0 \textbf{ Dollars} \leqq 23.35 \textbf{ Dollars}$$
  which boils down to the inequation in **Dollars**
  $$x \leqq 23.35$$
  which is a *basic* problem.

10.2 Solving Affine Problems

We now turn to the investigation of the solution subset of *affine* problems which we will do in accordance with the PASCH PROCEDURE. The investigation of *affine* problems will proceed much in the same way as that of *translation* and *dilation* problems. As usual, the only difficulty will be that, although similar in nature, problems may involve numerators of different kinds:

- *plain counting* numerators to represent *numbers* of items,
- *signed counting* numerators to represent *two-way numbers* of items,
- *plain decimal* numerators to represent *quantities* of stuff,
- *signed decimal* numerators to represent *two-way quantities* of stuff.

1. We locate the *boundary point* of the solution subset. This involves the following steps:

i. We write the associated equation for the given problem:

EXAMPLE 5. Given the affine problem in **Dollars** in EXAMPLE 1

$$3 \times x + 4.5 \leqq 23.34$$

the associated equation in **Dollars** is

$$3 \times x + 4.5 = 23.34$$

ii. We try to solve the associated equation in *two* stages by way of the REDUCTION APPROACH:

i. We try to reduce the *affine* problem to a *dilation* problem by subtracting the fixed term from *both* sides so as to be able to invoke the **Fairness Theorem**,

ii. We then try to reduce the resulting *dilation* problem to a *basic* problem by dividing by the coefficient of x *both* sides so as to be able to invoke the **Fairness Theorem**.

EXAMPLE 6. Given the affine equation in **Dollars** in EXAMPLE 2

$$3 \times x + 4.5 = 23.34$$

i. We *subtract* 4.5 from *both* sides:

$$3 \times x + 4.5 \; -4.5 = 23.34 \; -4.5$$

which boils down to the dilation equation in **Dollars**

$$3 \times x = 18.84$$

ii. We *divide both* sides by 3

$$3 \times x \; \div 3 = 18.84 \; \div 3$$

which boils down to the *basic* equation in **Dollars**

$$x = 6.28$$

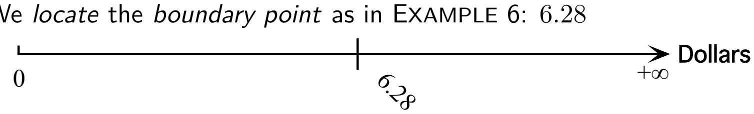
2. We locate the *interior* of the solution subset according to the GENERAL PROCEDURE. (For the sake of completion, we include in the EXAMPLE the step in which we get the *boundary point*.

EXAMPLE 7. Given the *affine* problem in **Dollars** in EXAMPLE 1:

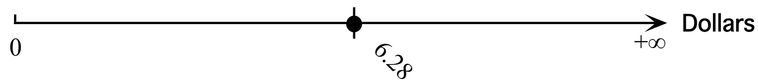
$$3 \times x + 4.5 \leqq 23.34$$

i. To get the *boundary* of the solution subset

i. We *locate* the *boundary point* as in EXAMPLE 6: 6.28

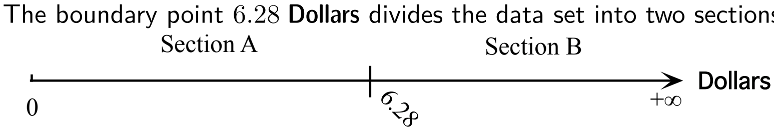


ii. Since the inequation is *lenient*, the *boundary point* is *included* in the solution subset and so we graph it with a *solid* dot.



ii. We locate the *interior* of the solution subset

i. The boundary point 6.28 **Dollars** divides the data set into two sections:



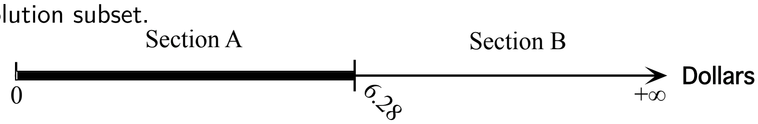
ii. We test Section A with, for instance, 0.1 and, since

$$3 \times x + 4.5 \leqq 23.34|_{x:=0.1} \text{ is TRUE}$$

we get that 0.1 is a *solution* of the inequation in **Dollars**

$$3 \times x + 4.5 \leqq 23.34$$

and **Pasch's Theorem** then tells us that *all* number-phrases in Section A are *included* in the solution subset.



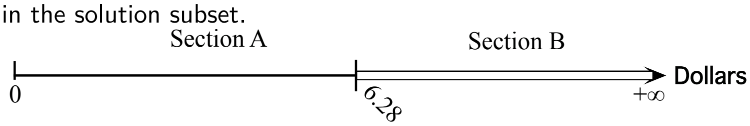
iii. We test Section B with, for instance, +5.0 and, since

$$3 \times x + 4.5 \leqq 23.34|_{x:=1000} \text{ is FALSE}$$

we get that 1000 is a *non-solution* of the inequation in **Dollars**

$$3 \times x + 4.5 \leqq 23.34$$

and **Pasch's Theorem** then tells us that *all* number-phrases in Section B are *non-included* in the solution subset.

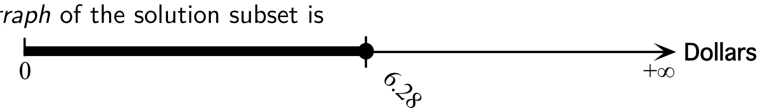


iii. *Altogether*, we represent the solution subset of the inequation in **Dollars**

$$3 \times x + 4.5 \leqq 23.34$$

as follows:

- The *graph* of the solution subset is



- The *name* of the solution subset is

$$(0, 6.28) \text{ Dollars}$$

double basic problem
connected
connector
condition

Chapter 11

Double Basic Problems

We now investigate **double basic problems**, that is problems that involve two basic formulas which can be

- two basic equations

or

- two basic inequations

or

- one basic equation and one basic inequation

These two formulas will be **connected** by one of the following **connectors**:

BOTH,
EITHER ONE OR BOTH,
EITHER ONE BUT NOT BOTH.

As we did with single problems, we will use the PASCH PROCEDURE, that is we will

i. Locate the *boundary* of the solution subset of the double problem,

ii. Locate the *interior* of the solution subset of the double problem using test points and the **Pasch Theorem**,

11.1 Double Basic Equation Problems

We begin with problems that involve two basic *equations* with one of the above *connectors* and with the **condition** that the two *gauge* number-phrases

OR
problem, double basic
equation

x_1 and x_2 be *different*.

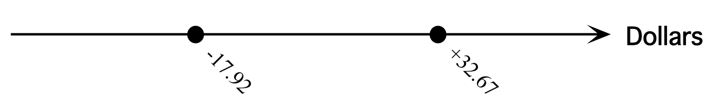
1. Since the connector used in a double basic problem can any one of three possible *connectors*, up front, there will be three types of double basic equation problems.

- Double basic equation problems involving the connector BOTH:
$$\text{BOTH}\begin{cases} x = x_1 \\ x = x_2 \end{cases} \quad \text{(with the } \textit{condition} \text{ that } x_1 \neq x_2\text{)}$$
But double problems of this type have *no* solution. (Why not?)
- Double basic equation problems involving the connector EITHER ONE AND BOTH:
$$\text{EITHER ONE OR BOTH}\begin{cases} x = x_1 \\ x = x_2 \end{cases} \quad \text{(with the } \textit{condition} \text{ that } x_1 \neq x_2\text{)}$$
Double problems of this type have *two* solutions, namely the two gauge numerators, x_1 and x_2.
- Double basic equation problems involving the connector EITHER ONE BUT NOT BOTH:
$$\text{EITHER ONE BUT NOT BOTH}\begin{cases} x = x_1 \\ x = x_2 \end{cases} \quad \text{(with the } \textit{condition} \text{ that } x_1 \neq x_2\text{)}$$
Double problems of this type have the same *two* solutions as above, namely the two gauge numerators, x_1 and x_2, since here the specification BUT NOT BOTH is superfluous. (Why?)

2. So, since in the case of double *basic equation* problems it makes no difference whether we use EITHER ONE AND BOTH or EITHER ONE BUT NOT BOTH, we will just write **OR** and what we will mean by **double basic equation problem** will be *only* problems of the type:
$$\text{OR}\begin{cases} x = x_1 \\ x = x_2 \end{cases} \quad \text{(with the } \textit{condition} \text{ that } x_1 \neq x_2\text{)}$$

EXAMPLE 1. We represent the solution subset of the double basic equation problem in **Dollars**
$$\text{OR}\begin{cases} x = +32.67 \\ x = -17.92 \end{cases}$$
as follows:

- The *graph* of the solution subset is



- The *name* of the solution subset is
$$\{-17.92, +32.67\} \text{ Dollars}$$

11.2 Problems of Type BETWEEN

between
gauge-numerators
problem, of type BETWEEN
interval
boundary (of an interval)
boundary points (of an interval)

These are the *first* of the two types of double basic *inequation* problems that we shall investigate in full in this chapter.

1. Given a set of *selectable* collections and given two *gauge* collections, we can specify a subset of the set of selectable collections by the requirement that the size of the collections be **between** the sizes of the two *gauge collections*.

EXAMPLE 2. The legal occupancy of a movie theater is that it can seat at most five hundred viewers but the the owner of the movie theater may decide that showing the movie to fewer than sixty viewers is not worth it. Thus, the collection of viewers in any show is *between* sixty and five hundred viewers.

In other words, we require that the size of the collections in the *subset* be BOTH

- *larger* than the size of the *smaller* of the two gauge collections

AND

- *smaller* than the size of the *larger* of the two gauge collections

2. We now discuss the paper representation in some generality.

a. We start with two **gauge-numerators**, x_1 and x_1, that is with the numerators of the number-phrases that represent the two *gauge* collections. One of the gauge numerators has of course to be smaller than the other and so, for the sake of convenience, we shall let

$$\boxed{x_1 < x_2}$$

so that here

- x_1 will be the *smaller* of the two gauge numerators
- x_2 will be the *larger* of the two gauge numerators

b. Since each one of the two verbs can be either *strict* of *lenient*, there will be four kinds of **problems of type BETWEEN**:

$$\text{BOTH}\begin{cases} x > x_1 \\ x < x_2 \end{cases} \qquad \text{BOTH}\begin{cases} x \geqq x_1 \\ x \leqq x_2 \end{cases} \qquad \text{BOTH}\begin{cases} x \geqq x_1 \\ x < x_2 \end{cases} \qquad \text{BOTH}\begin{cases} x > x_1 \\ x \leqq x_2 \end{cases}$$

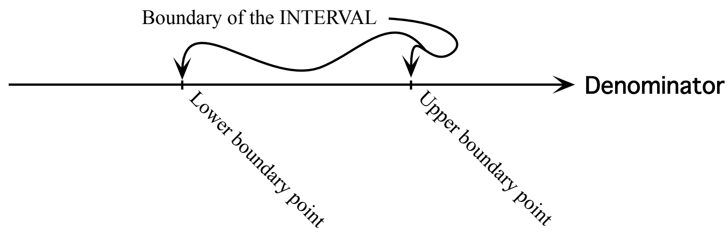
3. The *solution subset* of any problem of type BETWEEN is called an **interval**:

- The **boundary of an interval** consists of the two *gauge* numerators because they are solutions of the associated double equation problem

$$\text{OR}\begin{cases} x = x_1 \\ x = x_2 \end{cases}$$

The two *gauge* numerators are then called **boundary points** of the interval.

interior (of an interval)
segment



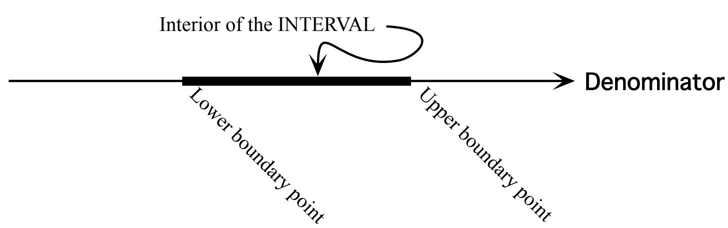
However, the double basic equation problem

$$\text{OR} \begin{cases} x = x_1 \\ x = x_2 \end{cases}$$

being associated with a double basic inequation problem, each one of the two *boundary points* may be *included* or *non-included* in the solution subset of the double inequation problem depending on whether the corresponding inequation is *strict* or *lenient*. So, we will have to check that.

We shall graph the boundary points as usual, that is with a *solid dot* for a boundary point that is *included* in the solution subset and a *hollow dot* for a boundary point that is *non-included* in the solution subset.

- The **interior of an interval** consists of all the numerators that are *between* the two *gauge* numerators, that is, the interior consists of all numerators that are BOTH larger than the smaller gauge numerator AND smaller than the larger gauge numerator. So, we represent the *interior* of the interval by a **segment**.



4. We now investigate an EXAMPLE of each one of the four kinds of problem of type BETWEEN.

I. Problems of type BETWEEN of the kind $\text{BOTH} \begin{cases} x > x_1 \\ x < x_2 \end{cases}$

EXAMPLE 3. Given the problem in **Dollars**

$$\text{BOTH} \begin{cases} x > -37.41 \\ x < +68.92 \end{cases}$$

this is a problem of type BETWEEN and we get its solution subset according to thePASCH PROCEDURE:

1. We locate the *boundary* of the solution subset. This involves the following steps:

i. We solve the double basic equation problem associated with the given problem

$$\text{OR}\begin{cases} x = -37.41 \\ x = +68.92 \end{cases}$$

which gives us the boundary points -37.41 and $+68.92$.

ii. We check if the *boundary points* are in the solution subset.

- Since we have

$$x > -37.41|_{x:=-37.41} \text{ is FALSE}$$
$$x < +68.92|_{x:=-37.41} \text{ is TRUE}$$

and since, in order for -37.41 to be a solution with the connector BOTH, -37.41 has to satisfy BOTH formulas, we have that

$$\text{BOTH}\begin{cases} x > -37.41|_{x:=-37.41} \\ x < +68.92|_{x:=-37.41} \end{cases} \text{ is FALSE}$$

so that -37.41 is *non-included* in the solution subset and we must graph -37.41 with a *hollow* dot.

- Since we have

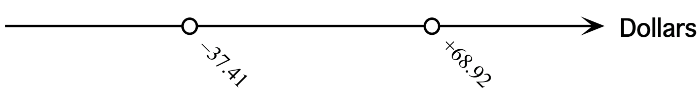
$$x > -37.41|_{x:=+68.92} \text{ is TRUE}$$
$$x < +68.92|_{x:=+68.92} \text{ is FALSE}$$

and since, in order for $+68.92$ to be a solution with the connector BOTH, $+68.92$ has to satisfy BOTH formulas, we have that

$$\text{BOTH}\begin{cases} x > -37.41|_{x:=+68.92} \\ x < +68.92|_{x:=+68.92} \end{cases} \text{ is FALSE}$$

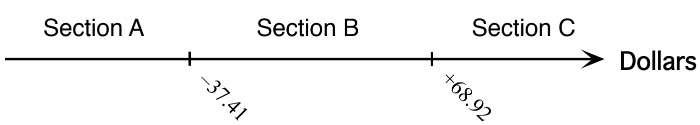
so that $+68.92$ is *non-included* in the solution subset and we must graph $+68.92$ with a *hollow* dot.

Altogether, we have



2. We locate the *interior* of the solution subset. This involves the following steps:

i. The boundary points divide the data set into three sections



ii. We test Section A with, for instance, -1000. Since we have

$$x > -37.41|_{x:=-1000} \text{ is FALSE}$$
$$x < +68.92|_{x:=-1000} \text{ is TRUE}$$

and since, in order for -1000 to be a solution with the connector BOTH, -1000 has to satisfy BOTH formulas, we have that

$$\text{BOTH}\begin{cases} x > -37.41|_{x:=-1000} \\ x < +68.92|_{x:=-1000} \end{cases} \text{ is FALSE}$$

so that -1000 is *non-included* in the solution subset. **Pasch's Theorem** then tells us that *all* number-phrases in Section A are *non-included* in the solution subset.

iii. We test Section B with, for instance, 0. Since we have

$$x > -37.41|_{x:=0} \text{ is TRUE}$$
$$x < +68.92|_{x:=0} \text{ is TRUE}$$

and since, in order for 0 to be a solution with the connector BOTH, 0 has to satisfy BOTH formulas, we have that

$$\text{BOTH}\begin{cases} x > -37.41|_{x:=0} \\ x < +68.92|_{x:=0} \end{cases} \text{ is TRUE}$$

so that 0 is *included* in the solution subset. **Pasch's Theorem** then tells us that *all* number-phrases in Section B are *included* in the solution subset.

iv. We test Section C with, for instance, $+1000$. Since we have

$$x > -37.41|_{x:=+1000} \text{ is TRUE}$$
$$x < +68.92|_{x:=+1000} \text{ is FALSE}$$

and since, in order for $+1000$ to be a solution with the connector BOTH, $+1000$ has to satisfy BOTH formulas, we have that

$$\text{BOTH}\begin{cases} x > -37.41|_{x:=+1000} \\ x < +68.92|_{x:=+1000} \end{cases} \text{ is FALSE}$$

so that $+1000$ is *non-included* in the solution subset. **Pasch's Theorem** then tells us that *all* number-phrases in Section A are *non-included* in the solution subset.

3. We *represent* and *describe* the solution subset of the problem of type BETWEEN in **Dollars**

$$\text{BOTH}\begin{cases} x > -37.41 \\ x < +68.92 \end{cases}$$

- The *graph* of the solution subset is the *lower-open*, *upper-open segment*

- The *name* of the solution subset is the *lower-open*, *upper-open interval*

$$\big(-37.41, +68.92\big) \text{ Dollars}$$

II. Problems of type BETWEEN of the kind $\text{BOTH}\begin{cases} x \geqq x_1 \\ x \leqq x_2 \end{cases}$

EXAMPLE 4. Given the problem in **Dollars**

$$\text{BOTH}\begin{cases} x \geqq -37.41 \\ x \leqq +68.92 \end{cases}$$

this is a problem of type BETWEEN and we get its solution subset according to thePASCH PROCEDURE:

1. We locate the *boundary* of the solution subset. This involves the following steps:

i. We solve the double basic equation problem associated with the given problem:

$$\text{OR}\begin{cases} x = -37.41 \\ x = +68.92 \end{cases}$$

which gives us the boundary points -37.41 and $+68.92$.

ii. We check if the *boundary points* are in the solution subset.

- Since we have

$$x \geqq -37.41|_{x:=-37.41} \text{ is TRUE}$$
$$x \leqq +68.92|_{x:=-37.41} \text{ is TRUE}$$

and since, in order for -37.41 to be a solution with the connector BOTH, -37.41 has to satisfy BOTH formulas, we have that

$$\text{BOTH}\begin{cases} x \geqq -37.41|_{x:=-37.41} \\ x \leqq +68.92|_{x:=-37.41} \end{cases} \text{ is TRUE}$$

so that -37.41 is *included* in the solution subset and we must graph -37.41 with a *sokid* dot.

- Since we have

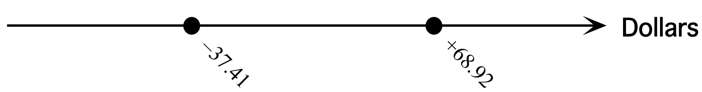
$$x \geqq -37.41|_{x:=+68.92} \text{ is TRUE}$$
$$x \leqq +68.92|_{x:=+68.92} \text{ is TRUE}$$

and since, in order for $+68.92$ to be a solution with the connector BOTH, $+68.92$ has to satisfy BOTH formulas, we have that

$$\text{BOTH}\begin{cases} x \geqq -37.41|_{x:=+68.92} \\ x \leqq +68.92|_{x:=+68.92} \end{cases} \text{ is TRUE}$$

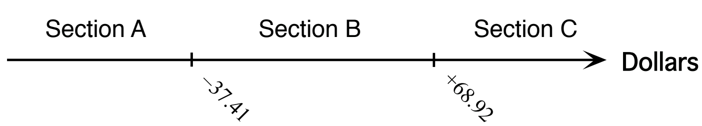
so that $+68.92$ is *included* in the solution subset and we must graph $+68.92$ with a *solid* dot.

Altogether, we have



2. We locate the *interior* of the solution subset. This involves the following steps:

i. The boundary points divide the data set into three sections



ii. We test Section A with, for instance, -1000. Since we have

$$x \geqq -37.41|_{x:=-1000} \text{ is FALSE}$$
$$x \leqq +68.92|_{x:=-1000} \text{ is TRUE}$$

and since, in order for -1000 to be a solution with the connector BOTH, -1000 has to satisfy BOTH formulas, we have that

$$\text{BOTH}\begin{cases} x \geqq -37.41|_{x:=-1000} \\ x \leqq +68.92|_{x:=-1000} \end{cases} \text{ is FALSE}$$

so that -1000 is *non-included* in the solution subset. **Pasch's Theorem** then tells us that *all* number-phrases in Section A are *non-included* in the solution subset.

iii. We test Section B with, for instance, 0. Since we have

$$x \geqq -37.41|_{x:=0} \text{ is TRUE}$$
$$x \leqq +68.92|_{x:=0} \text{ is TRUE}$$

and since, in order for 0 to be a solution with the connector BOTH, 0 has to satisfy BOTH formulas, we have that

$$\text{BOTH}\begin{cases} x \geqq -37.41|_{x:=0} \\ x \leqq +68.92|_{x:=0} \end{cases} \text{ is TRUE}$$

so that 0 is *included* in the solution subset. **Pasch's Theorem** then tells us that *all* number-phrases in Section B are *included* in the solution subset.

iv. We test Section C with, for instance, $+1000$. Since we have

$$x \geqq -37.41|_{x:=+1000} \text{ is TRUE}$$
$$x \leqq +68.92|_{x:=+1000} \text{ is FALSE}$$

and since, in order for $+1000$ to be a solution with the connector BOTH, $+1000$ has to satisfy BOTH formulas, we have that

$$\text{BOTH}\begin{cases} x \geqq -37.41|_{x:=+1000} \\ x \leqq +68.92|_{x:=+1000} \end{cases} \quad \text{is FALSE}$$

so that $+1000$ is *non-included* in the solution subset. **Pasch's Theorem** then tells us that *all* number-phrases in Section A are *non-included* in the solution subset.

3. We *represent* and *describe* the solution subset of the problem of type BETWEEN in **Dollars**

$$\text{BOTH}\begin{cases} x \geqq -37.41 \\ x \leqq +68.92 \end{cases}$$

- The *graph* of the solution subset is the *lower-closed*, *upper-closed segment*

- The *name* of the solution subset is the *lower-closed*, *upper-closed interval*

$$\big[-37.41, +68.92\big] \text{ Dollars}$$

III. Problems of type BETWEEN of the kind $\text{BOTH}\begin{cases} x \geqq x_1 \\ x < x_2 \end{cases}$

EXAMPLE 5. Given the problem in **Dollars**

$$\text{BOTH}\begin{cases} x \geqq -37.41 \\ x < +68.92 \end{cases}$$

this is a problem of type BETWEEN and we get its solution subset according to thePASCH PROCEDURE:

1. We locate the *boundary* of the solution subset. This involves the following steps:

i. We solve the double basic equation problem associated with the given problem:

$$\text{OR}\begin{cases} x = -37.41 \\ x = +68.92 \end{cases}$$

which gives us the boundary points -37.41 and $+68.92$.

ii. We check if the *boundary points* are in the solution subset.

- Since we have

$$x \geqq -37.41|_{x:=-37.41} \text{ is TRUE}$$
$$x < +68.92|_{x:=-37.41} \text{ is TRUE}$$

and since, in order for -37.41 to be a solution with the connector BOTH, -37.41 has to satisfy BOTH formulas, we have that

$$\text{BOTH}\begin{cases} x \geqq -37.41|_{x:=-37.41} \\ x < +68.92|_{x:=-37.41} \end{cases} \quad \text{is TRUE}$$

so that -37.41 is *included* in the solution subset and we must graph -37.41 with a *solid* dot.

- Since we have

$$x \geqq -37.41|_{x:=+68.92} \text{ is TRUE}$$
$$x < +68.92|_{x:=+68.92} \text{ is FALSE}$$

and since, in order to be a solution with the connector BOTH, $+68.92$ has to satisfy BOTH formulas, we have that

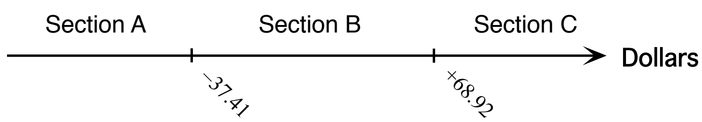
$$\text{BOTH}\begin{cases} x \geqq -37.41|_{x:=+68.92} \\ x < +68.92|_{x:=+68.92} \end{cases} \text{ is FALSE}$$

so that $+68.92$ is *non-included* in the solution subset and we must graph $+68.92$ with a *hollow* dot.

Altogether, we have

2. We locate the *interior* of the solution subset. This involves the following steps:

i. The boundary points divide the data set into three sections



ii. We test Section A with, for instance, -1000. Since we have

$$x \geqq -37.41|_{x:=-1000} \text{ is FALSE}$$
$$x < +68.92|_{x:=-1000} \text{ is TRUE}$$

and since, in order for -1000 to be a solution with the connector BOTH, -1000 has to satisfy BOTH formulas, we have that

$$\text{BOTH}\begin{cases} x \geqq -37.41|_{x:=-1000} \\ x < +68.92|_{x:=-1000} \end{cases} \text{ is FALSE}$$

so that -1000 is *non-included* in the solution subset. **Pasch's Theorem** then tells us that *all* number-phrases in Section A are *non-included* in the solution subset.

iii. We test Section B with, for instance, 0. Since we have

$$x \geqq -37.41|_{x:=0} \text{ is TRUE}$$
$$x < +68.92|_{x:=0} \text{ is TRUE}$$

and since, in order for 0 to be a solution with the connector BOTH, 0 has to satisfy BOTH formulas, we have that

$$\text{BOTH}\begin{cases} x \geqq -37.41|_{x:=0} \\ x < +68.92|_{x:=0} \end{cases} \text{ is TRUE}$$

so that 0 is *included* in the solution subset. **Pasch's Theorem** then tells us that *all* number-phrases in Section B are *included* in the solution subset.

iv. We test Section C with, for instance, $+1000$. Since we have

$$x \geqq -37.41|_{x:=+1000} \text{ is TRUE}$$
$$x < +68.92|_{x:=+1000} \text{ is FALSE}$$

and since, in order for $+1000$ to be a solution with the connector BOTH, $+1000$ has to satisfy BOTH formulas, we have that

$$\text{BOTH}\begin{cases} x \geqq -37.41|_{x:=+1000} \\ x < +68.92|_{x:=+1000} \end{cases} \quad \text{is FALSE}$$

so that $+1000$ is *non-included* in the solution subset. **Pasch's Theorem** then tells us that *all* number-phrases in Section A are *non-included* in the solution subset.

3. We *represent* and *describe* the solution subset of the problem of type BETWEEN in **Dollars**

$$\text{BOTH}\begin{cases} x \geqq -37.41 \\ x < +68.92 \end{cases}$$

- The *graph* of the solution subset is the *lower-closed*, *upper-open segment*

- The *name* of the solution subset is the *lower-closed*, *upper-open interval*

$$\big[-37.41, +68.92\big) \text{ Dollars}$$

IV. Problems of type BETWEEN of the kind $\text{BOTH}\begin{cases} x > x_1 \\ x \leqq x_2 \end{cases}$

EXAMPLE 6. Given the problem in **Dollars**

$$\text{BOTH}\begin{cases} x > -37.41 \\ x \leqq +68.92 \end{cases}$$

this is a problem of type BETWEEN and we get its solution subset according to thePASCH PROCEDURE:

1. We locate the *boundary* of the solution subset. This involves the following steps:

i. We solve the double basic equation problem associated with the given problem:

$$\text{OR}\begin{cases} x = -37.41 \\ x = +68.92 \end{cases}$$

which gives us the boundary points -37.41 and $+68.92$.

ii. We check if the *boundary points* are in the solution subset.

- Since we have

$$x > -37.41|_{x:=-37.41} \text{ is FALSE}$$
$$x \leqq +68.92|_{x:=-37.41} \text{ is TRUE}$$

and since, in order for -37.41 to be a solution with the connector BOTH, -37.41 has to satisfy BOTH formulas, we have that

$$\text{BOTH}\begin{cases} x > -37.41|_{x:=-37.41} \\ x \leqq +68.92|_{x:=-37.41} \end{cases} \quad \text{is FALSE}$$

so that -37.41 is *non-included* in the solution subset and we must graph -37.41 with a *hollow* dot.

- Since we have

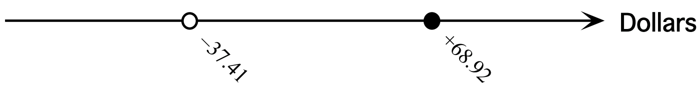
$$x > -37.41|_{x:=+68.92} \text{ is TRUE}$$
$$x \leqq +68.92|_{x:=+68.92} \text{ is TRUE}$$

and since, in order for $+68.92$ to be a solution with the connector BOTH, $+68.92$ has to satisfy BOTH formulas, we have that

$$\text{BOTH}\begin{cases} x > -37.41|_{x:=+68.92} \\ x \leqq +68.92|_{x:=+68.92} \end{cases} \quad \text{is TRUE}$$

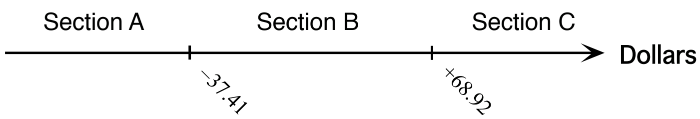
so that $+68.92$ is *included* in the solution subset and we must graph $+68.92$ with a *solid* dot.

Altogether, we have



2. We locate the *interior* of the solution subset. This involves the following steps:

i. The boundary points divide the data set into three sections



ii. We test Section A with, for instance, -1000. Since we have

$$x > -37.41|_{x:=-1000} \quad \text{is FALSE}$$
$$x \leqq +68.92|_{x:=-1000} \quad \text{is TRUE}$$

and since, in order for -1000 to be a solution with the connector BOTH, -1000 has to satisfy BOTH formulas, we have that

$$\text{BOTH}\begin{cases} x > -37.41|_{x:=-1000} \\ x \leqq +68.92|_{x:=-1000} \end{cases} \quad \text{is FALSE}$$

so that -1000 is *non-included* in the solution subset. **Pasch's Theorem** then tells us that *all* number-phrases in Section A are *non-included* in the solution subset.

iii. We test Section B with, for instance, 0. Since we have

$$x > -37.41|_{x:=0} \quad \text{is TRUE}$$
$$x \leqq +68.92|_{x:=0} \quad \text{is TRUE}$$

and since, in order for 0 to be a solution with the connector BOTH, 0 has to satisfy BOTH formulas, we have that

$$\text{BOTH}\begin{cases} x > -37.41|_{x:=0} \\ x \leqq +68.92|_{x:=0} \end{cases} \quad \text{is TRUE}$$

so that 0 is *included* in the solution subset. **Pasch's Theorem** then tells us that *all* number-phrases in Section B are *included* in the solution subset.

iv. We test Section C with, for instance, $+1000$. Since we have

$$x > -37.41|_{x:=+1000} \quad \text{is TRUE}$$
$$x \leqq +68.92|_{x:=+1000} \quad \text{is FALSE}$$

and since, in order for $+1000$ to be a solution with the connector BOTH, $+1000$ has to satisfy BOTH formulas, we have that

$$\text{BOTH}\begin{cases} x > -37.41|_{x:=+1000} \\ x \leqq +68.92|_{x:=+1000} \end{cases} \quad \text{is FALSE}$$

so that $+1000$ is *non-included* in the solution subset. **Pasch's Theorem** then tells us that *all* number-phrases in Section A are *non-included* in the solution subset.

3. We *represent* and *describe* the solution subset of the problem of type BETWEEN in **Dollars**

beyond
OR

$$\text{BOTH}\begin{cases} x > -37.41 \\ x \leqq +68.92 \end{cases}$$

- The *graph* of the solution subset is the *lower-open*, *upper-closed segment*

- The *name* of the solution subset is the *lower-open*, *upper-closed interval*

$$\big(-37.41, +68.92\big] \text{ Dollars}$$

11.3 Problems of Type BEYOND

These are the *second* of the two types of double basic inequation problems that we shall investigate in full in this chapter but the development of this investigation will be completely similar to that for the problems of type BETWEEN.

1. Given a set of *selectable* collections and given two *gauge* collections, we can specify a subset of collections by the requirement that the size of the collections be **beyond** the sizes of the two *gauge collections*.

EXAMPLE 7. It is often said that in order to qualify for a one million dollar loan, you must be worth either more than one hundred million dollars or already be in debt for one hundred millions dollars. Thus, your worth must be beyond minus one hundred million dollars and plus one hundred millions dollars

In other words, we require that the size of the collections in the *subset* be EITHER

- *smaller* than the size of the *smaller* of the two gauge collections

OR

- *larger* than the size of the *larger* of the two gauge collections

NOTE. Here we don't have to say whether AND BOTH or BUT NOT BOTH since a collection cannot be at the same time *larger* than the *larger* of the two gauge collections and *smaller* than the *smaller* of the two gauge collections. So, here again, we will just sat **OR**

2. We now discuss the paper representation in some generality.

a. We start with two *gauge-numerators*, x_1 and x_1, that is with the numerators of the number-phrases that represent the two *gauge* collections. One of the gauge numerators has of course to be smaller than the other and so, for the sake of convenience, we shall call let

$$\boxed{x_1 < x_2}$$

so that

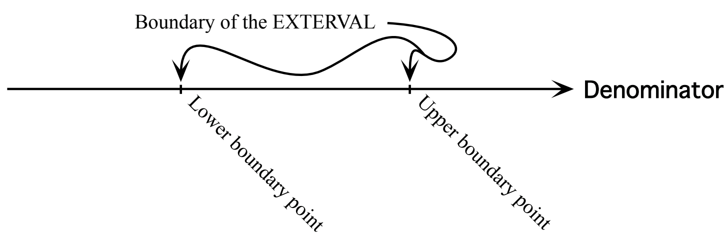
- x_1 will be the *smaller* of the two gauge numerators
- x_2 will be the *larger* of the two gauge numerators

b. Since each one of the two verbs can be either *strict* of *lenient*, there will be four kinds of **problems of type BEYOND**:

$$\text{OR}\begin{cases} x < x_1 \\ x > x_2 \end{cases} \qquad \text{OR}\begin{cases} x \leqq x_1 \\ x \geqq x_2 \end{cases} \qquad \text{OR}\begin{cases} x \leqq x_1 \\ x > x_2 \end{cases} \qquad \text{OR}\begin{cases} x < x_1 \\ x \geqq x_2 \end{cases}$$

3. The *solution subset* of any problem of type BEYOND is called an **exterval**[1]:

- The **boundary** of an *exterval* consists of the two *gauge* numerators which are called **boundary points** of the exterval.



However, the double basic equation problem

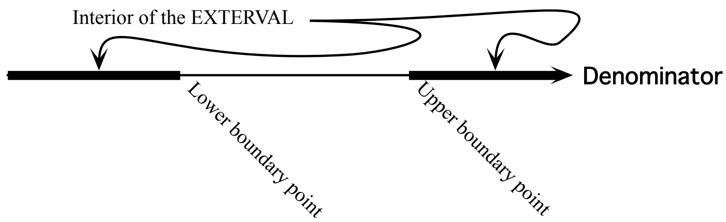
$$\text{OR}\begin{cases} x = x_1 \\ x = x_2 \end{cases}$$

being associated with a double basic *inequation* problem, each one of the two *boundary points* may be *included* or *non-included* in the solution subset of the double inequation problem depending on whether the corresponding inequation is *strict* or *lenient*.
We shall graph the boundary points as usual, that is with a *solid dot* for a boundary point that is *included* in the solution subset and a *hollow dot* for a boundary point that is *non-included* in the solution subset.

- The **interior** of an *exterval* consists of all the numerators that are *beyond* the two *gauge* numerators, that is, the interior consists of all numerators that are EITHER larger than the larger gauge numerator OR smaller than the smaller gauge numerator. So, we represent the interior of the exterval by a **double ray**. Since an exterval is made of two rays, we will use the symbol $\cup$, read "**union**", to *name* the assembly.

problem (of type BEYOND
exterval
boundary (of an exterval)
boundary points (of an exterval)
interior (of an exterval)
double ray
$\cup$
union

[1]The author fervently hopes that Educologists will not object to this term. While decidedly unheard of—so far, it makes perfect sense, at least etymologically.



4. We now investigate an EXAMPLE of each one of the four kinds of problem of type BEYOND.

I. Problems of type BEYOND of the kind $\text{OR}\begin{cases} x < x_1 \\ x > x_2 \end{cases}$

EXAMPLE 8. Given the problem in **Dollars**

$$\text{OR}\begin{cases} x < -37.41 \\ x > +68.92 \end{cases}$$

this is a problem of type BEYOND. We get the solution subset as usual, that is according to thePASCH PROCEDURE:

1. We locate the *boundary* of the solution subset. This involves the following steps:

i. We solve the double basic equation problem associated with the given problem:

$$\text{OR}\begin{cases} x = -37.41 \\ x = +68.92 \end{cases}$$

which gives us the boundary points -37.41 and $+68.92$.

ii. We check if the *boundary points* are in the solution subset.

- Since we have

$$x < -37.41|_{x:=-37.41} \text{ is FALSE}$$
$$x > +68.92|_{x:=-37.41} \text{ is FALSE}$$

and since, in order for -37.41 to be a solution with the connector OR, -37.41 has to satisfy AT LEAST ONE formula, we have that

$$\text{OR}\begin{cases} x < -37.41|_{x:=-37.41} \\ x > +68.92|_{x:=-37.41} \end{cases} \text{ is FALSE}$$

so that -37.41 is *non-included* in the solution subset and we must graph -37.41 with a *hollow* dot.

- Since we have

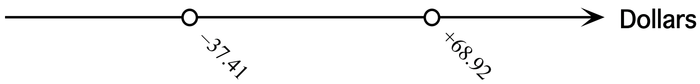
$$x < -37.41|_{x:=+68.92} \text{ is FALSE}$$
$$x > +68.92|_{x:=+68.92} \text{ is FALSE}$$

and since, in order for $+68.92$ to be a solution with the connector OR, $+68.92$ has to satisfy AT LEAST ONE formula, we have that

$$\text{OR}\begin{cases} x < -37.41|_{x:=+68.92} \\ x > +68.92|_{x:=+68.92} \end{cases} \text{ is FALSE}$$

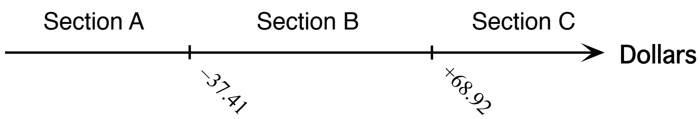
so that $+68.92$ is *non-included* in the solution subset and we must graph $+68.92$ with a *hollow* dot.

Altogether, we have



2. We locate the *interior* of the solution subset. This involves the following steps:

i. The boundary points divide the data set into three sections



ii. We test Section A with, for instance, -1000. Since we have

$$x < -37.41|_{x:=-1000} \text{ is TRUE}$$
$$x > +68.92|_{x:=-1000} \text{ is FALSE}$$

and since, in order for -1000 to be a solution with the connector OR, -1000 has to satisfy AT LEAST ONE formula, we have that

$$\text{OR}\begin{cases} x < -37.41|_{x:=-1000} \\ x > +68.92|_{x:=-1000} \end{cases} \quad \text{is TRUE}$$

so that -1000 is *included* in the solution subset. **Pasch's Theorem** then tells us that *all* number-phrases in Section A are *included* in the solution subset.

iii. We test Section B with, for instance, 0. Since we have

$$x < -37.41|_{x:=0} \text{ is FALSE}$$
$$x > +68.92|_{x:=0} \text{ is FALSE}$$

and since, in order for 0 to be a solution with the connector OR, 0 has to satisfy AT LEAST ONE formula, we have that

$$\text{OR}\begin{cases} x < -37.41|_{x:=0} \\ x > +68.92|_{x:=0} \end{cases} \quad \text{is FALSE}$$

so that 0 is *non-included* in the solution subset. **Pasch's Theorem** then tells us that *all* number-phrases in Section B are *non-included* in the solution subset.

iv. We test Section C with, for instance, $+1000$. In order for $+1000$ to be a solution of the double problem, $+1000$ has to satisfy ONE of the inequations. Since we have

$$x < -37.41|_{x:=+1000} \text{ is FALSE}$$
$$x > +68.92|_{x:=+1000} \text{ is TRUE}$$

and since, in order for $+1000$ to be a solution with the connector OR, $+1000$ has to satisfy AT LEAST ONE formula, we have that

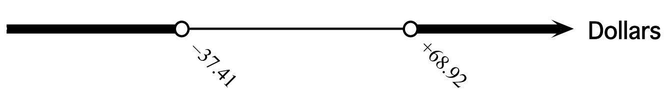
$$\text{OR}\begin{cases} x < -37.41|_{x:=+1000} \\ x > +68.92|_{x:=+1000} \end{cases} \quad \text{is TRUE}$$

so that $+1000$ is *included* in the solution subset. **Pasch's Theorem** then tells us that *all* number-phrases in Section A are *included* in the solution subset.

3. We *represent* and *describe* the solution subset of the problem of type BEYOND in **Dollars**

$$\text{BOTH}\begin{cases} x < -37.41 \\ x > +68.92 \end{cases}$$

- The *graph* of the solution subset is the *lower-open*, *upper-open double ray*



- The *name* of the solution subset is the *lower-open*, *upper-open exterval*

$$\big(-\infty, -37.41\big) \cup \big(+68.92, +\infty\big) \text{ **Dollars**}$$

II. Problems of type BEYOND of the kind $\text{OR}\begin{cases} x \leqq x_1 \\ x \geqq x_2 \end{cases}$

EXAMPLE 9. Given the problem in **Dollars**

$$\text{OR}\begin{cases} x \leqq -37.41 \\ x \geqq +68.92 \end{cases}$$

this is a problem of type BEYOND. We get the solution subset as usual, that is according to thePASCH PROCEDURE:

1. We locate the *boundary* of the solution subset. This involves the following steps:

i. We solve the double basic equation problem associated with the given problem:

$$\text{OR}\begin{cases} x = -37.41 \\ x = +68.92 \end{cases}$$

which gives us the boundary points -37.41 and $+68.92$.

ii. We check if the *boundary points* are in the solution subset.

- Since we have

$$x \leqq -37.41\big|_{x:=-37.41} \text{ is TRUE}$$
$$x \geqq +68.92\big|_{x:=-37.41} \text{ is FALSE}$$

and since, in order for -37.41 to be a solution with the connector OR, -37.41 has to satisfy AT LEAST ONE formula, we have that

$$\text{OR}\begin{cases} x \leqq -37.41\big|_{x:=-37.41} \\ x \geqq +68.92\big|_{x:=-37.41} \end{cases} \text{ is TRUE}$$

so that -37.41 is *included* in the solution subset and we must graph -37.41 with a *solid* dot.

- Since we have

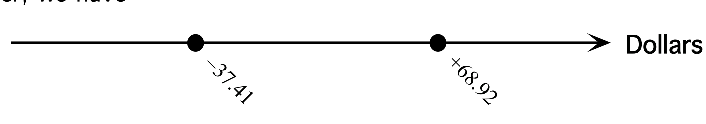
$$x \leqq -37.41\big|_{x:=+68.92} \text{ is FALSE}$$
$$x \geqq +68.92\big|_{x:=+68.92} \text{ is TRUE}$$

and since, in order for $+68.92$ to be a solution with the connector OR, $+68.92$ has to satisfy AT LEAST ONE formula, we have that

$$\text{OR}\begin{cases} x \leqq -37.41\big|_{x:=+68.92} \\ x \geqq +68.92\big|_{x:=+68.92} \end{cases} \text{ is TRUE}$$

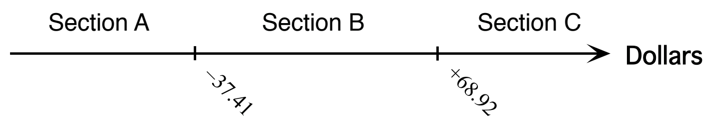
so that $+68.92$ is *included* in the solution subset and we must graph $+68.92$ with a *solid* dot.

Altogether, we have



2. We locate the *interior* of the solution subset. This involves the following steps:

i. The boundary points divide the data set into three sections



ii. We test Section A with, for instance, -1000. Since we have

$$x \leqq -37.41|_{x:=-1000} \text{ is TRUE}$$
$$x \geqq +68.92|_{x:=-1000} \text{ is FALSE}$$

and since, in order for -1000 to be a solution with the connector OR, -1000 has to satisfy AT LEAST ONE formula, we have that

$$\text{OR}\begin{cases} x \leqq -37.41|_{x:=-1000} \\ x \geqq +68.92|_{x:=-1000} \end{cases} \text{ is TRUE}$$

so that -1000 is *included* in the solution subset. **Pasch's Theorem** then tells us that *all* number-phrases in Section A are *included* in the solution subset.

iii. We test Section B with, for instance, 0. Since we have

$$x \leqq -37.41|_{x:=0} \text{ is FALSE}$$
$$x \geqq +68.92|_{x:=0} \text{ is FALSE}$$

and since, in order for 0 to be a solution with the connector OR, 0 has to satisfy AT LEAST ONE formula, we have that

$$\text{OR}\begin{cases} x \leqq -37.41|_{x:=0} \\ x \geqq +68.92|_{x:=0} \end{cases} \text{ is FALSE}$$

so that 0 is *non-included* in the solution subset. **Pasch's Theorem** then tells us that *all* number-phrases in Section B are *non-included* in the solution subset.

iv. We test Section C with, for instance, $+1000$. Since we have

$$x \leqq -37.41|_{x:=+1000} \text{ is FALSE}$$
$$x \geqq +68.92|_{x:=+1000} \text{ is TRUE}$$

and since, in order for $+1000$ to be a solution with the connector OR, $+1000$ has to satisfy AT LEAST ONE formula, we have that

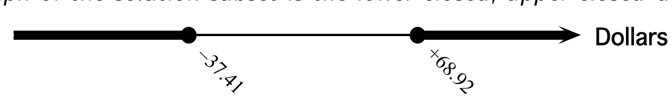
$$\text{OR}\begin{cases} x \leqq -37.41|_{x:=+1000} \\ x \geqq +68.92|_{x:=+1000} \end{cases} \text{ is TRUE}$$

so that $+1000$ is *included* in the solution subset. **Pasch's Theorem** then tells us that *all* number-phrases in Section A are *included* in the solution subset.

3. We *represent* and *describe* the solution subset of the problem of type BEYOND in **Dollars**

$$\text{OR}\begin{cases} x \leqq -37.41 \\ x \geqq +68.92 \end{cases}$$

- The *graph* of the solution subset is the *lower-closed*, *upper-closed double ray*



- The *name* of the solution subset is the *lower-closed*, *upper-closed exterval*

$$\big(-\infty, -37.41\big] \cup \big[+68.92, +\infty\big) \text{ Dollars}$$

III. Problems of type BEYOND of the kind $\text{OR}\begin{cases} x \leqq x_1 \\ x > x_2 \end{cases}$

EXAMPLE 10. Given the problem in **Dollars**

$$\text{OR}\begin{cases} x \leqq -37.41 \\ x > +68.92 \end{cases}$$

this is a problem of type BEYOND. We get the solution subset as usual, that is according to thePASCH PROCEDURE:

1. We locate the *boundary* of the solution subset. This involves the following steps:

i. We solve the double basic equation problem associated with the given problem:

$$\text{OR}\begin{cases} x = -37.41 \\ x = +68.92 \end{cases}$$

which gives us the boundary points -37.41 and $+68.92$.

ii. We check if the *boundary points* are in the solution subset.

- Since we have

$$x \leqq -37.41|_{x:=-37.41} \text{ is TRUE}$$
$$x > +68.92|_{x:=-37.41} \text{ is FALSE}$$

and since, in order for -37.41 to be a solution with the connector OR, -37.41 has to satisfy AT LEAST ONE formula, we have that

$$\text{OR}\begin{cases} x \leqq -37.41|_{x:=-37.41} \\ x > +68.92|_{x:=-37.41} \end{cases} \text{ is TRUE}$$

so that -37.41 is *included* in the solution subset and we must graph -37.41 with a *solid* dot.

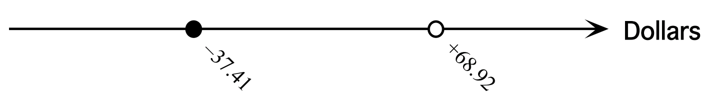
- Since we have

$$x \leqq -37.41|_{x:=+68.92} \text{ is FALSE}$$
$$x > +68.92|_{x:=+68.92} \text{ is FALSE}$$

and since, in order for $+68.92$ to be a solution with the connector OR, $+68.92$ has to satisfy AT LEAST ONE formula, we have that

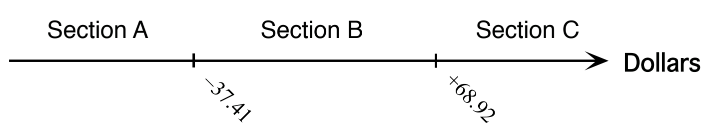
$$\text{OR}\begin{cases} x \leqq -37.41|_{x:=+68.92} \\ x > +68.92|_{x:=+68.92} \end{cases} \text{ is FALSE}$$

so that $+68.92$ is *non-included* in the solution subset and we must graph $+68.92$ with a *hollow* dot.



2. We locate the *interior* of the solution subset. This involves the following steps:

i. The boundary points divide the data set into three sections



ii. We test Section A with, for instance, -1000. Since we have

$$x \leqq -37.41|_{x:=-1000} \text{ is TRUE}$$
$$x > +68.92|_{x:=-1000} \text{ is FALSE}$$

and since, in order for -1000 to be a solution with the connector OR, -1000 has to satisfy AT LEAST ONE formula, we have that

$$\text{OR}\begin{cases} x \leqq -37.41\big|_{x:=-1000} \\ x > +68.92\big|_{x:=-1000} \end{cases} \quad \text{is TRUE}$$

so that -1000 is *included* in the solution subset. **Pasch's Theorem** then tells us that *all* number-phrases in Section A are *included* in the solution subset.

iii. We test Section B with, for instance, 0. Since we have

$$x \leqq -37.41\big|_{x:=0} \text{ is FALSE}$$
$$x > +68.92\big|_{x:=0} \text{ is FALSE}$$

and since, in order for 0 to be a solution with the connector OR, 0 has to satisfy AT LEAST ONE formula, we have that

$$\text{OR}\begin{cases} x \leqq -37.41\big|_{x:=0} \\ x > +68.92\big|_{x:=0} \end{cases} \quad \text{is FALSE}$$

so that 0 is *non-included* in the solution subset. **Pasch's Theorem** then tells us that *all* number-phrases in Section B are *non-included* in the solution subset.

iv. We test Section C with, for instance, $+1000$. Since we have

$$x \leqq -37.41\big|_{x:=+1000} \text{ is FALSE}$$
$$x > +68.92\big|_{x:=+1000} \text{ is TRUE}$$

and since, in order for $+1000$ to be a solution with the connector OR, $+1000$ has to satisfy AT LEAST ONE formula, we have that

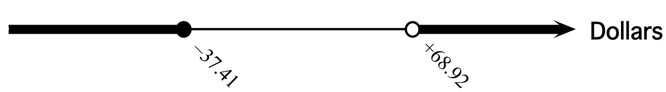
$$\text{OR}\begin{cases} x \leqq -37.41\big|_{x:=+1000} \\ x > +68.92\big|_{x:=+1000} \end{cases} \quad \text{is TRUE}$$

so that $+1000$ is *included* in the solution subset. **Pasch's Theorem** then tells us that *all* number-phrases in Section A are *included* in the solution subset.

3. We *represent* and *describe* the solution subset of the problem of type BEYOND in **Dollars**

$$\text{OR}\begin{cases} x \leqq -37.41 \\ x > +68.92 \end{cases}$$

- The *graph* of the solution subset is the *lower-open*, *upper-open double ray*



- The *name* of the solution subset is the *lower-open*, *upper-open exterval*

$$\big(-\infty, -37.41\big] \cup \big(+68.92, +\infty\big) \text{ Dollars}$$

IV. Problems of type BEYOND of the kind $\text{OR}\begin{cases} x < x_1 \\ x \geqq x_2 \end{cases}$

EXAMPLE 11. Given the problem in **Dollars**

$$\text{OR}\begin{cases} x < -37.41 \\ x \geqq +68.92 \end{cases}$$

this is a problem of type BEYOND. We get the solution subset as usual, that is according to thePASCH PROCEDURE:

1. We locate the *boundary* of the solution subset. This involves the following steps:

i. We solve the double basic equation problem associated with the given problem:

$$\text{OR}\begin{cases} x = -37.41 \\ x = +68.92 \end{cases}$$

which gives us the boundary points -37.41 and $+68.92$.

ii. We check if the *boundary points* are in the solution subset.

- Since we have

$$x < -37.41|_{x:=-37.41} \text{ is FALSE}$$
$$x \geqq +68.92|_{x:=-37.41} \text{ is FALSE}$$

and since, in order for -37.41 to be a solution with the connector OR, -37.41 has to satisfy AT LEAST ONE formula, we have that

$$\text{OR}\begin{cases} x < -37.41|_{x:=-37.41} \\ x \geqq +68.92|_{x:=-37.41} \end{cases} \text{ is FALSE}$$

so that -37.41 is *non-included* in the solution subset and we must graph -37.41 with a *hollow* dot.

- Since we have

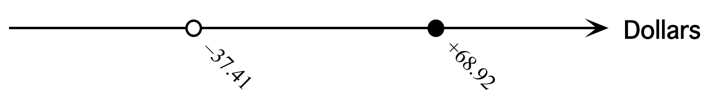
$$x < -37.41|_{x:=+68.92} \text{ is FALSE}$$
$$x \geqq +68.92|_{x:=+68.92} \text{ is TRUE}$$

and since, in order for $+68.92$ to be a solution with the connector OR, $+68.92$ has to satisfy AT LEAST ONE formula, we have that

$$\text{OR}\begin{cases} x < -37.41|_{x:=+68.92} \\ x \geqq +68.92|_{x:=+68.92} \end{cases} \text{ is TRUE}$$

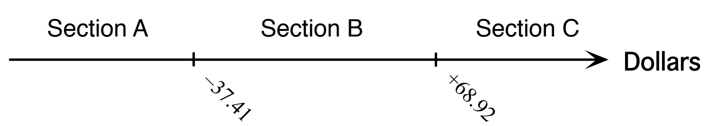
so that $+68.92$ is *included* in the solution subset and we must graph $+68.92$ with a *solid* dot.

Altogether, we have



2. We locate the *interior* of the solution subset. This involves the following steps:

i. The boundary points divide the data set into three sections



ii. We test Section A with, for instance, -1000. Since we have

$$x < -37.41|_{x:=-1000} \text{ is TRUE}$$
$$x \geqq +68.92|_{x:=-1000} \text{ is FALSE}$$

and since, in order for -1000 to be a solution with the connector OR, -1000 has to satisfy AT LEAST ONE formula, we have that

$$\text{OR}\begin{cases} x < -37.41|_{x:=-1000} \\ x \geqq +68.92|_{x:=-1000} \end{cases} \text{ is TRUE}$$

so that -1000 is *included* in the solution subset. **Pasch's Theorem** then tells us that *all* number-phrases in Section A are *included* in the solution subset.

iii. We test Section B with, for instance, 0. Since we have

$$x < -37.41|_{x:=0} \text{ is FALSE}$$
$$x \geqq +68.92|_{x:=0} \text{ is FALSE}$$

and since, in order for 0 to be a solution with the connector OR, 0 has to satisfy AT LEAST ONE formula, we have that

$\backslash$ except

$$\text{OR}\begin{cases} x < -37.41\big|_{x:=0} \\ x \geqq +68.92\big|_{x:=0} \end{cases} \quad \text{is FALSE}$$

so that 0 is *non-included* in the solution subset. **Pasch's Theorem** then tells us that *all* number-phrases in Section B are *non-included* in the solution subset.

iv. We test Section C with, for instance, $+1000$. Since we have

$$x < -37.41\big|_{x:=+1000} \text{ is FALSE}$$
$$x \geqq +68.92\big|_{x:=+1000} \text{ is TRUE}$$

and since, in order for $+1000$ to be a solution with the connector OR, $+1000$ has to satisfy AT LEAST ONE formula, we have that

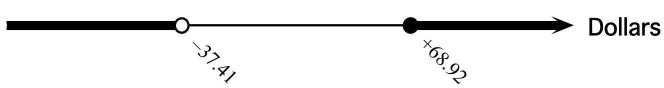
$$\text{OR}\begin{cases} x < -37.41\big|_{x:=+1000} \\ x \geqq +68.92\big|_{x:=+1000} \end{cases} \quad \text{is TRUE}$$

so that $+1000$ is *included* in the solution subset. **Pasch's Theorem** then tells us that *all* number-phrases in Section A are *included* in the solution subset.

3. We *represent* and *describe* the solution subset of the problem of type BEYOND in **Dollars**

$$\text{BOTH}\begin{cases} x < -37.41 \\ x \geqq +68.92 \end{cases}$$

- The *graph* of the solution subset is the *lower-open*, *upper-open double ray*



- The *name* of the solution subset is the *lower-open*, *upper-open exterval*

$$\big(-\infty, -37.41\big) \cup \big[+68.92, +\infty\big) \text{ Dollars}$$

11.4 Other Double Basic Problems

Even with just *basic* inequations and equations, there is large number of possible double problems and it is not possible to memorize them. On the other hand, thePasch Procedure that we used in the case of problems of type BETWEEN of type and problems of type BEYOND of type continues to work.

Here, though, we will usually not be able to just say OR and we usually will have to specify EITHER ONE OR BOTH or EITHER ONE BUT NOT BOTH.

While we will continue to use the symbol $\cup$, it will also be occasionally convenient to use the symbol $\backslash$, read **except** when *naming* the solution subset.

Example 12. Given the double basic inequation problem in **Dollars**

$$\text{EITHER ONE OR BOTH}\begin{cases} x < -37.41 \\ x = +68.92 \end{cases}$$

we get its solution subset according to thePASCH PROCEDURE.

1. We locate the *boundary* of the solution subset. This involves the following steps.

i. We solve the double basic equation problem associated with the given problem:

$$\text{OR}\begin{cases} x = -37.41 \\ x = +68.92 \end{cases}$$

which gives us the boundary point -37.41 and the potential solution $+68.92$.

ii. We check if the *boundary point* -37.41 and the potential solution $+68.92$ are in the solution subset.

- Since we have

  $$x < -37.41|_{x:=-37.41} \text{ is FALSE}$$
  $$x = +68.92|_{x:=-37.41} \text{ is FALSE}$$

  and since, in order for -37.41 to be a solution with the connector EITHER ONE OR BOTH, -37.41 has to satisfy AT LEAST ONE formula, we have that

  $$\text{EITHER ONE OR BOTH}\begin{cases} x < -37.41|_{x:=-37.41} \\ x = +68.92|_{x:=-37.41} \end{cases} \text{ is FALSE}$$

  so that -37.41 is *non-included* in the solution subset and we must graph -37.41 with a *hollow* dot.

- Since we have

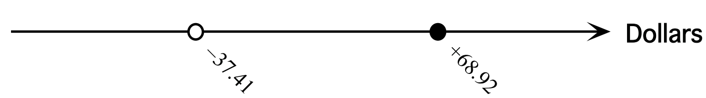
  $$x < -37.41|_{x:=+68.92} \text{ is FALSE}$$
  $$x = +68.92|_{x:=+68.92} \text{ is TRUE}$$

  and since, in order to be a solution with the connector EITHER ONE OR BOTH, $+68.92$ has to satisfy AT LEAST ONE formula, we have that

  $$\text{EITHER ONE OR BOTH}\begin{cases} x < -37.41|_{x:=+68.92} \\ x = +68.92|_{x:=+68.92} \end{cases} \text{ is TRUE}$$

  so that $+68.92$ is *included* in the solution subset and we must graph $+68.92$ with a *sokid* dot.

Altogether, we have



2. We locate the *interior* of the solution subset. This involves the following steps.

i. The boundary point divides the data set in two sections

ii. We test Section A with, for instance, -1000. Since we have

$$x < -37.41|_{x:=-1000} \text{ is TRUE}$$
$$x = +68.92|_{x:=-1000} \text{ is FALSE}$$

and since in order for -1000 to be a solution with the connector EITHER ONE OR BOTH, -1000 has to satisfy AT LEAST ONE formula, we have that

$$\text{EITHER ONE OR BOTH}\begin{cases} x \geqq -37.41|_{x:=-1000} \\ x < +68.92|_{x:=-1000} \end{cases} \text{ is TRUE}$$

so that -1000 is *included* in the solution subset. **Pasch's Theorem** then tells us that *all* number-phrases in Section A are *included* in the solution subset.

iii. We test Section B with, for instance, $+1000$. Since we have

$$x < -37.41|_{x:=+1000} \text{ is FALSE}$$
$$x = +68.92|_{x:=+1000} \text{ is FALSE}$$

and since in order for $+1000$ to be a solution with the connector EITHER ONE OR BOTH, $+1000$ has to satisfy AT LEAST ONE formula, we have that

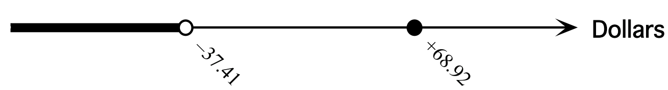
$$\text{EITHER ONE OR BOTH}\begin{cases} x \geqq -37.41|_{x:=+1000} \\ x < +68.92|_{x:=+1000} \end{cases} \text{ is FALSE}$$

so that $+1000$ is *non-included* in the solution subset. **Pasch's Theorem** then tells us that *all* number-phrases in Section B (other than $+68.82$ which was dealt with separately above) are *non-included* in the solution subset.

3. We *represent* and *describe* the solution subset of the problem in **Dollars**

$$\text{EITHER ONE OR BOTH}\begin{cases} x < -37.41 \\ x = +68.92 \end{cases}$$

- The *graph* of the solution subset is



- The *name* of the solution subset is

$$\big(-\infty, -37.41\big) \cup \big\{+68.92\big\} \text{ Dollars}$$

EXAMPLE 13. Given the double basic inequation problem in **Dollars**

$$\text{EITHER ONE BUT NOT BOTH}\begin{cases} x < -37.41 \\ x \leqq +68.92 \end{cases}$$

we get the solution subset according to thePASCH PROCEDURE.

1. We locate the *boundary* of the solution subset. This involves the following steps.

i. We solve the double basic equation problem associated with the given problem:

$$\text{OR}\begin{cases} x = -37.41 \\ x = +68.92 \end{cases}$$

which gives us the boundary points -37.41 and $+68.92$.

ii. We check if the *boundary points* are in the solution subset.

- Since we have

$$x < -37.41|_{x:=-37.41} \text{ is FALSE}$$
$$x \leqq +68.92|_{x:=-37.41} \text{ is TRUE}$$

and since, in order for -37.41 to be a solution with the connector EITHER ONE BUT NOT BOTH, -37.41 has to satisfy EXACTLY ONE formula, we have that

$$\text{EITHER ONE BUT NOT BOTH}\begin{cases} x < -37.41|_{x:=-37.41} \\ x \leqq +68.92|_{x:=-37.41} \end{cases} \text{ is TRUE}$$

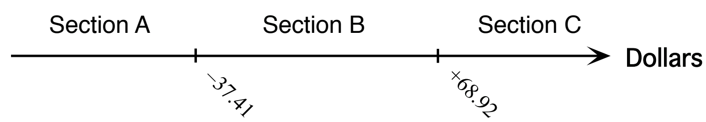
so that -37.41 is *included* in the solution subset and we must graph -37.41 with a *solid* dot.

- Since we have
$$x < -37.41|_{x:=+68.92} \text{ is FALSE}$$
$$x \leqq +68.92|_{x:=+68.92} \text{ is TRUE}$$
and since, in order to be a solution with the connector EITHER ONE BUT NOT BOTH, $+68.92$ has to satisfy EXACTLY ONE formula, we have that
$$\text{EITHER ONE BUT NOT BOTH} \begin{cases} x < -37.41|_{x:=+68.92} \\ x \leqq +68.92|_{x:=+68.92} \end{cases} \text{ is TRUE}$$
so that $+68.92$ is *included* in the solution subset and we must graph $+68.92$ with a *solid* dot.

Altogether, we have

2. We locate the *interior* of the solution subset. This involves the following steps.

i. The boundary points divide the data set into three sections



ii. We test Section A with, for instance, -1000. Since we have
$$x < -37.41|_{x:=-1000} \text{ is TRUE}$$
$$x \leqq +68.92|_{x:=-1000} \text{ is TRUE}$$
and since, in order for -1000 to be a solution with the connector EITHER ONE BUT NOT BOTH, -1000 has to satisfy EXACTLY ONE formula, we have that
$$\text{EITHER ONE BUT NOT BOTH} \begin{cases} x < -37.41|_{x:=-1000} \\ x \leqq +68.92|_{x:=-1000} \end{cases} \text{ is FALSE}$$
so that -1000 is *non-included* in the solution subset. **Pasch's Theorem** then tells us that *all* number-phrases in Section A are *non-included* in the solution subset.

iii. We test Section B with, for instance, 0. Since we have
$$x < -37.41|_{x:=0} \text{ is FALSE}$$
$$x \leqq +68.92|_{x:=0} \text{ is TRUE}$$
and since, in order for 0 to be a solution with the connector EITHER ONE BUT NOT BOTH, 0 has to satisfy EXACTLY ONE formula, we have that
$$\text{EITHER ONE BUT NOT BOTH} \begin{cases} x < -37.41|_{x:=0} \\ x \leqq +68.92|_{x:=0} \end{cases} \text{ is TRUE}$$
so that 0 is *included* in the solution subset. **Pasch's Theorem** then tells us that *all* number-phrases in Section B are *included* in the solution subset.

iv. We test Section C with, for instance, $+1000$. Since we have
$$x < -37.41|_{x:=+1000} \text{ is FALSE}$$
$$x \leqq +68.92|_{x:=+1000} \text{ is FALSE}$$
and since, in order for $+1000$ to be a solution with the connector EITHER ONE BUT NOT BOTH, $+1000$ has to satisfy EXACTLY ONE formula, we have that

$$\text{EITHER ONE BUT NOT BOTH} \begin{cases} x < -37.41\big|_{x:=+1000} \\ x \leqq +68.92\big|_{x:=+1000} \end{cases} \quad \text{is FALSE}$$

so that $+1000$ is *non-included* in the solution subset. **Pasch's Theorem** then tells us that *all* number-phrases in Section C are *non-included* in the solution subset.

3. We *represent* and *describe* the solution subset of the double inequation problem in **Dollars**

$$\text{EITHER ONE BUT NOT BOTH} \begin{cases} x < -37.41 \\ x \leqq +68.92 \end{cases}$$

- The *graph* of the solution subset is

- The *name* of the solution subset is

$$\big[-37.41, +68.92\big] \text{ Dollars}$$

Chapter 12

Double Affine Problems

We conclude Part Two with double problems which are just like those in the preceding chapter but with *affine* problems instead of *basic* problems.

Conceptually, since *affine* problems can be reduced to *basic* problems, there will be absolutely nothing new in this chapter which serves only to show how much our investment in thePASCH PROCEDURE and the REDUCTION APPROACH will pay.

As a result, the only difficulty will be the "staying power" that will be required by the length of some of the computations.

EXAMPLE 1. Solve the double problem in **Dollars**

$$\text{BOTH} \begin{cases} +3x + 4.51 \leqq +23.35 \\ +2.34 < +2x \end{cases}$$

1. The formula $+3x + 4.51 \leqq +23.35$ is an *affine* inequation and the formula $+2.34 < +2x$ is a *basic* inequation so we should be able to find the solution subset on the basis of our previous work. At this point, though, we are not in a position to tell what "named" type of problem this is, if any.

2. We locate the *boundary* of the double problem by looking for the boundary point of each inequation, that is by solving the equation *associated* with each inequation.

a. The equation associated with the inequation $3x + 4.51 \leqq +23.35$ is

$$+3x + 4.51 = +23.35$$

i. In order to reduce this *affine* equation to a *basic* equation, we must get rid of $+4.51$ on the right side which we do by adding its *opposite* -4.51 on both sides so as to be able to invoke the FAIRNESS THEOREM:

$$+3x + 4.51 \boxed{-4.51} = +23.35 \boxed{-4.51}$$

$$+3x = +18.84$$

Then, dividing by $+3$ on both sides

$$+3x \div (+3) = +18.84 \div (+3)$$

gives the basic equation

$$x = +6.28$$

and therefore the boundary point $+6.28$.
ii. We check if the boundary point $+6.28$ is included or non-included in the solution subset.
Since we have

$$+3x + 4.51 \leqq +23.35|_{x:=+6.28} \text{ is TRUE}$$
$$+2.34 < +2x|_{x:=+6.28} \text{ is TRUE}$$

and since, in order for $+6.28$ to be a solution with the connector BOTH, $+6.28$ has to satisfy BOTH formulas, we have that

$$\text{BOTH} \begin{cases} +3x + 4.51 \leqq +23.35|_{x:=+6.28} \\ +2.34 < +2x|_{x:=+6.28} \end{cases} \text{ is TRUE}$$

so that $+6.28$ is *included* in the solution subset and we must graph $+6.28$ with a *solid* dot.

b. The equation associated with the inequation $+2.34 < +2x$ is:

$$+2.34 = +2x$$

i. We reduce to a *basic* equation by dividing both sides by $+2$

$$x = +1.17$$

and therefore the boundary point is $+1.17$
ii. We check if the boundary point $+1.17$ is included or non-included in the solution subset.
Since we have

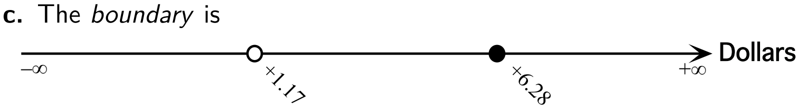
$$+3x + 4.51 \leqq +23.35|_{x:=+1.17} \text{ is TRUE}$$
$$+2.34 < +2x|_{x:=+1.17} \text{ is FALSE}$$

and since, in order for $+1.17$ to be a solution with the connector BOTH, $+1.17$ has to satisfy BOTH formulas, we have that

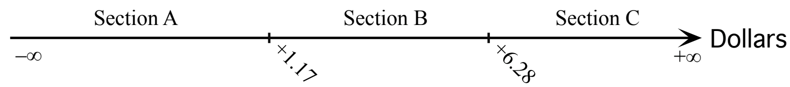
$$\text{BOTH} \begin{cases} +3x + 4.51 \leqq +23.35|_{x:=+1.17} \\ +2.34 < +2x|_{x:=+1.17} \end{cases} \text{ is FALSE}$$

so that $+1.17$ is *non-included* in the solution subset and we must graph $+6.28$ with a *hollow* dot.

c. The *boundary* is



3. We locate the *interior* of the double problem by testing each one of the three sections determined by the two boundary points:



- We test Section A with, for instance, -1000. That is, we must *evaluate* the two formulas in the given problem with -1000.

$$+3x + 4.51 \leqq +23.35|_{x:=-1000}$$
$$+2.34 < +2x|_{x:=-1000}$$

that is

$$+3 \cdot (-1000) + 4.51 \leqq +23.35$$
$$+2.34 < +2 \cdot (-1000)$$

that is

$$-3000 + 4.51 \leqq +23.35$$
$$+2.34 < -2000$$

that is

$$-2995.49 \leqq +23.35 \quad \text{which is TRUE}$$
$$+2.34 < -2000 \quad \text{which is FALSE}$$

Since, in order for -1000 to be a solution with the connector BOTH, -1000 has to satisfy BOTH formulas, we have that

$$\text{BOTH}\begin{cases} +3x + 4.51 \leqq +23.35|_{x:=-1000} \\ +2.34 < \ +2x|_{x=-1000} \end{cases} \qquad \text{is FALSE}$$

so that -1000 is *non-included* in the solution subset. **Pasch's Theorem** then tells us that all number-phrases in Section A are *non-included* in the solution subset.

- We test Section B with, for instance, $+2$. (We cannot test with 0 since 0 is not in Section B.) That is, we must *evaluate* the two formulas in the given problem with $+2$.

$$+3x + 4.51 \leqq +23.35|_{x:=+2}$$
$$+2.34 < +2x|_{x:=+2}$$

that is

$$+3 \cdot (+2) + 4.51 \leqq +23.35$$
$$+2.34 < +2 \cdot (+2)$$

that is

$$+6 + 4.51 \leqq +23.35$$
$$+2.34 < +4$$

that is

$$+10.51 \leqq +23.35 \quad \text{which is TRUE}$$
$$+2.34 < +4 \quad \text{which is TRUE}$$

Since, in order for $+2$ to be a solution with the connector BOTH, $+2$ has to satisfy BOTH formulas, we have that

$$\text{BOTH}\begin{cases} +3x + 4.51 \leqq +23.35|_{x:=+2} \\ +2.34 < \ +2x|_{x=+2} \end{cases} \qquad \text{is TRUE}$$

so that $+2$ is *included* in the solution subset. **Pasch's Theorem** then tells us that all number-phrases in Section B are *included* in the solution subset.

- We test Section C with, for instance, $+1000$. That is, we must *evaluate* the two formulas in the given problem with $+1000$.

$$+3x + 4.51 \leqq +23.35\big|_{x:=+1000}$$
$$+2.34 < +2x\big|_{x:=+1000}$$

that is

$$+3 \cdot (+1000) + 4.51 \leqq +23.35$$
$$+2.34 < +2 \cdot (+1000)$$

that is

$$+3000 + 4.51 \leqq +23.35$$
$$+2.34 < +2000$$

that is

$$+3004.51 \leqq +23.35 \quad \text{which is FALSE}$$
$$+2.34 < +2000 \quad \text{which is TRUE}$$

Since, in order for $+1000$ to be a solution with the connector BOTH, $+1000$ has to satisfy BOTH formulas, we have that

$$\text{BOTH}\begin{cases} +3x + 4.51 \leqq +23.35\big|_{x:=+1000} \\ +2.34 < \ +2x\big|_{x=+1000} \end{cases} \qquad \text{is FALSE}$$

so that -1000 is *non-included* in the solution subset. **Pasch's Theorem** then tells us that all number-phrases in Section A are *non-included* in the solution subset.

4. We represent and describe the solution subset of the problem in **Dollars**

$$\text{BOTH}\begin{cases} +3x + 4.51 \leqq +23.35 \\ +2.34 < +2x \end{cases}$$

- The *graph* of the solution subset is

- The *name* of the solution subset is

$$\big(+1.17, +6.28\big]\ \textbf{Dollars}$$

Part III

Laurent Polynomial Algebra

repeat
involution

Chapter 13

Repeated Multiplications and Divisions

Given a *number-phrase* we investigate what is involved in **repeated** multiplications or *repeated* divisions by a given *numerator*, something which used to be called **involution**[1].

13.1 A Problem With English

English can be confusing when we want to *indicate* "how many times" an operation is to be repeated.

1. One source of confusion is the word "times" because *multiplication* may not be involved at all.

EXAMPLE 1. When we tell someone

Divide 375 **Dollars** 3 times by 5

multiplication is not involved and we just mean:

Divide 375 **Dollars**

i. a first time by 5—which gives 75 **Dollars** as a result,

ii. a second time by 5—which gives 15 **Dollars** as a result,

iii. a third time by 5—which gives 3 **Dollars** as a result.

[1]Educologists will surely deplore this departure from the usual "modern" treatment. Yet, it is difficult to see how conflating *unary operators* and *binary operations* can be helpful.

coefficient
base
plain exponent

NOTE. In fact, the use of "first time", "second time", etc is also a bit misleading since, when we "divide for the second time", we are not dividing the *initial number-phrase* a second time but *the result of the first division* for the first time. Etc.

2. Another source of confusion is when we do not pay attention to the exact place of the word "by".

EXAMPLE 2. While, as we saw in **EXAMPLE** 1,

Divide 375 **Dollars** 3 times *by* 5

results in

3 **Dollars**

it is easily confused with

Divide 375 **Dollars** *by* 3 times 5

that is

Divide 375 **Dollars** *by* 15

whose results is

25 **Dollars**

3. A workaround would seem just to avoid using the word "by" but it is awkward and even misleading when we *say* it and downright dangerous when we *write* it.

EXAMPLE 3. To *say*

multiply 7 **Dollars** by 2, 3 times

can be correctly understood but requires to stop markedly after the 2 as, otherwise, it will be understood to mean

multiply 7 **Dollars** by 2 OR by 3.

EXAMPLE 4. To *write*

multiply 7 **Dollars** by 2, 3 times

can be correctly understood but requires paying attention to the comma between the 2 and the 3 as otherwise it will be understood to mean

Multiply 7 **Dollars** by 23

4. What we will now do will be to develop a *specialized language* to deal with repeated operations. Perhaps surprisingly, though, writing specifying-phrases for *repeated* operations is not quite a simple matter.

13.2 Templates

We begin by looking at the way we actually go about repeating operations.

1. Given a *number-phrase*, whose *numerator* we will refer to as the **coefficient**, and:

- given a *numerator*, called the **base**, by which the given number-phrase is to be repeatedly multiplied or repeatedly divided,
- given a *numerator*, called the **plain exponent**, to indicate how many multiplications or how many divisions we want done on the *coefficient*,

the simplest way to *specify* how many repeated multiplications or how many divisions we want done on the *number-coefficient* is to use a **staggered template** in which each operation is done on a separate line with a separate **copy** of the base.

staggered template
copy

EXAMPLE 5. When we want the number-phrase $+7$ **Dollars** *multiplied* by 6 copies of -2, we say that

- the *coefficient* is $+7$,
- the *base* from which we make the *copies* is -2,
- the *plain exponent* is 6

and we write the following *staggered template*:

$\underbrace{+7 \textbf{ Dollars} \otimes -2}$ (1st multiplication by -2)

$\underbrace{\boxed{} \otimes -2}$ (2nd multiplication by -2)

$\underbrace{\boxed{} \otimes -2}$ (3rd multiplication by -2)

$\underbrace{\boxed{} \otimes -2}$ (4th multiplication by -2)

$\underbrace{\boxed{} \otimes -2}$ (5th multiplication by -2)

$\underbrace{\boxed{} \otimes -2}$ (6th multiplication by -2)

$\boxed{}$ (Result of the repeated multiplications)

The staggered template specifies what is to be done at each stage and therefore what the result will be:

$\underbrace{+7 \textbf{ Dollars} \otimes -2}$ (1st multiplication by -2)

$\underbrace{\boxed{-14 \textbf{ Dollars}} \otimes -2}$ (2nd multiplication by -2)

$\underbrace{\boxed{+28 \textbf{ Dollars}} \otimes -2}$ (3rd multiplication by -2)

$\underbrace{\boxed{-56 \textbf{ Dollars}} \otimes -2}$ (4th multiplication by -2)

$\underbrace{\boxed{+112 \textbf{ Dollars}} \otimes -2}$ (5th multiplication by -2)

$\underbrace{\boxed{-224 \textbf{ Dollars}} \otimes -2}$ (6th multiplication by -2)

$\boxed{+448 \textbf{ Dollars}}$ (Result of the repeated multiplications)

EXAMPLE 6. When we want the number-phrase $+112$ **Dollars** *divided* by 4 copies of -2, we say that

- the *coefficient* is $+112$,
- the *base* from which we make the *copies* is -2,
- the *plain exponent* is 4

and we write the following *staggered template*:

Divide $+112$ **Dollars** by 4 *copies* of -2

we use the *staggered template*:

$$\underbrace{+112 \textbf{ Dollars} \mathbin{⨸} -2} \qquad (1^{st} \text{ division by } -2)$$
$$\underbrace{\boxed{} \mathbin{⨸} -2} \qquad (2^{nd} \text{ division by } -2)$$
$$\underbrace{\boxed{} \mathbin{⨸} -2} \qquad (3^{rd} \text{ division by } -2)$$
$$\underbrace{\boxed{} \mathbin{⨸} -2} \qquad (4^{th} \text{ division by } -2)$$
$$\boxed{} \qquad (\text{Result of the repeated divisions})$$

The staggered template specifies what is to be done at each stage and therefore what the result will be:

$$\underbrace{+112 \textbf{ Dollars} \mathbin{⨸} -2} \qquad (1^{st} \text{ division by } -2)$$
$$\underbrace{\boxed{-56 \textbf{ Dollars}} \mathbin{⨸} -2} \qquad (2^{nd} \text{ division by } -2)$$
$$\underbrace{\boxed{+28 \textbf{ Dollars}} \mathbin{⨸} -2} \qquad (3^{rd} \text{ division by } -2)$$
$$\underbrace{\boxed{-14 \textbf{ Dollars}} \mathbin{⨸} -2} \qquad (4^{th} \text{ division by } -2)$$
$$\boxed{+7 \textbf{ Dollars}} \qquad (\text{Result of the repeated divisions})$$

2. As usual, instead of writing the denominator on each line, we can *declare* the denominator up front and then write the staggered template just for the *numerators*.

EXAMPLE 7. When we want the number-phrase $+7$ **Dollars** *multiplied* by 6 copies of -2, we can

i. *declare* that the template is in **Dollars**

ii. write the staggered template just for the *numerators*

$$\underbrace{+7 \otimes -2} \qquad (1^{st} \text{ multiplication by } -2)$$
$$\underbrace{\boxed{} \otimes -2} \qquad (2^{nd} \text{ multiplication by } -2)$$
$$\underbrace{\boxed{} \otimes -2} \qquad (3^{rd} \text{ multiplication by } -2)$$
$$\underbrace{\boxed{} \otimes -2} \qquad (4^{th} \text{ multiplication by } -2)$$
$$\underbrace{\boxed{} \otimes -2} \qquad (5^{th} \text{ multiplication by } -2)$$
$$\underbrace{\boxed{} \otimes -2} \qquad (6^{th} \text{ multiplication by } -2)$$
$$\boxed{} \qquad (\text{Result of the repeated multiplications})$$

The staggered template specifies what the *numerator* of the result will be and the declaration specifies that the *denominator* is **Dollars**.

EXAMPLE 8. When we want the number-phrase $+112$ **Dollars** *divided* by 4 copies of -2, we say that

- the *coefficient* is $+112$,

in-line template

- the *base* from which we make the *copies* is -2,
- the *plain exponent* is 4

and we write the following *staggered template* in **Dollars**:

$$\underbrace{+112 \; \oplus - 2}$$ (1st division by -2)

$$\underbrace{\boxed{} \oplus - 2}$$ (2nd division by -2)

$$\underbrace{\boxed{} \oplus - 2}$$ (3rd division by -2)

$$\underbrace{\boxed{} \oplus - 2}$$ (4th division by -2)

$$\boxed{}$$ (Result of the repeated divisions)

The staggered template specifies what is to be done at each stage and therefore what the numerator of the result in **Dollars** will be.

3. Quite often, though, we will not want to *get* the actual result but just be able to *discuss* the repeated operations and, in that case, the use of *staggered* templates is cumbersome. So, what we will do is to let the boxes "go without saying" which will allow us to write an **in-line template**, that is:

i. For the *numerators*, we write on a single line:

i. The *coefficient*,

ii. The *operation symbol* followed by the 1st *copy* of the *base*

iii. The *operation symbol* followed by the 2nd *copy* of the *base*

iv. The *operation symbol* followed by the 3rd *copy* of the *base*

v. Etc until all *copies* specified by the *plain exponent* have been written.

ii. For the *denominator*, we have a choice:

– We can *declare* the *denominator* up front and then write the in-line template for the *numerators*,
– We can write the in-line template for the *numerators within square brackets* and then write the *denominator*.

EXAMPLE 9. Instead of writing the *staggered* template in **Dollars**

$$\underbrace{+7 \otimes -2}$$ (1st multiplication by -2)
$$\underbrace{\square \otimes -2}$$ (2nd multiplication by -2)
$$\underbrace{\square \otimes -2}$$ (3rd multiplication by -2)
$$\underbrace{\square \otimes -2}$$ (4th multiplication by -2)
$$\underbrace{\square \otimes -2}$$ (5th multiplication by -2)
$$\underbrace{\square \otimes -2}$$ (6th multiplication by -2)
$$\square$$ (Result of the repeated multiplications)

we can:

- *Declare up front* that the *in-line* template is in **Dollars** and then write:

$$+17 \otimes -2 \;\otimes -2 \;\otimes -2 \;\otimes -2 \;\otimes -2 \;\otimes -2$$

or

- Write the in-line template for the numerators *within square brackets* and then write the denominator **Dollars**

$$\left[+17 \otimes -2 \;\otimes -2 \;\otimes -2 \;\otimes -2 \;\otimes -2 \;\otimes -2\right] \textbf{ Dollars}$$

EXAMPLE 10. Instead of writing the *staggered* template in **Dollars**

$$\underbrace{-208 \odiv -2}$$ (1st division by -2)
$$\underbrace{\square \odiv -2}$$ (2nd division by -2)
$$\underbrace{\square \odiv -2}$$ (3rd division by -2)
$$\underbrace{\square \odiv -2}$$ (4th division by -2)
$$\square$$ (Result of the repeated divisions)

we can

- *Declare up front* that the *in-line* template is in **Dollars** and then write:

$$-208 \odiv -2 \;\odiv -2 \;\odiv -2 \;\odiv -2$$

- Write the in-line template for the numerators *within square brackets* and then write the denominator **Dollars**

$$\left[-208 \odiv -2 \;\odiv -2 \;\odiv -2 \;\odiv -2\right] \textbf{ Dollars}$$

13.3 The Order of Operations

The use of *in-line* templates for repeated operations, though, poses a problem: how do we know for sure in what order the recipient of an *in-line* template is going to do the operations?

The reason this can be a problem is that this order can make all the difference between the recipient arriving at the intended result and the recipient arriving at something completely irrelevant.

1. When the operation being repeated is *multiplication*, it turns out that the order in which the operations are done does *not* matter

EXAMPLE 11. Given the in-line template in **Dollars**

$$17 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2$$

the recipient might choose to compute it as

$$\underbrace{\underbrace{\underbrace{\underbrace{\underbrace{\underbrace{17\times 2}_{34}\times 2}_{68}\times 2}_{136}\times 2}_{272}\times 2}_{544}\times 2}_{1088}$$

or the recipient might choose to compute it as

$$\underbrace{17\times\underbrace{2\times\underbrace{2\times\underbrace{2\times\underbrace{2\times\underbrace{2\times 2}_{4}}_{8}}_{16}}_{32}}_{64}}_{1088}$$

or as

$$\underbrace{17\times\underbrace{2\times\underbrace{2\times\underbrace{\underbrace{\underbrace{2\times 2}_{4}\times 2}_{8}\times 2}_{16}}_{32}}_{64}}_{1088}$$

etc but, it does not matter as the result will always be 1088.

However, proving *in general* that the *order* in which the *multiplications* are done does *not* matter takes some work because, as the number of copies gets large, the number of ways in which the multiplications could be done gets even larger and yet, to be able to make a *general statement*, we would have to make sure that *all* of these ways have been accounted for. So, for the sake of time, in the case of repeated *multiplications*, we will take the following for granted:

THEOREM 6. *The order in which* multiplications *are done does* not *matter.*

2. In the case of repeated *division*, though, the order usually makes a *huge* difference.

Example 12. Given the in-line template in **Dollars**

$$448 \div 2 \div 2 \div 2 \div 2 \div 2 \div 2$$

and while the recipient might indeed choose to compute is as

$$\underbrace{\underbrace{\underbrace{\underbrace{\underbrace{\underbrace{448 \div 2}_{224} \div 2}_{112} \div 2}_{56} \div 2}_{28} \div 2}_{14} \div 2}_{7}$$

the recipient might also choose to compute it as

$$\underbrace{448 \div \underbrace{2 \div \underbrace{2 \div \underbrace{2 \div \underbrace{2 \div \underbrace{2 \div 2}_{1}}_{2}}_{1}}_{2}}_{1}}_{448}$$

or as

$$\underbrace{448 \div \underbrace{2 \div \underbrace{2 \div \underbrace{\underbrace{\underbrace{2 \div 2}_{1} \div 2}_{0.5} \div 2}_{0.25}}_{8}}_{0.25}}_{1796}$$

etc

Thus, in the case of repeated *divisions* it is crucial to agree on the order in which to do them and so, in the absence of any instructions to that effect, we will use

DEFAULT RULE # 5. *The order in which* divisions *are to be done is* from left to right.

13.4 The Way to Powers

Eventually, we will devise a very powerful language to deal both with repeated multiplications and repeated divisions but, before we can do that, we need to clear the way.

1. While, as we have seen, 1 does tend to "go without saying", what we can do when the *coefficient* in a repeated operation is 1 depends on whether the operation being repeated is *multiplication* or *division*.

a. When it is *multiplication* that is being repeated, we can let the coefficient 1 go without saying. However, the number of multiplications is then one less than the number of copies[2].

EXAMPLE 13. Given the in-line template in **Dollars**

$$1 \times 3 \times 3 \times 3 \times 3 \times 3$$

we *can* write instead

$$3 \times 3 \times 3 \times 3 \times 3$$

because we get 243 either way.
However, while we still have five copies of 3, we now have only four multiplications.

b. When it is *division* that is being repeated, we *must* write the coefficient 1 as, if we did not, we would be getting a different result.

EXAMPLE 14. Given the in-line template in **Dollars**

$$1 \div 2 \div 2 \div 2 \div 2 \div 2$$

the 1 cannot go without saying because, while the *given* in-line template computes to $\frac{1}{32}$, if we don't write the coefficient 1, we get an in-line template with coefficient 2 to be divided by four copies of 2:

$$2 \div 2 \div 2 \div 2 \div 2$$

which computes to $\frac{1}{8}$.

2. *Repeated divisions* are related to *repeated multiplications*. Indeed,

- instead of dividing a coefficient by a number of copies of the *base*,
- we can[3]:
  i. multiply 1 repeatedly by the number of copies of the base,
  ii. divide the coefficient by the *result* of the repeated multiplication.

EXAMPLE 15. Given the in-line template in **Dollars**

$$448 \div 2 \div 2 \div 2 \div 2 \div 2 \div 2$$

[2]Educologists will correctly point out that while 1× can "go without saying", this is really where multiplication as a *binary* operation comes in.

[3]Educologist will point out that, essentially, this is just the fact that, instead of dividing by a numerator, we can multiply by its *reciprocal*.

instead of computing it as follows:

$$\underbrace{\underbrace{\underbrace{\underbrace{\underbrace{\underbrace{448 \div 2}_{224} \div 2}_{112} \div 2}_{56} \div 2}_{28} \div 2}_{14} \div 2}_{\boxed{7}}$$

we can proceed as follows:

i. We *multiply* 1 by the 6 copies of 2

$$\underbrace{\underbrace{\underbrace{\underbrace{\underbrace{\underbrace{1 \times 2}_{2} \times 2}_{4} \times 2}_{8} \times 2}_{16} \times 2}_{32} \times 2}_{64}$$

ii. We *divide* the coefficient 448 by the *result* of this repeated multiplications:

$$448 \div 64 = \boxed{7}$$

which indeed gives us the same result as the repeated division.

The advantage of this second way of computing in-line templates involving repeated *divisions* is that while we now have one more *operation* than we had *divisions*, the first multiplication, multiplying the coefficient 1 by the first copy of the base, is no work and, as we saw above, need in fact not even be written so that the number of operations *requiring work* is the same in both cases. But now all operations except one are *multiplications* which are a lot less work than *divisions*.

However, here again, proving *in general* that the results are always the same takes some work so that, for the sake of saving time, we will take for granted that:

THEOREM 7. *A repeated division is the same as a single division of the coefficient by the result of* 1 *multiplied repeatedly by the same number of copies of the base.*

$$\boxed{\text{Coefficient} \mathbin{\oplus\!\!\!\div} \text{copies} = \text{Coefficient} \mathbin{\oplus\!\!\!\div} [1 \otimes \text{copies}]}$$

3. In order to *specify* the second way of computing, we can write either:

bracket in-line template
fraction-like template
fraction bar

- A **bracket in-line template** where we write:
  i. The coefficient followed by a division symbol,
  ii. A pair of square brackets within which we write
  iii. 1 repeatedly multiplied by the same number of copies of the base.

EXAMPLE 16. Instead of writing the in-line template in **Dollars** as

$$+448 ⨸ -2 ⨸ -2 ⨸ -2 ⨸ -2 ⨸ -2 ⨸ -2$$

we can write the *bracket in-line template* in **Dollars** as

$$+448 \div \left[+1 \otimes -2 \otimes -2 \otimes -2 \otimes -2 \otimes -2 \otimes -2\right]$$

or as

$$+448 \div \left[-2 \otimes -2 \otimes -2 \otimes -2 \otimes -2 \otimes -2\right]$$

or

- A **fraction-like template** where we write:
  i. The coefficient and, underneath,
  ii. A **fraction bar** and, underneath
  iii. 1 repeatedly multiplied by the same number of copies of the base

  with the 1 able to "go without saying".
  underneath and the repeated multiplication underneath the *bar*,

EXAMPLE 17. Instead of writing the in-line template in **Dollars**

$$+448 \div -2 \div -2 \div -2 \div -2 \div -2 \div -2$$

we can write the in-line template in **Dollars** as

$$\frac{+448}{+1 \otimes -2 \otimes -2 \otimes -2 \otimes -2 \otimes -2 \otimes -2}$$

or as

$$\frac{+448}{-2 \otimes -2 \otimes -2 \otimes -2 \otimes -2 \otimes -2}$$

NOTE. Whether we use a *bracket in-line template* or a *fraction-like template*, we need not write the 1 as, either way, there is something to remind us that the multiplications have to be done *first*:

- The *square brackets*

or

- The *fraction bar*

In general, though, we will prefer to use *fraction-like* templates with the 1 "going without saying".

In other words, instead of:

$$\boxed{\text{Coefficient } ⨸ \text{ copies} = \text{Coefficient} ⨸ [1 \otimes \text{ copies}]}$$

we prefer to write

monomial specifying-phrase
separator
signed exponent
superscript
signed power

$$\boxed{\text{Coefficient } \oplus\!\!\!\!-\ \text{copies} = \frac{\text{Coefficient}}{\text{copies}}}$$

but, even though both sides are *read* as

"Coefficient divided by copies"

- the *division symbol* $\oplus\!\!\!\!-$ on the left side of =

  $$\boxed{\text{Coefficient } \oplus\!\!\!\!-\ \text{copies} =}$$

  says that the coefficient is to be divided *repeatedly* by the copies of the base
- the *fraction bar* on the right side of =

  $$\boxed{= \frac{\text{Coefficient}}{\text{copies}}}$$

  says that the coefficient is to be divided by the *result* of the multiplication of 1 by the copies of the base.

13.5 Power Language

We are now ready to introduce a way of writing specifying-phrases that will work both for *repeated multiplications* and for *repeated divisions*.

1. The idea is to *write* just the *coefficient*, the *base*, the *number* of copies and whether the coefficient should be *multiplied* or *divided* by the copies. More precisely, in order to write a new kind of specifying-phrase which we will call a **monomial specifying-phrase**,

i. We write its *numerator*, that is we write:

i. The *coefficient*,

ii. The *multiplication* symbol $\times$ or $\otimes$ (depending on whether the numerators are *plain* or *signed*) as **separator** followed by the *base*,

iii. A **signed exponent**, that is a signed numerator

- whose *sign* is

  + when the coefficient is to be *multiplied* by the copies

  − when the coefficient is to be *divided* by the copies
- whose *size* is the number of copies

In order to be *separated* from the *base*, the *signed exponent* must be written as a **superscript**, that is small and raised a bit above the base line.

ii. We write its *denominator* if it has not been *declared* up front.

The *base* together with the *signed-exponent* is called a **signed power**.

We then *read* monomial specifying-phrases as

"Coefficient *multiplied/divided* by number of *copies* of the base"

EXAMPLE 18. Given the in-line *template* in **Dollars**

$$17 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2$$

- In order to *write* the monomial specifying-phrase,
  i. We write the *coefficient* 17:
  $$17$$
  ii. We write the *multiplication* symbol $\times$ as *separator* followed by the *base* 2:
  $$17 \times 2$$
  iii. We write the *signed exponent* as a *superscript* with $+$ to indicate that the coefficient is to be *multiplied* by the 6 copies of the base 2:
  $$17 \times 2^{+6}$$
- We *read* the monomial specifying-phrase
  $$\boxed{17 \times 2^{+6}}$$
  as
  $$\boxed{\text{17 multiplied by 6 copies of 2}}$$

EXAMPLE 19. Given the in-line *template*

$$448 \div \left[2 \times 2 \times 2 \times 2 \times 2 \times 2\right]$$

- In order to *write* the monomial specifying-phrase,
  i. We write the *coefficient* 448:
  $$448$$
  ii. We write the *multiplication* symbol $\times$ as *separator* followed by the *base* 2:
  $$448 \times 2$$
  iii. We write the *signed exponent* with $-$ to indicate that the coefficient is to be *divided* by 6 copies of the base 2:
  $$448 \times 2^{-6}$$
- We *read* the monomial specifying-phrase
  $$\boxed{448 \times 2^{-6}}$$
  as
  $$\boxed{\text{448 divided by 6 copies of 2}}$$

NOTE. In other words, here, $\times$ is really only a *separator* and has nothing to do with the kind of *repeated operation* we are specifying. While this way of writing things might seem rather strange, we will see in the next section how it turns out to make excellent sense.

2. As it happens, though, there is *no* procedure for identifying *monomial specifying-phrases* other than the procedures corresponding to *staggered templates*.

EXAMPLE 20. Given the following monomial specifying-phrase in **Dollars**

$$17 \times 2^{+6}$$

there is *no* way to identify it other than doing

$$\underbrace{17 \times 2}$$
$$\underbrace{34 \times 2}$$

Laurent monomial specifying-phrase
plain monomial specifying-phrase

$$\underbrace{\underbrace{\underbrace{\underbrace{68 \times 2}_{136} \times 2}_{272} \times 2}_{544} \times 2}_{1088}$$

This is in sharp contrast with the case of *repeated additions* for which there is a much shorter procedure for getting the result of repeated additions that is based on *multiplication* and with the case of *repeated subtractions* for which there is a much shorter procedure for getting the result based on *division*.

3. It is customary to distinguish monomial specifying-phrases in which the exponent has to be positive or 0 from monomial specifying-phrases in which the exponent can have any sign.

We will use the following names:

- A **Laurent monomial specifying-phrase** is a monomial specifying-phrases in which the exponent is a numerator that can have any sign.
- A **plain monomial specifying-phrase** is a monomial specifying-phrases in which the exponent is a numerator that can be only positive or 0 or, in other words, that can only be a plain numerator.

split equality
common base

Chapter 14

Laurent Monomials

Because of the lack of a *short* procedure for identifying monomial specifying-phrases, when working with monomial specifying-phrases, we tend to delay identifying them as much as possible and, instead, to *compute* with the monomial specifying-phrases themselves as long as possible, that is until there is nothing else to do but to *identify* the resulting monomial specifying-phrase. **NOTE.** The format that we will use to write these computations is called **split equality**: We will write on the left the (compound) specifying-phrase that we want to identify and we will write on the right the successive stages of the *computation* on separate lines.

14.1 Multiplying Monomial Specifying-Phrases

When we *multiply* two monomial specifying-phrases with a **common base**, that is when we multiply a first monomial specifying-phrase by a second monomial specifying-phrase with the same base, the result turns out to be a monomial specifying-phrase with the *common base*[1].

1. We can get this result either one of two ways:

- We can go back to the *in-line templates*:

  i. We replace each *monomial specifying-phrase* by the corresponding *in-line* template,

  ii. We change the order of the multiplications,

[1]Educologists will have recognized multiplication as a binary operation.

iii. We write the resulting monomial specifying phrase.

Example 1. In order to identify

$$\big[17 \times 2^{+5}\big] \times \big[11 \times 2^{+4}\big]$$

we replace each *monomial specifying-phrase* by the corresponding *in-line* template, we change the order of the multiplications and we write the resulting monomial specifying-phrase:

$$\begin{aligned}
\big[17 \times 2^{+5}\big] \times \big[11 \times 2^{+4}\big] &= \big[17 \times 2 \times 2 \times 2 \times 2 \times 2\big] \times \big[11 \times 2 \times 2 \times 2 \times 2\big] \\
&= 17 \times 2 \times 2 \times 2 \times 2 \times 2 \ \times \ 11 \times 2 \times 2 \times 2 \times 2 \\
&= 17 \times 11 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \\
&= \big[17 \times 11\big] \times 2^{+(5+4)} \\
&= 187 \times 2^{+12}
\end{aligned}$$

- We can build the resulting monomial specifying-phrase right from the given monomial specifying-phrases using the following procedure:

  i. We get the *coefficient* of the resulting monomial specifying-phrase by multiplying the coefficients of the given monomial specifying-phrases,

  ii. We get the *base* of the resulting monomial specifying-phrase by taking the base common to the given monomial specifying-phrases,

  iii. We get the signed exponent of the resulting monomial specifying-phrase by "oplussing" the signed exponents of the given monomial specifying-phrases.

  Example 2. In order to identify

  $$\big[17 \times 2^{+5}\big] \times \big[11 \times 2^{+4}\big]$$

  we multiply the coefficients and we "oplus" the signed exponents:

  $$\begin{aligned}
  \big[17 \times 2^{+5}\big] \times \big[11 \times 2^{+4}\big] &= \big[17 \times 11\big] \times 2^{+5 \oplus +4} \\
  &= 187 \times 2^{+12}
  \end{aligned}$$

2. In order to see why both ways give the same result, we now look at three more examples in which we will get the result both ways[2].

Example 3. We identify

$$\big[17 \times 2^{+5}\big] \times \big[11 \times 2^{-2}\big]$$

both ways:

- We replace each *monomial specifying-phrase* by the corresponding *in-line* template, change the order of the multiplications and write the resulting monomial specifying-

[2]Educologists will of course approve of letting the students "experience" the amount of work being saved by having them do it both ways for a while.

phrase:

$$\begin{aligned}
\big[17 \times 2^{+5}\big] \times \big[11 \times 2^{-2}\big] &= \big[17 \times 2 \times 2 \times 2 \times 2 \times 2\big] \times \left[\frac{11}{2 \times 2}\right] \\
&= \frac{17 \times 2 \times 2 \times 2 \times 2 \times 2 \times 11}{2 \times 2} \\
&= \frac{17 \times 11 \times 2 \times 2 \times 2 \times 2 \times 2}{2 \times 2} \\
&= \frac{17 \times 11 \times \not{2} \times \not{2} \times 2 \times 2 \times 2}{\not{2} \times \not{2}} \\
&= 17 \times 11 \times 2 \times 2 \times 2 \\
&= 17 \times 11\big] \times 2^{+(5-2)} \\
&= 187 \times 2^{+3}
\end{aligned}$$

- We multiply the coefficients and we "oplus" the signed exponents:

$$\begin{aligned}
\big[17 \times 2^{+5}\big] \times \big[11 \times 2^{-2}\big] &= \big[17 \times 11\big] \times 2^{+5 \oplus -2} \\
&= 187 \times 2^{+3}
\end{aligned}$$

EXAMPLE 4. We identify

$$\big[17 \times 2^{-6}\big] \times \big[11 \times 2^{+2}\big]$$

both ways:

- We replace each *monomial specifying-phrase* by the corresponding *in-line* template, change the order of the multiplications and write the resulting monomial specifying-phrase:

$$\begin{aligned}
\big[17 \times 2^{-6}\big] \times \big[11 \times 2^{+2}\big] &= \left[\frac{17}{2 \times 2 \times 2 \times 2 \times 2 \times 2}\right] \times \big[11 \times 2 \times 2\big] \\
&= \frac{17 \times 11 \times 2 \times 2}{2 \times 2 \times 2 \times 2 \times 2 \times 2} \\
&= \frac{17 \times 11 \times \not{2} \times \not{2}}{\not{2} \times \not{2} \times 2 \times 2 \times 2 \times 2} \\
&= \frac{17 \times 11}{2 \times 2 \times 2 \times 2} \\
&= \big[17 \times 11\big] \times 2^{-(6-2)} \\
&= 187 \times 2^{-4}
\end{aligned}$$

- We multiply the coefficients and we "oplus" the signed exponents:

$$\big[17 \times 2^{-6}\big] \times \big[11 \times 2^{+2}\big] = \big[17 \times 11\big] \times 2^{-6 \oplus +2}$$

common base

$$= 187 \times 2^{-4}$$

EXAMPLE 5. We identify

$$\left[17 \times 2^{-4}\right] \times \left[11 \times 2^{-3}\right]$$

both ways:

- We replace each *monomial specifying-phrase* by the corresponding *in-line* template, change the order of the multiplications and write the resulting monomial specifying-phrase:

$$\begin{aligned}\left[17 \times 2^{-4}\right] \times \left[11 \times 2^{-3}\right] &= \left[\frac{17}{2 \times 2 \times 2 \times 2}\right] \times \left[\frac{11}{2 \times 2 \times 2}\right] \\ &= \frac{17 \times 11}{2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2} \\ &= \left[17 \times 11\right] \times 2^{-(4+3)} \\ &= 187 \times 2^{-7}\end{aligned}$$

- We multiply the coefficients and we "oplus" the signed exponents:

$$\begin{aligned}\left[17 \times 2^{-4}\right] \times \left[11 \times 2^{-3}\right] &= \left[17 \times 11\right] \times 2^{-4 \oplus -3} \\ &= 187 \times 2^{-7}\end{aligned}$$

3. Thus, from the above examples, we see that the "power language" is indeed powerful as it allows for a single procedure since the "oplus" automatically takes care of the different cases whereas, when we use in-line templates, we need different procedures depending on whether the coefficients are to be repeatedly multiplied or divided by the copies of the base and also on the relative number of copies when one coefficient is to be repeatedly multiplied while the other coefficient is to be repeatedly divided.

14.2 Dividing Monomial Specifying-Phrases

When we divide two monomial specifying-phrases with a **common base**, that is when we divide a first monomial specifying-phrase by a second monomial specifying-phrase with the same base, the result turns out to be a monomial specifying-phrase with the same base.

1. We can get the result either one of two ways:

- We can go back to the *in-line templates*:

  i. We replace each *monomial specifying-phrase* by the corresponding *in-line* template, using fraction bars,

ii. We "invert and multiply", change the order of the multiplications, cancel, etc

iii. We write the resulting monomial specifying phrase.

EXAMPLE 6. In order to identify

$$\left[17 \times 2^{+7}\right] \div \left[11 \times 2^{+3}\right]$$

We replace each *monomial specifying-phrase* by the corresponding *in-line* template using fraction bars, "invert and multiply", change the order of the multiplications, cancel and write the resulting monomial specifying-phrase:

$$\begin{aligned}
\left[17 \times 2^{+7}\right] \div \left[11 \times 2^{+3}\right] &= \frac{17 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2}{1} \div \frac{11 \times 2 \times 2 \times 2}{1} \\
&= \frac{17 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2}{1} \times \frac{1}{11 \times 2 \times 2 \times 2} \\
&= \frac{17 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2}{11 \times 2 \times 2 \times 2} \\
&= \frac{17}{11} \times \frac{2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2}{2 \times 2 \times 2} \\
&= \frac{17}{11} \times \frac{\not{2} \times \not{2} \times \not{2} \times 2 \times 2 \times 2 \times 2}{\not{2} \times \not{2} \times \not{2}} \\
&= \frac{17}{11} \times \frac{2 \times 2 \times 2 \times 2}{1} \\
&= \frac{17}{11} \times 2 \times 2 \times 2 \times 2 \\
&= \frac{17}{11} \times 2^{+(7-3)} \\
&= \frac{17}{11} \times 2^{+4}
\end{aligned}$$

- We can build the resulting monomial specifying-phrase right from the given monomial specifying-phrases:

  i. We get the *coefficient* of the resulting monomial specifying-phrase by dividing the coefficients of the given monomial specifying-phrases,

  ii. We get the *base* of the resulting monomial specifying-phrase by taking the base common to the given monomial specifying-phrases,

  iii. We get the signed exponent of the resulting monomial specifying-phrase by "ominussing" the signed exponent of the second given monomial specifying-phrase from the signed exponent of the first given monomial specifying-phrase, that is by "oplussing" the *opposite* of the signed exponent of the second given monomial specifying-phrase to the signed exponent of the first given monomial specifying-phrase.

EXAMPLE 7. In order to identify

$$\left[17 \times 2^{+7}\right] \div \left[11 \times 2^{+3}\right]$$

We divide the coefficients and we "ominus" the signed exponents:

$$\begin{aligned}
\left[17 \times 2^{+7}\right] \div \left[11 \times 2^{+3}\right] &= \left[17 \div 11\right] \times 2^{+7 \ominus +3} \\
&= \frac{17}{11} \times 2^{+7 \oplus -3} \\
&= \frac{17}{11} \times 2^{+4}
\end{aligned}$$

2. In order to see why both ways give the same result, we now look at three more examples the result of each of which we will get both ways.

EXAMPLE 8. We identify

$$\left[17 \times 2^{+7}\right] \div \left[11 \times 2^{+3}\right]$$

both ways:

- We replace each *monomial specifying-phrase* by the corresponding *in-line* template using fraction bars, "invert and multiply", change the order of the multiplications, cancel and write the resulting monomial specifying-phrase:

$$\begin{aligned}
\left[17 \times 2^{+7}\right] \div \left[11 \times 2^{+3}\right] &= \frac{17 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2}{1} \div \frac{11 \times 2 \times 2 \times 2}{1} \\
&= \frac{17 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2}{1} \times \frac{1}{11 \times 2 \times 2 \times 2} \\
&= \frac{17 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2}{11 \times 2 \times 2 \times 2} \\
&= \frac{17}{11} \times \frac{2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2}{2 \times 2 \times 2} \\
&= \frac{17}{11} \times \frac{\cancel{2} \times \cancel{2} \times \cancel{2} \times 2 \times 2 \times 2 \times 2}{\cancel{2} \times \cancel{2} \times \cancel{2}} \\
&= \frac{17}{11} \times \frac{2 \times 2 \times 2 \times 2}{1} \\
&= \frac{17}{11} \times 2 \times 2 \times 2 \times 2 \\
&= \frac{17}{11} \times 2^{+(7-3)} \\
&= \frac{17}{11} \times 2^{+4}
\end{aligned}$$

- We divide the coefficients and we "ominus" the signed exponents:

$$\begin{aligned}
\left[17 \times 2^{+7}\right] \div \left[11 \times 2^{+3}\right] &= \left[17 \div 11\right] \times 2^{+7 \ominus +3} \\
&= \frac{17}{11} \times 2^{+7 \oplus -3} \\
&= \frac{17}{11} \times 2^{+4}
\end{aligned}$$

EXAMPLE 9. We identify

$$\left[17 \times 2^{+3}\right] \div \left[11 \times 2^{+7}\right]$$

both ways:

- We replace each *monomial specifying-phrase* by the corresponding *in-line* template using a fraction bar, change the order of the multiplications, cancel and write the resulting monomial specifying-phrase:

$$\begin{aligned}
\left[17 \times 2^{+7}\right] \div \left[11 \times 2^{+3}\right] &= \frac{17 \times 2 \times 2 \times 2}{1} \div \frac{11 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2}{1} \\
&= \frac{17 \times 2 \times 2 \times 2}{1} \div \frac{1}{11 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2} \\
&= \frac{17 \times 2 \times 2 \times 2}{11 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2} \\
&= \frac{17}{11} \times \frac{2 \times 2 \times 2}{2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2} \\
&= \frac{17}{11} \times \frac{\cancel{2} \times \cancel{2} \times \cancel{2}}{\cancel{2} \times \cancel{2} \times \cancel{2} \times 2 \times 2 \times 2 \times 2} \\
&= \frac{17}{11} \times \frac{1}{2 \times 2 \times 2 \times 2} \\
&= \frac{17}{11} \times 2^{-(7-3)} \\
&= \frac{17}{11} \times 2^{-4}
\end{aligned}$$

- We divide the coefficients and we "ominus" the signed exponents:

$$\begin{aligned}
\left[17 \times 2^{+3}\right] \div \left[11 \times 2^{+7}\right] &= \left[17 \div 11\right] \times 2^{+3 \ominus +7} \\
&= \frac{17}{11} \times 2^{+3 \oplus -7} \\
&= \frac{17}{11} \times 2^{-4}
\end{aligned}$$

EXAMPLE 10. We identify

$$\left[17 \times 2^{-5}\right] \div \left[11 \times 2^{+3}\right]$$

both ways:

- We replace each *monomial specifying-phrase* by the corresponding *in-line* template using a fraction bar, change the order of the multiplications, cancel and write the resulting monomial specifying-phrase:

$$\left[17 \times 2^{-5}\right] \div \left[11 \times 2^{+3}\right] = \frac{17}{2 \times 2 \times 2 \times 2 \times 2} \div \frac{11 \times 2 \times 2 \times 2}{1}$$

$$
\begin{aligned}
&= \frac{17}{2 \times 2 \times 2 \times 2 \times 2} \times \frac{1}{11 \times 2 \times 2 \times 2} \\
&= \frac{17 \times 1}{11 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2} \\
&= \frac{17}{11} \times \frac{1}{2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2} \\
&= \frac{17}{11} \times 2^{-(5+3)} \\
&= \frac{17}{11} \times 2^{-8}
\end{aligned}
$$

- We divide the coefficients and we "ominus" the signed exponents:

$$
\begin{aligned}
\left[17 \times 2^{-5}\right] \div \left[11 \times 2^{+3}\right] &= \left[17 \div 11\right] \times 2^{-5 \ominus +3} \\
&= \frac{17}{11} \times 2^{-5 \oplus -3} \\
&= \frac{17}{11} \times 2^{-8}
\end{aligned}
$$

3. Thus, from the above examples, we see that the "power language" is even more spectacular in the case of division as the "ominus" still takes automatically care of the different cases while, whereas, we use in-line templates, we need different procedures depending on whether the coefficients are to be repeatedly multiplied or divided by the copies of the base and also on the relative number of copies when one coefficient is to be repeatedly multiplied while the other coefficient is to be repeatedly divided.

4. The reason we are using *Laurent* monomial specifying-phrases rather than just *plain* monomial specifying-phrases is that we cannot always divide a first *plain* monomial specifying-phrase by a second *plain* monomial specifying-phrase and get as a result a *plain* monomial specifying-phrase. On the other hand, we can always multiply or divide a first *Laurent* monomial specifying-phrase by a second *Laurent* monomial specifying-phrase and get as a result a *Laurent* monomial specifying-phrase.

14.3 Terms

We now take a major step in the development of the "power language" by allowing *unspecified numerators* when writing monomials.

1. We begin by going back to the distinction between a *formula* and a *sentence*. Recall that by itself a *formula*, for instance an *inequation* or an *equation*, is neither TRUE nor FALSE and that only a *sentence* can represent a relationship among collections in the real-world. term

EXAMPLE 11. The inequation in **Apples**

$$x < 5$$

is neither TRUE nor FALSE because it does not represent a relationship among collections in the real world. (2 **Apples** represent a collection in the real world but x **Apples** does not represent a collection in the real world.)

Given a formula, it is only when we replace the *unspecified numerator* by a *specific numerator* that we get a *sentence* which is then either TRUE or FALSE depending on whether it fits the real world or not.

EXAMPLE 12. Given the *formula* in **Apples**

$$x < 5$$

when we replace the unspecified numerator x by the specific numerator 8 we get the sentence in **Apples**

$$x{<}5|_{x:=8}$$

that is the sentence

$$8 \textbf{ Apples} < 5 \textbf{ Apples}$$

which is FALSE but if, instead, we replace the unspecified numerator x by the specific numerator 3 we get the sentence in **Apples**

$$x{<}5|_{x:=3}$$

that is the sentence

$$3 \textbf{ Apples} < 5 \textbf{ Apples}$$

which is TRUE

2. Similarly, just as a *formula* can be viewed as an "incomplete" *sentence*, a **term** will be an "incomplete" *specifying-phrase*.

EXAMPLE 13. Given the *term* in **Apples**

$$x + 5$$

when we replace the unspecified numerator x by the specific numerator 8 we get the specifying-phrase in **Apples**

$$x + 5|_{x:=8}$$

that is the specifying-phrase

$$8 \textbf{ Apples} + 5 \textbf{ Apples}$$

which we may or may not chose to *identify*.

Of course, an *unspecified numerator* is the simplest possible kind of *term*.

EXAMPLE 14. Given the *term* in **Apples**

$$x$$

when we replace the unspecified numerator x by the specific numerator 8

$$x + 5|_{x:=8}$$

we get

$$8 \textbf{ Apples}$$

3. When replacing in a *monomial* specifying-phrase a specific numerator by an unspecified numerator to get a *term*, we will use

monomial term
monomial

- The letters $a, b, c, d \ldots$ for unspecified *signed* coefficients,
- The letters $x, y, x \ldots$ for unspecified *signed* bases,
- The letters $m, n, p \ldots$ for unspecified *plain* exponents.

EXAMPLE 15.

$$a \times x^{+n}$$
$$c \times y^{-m}$$

The reason we will use the letters $m, n, p \ldots$ to stand only for *plain* exponents (rather than for *signed* exponents) is that the *sign* of a exponent is most important since it distinguishes between multiplication and division and we will almost always have to specify it as in the above example.

In the rare cases when the sign of the exponent will not matter, we will write the symbol $\pm$, read "plus or minus" in front of the letter as in the following example.

EXAMPLE 16.

$$c \times x^{\pm n}$$

is intended to cover both the case

$$c \times x^{+n}$$

and the case

$$c \times x^{-n}$$

It is also customary to let the separator $\times$ go without saying. However, this tends to cause mistakes unless we make sure we read the monomial specifying-phrase according to whether the signed exponent is *positive* or *negative*, as

- "Coefficient *multiplied* by number of copies of the base" when the exponent is *positive*,
- "Coefficient *divided* by number of copies of the base" when the exponent is *negative*.

EXAMPLE 17.

- We read cx^{+n} as "c multiplied by n copies of x" because the exponent is *positive*,
- We read ay^{-p} as "a divided by p copies of y" because the exponent is *negative*.

14.4 Monomials

In the rest of this text, coefficients and exponents will always be specified and only the base will remain unspecified. Out of habit, we shall mostly use the letter x for the base.

1. Monomial specifying-phrases in which the *base* is unspecified are called **monomial terms** or **monomials** for short.

EXAMPLE 18. The following

$$-3x^{+5}$$
$$+5.23x^{-3}$$
$$-1600x^{-4}$$
$$+4x^{+2}$$

Laurent monomial
plain monomial
coefficient
power

are monomials but

$$+4x^{+2.5}$$

is not a monomial because 2.5 copies doesn't make sense.

a. Just as, earlier on, we distinguished *Laurent* monomial specifying-phrases (those whose exponent can have any sign) from *plain* monomial specifying phrases (those whose exponent can be only positive or 0), we could distinguish in the same manner **Laurent monomials** from **plain monomials**. However, since we will be using mostly *Laurent* monomials, we will just use *monomial* to mean Laurent monomial.

b. In a *monomial* we will distinguish:

- the **coefficient**, which is the number to be multiplied or divided by the copies of the base
- the **power**, which is the base together with the exponent.

In other words, the *separator* $\times$, whether it is actually written or goes without saying, separates the *coefficient* from the *power*.

EXAMPLE 19. In the monomial $-3x^{+4}$, -3 is the *coefficient* and x^{+4} is the *power*.

c. Thus, *monomials*, as well as *monomial specifying-phrases*, look very much like *ordinary* number-phrases (as opposed to *specifying* number-phrases):

- The *coefficient* in a monomial—or monomial specifying-phrase—is like the *numerator* in an ordinary number-phrase,
- The *power* in a monomial—or monomial specifying-phrase—is like the *denominator* in an ordinary number-phrase.

EXAMPLE 20. Monomial specifying-phrases like

$17.52 \times 2^{+3}$ (with $\times$ as *separator*)

and monomials like

$17.52\, x^{+3}$ (without *separator*)

look, and to a large extent will behave, very much like:

- Ordinary number-phrases like

  17.52 Meters

  in which there is no need for a *separator* between the *numerator* and the *denominator*,
- Metric number-phrases like

  17.52 KILO Meters

  in which there is no need for a *separator* between the *numerator* and the *denominator*,
- Base TEN number-phrases like

  17.52 x TEN^{+3} Meters

where × is a *separator* between the *numerator* and the *denominator*,

- Exponential number-phrases like

$$17.52 \text{ x } 10^{+3} \text{ Meters}$$

where × is a *separator* between the *numerator* and the *denominator*.

We will investigate how far the similarity goes in the following chapters.

2. When we multiply or divide a first monomial by a second monomial, we proceed just as we did with monomial specifying-phrases, that is we can proceed either:

- The long way which is to go back to in-line templates and then proceed according to whether we are dealing with *multiplication* or *division*
- The short way which is to use the following

THEOREM 8 (EXPONENT THEOREM). *In order to:*

i. Multiply *two monomials* $ax^{\pm m}$ *and* $bx^{\pm n}$, *we* multiply the coefficients *and* oplus the exponents*:*

$$ax^{\pm m} \times bx^{\pm n} = abx^{\pm m \oplus \pm n}$$

ii. Divide *two monomials* $ax^{\pm m}$ *and* $bx^{\pm n}$, *we* divide the coefficients *and* ominus the exponents*:*

$$ax^{\pm m} \div bx^{\pm n} = \frac{a}{b}x^{\pm m \ominus \pm n}$$

We now look at a few examples.

EXAMPLE 21. Given

$$\left[-17.89 \times x^{+547}\right] \times \left[-11.06 \times x^{+312}\right]$$

instead of replacing each *monomial* by the corresponding *in-line* template, change the order of the multiplications and write the resulting monomial:

$$\begin{aligned}
\left[-17.89 \times x^{+547}\right] \times \left[-11.06 \times x^{+312}\right] &= \left[-17.89 \times \underbrace{x \times x \times \cdots \times x}_{547 \text{ copies of } x}\right] \times \left[-11.06 \times \underbrace{x \times x \times \cdots \times x}_{312 \text{ copies of } x}\right] \\
&= -17.89 \times -11.06 \times \underbrace{x \times x \times \cdots \times x}_{547+312 \text{ copies of } x} \\
&= \left[-17.89 \times -11.06\right] \times x^{+(547+312)} \\
&= +\left[17.89 \times 11.06\right] \times x^{+859}
\end{aligned}$$

we can use the EXPONENT THEOREM:

$$\begin{aligned}
\left[-17.89 \times x^{+547}\right] \times \left[-11.06 \times x^{+312}\right] &= \left[-17.89 \times -11.06\right] \times x^{+547 \oplus +312} \\
&= +\left[17.89 \times 11.06\right] \times x^{+859}
\end{aligned}$$

EXAMPLE 22. Given

$$\left[+17.89 \times x^{+547}\right] \times \left[-11.06 \times x^{-312}\right]$$

instead of replacing each *monomial* by the corresponding *in-line* template, change the order of the multiplications and write the resulting monomial:

$$\begin{aligned}
\left[+17.89 \times x^{+547}\right] \times \left[-11.06 \times x^{-312}\right] &= \left[+17.89 \times \underbrace{x \times x \times \cdots \times x}_{547 \text{ copies of } x}\right] \times \left[\frac{-11.06}{\underbrace{x \times x \times \cdots \times x}_{312 \text{ copies of } x}}\right] \\
&= \left[+17.89 \times -11.06\right] \times \left[\frac{\underbrace{x \times x \times \cdots \times x}_{547 \text{ copies of } x}}{\underbrace{x \times x \times \cdots \times x}_{312 \text{ copies of } x}}\right] \\
&= -\left[17.89 \times 11.06\right] \times \left[\frac{\cancel{\underbrace{x \times x \times \cdots \times x}_{312 \text{ copies of } x}} \times \underbrace{x \times x \times \cdots \times x}_{547-312 \text{ copies of } x}}{\cancel{\underbrace{x \times x \times \cdots \times x}_{312 \text{ copies of } x}}}\right] \\
&= -\left[17.89 \times 11.06\right] \times x^{+(547-312)} \\
&= -\left[17.89 \times 11.06\right] \times x^{+235}
\end{aligned}$$

we can use the EXPONENT THEOREM:

$$\begin{aligned}
\left[+17.89 \times x^{+547}\right] \times \left[-11.06 \times x^{-312}\right] &= \left[+17.89 \times -11.06\right] \times x^{+547 \oplus -312} \\
&= -\left[17.89 \times 11.06\right] \times x^{+(547-312)} \\
&= -\left[17.89 \times 11.06\right] \times x^{+235}
\end{aligned}$$

EXAMPLE 23. Given

$$\left[-17.89 \times x^{-547}\right] \times \left[+11.06 \times x^{+312}\right]$$

instead of replacing each *monomial* by the corresponding *in-line* template, change the order of the multiplications and write the resulting monomial:

$$\begin{aligned}
\left[-17.89 \times x^{-547}\right] \times \left[+11.06 \times x^{+312}\right] &= \left[\frac{-17.89}{\underbrace{x \times x \times \cdots \times x}_{547 \text{ copies of } x}}\right] \times \left[+11.06 \times \underbrace{x \times x \times \cdots \times x}_{312 \text{ copies of } x}\right] \\
&= \left[-17.89 \times +11.06\right] \times \left[\frac{\underbrace{x \times x \times \cdots \times x}_{312 \text{ copies of } x}}{\underbrace{x \times x \times \cdots \times x}_{547 \text{ copies of } x}}\right]
\end{aligned}$$

$$
\begin{aligned}
&= -\Big[17.89 \times 11.06\Big] \times \left[\frac{\cancel{\underbrace{x \times x \times \cdots \times x}_{312 \text{ copies of } x}}}{\cancel{\underbrace{x \times x \times \cdots \times x}_{312 \text{ copies of } x}} \times \underbrace{x \times x \times \cdots \times x}_{547-312 \text{ copies of } x}}\right] \\
&= -\Big[17.89 \times 11.06\Big] \times x^{-(547-312)} \\
&= -\Big[17.89 \times 11.06\Big] \times x^{-235}
\end{aligned}
$$

we can use the EXPONENT THEOREM:

$$
\begin{aligned}
\Big[-17.89 \times x^{-547}\Big] \times \Big[11.06 \times x^{+312}\Big] &= \Big[-17.89 \times +11.06\Big] \times x^{-547 \oplus +312} \\
&= -\Big[17.89 \times 11.06\Big] \times x^{-(547-312)} \\
&= -\Big[17.89 \times 11.06\Big] \times x^{-235}
\end{aligned}
$$

EXAMPLE 24. Given

$$\Big[+17.89 \times x^{+547}\Big] \div \Big[+11.06 \times x^{+312}\Big]$$

instead of replacing each *monomial* by the corresponding *in-line* template, change the order of the multiplications, rewrite as fraction, multiply by the reciprocal instead of divide, and write the resulting monomial:

$$
\begin{aligned}
\Big[+17.89 \times x^{+547}\Big] \div \Big[+11.06 \times x^{+312}\Big] &= \left[+17.89 \times \underbrace{x \times x \times \cdots \times x}_{547 \text{ copies of } x}\right] \div \left[\frac{+11.06 \times \underbrace{x \times x \times \cdots \times x}_{312 \text{ copies of } x}}{1}\right] \\
&= \left[+17.89 \times \underbrace{x \times x \times \cdots \times x}_{547 \text{ copies of } x}\right] \times \left[\frac{1}{+11.06 \times \underbrace{x \times x \times \cdots \times x}_{312 \text{ copies of } x}}\right] \\
&= \left[\frac{+17.89}{+11.06}\right] \times \left[\frac{\underbrace{x \times x \times \cdots \times x}_{547 \text{ copies of } x}}{\underbrace{x \times x \times \cdots \times x}_{312 \text{ copies of } x}}\right] \\
&= +\left[\frac{17.89}{11.06}\right] \times \left[\frac{\cancel{\underbrace{x \times x \times \cdots \times x}_{312 \text{ copies of } x}} \times \underbrace{x \times x \times \cdots \times x}_{547-312 \text{ copies of } x}}{\cancel{\underbrace{x \times x \times \cdots \times x}_{312 \text{ copies of } x}}}\right] \\
&= +\left[\frac{17.89}{11.06}\right] \times x^{+(547-312)}
\end{aligned}
$$

$$= + \left[\frac{17.89}{11.06}\right] \times x^{+235}$$

it is easier to use the EXPONENT THEOREM:

$$\left[+17.89 \times x^{+547}\right] \div \left[+11.06 \times x^{+312}\right] = \left[\frac{+17.89}{+11.06}\right] \times x^{+547 \ominus +312}$$
$$= + \left[\frac{17.89}{11.06}\right] \times x^{+547 \oplus -312}$$
$$= + \left[\frac{17.89}{11.06}\right] \times x^{+(547-312)}$$
$$= + \left[\frac{17.89}{11.06}\right] \times x^{+235}$$

Example 25. Given

$$\left[17.89 \times x^{-547}\right] \div \left[11.06 \times x^{-312}\right]$$

instead of replacing each *monomial* by the corresponding *in-line* template, change the order of the multiplications, rewrite as fraction, multiply by the reciprocal instead of divide, and write the resulting monomial:

$$\left[17.89 \times x^{-547}\right] \div \left[11.06 \times x^{-312}\right] = \left[\frac{17.89}{\underbrace{x \times x \times \cdots \times x}_{547 \text{ copies of } x}}\right] \div \left[\frac{11.06}{\underbrace{x \times x \times \cdots \times x}_{312 \text{ copies of } x}}\right]$$
$$= \left[\frac{17.89}{\underbrace{x \times x \times \cdots \times x}_{547 \text{ copies of } x}}\right] \times \left[\frac{\underbrace{x \times x \times \cdots \times x}_{312 \text{ copies of } x}}{11.06}\right]$$
$$= \left[\frac{17.89}{11.06}\right] \times \left[\frac{\underbrace{x \times x \times \cdots \times x}_{312 \text{ copies of } x}}{\underbrace{x \times x \times \cdots \times x}_{547 \text{ copies of } x}}\right]$$
$$= \left[\frac{17.89}{11.06}\right] \times \left[\frac{\cancel{\underbrace{x \times x \times \cdots \times x}_{312 \text{ copies of } x}}}{\cancel{\underbrace{x \times x \times \cdots \times x}_{312 \text{ copies of } x}} \times \underbrace{x \times x \times \cdots \times x}_{547-312 \text{ copies of } x}}\right]$$
$$= \left[\frac{17.89}{11.06}\right] \times x^{-(547-312)}$$

$$= \left[\frac{17.89}{11.06}\right] \times x^{-235}$$

it is easier to use the EXPONENT THEOREM:

$$\begin{aligned}\left[17.89 \times x^{-547}\right] \div \left[11.06 \times x^{-312}\right] &= \left[\frac{17.89}{11.06}\right] \times x^{-547 \ominus -312} \\ &= \left[\frac{17.89}{11.06}\right] \times x^{-547 \oplus +312} \\ &= \left[\frac{17.89}{11.06}\right] \times x^{-(547-312)} \\ &= \left[\frac{17.89}{11.06}\right] \times x^{-235}\end{aligned}$$

Chapter 15

Polynomials 1: Addition, Subtraction

While, as we saw in the preceding chapter, monomials behave very well with respect to *multiplication* and *division* in the sense that we can always multiply or divide a first monomial by a second monomial and get a monomial as a result, we will see that monomials behave very badly with respect to *addition* and *subtraction*. This, though, gives raise to a new type of *term* which will in fact play a fundamental role—to be described in the Epilogue at the end of this text—in the investigation of FUNCTIONS.

In the rest of this text, we will introduce and discuss the way this new type of terms behaves with respect to the four operations. These are the basics of what is called POLYNOMIAL ALGEBRA.

15.1 Monomials and Addition

We begin by looking at the way monomials behave with regard to *addition*. The short of it is that, most of the time, monomials *cannot* be added.

1. One way to look at why monomials usually cannot be added is to observe that powers are to monomials much the same as denominators are to number-phrases.

- Just like ordinary number-phrases need to involve the *same denominator* in order to be added, monomials need to involve the *same power* to be added.

EXAMPLE 1. Just like

$$17.52 \textbf{ Meters} + 4.84 \textbf{ Meters} = 22.36 \textbf{ Meters}$$

we have that

$$17.52x^{+6} + 4.84x^{+6} = 22.36x^{+6}$$

- Just like ordinary number-phrases that involve *different denominators* cannot be added and just make up a *combination*, monomials that involve *different powers* cannot be added and just make up a *combination*.

  EXAMPLE 2. Just like

  $$17.52 \textbf{ Feet} + 4.84 \textbf{ Inches} \text{ is a } \textit{combination}$$

  we have that

  $$17.52x^{+6} + 4.84x^{+4} \text{ is a } \textit{combination}$$

2. A more technical way to look at why monomials cannot be added when the powers are different is to try various ways of "adding" monomials and then to see what the results would be when we replace the *unspecified numerator* x by *specific* numerators.

EXAMPLE 3. Suppose we think that the rule for adding the monomials should be "add the coefficients and add the exponents".
Then, given for instance the monomials

$$+7x^{-2} \text{ and } -3x^{+3}$$

the rule "add the coefficients and add the exponents" would give us the following monomial as a result:

$$(+7 \oplus -3)x^{-2 \oplus +3}$$

that is

$$+4x^{+1}$$

Now while, on the one hand, there is no obvious reason why this should not be an acceptable result, on the other hand, monomials are waiting for x to be replaced by some specific numerator.
So, say we replace x by $+4$. The given monomials would then give:

$$\begin{aligned} +7x^{-2}\big|_{x:=+4} &= \frac{+7}{(+4) \bullet (+4)} \\ &= \frac{+7}{+16} \\ &= 0.4375 \end{aligned}$$

and

$$\begin{aligned} -3x^{+3}\big|_{x:=+4} &= -3 \bullet (+4) \bullet (+4) \bullet (+4) \\ &= -192 \end{aligned}$$

which, when we add them up, gives us

Laurent polynomial reduced

$$-191.5625$$

But, when we replace x by $+4$ in the supposed result, we get

$$\begin{aligned} +4x^{+1}\big|_{x:=+4} &= +4 \bullet (+4) \\ &= +16 \end{aligned}$$

So, in the end, the rule "add the coefficients and add the exponents" would not produce an acceptable result.

Even though, as it happens, no rule for adding monomials will survive replacement of x by a specific numerator, the reader is encouraged to try so as to convince her/him self that this is really the case.

15.2 Laurent Polynomials

A **Laurent polynomial** is a *combination* of *powers* involving:

- *exponents* that can be any *signed counting* numerator (including 0).
- *coefficients* that can be any *signed decimal* numerator

EXAMPLE 4. All of the following are Laurent polynomials:

$$+22.71x^{+3} + 0.3x^{0} - 47.03x^{+2} + 57.89x^{-3}$$
$$+21.09x^{-4} - 33.99x^{+2} + 45.02x^{-1} + 52.74x^{+1} - 34.82x^{+7}$$
$$-30.18x^{+6} - 41.02x^{+5} + 5.6x^{+4}$$
$$+20.13x^{+3} + 0.03x^{+5} + 50.01x^{0} - 0.04x^{+1}$$
$$-0.02x^{-7} + 18.03x^{+6}$$

1. While there is nothing difficult about what Laurent polynomials *are*, we need to agree on a few rules to make them easier to *work* with since, otherwise, it is not always easy even just to see if two Laurent polynomials are the same or not.

EXAMPLE 5. The following two Laurent polynomials are the same

$$+0.3x^{0} - 47.03x^{+2} + 22.71x^{+3} + 57.89x^{-3}$$
$$+57.89x^{-3} + 22.71x^{+3} + 0.3x^{0} - 47.03x^{+2}$$

but the following two Laurent polynomials are not the same

$$+0.3x^{0} - 47.03x^{+2} - 22.71x^{+3} + 57.89x^{-3}$$
$$+57.89x^{-3} + 22.71x^{+3} + 0.3x^{0} - 47.03x^{+2}$$

EXAMPLE 6. The following two Laurent polynomials are in fact the same

$$+2x^{+3} + 6x^{-4}$$
$$-6x^{+3} + 4x^{-4} + 8x^{+3} + 2x^{-4}$$

a. The first thing we have to agree on is that Laurent polynomials must always be **reduced**, that is that monomials in a given Laurent polynomial

ascending order of exponents
descending order of exponents

that *can* be added (because they involve the same power) *must* in fact be added.

Example 7. Given the following Laurent polynomial

$$-6x^{+3} + 4x^{-4} + 8x^{+3} + 2x^{-4}$$

it must be *reduced* to

$$+2x^{+3} + 6x^{-4}$$

before we do anything else.

b. The second thing we have to do is to agree on some order in which to write the monomials in a Laurent polynomial.

i. We will agree that:
The monomials in a Laurent polynomial will and can only be written in either one of two orders:

- **ascending order of exponents**, that is, as we read or write a Laurent polynomial from left to right, the *exponents* must get *larger and larger* regardless of the *coefficients*.
- **descending order of exponents**, that is, as we read or write a Laurent polynomial from left to right, the *exponents* must get *smaller and smaller* regardless of the *coefficients*.

Example 8. The following Laurent polynomial

$$-47.03x^{+2} + 57.89x^{-3} + 22.71x^{+4} + 0.3x^{0}$$

can only be written either in *ascending* order of *exponents*

$$+57.89^{-3} + 0.3^{0} - 47.03x^{+2} + 22.71x^{+4}$$

or in *descending* order of *exponents*

$$+22.71x^{+4} - 47.03x^{+2} + 0.3x^{0} + 57.89x^{-3}$$

regardless of the *coefficients*.

ii. Which of the two orders is to be used depends on the *size* of the numerators with which x can be replaced:

- The *ascending* order must be used when x can be replaced only by *small* numerators,
- The *descending* order must be used when when x can be replaced only by for *large* numerators.

We will see the reason in a short while.

Note. When the size of what x stands for is *unknown*, it is customary, even if for no special reason, to use the *descending* order of exponents.

c. The third thing we have to do is to introduce *customary practices* even though these practices will not be followed here.

i. It is usual to write just *plain* exponents instead of *positive* exponents.

Example 9. Instead of writing

$$+57.89x^{-3} + 0.3x^{0} - 47.03x^{+2} + 22.71x^{+4}$$

it is usual to write

$$+57.89x^{-3} + 0.3x^{0} - 47.03x^{2} + 22.71x^{4}$$

consecutive
missing power
evaluate

ii. It is usual not to write the exponent +1 at all.

Example 10. Instead of writing

$$+57.89x^{+3} + 0.3x^{+2} - 47.03x^{+1} + 29.77x^{+4}$$

it is usual to write

$$+57.89x^{+3} + 0.3x^{+2} - 47.03x \quad + 29.77x^{+4}$$

iii. It is usual not to write the power x^0 at all.

Example 11. Instead of

$$+57.89x^{-3} + 0.3x^{0} - 47.03x^{+2} + 22.71x^{+4}$$

it is usual to write

$$+57.89x^{-3} + 0.3 \quad - 47.03x^{+2} + 22.71x^{+4}$$

iv. Most of the time, the exponents of the powers will be **consecutive** but occasionally there can be **missing powers**.

Example 12. The following Laurent polynomials in which the powers are *consecutive* are fairly typical of those that we will usually encounter.

$$-47.03x^{+3} + 57.89x^{+2} + 22.71x^{+1} + 0.3x^{0}$$
$$-47.03x^{+1} + 57.89x^{0} + 22.71x^{-1}$$
$$-47.03x^{-1} + 57.89x^{0} + 22.71x^{+1} + 0.3x^{+2}$$

Example 13. The following Laurent polynomials in which at least one power is *missing* are fairly typical of those that we will occasionally encounter.

$$-47.03x^{+3} + 0.3x^{0}$$
$$-47.03x^{+2} + 57.89x^{0} + 22.71x^{-1}$$
$$-47.03x^{-1} + 57.89x^{0} + 22.71x^{+1} + 0.3x^{+3}$$

When working with a Laurent polynomial in which powers are *missing*, it is much safer to insert in their place powers with coefficient 0.

Example 14. Instead of working with

$$-47.03x^{+3} + 13.3x^{0}$$

it is much safer to work with

$$-47.03x^{+3} + 0x^{+2} + 0x^{+1} + 13.3x^{0}$$

2. Laurent polynomials are specifying-phrases and we **evaluate** Laurent polynomials in the usual manner, that is we replace x by the required numerator and we then compute the result.

a. Example 15. Given the Laurent polynomial

$$-47.03x^{+2} \oplus +13.3x^{-3}$$

when $x := -5$

$$\begin{aligned}
-47.03x^{+2} \oplus +13.3x^{-3}\Big|_{x:=-5} &= -47.03(-5)^{+2} \oplus +13.3(-5)^{-3} \\
&= [-47.03 \otimes (-5)(-5)] \oplus \left[\frac{+13.3}{(-5)(-5)(-5)}\right] \\
&= [-47.03 \otimes +25] \oplus \left[\frac{+13.3}{-125}\right]
\end{aligned}$$

diminishing

$$= -1175.75 \oplus +0.1064$$
$$= -1175.6436$$

b. When the coefficients are all single-digit counting numerators and we replace x by TEN, the result shows an interesting connection between Laurent polynomials and decimal numbers.

EXAMPLE 16. Given the Laurent polynomial

$$4x^{+3} + 7x^{+2} + 9x^{+1} + 8x^{0} + 2x^{-1} + 5x^{-2} + 6x^{-3}$$

when $x := 10$ we get:

$$\begin{aligned} 4x^{+3} + 7x^{+2} + 9x^{+1} + 4x^{0} + 2x^{-1} + 7x^{-2} + 7x^{-3}\big|_{x=10} &= \\ &= 4 \times 10^{+3} + 7 \times 10^{+2} + 9 \times 10x^{+1} + 8 \times 10^{0} + 2 \times 10^{-1} + 5 \times 10^{-2} + 6 \times 10^{-3} \\ &= 4 \times 1000. + 7 \times 100. + 9 \times 10. + 8 \times 1. + 2 \times 0.1 + 5 \times 0.01 + 6 \times 0.001 \\ &= 4000. + 700. + 90. + 8. + 0.2 + 0.05 + 0.006 \\ &= 4\ 7\ 9\ 8.\ 2\ 5\ 6 \end{aligned}$$

which is the decimal number whose digits are the coefficients of the Laurent polynomial.

3. We are now in a position at least to state the reason for allowing only the *ascending* order of exponents and the *descending* order of exponents:

When we replace x by a specific numerator and go about evaluating the Laurent polynomial, we evaluate, one by one, each one of the monomials in the Laurent polynomial. But what happens is that

- When x is replaced by a numerator that is *large in size*, the more copies there are in a monomial, the *larger in size* the result will be.
- When x is replaced by a numerator that is *small in size*, the more copies there are in a monomial, the *smaller in size* the result will be.

But what we want, no matter what, is that the size of the successive results go **diminishing**. So,

- When x is to be replaced by a numerator that is going to be *large in size*, we will want the Laurent polynomial to be written in *descending order of exponents*.
- When x is to be replaced by a numerator that is going to be *small in size*, we will want the Laurent polynomial to be written in *ascending order of exponents*.

For lack of time, we cannot go here into any more detail but the interested reader will find this discussed at some length in the Epilogue.

15.3 Plain Polynomials

plain polynomial
polynomial

A **plain polynomial** is a combination of *powers* involving:

- *exponents* that can be any *positive* counting numerator as well as 0.
- *coefficients* that can be any signed decimal numerator

In other words, a *plain* polynomial is a combination of *powers* that do not involve any *negative* exponent—but can involve the exponent 0.

Example 17. The following are *plain* polynomials:

$$-47.03x^{+3} + 57.89x^{+2} + 22.71x^{+1} + 0.3x^{0}$$
$$0.3x^{0} - 47.03x^{+1} + 57.89x^{+2} + 22.71x^{+3}$$

The following are *not plain* polynomials:

$$-47.03x^{+3} + 57.89x^{+2} + 22.71x^{+1} + 0.3x^{0} - 22.43x^{-1}$$
$$-22.43x^{-1} + 0.3x^{0} - 47.03x^{+1} + 57.89x^{+2} + 22.71x^{+3}$$

1. When we replace x by TEN in a *plain* polynomial whose coefficients are all single-digit counting numerators, the result is a *counting* number.

Example 18. Given the *plain* polynomial

$$4x^{+3} + 7x^{+2} + 9x^{+1} + 8x^{0}$$

when $x := 10$ we get:

$$\begin{aligned} 4x^{+3} + 7x^{+2} + 9x^{+1} + 4x^{0}\big|_{x:=10} &= 4 \times 10^{+3} + 7 \times 10^{+2} + 9 \times 10x^{+1} + 8 \times 10^{0} \\ &= 4 \times 1000 + 7 \times 100 + 9 \times 10 + 8 \times 1 \\ &= 4000 + 700 + 90 + 8 \\ &= 4\ 7\ 9\ 8 \end{aligned}$$

which is the *counting* number whose digits are the coefficients of the *plain* polynomial.

2. Just like decimal numerators are not really more difficult to use than just counting numerators—they just require understanding that the decimal point indicates which of the digits in the decimal numerator corresponds to the denominator[1], Laurent polynomials are just as easy to use as just plain polynomials. This is particularly the case since, in the case of polynomials, we do not have to worry about the "place" of a monomial in a polynomial since the place is always given by the exponent

3. Just like decimal numbers are vastly more useful than just counting numbers, Laurent polynomials will be vastly more useful than plain polynomials for our purposes as the discussion in the EPILOGUE will show.

4. Since, from the point of view of handling them, there is not going to be any difference between Laurent polynomials and plain polynomials, we will just the word **polynomial**.

[1] But then of course, since Educologists have a deep aversion to denominators, they are sure to disagree.

add
like monomials
$\boxplus$
addition of polynomials

15.4 Addition

Just like *combinations* can always be added to give another combination, *polynomials* can always be added to give another polynomial.

EXAMPLE 19. Just like the combinations

17 **Apples** & 4 **Bananas** and 7 **Bananas** & 8 **Carrots**

can be added to give another combination:

17 **Apples** &	4 **Bananas**	
	7 **Bananas** &	8 **Carrots**
17 **Apples** &	11 **Bananas** &	8 **Carrots**

the polynomials

$$-17x^{+6} + 4x^{-3} \quad \text{and} \quad +7x^{-3} + 8x^{+2}$$

can be added to give another polynomial:

$$\begin{array}{lll} -17x^{+6} & +4x^{-3} & \\ & +7x^{-3} & +8x^{+2} \\ \hline -17x^{+6} & +11x^{-3} & +8x^{+2} \end{array}$$

1. To **add** two polynomials with signed coefficients, we *oplus* the coefficients of **like monomials** that is of monomials with the same exponent. We will use the symbol $\boxplus$ to write the specifying-phrase that corresponds to the **addition of polynomials**.

EXAMPLE 20. Given the polynomials

$$-17x^{+6} + 4x^{-3} \quad \text{and} \quad +7x^{-3} + 8x^{+2}$$

the specifying-phrase for addition will be

$$-17x^{+6} + 4x^{-3} \boxplus +7x^{-3} + 8x^{+2}$$

and to identify it, we will write

$$\begin{aligned} -17x^{+6} + 4x^{-3} \boxplus +7x^{-3} + 8x^{+2} &= -17x^{+6} + [+4 \oplus +7]x^{-3} + 8x^{+2} \\ &= -17x^{+6} + 11x^{-3} + 8x^{+2} \end{aligned}$$

2. The only difficulties when adding polynomials occur when one is not careful to write them:

- in order—whether ascending or descending
- with missing monomials written-in with 0 coefficient

EXAMPLE 21. Given the polynomials

$$-17x^{+3} - 14x^{+2} - 8x^{0} + 4x^{-1} \quad \text{and} \quad +7x^{+4} + 8x^{+3} - 11x^{+1} - 4x^{-2}$$

consider the difference between the following two ways to write the addition of two polynomials:

- When we *do not* write the polynomials in order and *do not* write-in missing monomials with a 0 coefficient, we get:

$$\begin{array}{l} -17x^{+3} - 14x^{+2} \quad - 8x^{0} \quad + 4x^{-1} \\ \ +7x^{+4} \quad + 8x^{+3} - 11x^{+1} - 4x^{-2} \\ \hline \end{array}$$

  and it is not easy to do the addition and get the result:

$$+7x^{+4} - 9x^{+3} - 14x^{+2} - 11x^{+1} - 8x^{0} + 4x^{-1} - 4x^{-2}$$

- When we *do* write the polynomials in order and we *do* write-in the missing monomials with a 0 coefficient, we get:

$$\begin{array}{lllllll} 0x^{+4} & - 17x^{+3} & - 14x^{+2} & + 0x^{+1} & - 8x^{0} & + 4x^{-1} & + 0x^{-2} \\ +7x^{+4} & + 8x^{+3} & + 0x^{+2} & - 11x^{+1} & + 0x^{0} & + 0x^{-1} & - 4x^{-2} \\ \hline +7x^{+4} & - 9x^{+3} & - 14x^{+2} & - 11x^{+1} & - 8x^{0} & + 4x^{-1} & - 4x^{-2} \end{array}$$

  where the result is much easier to get.

3. One way in which *polynomials* are easier than *numerators* to deal with is that when we add them there is no so-called "carry-over".

The reason we have "carry-over" in ARITHMETIC is that when dealing with combinations of powers of TEN, the coefficients can only be *digits*. So, when we add, say, the hundreds, if the result is still a single digit, we can can write it down but if the result is more than *nine*, we have no single digit to write the result down and we must *change* TEN of the hundreds for a thousand which is what the "carry-over" is.

But in ALGEBRA, with combinations of powers of x, there is no such restriction on the coefficients which can be any numerator and so, when we add, we can write down the result whatever it is.

EXAMPLE 22.

- When we add the numerators 756.92 and 485.57 we get:

$$\begin{array}{rr} & {\scriptstyle 1\,1\,1} \\ & 756.92 \\ + & 485.57 \\ \hline & 1242.49 \end{array}$$

  in which there are three "carry-overs" because there are three places where we couldn't write the result with a single digit.

- When we add the corresponding single-digit coefficient polynomials, we get:

$$\begin{array}{llllll} & +7x^{+2} & + 5x^{+1} & + 6x^{0} & + 9x^{-1} & + 2x^{-2} \\ \boxplus & +4x^{+2} & + 8x^{+1} & + 5x^{0} & + 5x^{-1} & + 7x^{-2} \\ \hline \end{array}$$

$$+11x^{+2} + 13x^{+1} + 11x^{0} + 14x^{-1} + 9x^{-2}$$

in which there is no" carry-over" since we can write two-digit coefficients.

15.5 Subtraction

Subtraction "works" essentially the same way as addition except of course that while, in the case of *addition*, we *oplus* the monomials of the second polynomial, in the case of *subtraction*, we *ominus* the monomials of the second polynomial, that is we *oplus the opposite* of the monomials of the second polynomial.

EXAMPLE 23. In order to subtract the second polynomial from the first:

$$\begin{array}{rrrrrr} & +2x^{+2} & +4x^{+1} & +6x^{0} & -6x^{-1} & -5x^{-2} \\ \boxminus & \boxed{-9}\,x^{+2} & \boxed{-3}\,x^{+1} & \boxed{+3}\,x^{0} & \boxed{-5}\,x^{-1} & \boxed{+7}\,x^{-2} \\ \hline \end{array}$$

we add the opposite of the second polynomial to the first polynomial, that is we *oplus the opposite* of each monomial in the second polynomial to the corresponding monomial in the first polynomial:

$$\begin{array}{rrrrrr} & +2x^{+2} & +4x^{+1} & +6x^{0} & -6x^{-1} & -5x^{-2} \\ \boxplus & \boxed{+9}\,x^{+2} & \boxed{+3}\,x^{+1} & \boxed{-3}\,x^{0} & \boxed{+5}\,x^{-1} & \boxed{-7}\,x^{-2} \\ \hline & +11x^{+2} & +7x^{+1} & +3x^{0} & -1x^{-1} & -12x^{-2} \end{array}$$

Here again, things are easier with polynomials than with numerals since there is no "borrowing".

Chapter 16

Polynomials 2: Multiplication

Multiplication of polynomials is very close to multiplication of decimal number-phrases so, we begin by discussion of multiplication in ARITHMETIC.

16.1 Multiplication in Arithmetic

In ARITHMETIC, *multiplication* is an operation that is very different from *addition* in many ways.

1. While number-phrases (with a common denominator) can always be added, number-phrases, even with a common denominator, usually cannot be multiplied.

EXAMPLE 1.
While

$$2\ \textbf{Apples} + 3\ \textbf{Apples} = 5\ \textbf{Apples}$$

the following

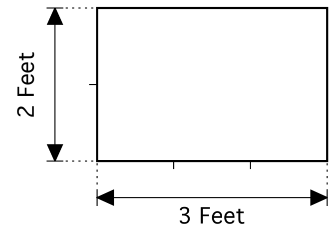
$$2\ \textbf{Apples} \times 3\ \textbf{Apples}$$

makes no sense whatsoever.
(2 **Apples** $\times$ 3 **Apples** is not the same as $2(3$ **Apples**$)$ which is equal to 6 **Apples**)

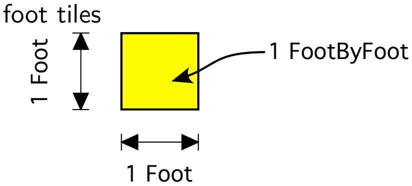
2. Even when number-phrases can be multiplied, the result involves a *different* denominator.

EXAMPLE 2.

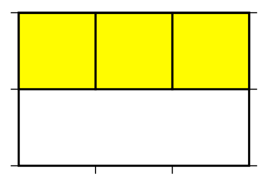
- Say that, in the real world, we want to tile a table three feet long by two feet wide



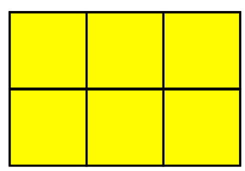
with one foot by one foot tiles



We need three tiles to tile the first row:



and another three tiles to tile the second row:



Altogether then, we used six one foot by one foot tiles.

- On paper, the *specifying-phrase* that represents the area of the *table* is

  $$2\ \textbf{Feet} \times 3\ \textbf{Feet}$$

  and the *number-phrase* that represents the area of a *tile* is

  $$1\ \textbf{FootByFoot}$$

  also known as

  $$1\ \textbf{SquareFoot}$$

  We then represent the fact that we used two rows of three tiles by

  $$\begin{aligned} 2(3\ \textbf{FootByFoot}) &= (2 \times 3)\ \textbf{FootByFoot} \\ &= 6\ \textbf{FootByFoot} \end{aligned}$$

  Altogether, we represent the real world tiling *process* by the paper *procedure*

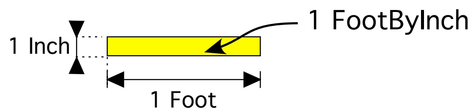
  $$2\ \textbf{Feet} \times 3\ \textbf{Feet} = (2 \times 3)\ \textbf{FootByFoot}$$

3. While number-phrases involving different denominators can never be added, number-phrases involving different denominators can occasionally be multiplied.

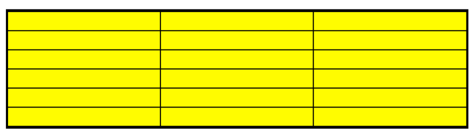
EXAMPLE 3.

- Say that, in the real world, we want to tile a shelf three feet long by six inches wide

with one foot by one inch tiles



We need three tiles to tile each row and since there are six rows



we need eighteen one foot by one inch tiles.

- On paper, the *specifying-phrase* that represent the area of the *table* is

$$3\ \textbf{Feet} \times 6\ \textbf{Inches}$$

and the *number-phrase* that represents the area of a *tile* is

$$1\ \textbf{FootByInch}$$

We then represent the fact that we used six rows of three tiles by

$$\begin{aligned} 6(3\ \textbf{FootByInch}) &= (6 \times 3)\ \textbf{FootByInch} \\ &= 18\ \textbf{FootByInch} \end{aligned}$$

Altogether, we represent the real world tiling *process* by the paper *procedure*

$$6\ \textbf{Feet} \times 3\ \textbf{Inches} = (6 \times 3)\ \textbf{FootByInch}$$

16.2 Multiplication of Polynomials

In POLYNOMIAL ALGEBRA, things are much simpler: Because we can always multiply monomials, it turns out that we can multiply polynomials. We will use the symbol ⊠ to denote multiplication of polynomials.

1. In order to multiply a given polynomial by a given monomial, we multiply each and every monomial in the given polynomial by the given monomial and the result is another polynomial.

EXAMPLE 4. Given the polynomial

$$+2x^{+2} \quad +4x^{+1} \quad +6x^{0} \quad -6x^{-1} \quad -5x^{-2}$$

and the monomial

$$-4x^{+3}$$

In order to identify the specifying phrase

$$\left[+2x^{+2} \quad +4x^{+1} \quad +6x^{0} \quad -6x^{-1} \quad -5x^{-2}\right] \boxtimes \left[-4x^{+3}\right]$$

i. We set up as in ARITHMETIC

$$\begin{array}{lccccc} & +2x^{+2} & +4x^{+1} & +6x^{0} & -6x^{-1} & -5x^{-2} \\ \boxtimes & & & -9x^{+2} & & \\ \hline \end{array}$$

ii. We multiply each and every monomial in the given polynomial by the given monomial:

$$\begin{array}{lccccc} & +2x^{+2} & +4x^{+1} & +6x^{0} & -6x^{-1} & -5x^{-2} \\ \boxtimes & & & -9x^{+2} & & \\ \hline \end{array}$$

$$(+2)(-9)x^{+2\oplus+2} \quad (+4)(-9)x^{+1\oplus+2} \quad (+6)(-9)x^{0\oplus+2} \quad (-6)(-9)x^{-1\oplus+2} \quad (-5)(-9)x^{-2\oplus+2}$$

iii. We get

$$\begin{array}{lccccc} & +2x^{+2} & +4x^{+1} & +6x^{0} & -6x^{-1} & -5x^{-2} \\ \boxtimes & & & -9x^{+2} & & \\ \hline & -18x^{+4} & -36x^{+3} & -54x^{+2} & -54x^{+1} & +45x^{0} \end{array}$$

2. In order to multiply a first polynomial by a second polynomial, we multiply each and every monomial in the first polynomial by each and every monomial in the first polynomial and the result is another polynomial.
In order to keep some order in the procedure,
i. We set up the multiplication pretty much as in ARITHMETIC:
a. We write the first polynomial on the first line with missing monomials written-in with a 0 coefficient
b. We write the second polynomial on the second line *without* writing-in the missing monomials with a 0 coefficient. Also, the second polynomial need not be lined up exponent-wise with the first polynomial
ii. We write the results of the multiplication of the first polynomial by each monomial of the second polynomial on a separate line
iii. As we write the results of the multiplication of the first polynomial by the next monomial of the second polynomial, we make sure that the exponents are lined up vertically (to make the next step easier).
iv. We add the terms with same exponent (lined up vertically as a result of the previous step).

EXAMPLE 5. Given a first polynomial

$$+5x^{+3} \quad -4x^{+1} \quad +6x^{0} \quad -7x^{-2}$$

and a second polynomial

$$+2x^{+2} \quad -8x^{+1} \quad +3x^{-1}$$

In order to identify the specifying phrase

$$\left[+5x^{+3}\ -4x^{+1}\ +6x^{0}\ -7x^{-2}\right] \boxtimes \left[+2x^{+2}\ -8x^{+1}\ +3x^{-1}\right]$$

we proceed as follows:

i. We set up as usual, writing the monomials missing *in the first polynomial* with a 0 coefficient.

$$\begin{array}{lllllll} & +5x^{+3} & +0x^{+2} & -4x^{+1} & +6x^{0} & +0x^{-1} & -7x^{-2} \\ \boxtimes & & +2x^{+2} & -8x^{+1} & +3x^{-1} & & \\ \hline \end{array}$$

ii. We multiply each and every monomial in the first polynomial by the first monomial in the second polynomial, writing the missing monomials with a 0 coefficient.

$$\begin{array}{lllllll} & +5x^{+3} & +0x^{+2} & -4x^{+1} & +6x^{0} & +0x^{-1} & -7x^{-2} \\ \boxtimes & & +2x^{+2} & -8x^{+1} & +3x^{-1} & & \\ \hline \end{array}$$

$$+10x^{+5}\ +0x^{+4}\ -8x^{+3}\ +12x^{+2}\ +0x^{+1}\ -14x^{0}$$

iii. We multiply each and every monomial in the first polynomial by the second monomial in the second polynomial, writing the missing monomials with a 0 coefficient.

$$\begin{array}{lllllll} & +5x^{+3} & +0x^{+2} & -4x^{+1} & +6x^{0} & +0x^{-1} & -7x^{-2} \\ \boxtimes & & +2x^{+2} & -8x^{+1} & +3x^{-1} & & \\ \hline \end{array}$$

$$\begin{array}{lllllll} +10x^{+5} & +0x^{+4} & -8x^{+3} & +12x^{+2} & +0x^{+1} & -14x^{0} & \\ & -40x^{+4} & +0x^{+3} & +32x^{+2} & -48x^{+1} & +0x^{0} & +56x^{-1} \end{array}$$

iv. We multiply each and every monomial in the first polynomial by the third monomial in the second polynomial, writing the missing monomials with a 0 coefficient.

$$\begin{array}{lllllll} & +5x^{+3} & +0x^{+2} & -4x^{+1} & +6x^{0} & +0x^{-1} & -7x^{-2} \\ \boxtimes & & +2x^{+2} & -8x^{+1} & +3x^{-1} & & \\ \hline \end{array}$$

$$\begin{array}{lllllllll} +10x^{+5} & +0x^{+4} & -8x^{+3} & +12x^{+2} & +0x^{+1} & -14x^{0} & & & \\ & -40x^{+4} & +0x^{+3} & +32x^{+2} & -48x^{+1} & +0x^{0} & +56x^{-1} & & \\ & & & +15x^{+2} & +0x^{+1} & -12x^{0} & +18x^{-1} & +0x^{-2} & -21x^{-3} \end{array}$$

v. We add the terms with same exponent

$$\begin{array}{lllllll} & +5x^{+3} & +0x^{+2} & -4x^{+1} & +6x^{0} & +0x^{-1} & -7x^{-2} \\ \boxtimes & & +2x^{+2} & -8x^{+1} & +3x^{-1} & & \\ \hline \end{array}$$

$$\begin{array}{lllllllll} +10x^{+5} & +0x^{+4} & -8x^{+3} & +12x^{+2} & +0x^{+1} & -14x^{0} & & & \\ & -40x^{+4} & +0x^{+3} & +32x^{+2} & -48x^{+1} & +0x^{0} & +56x^{-1} & & \\ & & & +15x^{+2} & +0x^{+1} & -12x^{0} & +18x^{-1} & +0x^{-2} & -21x^{-3} \\ \hline \end{array}$$

$+10x^{+5} \quad -40x^{+4} \quad -8x^{+3} \quad +59x^{+2} \quad -48x^{+1} \quad -26x^{0} \quad +74x^{-1} \quad +0x^{-2} \quad -21x^{-3}$

v. We thus have:

$$\left[+5x^{+3} - 4x^{+1} + 6x^{0} - 7x^{-2}\right] \boxtimes \left[+2x^{+2} - 8x^{+1} + 3x^{-1}\right] =$$
$$= +10x^{+5} - 40x^{+4} - 8x^{+3} + 59x^{+2} - 48x^{+1} - 26x^{0} + 74x^{-1} + 0x^{-2} - 21x^{-3}$$

near
small

Chapter 17

Polynomials 3: Powers of $x_0 + h$

While it is easy to compute with powers of a *counting*-numerator, it is a lot more difficult to compute with powers of a *decimal*-numerator.

EXAMPLE 1. While it is easy to find that:

$$5 \bullet 3^4 = 405$$

it is a lot more difficult to find that

$$4 \bullet 3.14^4 = 388.84684864$$

But the main issue is that the result of a *repeated-multiplication* with a *base* that is a decimal numerator will usually involve a lot more decimals than are in the base and than we really want so that a lot of the work is wasted.

EXAMPLE 2. In

$$4 \bullet 3.14^4 = 388.84684864$$

the base, 3.14, has only two decimals but the result, 388.84684864, most probably has a lot more decimals than we want.

In this chapter we will investigate a procedure that will allow us to get only the number of decimals we want. It is based on the fact that any *decimal* numerator is always **near** a *counting* numerator in the sense that any *decimal* numerator is equal to a *counting*-numerator plus a **small** numerator

EXAMPLE 3. 3.14 is *near* 3 because $3.14 = 3 + 0.14$ and 0.14 is *small* binomial

We will thus investigate the powers of the **binomial** x_0+h. We will begin by investigating the case in which the repeated multiplication involves two copies of the *binomial* and then the case in which the repeated multiplication involves three copies of the *binomial*. Then we will develop a procedure for the cases in which the repeated multiplication involves at least three copies of the *binomial*.

17.1 The Second Power: $(x_0 + h)^2$

1. In this case, the *repeated-multiplication procedure* is simple enough. In order to compute the second power of $x_0 + h$, we write, keeping in mind that we want the monomials to appear in order of diminishing sizes and since x_0 and h both stand for *signed* numerators:

$$\begin{array}{ccccc} & & x_0 & \oplus & h \\ & & x_0 & \oplus & h \\ \hline & & x_0h & \oplus & h^2 \\ x_0^2 & \oplus & x_0h & & \\ \hline x_0^2 & \oplus & 2x_0h & \oplus & h^2 \end{array}$$

a. We begin by looking at what happens in ARITHMETIC which is that the multiplication procedure essentially keeps track and respects the sizes—but, because of carryovers, only roughly so.

EXAMPLE 4. In order to compute 3.2^2, we actually compute $(3 + 0.2)^2$ and write—since we are dealing with *plain* numerators:

$$\begin{array}{ccccc} & & 3 & + & 0.2 \\ & & 3 & + & 0.2 \\ \hline & & 3 \bullet 0.2 & + & 0.2^2 \\ 3^2 & + & 3 \bullet 0.2 & & \\ \hline 3^2 & + & 2 \bullet 3 \bullet 0.2 & + & 0.2^2 \end{array}$$

that is

$$\begin{array}{ccccc} & & 3 & + & 0.2 \\ & & 3 & + & 0.2 \\ \hline & & 0.6 & + & 0.04 \\ 9 & + & 0.6 & & \\ \hline 9 & + & 1.2 & + & 0.04 \end{array}$$

The multiplication procedure kept roughly track of the sizes except for what the carryover caused:

- All the way to the left are the "ones"
- In the middle are the "tenths"
- All the way to the right are the "hundredths"

so that if we want:

- No decimal, we write
$$3.2^2 = 10 + (...)$$
- One decimal, we write
$$3.2^2 = 10.2 + (...)$$
- Two decimals, we write
$$3.2^2 = 10.24$$

where $+ (...)$ is there to remind us that we are ignoring something too "in the tenths" to matter here.

EXAMPLE 5. In order to compute 2.8^2, we observe that 2.8 is nearer 3 than 2 so that we actually compute $(3 \oplus -0.2)^2$ and write—since we are now dealing with *signed* numerators:

$$\begin{array}{ccccc} & & +3 & \oplus & -0.2 \\ & & +3 & \oplus & -0.2 \\ \hline & & +3 \bullet -0.2 & \oplus & (-0.2)^2 \\ (+3)^2 & \oplus & +3 \bullet -0.2 & & \\ \hline (+3)^2 & \oplus & 2 \bullet +3 \bullet -0.2 & \oplus & (-0.2)^2 \end{array}$$

that is

$$\begin{array}{ccccc} & & +3 & \oplus & -0.2 \\ & & +3 & \oplus & -0.2 \\ \hline & & -0.6 & \oplus & +0.04 \\ +9 & \oplus & -0.6 & & \\ \hline +9 & \oplus & -1.2 & \oplus & +0.04 \end{array}$$

The multiplication procedure kept roughly track of the sizes except for what the carryover caused:

- All the way to the left are the "ones"
- In the middle are the "tenths"
- All the way to the right are the "hundredths"

so that if we want:

- No decimal, we write
$$2.8^2 = 8 + (...)$$
- One decimal, we write
$$2.8^2 = 7.8 + (...)$$
- Two decimals, we write
$$2.8^2 = 7.84$$

where $+ (...)$ is there to remind us that we are ignoring something, *positive or negative*, too "in the tenths" to matter here.

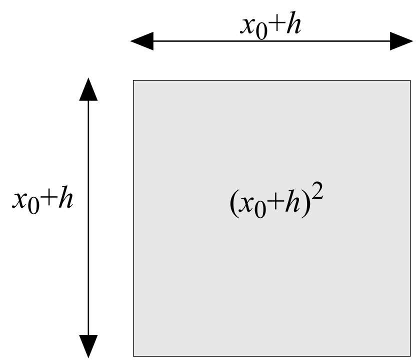
b. In algebra, a very frequent case is when we want a template for the power of any decimal-numerator in the neighborhood of a given x_0. In other words, we do not want yet to commit ourselves to how far the decimal-numerator is from the given x_0 and we use h to represent how far the decimal-numerator is from the given x_0.

Of course, when, ultimately, we replace h by some "in the tenths" number, there remains the possibility that a carryover will mess up the result a little bit. This though is something that we will not deal with here. (But see the Epilogue.)

EXAMPLE 6. In order to get a template for the second power of any decimal-numerator near 3, both above 3 and below 3, we write:

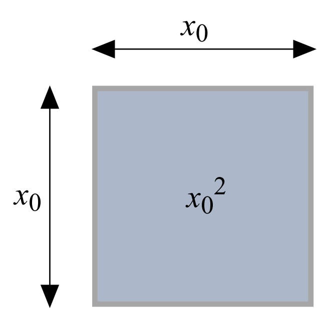
		3	$\oplus$	h
		3	$\oplus$	h
		$3h$	$\oplus$	h^2
3^2	$\oplus$	$3h$		
3^2	$\oplus$	$2 \bullet 3h$	$\oplus$	h^2

2. Another, much more fruitful to get the above template is to go back to the definition of multiplication in terms of the *area of a rectangle* so that $(x_0 + h)^2$ is the area of a $x_0 + h$ by $x_0 + h$ *square*:

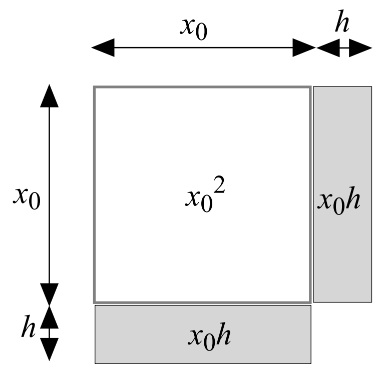


What we then do is to enlarge the sides of a x_0 by x_0 square by h but, for the sake of clarity, we will construct the enlarged square one step at a time:

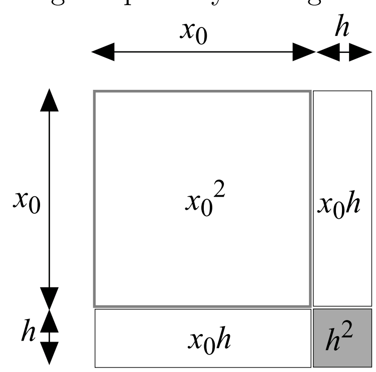
i. We start with x_0^2 as the area of a square with side x_0:



ii. We now enlarge the sides of the square by h in each dimension which adds two $x_0 + h$ by h rectangles:



iii. We complete the enlarged square by adding one h by h square:



EXAMPLE 7. In order to get a template to get the second power of any decimal-numerator near 3, both above 3 and below 3, we visualize the above picture and see in our mind that we need the area of:
i. the original square: 3^2
ii. the two rectangular strips: $2 \bullet 3 \bullet h$
iii. the little square: h^2
so that we have the template:

$$(3 \oplus h)^2 = 3^2 \oplus 2 \bullet 3 \bullet h \oplus h^2$$

This second approach has three major advantages over the first one:
i. The terms in the sum are clearly in order of *diminishing size*: Since x_0 is "in the ones" and h is "in the tenths",

- both dimensions of the "initial square" are "in the ones" so that x_0^2 is "in the ones",
- one dimension of the rectangles is "in the ones" but the other dimension is "in the tenths" so that $2x_0h$ is "in the tenths",
- both dimensions of the "little square" are "in the tenths" so that h^2 is "in the hundredths".

ii. When we will need formulas for $(x_0 + h)^3$, $(x_0 + h)^4$, etc, not only will repeated multiplication get out of hand but, as we shall see, we will never

need more than the first few monomials in the result.
iii. If all we need is a particular monomial in the result, which is often the case, we can get it from the picture without having to do the whole repeated multiplication.

EXAMPLE 8. If, for whatever reason, we need the h monomial in $(3 \oplus h)^2$, we visualize the two rectangular strips and we write:

$$2 \bullet 3 \bullet h$$

THEOREM 9 (Addition Formula for Quadratics).

$$(x_0 + h)^2 = x_0^2 + 2x_0h + h^2$$

17.2 The Third Power: $(x_0 + h)^3$

For the sake of brevity we omit the investigation of what happens in arithmetic.

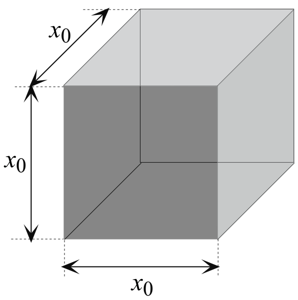
1. The repeated-multiplication procedure already begins to be painful: First we must multiply two copies of $x_0 + h$:

$$\begin{array}{ccccc} & & x_0 & + & h \\ & & x_0 & + & h \\ \hline & & x_0h & + & h^2 \\ x_0^2 & + & x_0h & & \\ \hline x_0^2 & + & 2x_0h & + & h^2 \end{array}$$

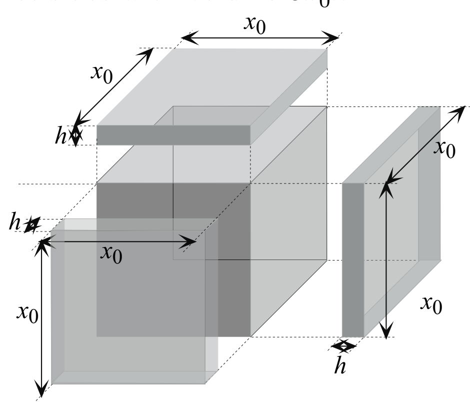
Then, we must multiply $x_0^2 + 2x_0h + h^2$ by the third copy of $x_0 + h$

$$\begin{array}{ccccccc} & & x_0^2 & + & 2x_0h & + & h^2 \\ & & & & x_0 & + & h \\ \hline & & x_0^2h & + & 2x_0h^2 & + & h^3 \\ x_0^3 & + & 2x_0^2h & + & x_0h^2 & & \\ \hline x_0^3 & + & 3x_0^2h & + & 3x_0h^2 & + & h^3 \end{array}$$

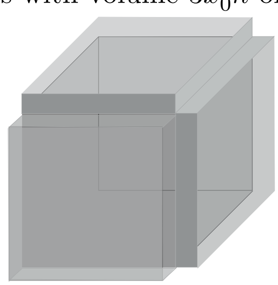
2. Another, much more fruitful approach to the addition formula is to go back to the definition of multiplication in terms of the area/volume of a rectangle so that $(x_0 + h)^3$ is the volume of a $x_0 + h$ by $x_0 + h$ by $x_0 + h$ cube:
What we then do is to enlarge the three sides of a x_0 by x_0 cube by h but, for the sake of clarity, we will construct the enlarged cube one step at a time:
i. We draw the initial cube with volume x_0^3:



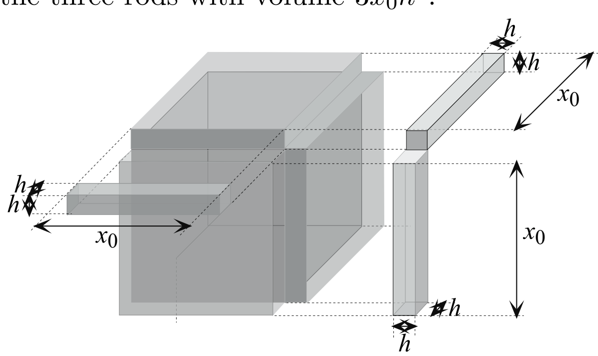
ii. We draw the three slabs with volume $3x_0^2h$:



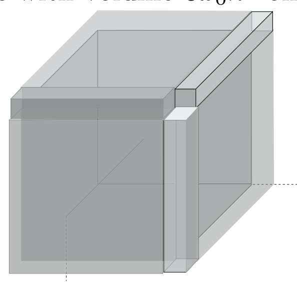
iii. We glue the three slabs with volume $3x_0^2h$ onto what we already glued:



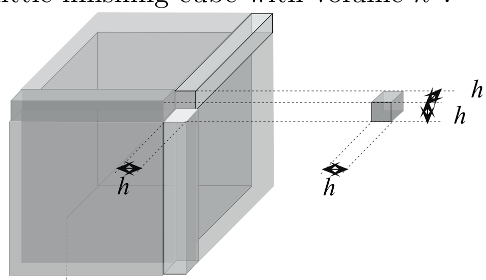
iv. We draw the three rods with volume $3x_0h^2$:



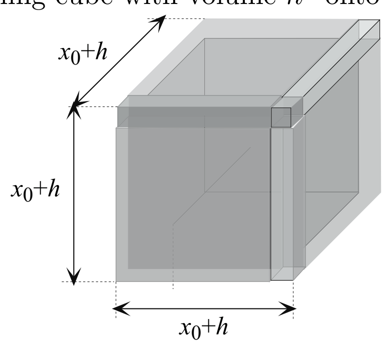
v. We glue the three rods with volume $3x_0h^2$ onto what we already glued:



vi. We draw the little finishing cube with volume h^3:



We glue the little finishing cube with volume h^3 onto what we already glued:



This approach has three major advantages over the *repeated-multiplication*:
i. The terms in the sum are in *order of diminishing size*. Since x_0 is "in the ones" and h is "in the tenths",

- all three dimensions of the "initial cube" are "in the ones" so that x_0^3 is "in the ones",
- two dimensions of the "slabs" are "in the ones" but the third dimension is "in the tenths" so that, if h is "in the tenths", then $3x_0^2h$ is "in the tenths",
- one dimensions of the "square rods" is "in the ones" so that, if h is "in the tenths", then $3x_0h^2$ is "in the hundredths".
- all three dimensions of the "little cube" are "in the tenths" so that, if h is "in the tenths", then x_0h^3 is "in the thousandths".

ii. If all we need is a particular one of the terms, which will often be the case, we can get it from the picture without having to do the whole multiplication. pattern
iii. Later on, when we shall need formulas for $(x_0 + h)^4$, etc, not only will repeated multiplication get out of hand but, as we shall see, we will never need more than the first few monomials of the result.
THEOREM 10 (ADDITION FORMULA for CUBICS).

$$(x_0 + h)^3 = x_0^3 + 3x_0^2h + 3x_0h^2 + h^3$$

17.3 Higher Powers: $(x_0 + h)^n$ when $n > 3$

Here of course:
- Repeated-multiplication is of course going to be ever more painful
- We cannot make pictures because we would need to be able to draw in more than 3 dimensions.

So, we need to find a *procedure*.

1. We begin by looking for a **pattern** in what we have so far. In order to see better what we are doing, we will not let anything go without saying.
a. When the exponent is 3, we had:

$$\begin{aligned}(x_0 + h)^3 &= x_0^3 + 3x_0^2h + 3x_0h^2 + h^3 \\ &= x_0^3h^0 + 3x_0^2h^1 + 3x_0^1h^2 + x_0^0h^3 \\ &= x_0 \bullet x_0 \bullet x_0 \;\&\; 3 \bullet x_0 \bullet x_0 \bullet h \;\&\; 3 \bullet x_0 h \bullet h \;\&\; h \bullet h \bullet h \\ &= 1 \bullet x_0 \bullet x_0 \bullet x_0 \;\&\; 3 \bullet x_0 \bullet x_0 \bullet h \;\&\; 3 \bullet x_0 h \bullet h \;\&\; 1 \bullet h \bullet h \bullet h\end{aligned}$$

Looking at the *factors* and the *coefficients* separately, we get the following:
- The *factors* are

  $$x_0 \bullet x_0 \bullet x_0 \qquad x_0 \bullet x_0 \bullet h \qquad x_0 \bullet h \bullet h \qquad h \bullet h \bullet h$$

  In other words, starting with 3 copies of x_0

  $$x_0 \bullet x_0 \bullet x_0$$

  we get the others by replacing one by one the copies of x_0 by copies of h.
- The *coefficients* are

  $$1 \qquad 3 \qquad 3 \qquad 1$$

  Here we cannot see the pattern,

b. When the exponent is 2, we have

$$\begin{aligned}(x_0 + h)^2 &= x_0^2 + 2x_0h + h^2 \\ &= x_0^2h^0 + 2x_0^1h^1 + x_0^0h^2 \\ &= x_0 \bullet x_0 \;\&\; 2 \bullet x_0 \bullet h \;\&\; h \bullet h\end{aligned}$$

$$= 1 \bullet x_0 \bullet x_0 \ \& \ 2 \bullet x_0 \bullet h \ \& \ 1 \bullet h \bullet h$$

Looking at the *factors* and the *coefficients* separately, we get the following:

- The *factors* are
$$x_0 \bullet x_0 \qquad x_0 \bullet h \qquad h \bullet h$$
In other words, starting with
$$x_0 \bullet x_0$$
we get the others by replacing one by one the copies of x_0 by copies of h.
- The *coefficients* are
$$1 \qquad 2 \qquad 1$$
Here again we cannot see the pattern.

c. When the exponent is 1, we have

$$\begin{aligned}(x_0 + h)^1 &= x_0 + h \\ &= x_0^1 + h^1 \\ &= x_0^1 h^0 + x_0^0 h^1 \\ &= x_0 \ \& \ h \\ &= 1 \bullet x_0 \ \& \ 1 \bullet h\end{aligned}$$

Looking at the *factors* and the *coefficients* separately, we get the following:

- The *factors* are
$$x_0 \qquad h$$
In other words, starting with
$$x_0$$
we get the others by replacing the one copy of x_0 by a copy of h.
- The *coefficients* are
$$1 \qquad 1$$
Here we cannot see the pattern,

2. Putting everything together, though,

- The procedure for finding the *powers* seems to be in all cases:
  i. Make as many copies of x_0 as what the exponent n in $(x_0 + h)^n$ indicates
  ii. Make as many copies plus 1 of what the exponent n in $(x_0 + h)^n$ indicates
  iii. Starting with the second copy, replace one by one the copies of x_0 by copies of h
- In order to see a pattern for the *coefficients*, we write them starting with exponent 1 and ending with exponent 3:

$$\begin{array}{ccccccc} & & 1 & & 1 & & \\ & 1 & & 2 & & 1 & \\ 1 & & 3 & & 3 & & 1 \end{array}$$

PASCAL TRIANGLE
parent-entries

The way things are arranged, we see that we get each entry in what is called the **PASCAL TRIANGLE** by adding its two **parent-entries** that is the two entries just above it.
Thus, the next line in the PASCAL TRIANGLE would be:

1 4 6 4 1

3. *Proving* that all this is indeed the case would involve more work than we are willing to do here and so we will take the following for granted:

THEOREM 11 (**BINOMIAL THEOREM**). *The addition formula for a binomial of degree n is:*

$$\begin{aligned}(x_0 + h)^n = x_0^n h^0 + \frac{n}{1} x_0^{n-1} h^{+1} &+ \frac{n(n-1)}{1 \cdot 2} x_0^{n-2} h^{+2} \\ &+ \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} x_0^{n-3} h^{+3} \\ &+ \ldots \\ &+ \frac{n(n-1)(n-2)\cdots(1)}{1 \cdot 2 \cdot 3 \cdots n} x_0^0 h^n\end{aligned}$$

4. According to the **BINOMIAL THEOREM**,

$$\begin{aligned}(x_0 + h)^0 &= x_0^0 h^0 \\ &= 1\end{aligned}$$

which is of course as it should be. Moreover, since the coefficient 1 goes without saying, this means that the very first line in the PASCAL TRIANGLE is 1 so that the "complete" PASCAL TRIANGLE is:

$n := 0$							1						
$n := 1$						1		**1**					
$n := 2$					1		**2**		1				
$n := 3$				1		**3**		3		1			
$n := 4$			1		**4**		6		4		1		
$n := 5$		1		**5**		10		10		5		1	
$n := 6$	1		**6**		15		20		15		6		1
……	…	…	…	…	…	…	…	…	…	…	…	…	…

- The numerators in the *second* slanted row (bold-faced) are the coefficients of the h^{+1} powers which shows that the coefficient of the h^{+1} power in x_0^n is n.
- We check that the *third* slanted row are the coefficients of the h^{+2} powers which shows that the coefficient of the h^{+2} power in x_0^n is $\frac{n(n-1)}{2}$.

constant approximation
affine approximation
quadratic approximation

- Etc

17.4 Approximations

Fortunately, most of the time we only need the very first few terms of the addition formulas.

1. Very often, we will need only the **constant approximation** of $(x_0 + h)^n$ which is just x_0^n. Indeed, very often h will be small enough that we will not need to consider any of the monomials that involve it and we will write:

$$(x_0 + h)^n = x_0^n + (...)$$

EXAMPLE 9. The *constant approximation* of 16.072^7 is 16^7 and we write

$$16.072^7 = 16^7 + (...)$$

More generally, the *constant approximation* of $(16 + h)^7$ is 16^7 and we write

$$(16 + h)^7 = 16^7 + (...)$$

2. When the *constant* approximation is too crude, we will often use the **affine approximation** of $(x_0 + h)^n$ which is $x_0^n + nx_0h$. Indeed, while h may not be small enough not to matter, the other powers, h^2, h^3 etc being smaller than h can often still be ignored and we will then write

$$(x_0 + h)^n = x_0^n + nx_0^{n-1}h + (...)$$

EXAMPLE 10. The *affine approximation* of 16.072^7 is $16^7 + 7 \cdot 16^6 \cdot 0.072$ and we write

$$16.072^7 = 16^7 + 7 \cdot 16^6 \cdot 0.072 + (...)$$

More generally, the *affine approximation* of $(16 + h)^7$ is $16^7 + 7 \cdot 16^6 \cdot h + (...)$ and we write

$$(16 + h)^7 = 16^7 + 7 \cdot 16^6 \cdot h + (...)$$

3. And finally we will also use the **quadratic approximation** of $(x_0 + h)^n$ which is $x_0^n + nx_0h + \frac{n(n-1)}{2}x_0h^2$ when we will need more precision that the affine approximation will be able to give us and we will then write

$$(x_0 + h)^n = x_0^n + nx_0h + \frac{n(n-1)}{2}x_0h^2 + (...)$$

EXAMPLE 11. The *quadratic approximation* of 16.072^7 is $16^7 + 7 \cdot 16^6 \cdot 0.072 + 21 \cdot 16^6 \cdot 0.072^2$ and we write

$$16.072^7 = 16^7 + 7 \cdot 16^6 \cdot 0.072 + 21 \cdot 16^6 \cdot 0.072^2 + (...)$$

More generally, the *quadratic approximation* of $(16 + h)^7$ is $16^7 + 7 \cdot 16^6 \cdot h + 21 \cdot 16^6 \cdot h^2$ and we write

$$(16 + h)^7 = 16^7 + 7 \cdot 16^6 \cdot h + 21 \cdot 16^6 \cdot h^2 + (...)$$

assign
round

Chapter 18

Polynomials 4: Division (In Descending & Ascending Powers)

We now turn to the last one of the four operation with polynomials: division. However, in order to understand the procedure, we must first take a look at the division procedure in ARITHMETIC.

18.1 Division In Arithmetic

We first look at the *real-world process* and then we look at the corresponding *paper-world procedure*.

1. In the real world, we often encounter situations in which we have to **assign** (equally) the items in a first collection to the items of another collection.

The *process* is to make **rounds** during each of which we *assign* one item of the first collection to each one of the items in the second collection. The process comes to an end when, after a round has been completed,

- there are items left unassigned but not enough to complete another round.

share
leftovers
division
dividend
divisor
quotient

The **share** is then the collection of items from the first collection that have been assigned to each item of the second collection and the **leftovers** are the collection of items from the first collection left unassigned after the process has come to an end.

EXAMPLE 1. In the real world, say we have a collection of seven dollar-bills which we want to assign to each and every person in a collection of three person. We want to know how many dollar-bills we will assign to each person and how many dollar-bills will be left-over.
i. We make a *first round* during which we hand-out one dollar-bill to each and every person in the collection. This uses three dollar-bills and leaves us with four dollar-bills after the first round.
ii. We make a *second round*, we hand-out one dollar-bill to each and every person in the collection. This uses another three dollar-bills and leaves us with one dollar-bill after the second round.
iii. If we try to make a *third round*, we find that we cannot complete the third round.
So, the *share* is two dollar-bills and the *leftovers* is one dollar-bill.

or,

- there is no item left unassigned. The *share* is again the collection of items from the first collection that have been assigned to each item of the second collection and there are no *leftovers*.

EXAMPLE 2. In the real world, say we have a collection of eight dollar-bills which we want to assign to each and every person in a collection of four person. We want to know how many dollar-bills we will assign to each person and how many dollar-bills will be left-over.
i. We make a *first round* during which we hand-out one dollar-bill to each and every person in the collection. This uses four dollar-bills and leaves us with four dollar-bills after the first round.
ii. We make a *second round*, we hand-out one dollar-bill to each and every person in the collection. This uses another four dollar-bills and leaves us with no dollar-bill after the second round.
iii. So, we cannot make a *third round*.
So, the *share* is two dollar-bills and there are no leftovers.

2. The paper *procedure* that corresponds to the real-world process is called **division**. *Division* will involve the following language:

- The number-phrase that represents the first collection, that is the collections of items *to be assigned* to the items of the second collection, is called the **dividend**,
- The number-phrase that represents the second collection, that is the collection of items *to which* the items of the first collection are to be assigned, is called the **divisor**,
- The number-phrase that represents the *share* is called the **quotient**,

- The number-phrase that represents the *leftovers* is called the **remainder**.

remainder
trial and error
try
partial product
partial remainder

EXAMPLE 3. Given a real-world situation with a collection of eight dollar-bills to be assigned to each and every person in a collection of four persons,

- The *dividend* is 7 **Dollars**
- The *divisior* is 3 **Persons**
- The *share* is $2 \frac{\text{Dollars}}{\text{Person}}$
- The *remainder* is 1 **Dollar**

18.2 Elementary School Procedure

The *division procedure* taught in elementary schools is a **trial and error** procedure which follows the real-world process closely inasmuch as each *round* is represented by a **try** in which:

i. We use the *multiplication procedure* to find the **partial product** which represents how many items *have been used* by the end of the corresponding *real-world round*.

ii. We use the *subtraction procedure* to find the **partial remainder** which represents how many items, if any, are *left over* by the end of the corresponding *real-world round*.

EXAMPLE 4. In order to divide 987 by 321, we go through the following *tries*:

First try:

i. We multiply the *divisor* 321 by 1 which gives the *partial product* 321:

$$\begin{array}{r|r} & 1 \\ \hline 321 & 987 \\ & 321 \end{array}$$

ii. We subtract the *partial product* 321 from the *dividend* 987 which leaves the *partial remainder* 666:

$$\begin{array}{r|r} & 1 \\ \hline 321 & 987 \\ & 321 \\ \hline & 666 \end{array}$$

Second try:

i. We multiply the *divisor* 321 by 2 which gives the *partial product* 642:

$$\begin{array}{r|r} & 2 \\ \hline 321 & 987 \\ & 642 \end{array}$$

ii. We subtract the *partial product* 642 from the *dividend* 987 which leaves the *partial remainder* 345:

rank

```
         2
321 ) 987
      642
      ---
      345
```

Third try:

i. We multiply the *divisor* 321 by 3 which gives the *partial product* 963:

```
         3
321 ) 987
      963
```

ii. We subtract the *partial product* 963 from the *dividend* 987 which leaves the *partial remainder* 24:

```
         3
321 ) 987
      963
      ---
       24
```

Fourth try:

i. We multiply the *divisor* 321 by 4 which gives the *partial product* 1284:

```
          4
321 )  987
      1284
```

ii. We cannot subtract the *partial product* 1284 from the *dividend* 987:

```
          4
321 )  987
      1284
      ----
```

Since we cannot complete the fourth try, we go back to the last complete try, that is the third try, and we get that the *quotient* is 3 and the *remainder* 24.

This procedure, though, has two severe shortcomings:

- All these *full multiplications* require a lot of work.
- This procedure will not extend to *polynomials*

18.3 Efficient Division Procedure

We now present a much more efficient procedure that, instead of *full multiplications* to find the digits of the quotient, uses only a *multiplication table*[1] and which, for us, has the further advantages that it extends easily to *polynomials*.

1. By the **rank** of a multiplication table, we will mean the *numerator* common to all the multiplications in that multiplication table. The **table**

[1]Educologists will surely claim that this procedure is way beyond the feeble mind of their students. Yet, it seems to be the one taught in most of the world and the procedure that uses "full multiplication" seems to be taught mostly, if not only, in the U.S..

multipliers correspond to the successive lines in the multiplication table and therefore always range from 1 to 9. The **table products** are the results of the successive multiplications in the multiplication table.

table multiplier
table product
cycles
step
stop
continue

Example 5. In the following multiplication table

7	× 1	=	7
7	× 2	=	14
7	× 3	=	21
7	× 4	=	28
7	× 5	=	35
7	× 6	=	42
7	× 7	=	49
7	× 8	=	56
7	× 9	=	63

- the *rank* is 7,
- the *table multipliers* range from 1 to 9 (as in all multiplication tables),
- the *table products* range from 7 to 63.

2. The *procedure* consists of successive **cycles**. During each of these *cycles*, we go through the following four **step**s:

Step I. We find a *single digit* of the *quotient* by *trial and error* using only the *multiplication table* whose *rank* is the *first digit* of the *divisor*.

Step II. We find the *partial product* by multiplying the *full divisor* by the *single digit* of the quotient we found in Step I.

Step III. We find the *partial remainder* by subtracting the *partial product* we found in Step II from the *full dividend*.

Step IV. We decide whether we want to:

- **stop** the division,
- **continue** the division.

Example 6. We want to compute

$$\frac{9974.}{312.}$$

so we need to divide 312. *into* 9974., that is

$$312. \overline{)\ 9974.}$$

Since the first digit in the *divisor* is 3, we will use the multiplication table of rank 3:

3×1	$=$	3
3×2	$=$	6
3×3	$=$	9
3×4	$=$	12
3×5	$=$	15
3×6	$=$	18
3×7	$=$	21
3×8	$=$	24
3×9	$=$	27

CYCLE 1. We look for the *first* digit of the *quotient*.

Step I. We divide by *trial and error* the *first* digit in the *divisor*, 312., into the *first* digit of the *dividend*, 9974.

Trial 1. We try the *table multiplier* 1

i. When we multiply the *first digit* of the *divisor*, 3, by the *table multiplier* 1 we get the *table product* 3:

$$\begin{array}{r|l} 312 & \overline{9974} \\ & 3 \end{array}$$

ii. We subtract the *table product* 3 from the *first digit* of the *dividend*, 9974., which leaves the *remainder* 6:

$$\begin{array}{r|l} 312. & \overline{9974.} \\ & \underline{3} \\ & 6 \end{array}$$

Trial 2. We try the *table multiplier* 2

i. When we multiply the *first digit* of the *divisor*, 3, by the *table multiplier* 2 we get the *table product* 6:

$$\begin{array}{r|l} 321 & \overline{9974} \\ & 6 \end{array}$$

ii. We subtract the *table product* 6 from the *first digit* of the *dividend*, 9974., which leaves the *remainder* 3

$$\begin{array}{r|l} 312. & \overline{9974.} \\ & \underline{6} \\ & 3 \end{array}$$

Trial 3. We try the *table multiplier* 3

i. When we multiply the *first digit* of the *divisor*, 3, by the *table multiplier*

3 we get the *table product* 9:

```
312. ) 9974.
        9
```

ii. We subtract the *table product* 9 from the *first digit* of the *dividend*, 9974., which leaves the *remainder* 0

```
312. ) 9974.
       9
       0
```

Trial 4. We try the *table multiplier* 4

i. When we multiply the *first digit* of the *divisor*, 3, by the *table multiplier* 4 we get the *table product* 12:

```
312. ) 9974.
       12
```

ii. We cannot subtract the *table product* 12 from the *first digit* of the *dividend*, 9974. .

```
312. ) 9974.
       12
```

Since we cannot complete Trial 4, we must go back to the last complete trial, that is Trial 3, from which we get that:

The *first digit of the quotient* will be 3 unless the resulting partial product exceeds the *dividend*.

Step II. We multiply the *full divisor*, 312., by the *first* digit in the quotient, 3:

```
         3
312. ) 9974.
       936
```

The *first partial product* is 936 **Tens**.

Step III. We subtract the *first partial product*, 936 **Tens**, from the dividend 9974.:

```
        3
312. ) 9974.
       936
        614
```

The *first remainder* is 614. and the first digit in the quotient is 3.

Step IV. We decide if we want to *stop* or to *continue* the division:

- If we decide to *stop* the division,
  - the *quotient* of the division is 30. since the first digit of the quotient, 3, refers to the **Tens** and the only denominator that goes without saying is the **Ones**.

– the *remainder* of the division is 614.

If we don't care about the *remainder*, we write:

$$\frac{9974}{312} = 30 + (...)$$

where we write $+ (...)$ as a reminder that $\frac{9974}{312}$ is not exactly equal to 30 since there was a *remainder*.

- If we decide to *continue* the division,
  i. we recall that the 3 in the quotient refers to the **Tens**

$$\begin{array}{r} 3 \\ 312 \enclose{longdiv}{9974} \\ 936 \\ \hline 614 \end{array}$$

  ii. we recall that the remainder is 614 **Ones**,

$$\begin{array}{r} 3. \\ 312 \enclose{longdiv}{9974} \\ 936 \\ \hline 614 \end{array}$$

  iii. we start a new cycle.

CYCLE 2. We look for the *second* digit of the *quotient*.

Step I. We divide by *trial and error* the *first* digit in the *divisor*, 312., into the *first* digit of the *first remainder*, 614. :

Trial 1. We try the *table multiplier* 1

i. When we multiply the *first digit* of the *divisor*, 3, by the *table multiplier* 1 we get the *table product* 3:

$$\begin{array}{r} 3 \\ 312. \enclose{longdiv}{9974.} \\ 936 \\ \hline 614 \\ 3 \end{array}$$

ii. We subtract the *table product* 3 from the *first digit* of the *first remainder*, 614., which leaves the *remainder* 3:

$$\begin{array}{r} 3 \\ 312. \enclose{longdiv}{9974.} \\ 936 \\ \hline 614 \\ 3 \\ \hline 6 \end{array}$$

Trial 2. We try the *table multiplier* 2

i. When we multiply the *first digit* of the *divisor*, 3, by the *table multiplier* 2 we get the *table product* 6:

```
          3
312. ) 9974.
        936
        ---
         614
         6
```

ii. We subtract the *table product* 6 from the *first digit* of the *first remainder*, 614., which leaves the *remainder* 0:

```
          3
312. ) 9974.
        936
        ---
         614
         6
         -
         0
```

Trial 3. We don't need to do Trial 3 since we obviously will not be able to subtract the *table product* from the *first remainder*.
The second digit of the quotient will be 2 unless the resulting partial product exceeds the *first remainder*.

Step II. We multiply the *full divisor*, 312., by the *second* digit in the quotient, 2:

```
         32
312 ) 9974
       936
       ---
        614
        624
```

The *second partial product* is 624 **Ones**

Step III. We cannot subtract the *second partial product*, 624 from the *first remainder*, 614:

```
        3.2
312 ) 9974
       936
       ---
        614
        624
```

What happened here is due to the carryover in the multiplication.
So, the *second* digit in the quotient is the *table multiplier* in Trial 1, 1, and we must redo **Step II** and **Step III**:

New **Step II.** We multiply the *full divisor*, 312., by the *second* digit in the quotient, 1:

```
          31
312. ) 9974.
        936
        ---
         614
         312
```

The *second partial product* is 312 **Ones**

New **Step III.** We subtract the *second partial product*, 312 **Ones**, from the *first remainder* 614. :

$$\begin{array}{r} 31 \\ 312. \;\big)\; 9974. \\ 936 \\ \hline 614. \\ 312 \\ \hline 302 \end{array}$$

The *second remainder* is 302. and the second digit in the quotient is 1.

Step IV. We decide if we want to *stop* or *continue* the division.

- If we decide to *stop* the division,
  - the *quotient* of the division is 31. since the second digit of the quotient, 1, refers to the **Ones**.
  - the *remainder* of the division is 302.

  If we don't care about the *remainder*, we write:

$$\frac{9974.}{312.} = 31. + (...)$$

  where we write $+ (...)$ as a reminder that $\frac{9974.}{312.}$ is not exactly equal to $31.$ since there was a *remainder*.
- If we decide to *continue* the division,
  i. we point the 1 in the quotient to indicate that it refers to the **Ones**

$$\begin{array}{r} 31. \\ 312. \;\big)\; 9974. \\ 936 \\ \hline 614 \end{array}$$

  ii. we change the remainder 302 **Ones** to 3020 **Tenths**

$$\begin{array}{r} 31. \\ 312. \;\big)\; 9974. \\ 936 \\ \hline 614 \\ 312 \\ \hline 3020 \end{array}$$

  iii. we start a new cycle.

CYCLE 3. We look for the *third* digit of the *quotient*.

Step I. We divide by *trial and error* the *first* digit in the *divisor*, 312., into the *first two* digits of the *second remainder*, 3020. :

Trial 1. We try the *table multiplier* 1

i. When we multiply the *first digit* of the *divisor*, 3, by the *table multiplier* 1 we get the *table product* 3:

```
            31.
312. ) 9974.
       936
        614
        312
        3020
        3
```

ii. We subtract the *table product* 3 from the *first two digits* of the *second remainder*, 3020., which leaves the *remainder* 27:

```
            31.
312. ) 9974.
       936
        614
        312
        3020
         3
        27
```

Trial 2. We try the *table multiplier* 9

i. When we multiply the *first digit* of the *divisor*, 3, by the *table multiplier* 9 we get the *table product* 27:

```
            31.
312. ) 9974.
       936
        614
        312
        3020
        27
```

ii. We subtract the *table product* 27 from the *first two digits* of the *second remainder*, 3020., which leaves the *remainder* 3:

```
            31.
312. ) 9974.
       936
        614
        312
        3020
        27
         3
```

The third digit of the quotient will be 9 unless the resulting partial product exceeds the third remainder.

Step II. We multiply the *full divisor*, 312., by the *third* digit in the quotient, 9:

$$
\begin{array}{rl}
 & 31.9 \\
312. & \overline{)\,9974.} \\
 & \underline{936} \\
 & 614 \\
 & \underline{312} \\
 & 3020 \\
 & 2808
\end{array}
$$

The *third partial product* is 2808 **Tenths**

Step III. We subtract the *third partial product*, 2808 from the *second remainder*, 3020 :

$$
\begin{array}{rl}
 & 31.9 \\
312. & \overline{)\,9974.} \\
 & \underline{936} \\
 & 614 \\
 & \underline{312} \\
 & 3020 \\
 & \underline{2808} \\
 & 212
\end{array}
$$

The *third remainder* is 31.2 and the third digit of the quotient is 9

Step IV. We decide if we want to *stop* or *continue* the division.

- If we decide to *stop* the division,
  - the *quotient* of the division is 31.9 since the third digit of the quotient, 9 , refers to the **Tenths**.
  - the *remainder* of the division is 21.2

  If we don't care about the *remainder*, we write:

  $$\frac{9974.}{312.} = 31.9 + (...)$$

  where we write $+ (...)$ as a reminder that $\frac{9974.}{312.}$ is not exactly equal to 31.9 since there was a *remainder*.
- If we decide to *continue* the division,
  i. we recall that the 9 in the quotient refers to the **Tenths**

$$
\begin{array}{rl}
 & 31.9 \\
312. & \overline{)\,9974.} \\
 & \underline{936} \\
 & 614 \\
 & \underline{312} \\
 & 3020 \\
 & \underline{2808} \\
 & 212
\end{array}
$$

  ii. we change the remainder 212 **Tenths** to 2120 **Hundredths**

```
         31.9
312. ) 9974.
       936
        614
        312
        3020
        2808
         2120
```

iii. we start a new cycle.

3. While this procedure certainly appears to be a lot more complicated than the elementary school procedure, it isn't really and it just requires getting used to and taking the time to get used to it is a good investment because, in the long run, this procedure is much more economical since:

- We find the digits of the quotient using only one *multiplication table*,
- We then usually need only do one full multiplication and one subtraction (per cycle) as opposed to one for each try.
- We can decide exactly where we want to stop and see how precise the quotient then would be.

18.4 Division of Polynomials

Since *decimal numerators* are combinations of powers of TEN, it should not be surprising that the above procedure should work for *polynomials* which are combinations of powers of x.

The *procedure* consists of successive *cycles*, one for each monomial in the quotient. During each of these *cycles*, we go through four *step*s:

Step I. We find each *monomial* of the *quotient* by dividing the *first monomial* in the divisor into the *first monomial* of the previous partial remainder.

Step II. We find the *partial product* by multiplying the *full divisor* by the *monomial* of the quotient we found in Step I.

Step III. We find the *partial remainder* by subtracting the *partial product* we found in Step II from the previous partial remainder or, if there is not yet a partial remainder, from the *full dividend*.

Step IV. We decide if we

- *stop* the division
- *continue* the division

Just as, in ARITHMETIC, we can stop the division anywhere we want and we need not stop a division when the quotient reaches a monomial with exponent 0 because we can always divide a monomial into another since we can have *negative* exponents. In fact, again just as in ARITHMETIC, there are cases where we absolutely need to go beyond the exponent 0 and use negative exponents. (See Epilogue.)

EXAMPLE 7. In order to compute $\dfrac{-12x^3 + 11x^2 - 17x + 1}{-3x^2 + 5 - 2}$, we divide $-3x^2 + 5 - 2$ into $-12x^3 + 11x^2 - 17x + 1$:

$$-3x^2 \; +5x \; -2 \;\big)\; \overline{-12x^3 \quad +11x^2 \quad -17x \quad +1}$$

we proceed as follows:

CYCLE 1. Step I. We find the *first monomial in the quotient* by dividing the *first monomial in the divisor*, $-3x^2$, into the *first monomial of the dividend*, $-12x^3$ that is $\dfrac{-12x^3}{-3x^2} = +4x$

$$\begin{array}{r|llll} & +4x & & & \\ \hline -3x^2 \; +5x \; -2 \;) & -12x^3 & +11x^2 & -17x & +1 \end{array}$$

Step II. We find the *first partial product* by multiplying the *full divisor* by the *first monomial in the quotient*:

$$\begin{array}{r|llll} & +4x & & & \\ \hline -3x^2 \; +5x \; -2 \;) & -12x^3 & +11x^2 & -17x & +1 \\ \text{First partial product:} & -12x^3 & +20x^2 & -8x & \end{array}$$

Step III. We find the *first partial remainder* by *subtracting* the first partial product from the full dividend:

$$\begin{array}{r|llll} & +4x & & & \\ \hline -3x^2 \; +5x \; -2 \;) & -12x^3 & +11x^2 & -17x & +1 \\ \ominus & -12x^3 & +20x^2 & -8x & \end{array}$$

But to *subtract* the first partial product means to *add the opposite* of the first partial product to the full dividend:

$$\begin{array}{r|llll} & +4x & & & \\ \hline -3x^2 \; +5x \; -2 \;) & -12x^3 & +11x^2 & -17x & +1 \\ \oplus & +12x^3 & -20x^2 & +8x & \\ \hline \text{First remainder:} & +0x^3 & -9x^2 & -9x & +1 \end{array}$$

Step IV. We decide if we want to *stop* or *continue* the division.

- If we decide to *stop* the division,
  - the *quotient* of the division is $+4x$.
  - the *remainder* of the division is $-9x^2 - 8x + 1$

  If we don't care about the *remainder*, we write:

  $$\frac{-12x^3 + 11x^2 - 17x + 1}{-3x^2 + 5x - 2} = +4x + (...)$$

  where we write $+ (...)$ as a reminder that $\dfrac{-12x^3 + 11x^2 - 17x + 1}{-3x^2 + 5x - 2}$ is not exactly equal to $+4x$ since there was a *remainder*.

- If we decide to *continue* the division, we start a new cycle

CYCLE 2. Step I. We find the *second monomial in the quotient* by dividing the *first monomial in the divisor*, $-3x^2$, into *the first monomial in the first partial remainder*, $-9x^2$, that is $\dfrac{-9x^2}{-3x^2} = +3$

$$\begin{array}{rrr|rrrr} & & & +4x & +3 & & \\ \cline{4-7} -3x^2 & +5x & -2 & -12x^3 & +11x^2 & -17x & +1 \\ & & & -12x^3 & +20x^2 & -8x & \\ \cline{4-7} & & & & -9x^2 & -9x & +1 \end{array}$$

Step II. We find the *second partial product* by multiplying the *full divisor* by the *second monomial in the quotient*:

$$\begin{array}{rrr|rrrr} & & & +4x & +3 & & \\ \cline{4-7} -3x^2 & +5x & -2 & -12x^3 & +11x^2 & -17x & +1 \\ & & & -12x^3 & +20x^2 & -8x & \\ \cline{4-7} & & & & -9x^2 & -9x & +1 \\ \text{Second partial product:} & & & & -9x^2 & +15x & -6 \end{array}$$

Step III. We find the *second partial remainder* by *subtracting* the second partial product from the first partial remainder:

$$\begin{array}{rrr|rrrr} & & & +4x & +3 & & \\ \cline{4-7} -3x^2 & +5x & -2 & -12x^3 & +11x^2 & -17x & +1 \\ & & & -12x^3 & +20x^2 & -8x & \\ \cline{4-7} & & & & -9x^2 & -9x & +1 \\ & & & \ominus & -9x^2 & +15x & -6 \\ \cline{5-7} \end{array}$$

But to *subtract* the second partial product means to *add the opposite* of the second partial product to the first partial remainder:

$$\begin{array}{rrr|rrrr} & & & +4x & +3 & & \\ \cline{4-7} -3x^2 & +5x & -2 & -12x^3 & +11x^2 & -17x & +1 \\ & & & -12x^3 & +20x^2 & -8x & \\ \cline{4-7} & & & & -9x^2 & -9x & +1 \\ & & & \oplus & +9x^2 & -15x & +6 \\ \cline{5-7} \text{Second remainder:} & & & & +0x^2 & -24x & +7 \end{array}$$

Step IV. We decide if we want to *stop* or *continue* the division.

- If we decide to *stop* the division,
  - the *quotient* of the division is $+4x + 3$.
  - the *remainder* of the division is $-24x + 7$

  If we don't care about the *remainder*, we write:

$$\frac{-12x^3 + 11x^2 - 17x + 1}{-3x^2 + 5x - 2} = +4x + 3 + (\ldots)$$

where we write $+\ (\ldots)$ as a reminder that $\dfrac{-12x^3 + 11x^2 - 17x + 1}{-3x^2 + 5x - 2}$ is not exactly equal to $+4x+3$ since there was a *remainder*.

- If we decide to *continue* the division, we start a new cycle

CYCLE 3. Step I. We find the *third monomial in the quotient* by dividing the *first monomial in the divisor*, $-3x^2$, into *the first monomial in the second partial remainder*, $-24x$ that is $\dfrac{-24x}{-3x^2} = +8x^{-1}$

$$\begin{array}{r|rrrr} & +4x & +3 & +8x^{-1} & \\ \hline -3x^2\ +5x\ -2 & -12x^3 & +11x^2 & -17x & +1 \\ & -12x^3 & +20x^2 & -8x & \\ \hline & & -9x^2 & -9x & +1 \\ & & +9x^2 & -15x & +6 \\ \hline & & & -24x & +7 \end{array}$$

Step II. We find the *third partial product* by multiplying the *full divisor* by the *third monomial in the quotient*:

$$\begin{array}{r|rrrrr} & +4x & +3 & +8x^{-1} & & \\ \hline -3x^2\ +5x\ -2 & -12x^3 & +11x^2 & -17x & +1 & \\ & -12x^3 & +20x^2 & -8x & & \\ \hline & & -9x^2 & -9x & +1 & \\ & & +9x^2 & -15x & +6 & \\ \hline & & & -24x & +7 & \\ \text{Third partial product:} & & & -24x & +40 & -16x^{-1} \end{array}$$

Step III. We find the *third partial remainder* by *subtracting* the third partial product from the first partial remainder:

$$\begin{array}{r|rrrrr} & +4x & +3 & +8x^{-1} & & \\ \hline -3x^2\ +5x\ -2 & -12x^3 & +11x^2 & -17x & +1 & \\ & -12x^3 & +20x^2 & -8x & & \\ \hline & & -9x^2 & -9x & +1 & \\ & & +9x^2 & -15x & +6 & \\ \hline & & & -24x & +7 & \\ \ominus & & & -24x & +40 & -16x^{-1} \end{array}$$

But to *subtract* the second partial product means to *add the opposite* of the second partial product to the first partial remainder:

$$\begin{array}{r|rrrrr} & +4x & +3 & +8x^{-1} & & \\ \hline -3x^2\ +5x\ -2 & -12x^3 & +11x^2 & -17x & +1 & \\ & -12x^3 & +20x^2 & -8x & & \\ \hline & & -9x^2 & -9x & +1 & \\ & & +9x^2 & -15x & +6 & \\ \hline & & & -24x & +7 & \\ \oplus & & & +24x & -40 & +16x^{-1} \\ \hline \text{Third remainder:} & & & 0x & -33 & +16x^{-1} \end{array}$$

Step IV. We decide if we want to *stop* or *continue* the division.

- If we decide to *stop* the division,

– the *quotient* of the division is $+4x + 3 + 8x^{-1}$.

– the *remainder* of the division is $-33 + 16x^{-1}$

If we don't care about the *remainder*, we write:

$$\frac{-12x^3 + 11x^2 - 17x + 1}{-3x^2 + 5x - 2} = +4x + 3 + 8x^{-1} + (\ldots)$$

where we write $+\ (\ldots)$ as a reminder that $\dfrac{-12x^3 + 11x^2 - 17x + 1}{-3x^2 + 5x - 2}$ is not exactly equal to $+4x + 3 + 8x^{-1}$ since there was a *remainder*.

- If we decide to *continue* the division, we start a new cycle

The procedure to divide polynomials is in fact a lot simpler than the procedure for dividing in ARITHMETIC:

- There is never any "carryover"
- The first term of each partial remainder has coefficient 0
- There are no Trials in **Step I** because, when we divide the first monomial in the divisor into the first monomial of a partial remainder, we always get a coefficient for the corresponding monomial in the quotient and the worst that can happen is that this coefficient is a fraction.

EXAMPLE 8. In order to divide $2x^3 + 5x^2 - 6$ by $3x - 1$ we write (in the *anglo-saxon* tradition):

$$\begin{array}{r|llll}
 & & \frac{2}{3}x^2 & +\frac{17}{9}x & +\frac{17}{27} \\ \hline
3x - 1 & 2x^3 & +5x^2 & & -6 \\
 & -2x^3 & +\frac{2}{3}x^2 & & \\ \cline{2-3}
 & & \frac{17}{3}x^2 & & \\
 & & -\frac{17}{3}x^2 & +\frac{17}{9}x & \\ \cline{3-4}
 & & & \frac{17}{9}x & -6 \\
 & & & -\frac{17}{9}x & +\frac{17}{27} \\ \cline{4-5}
 & & & & -\frac{145}{27}
\end{array}$$

The *quotient* is

$$+\frac{2}{3}x^2 + \frac{17}{9}x + \frac{17}{27}$$

The *remainder* is

$$-\frac{145}{27}$$

18.5 Default Rules for Division

Since mathematicians are lazy,

- mathematicians do not write the + sign in front of the coefficients of leading monomials,
- mathematicians do not write monomials with 0 coefficient,

and, most dangerously,

- mathematicians want to write only one stage in Step III but there are two traditions concerning what then to write, as a result, in Step II:
  - In the *latin* tradition, in Step II, we write the *partial product*, that is what we get it from the *multiplication*, and so in Step III, when it comes to subtracting, we visualize the *opposite of the partial product* we wrote in Step II and we oplus what we *visualize*. The advantage is that each line is exactly what we get from the previous operation.

    EXAMPLE 9.
    $$\begin{array}{r|rrrr} & +4x & +3 & & \\ \hline -3x^2 \; +5x \; -2 & -12x^3 & +11x^2 & -16x & +1 \\ & -12x^3 & +20x^2 & -8x & \\ \hline & & -9x^2 & -8x & +1 \end{array}$$
  - In the *anglo-saxon* tradition, we anticipate the subtraction to be done in Step III and in Step II *we write the opposite of the partial product* so in Step III we oplus what we *wrote* in Step II.

    EXAMPLE 10.
    $$\begin{array}{r|rrrr} & +4x & +3 & & \\ \hline -3x^2 \; +5x \; -2 & -12x^3 & +11x^2 & -16x & +1 \\ & +12x^3 & -20x^2 & +8x & \\ \hline & & -9x^2 & -8x & +1 \end{array}$$

From now on we will of course follow the *anglo-saxon* tradition.

EXAMPLE 11. In order to compute $\dfrac{6x^3 + 13x^2 + 13x + 7}{2x + 1}$, we divide $2x + 1$ into $6x^3 + 13x^2 + 13x + 7$:

CYCLE 1. Step I. We find the *first monomial* in the quotient by *short division*:

$$\begin{array}{r|l} & 3x^2 \\ \hline 2x + 1 & 6x^3 + 13x^2 + 13x + 7 \end{array}$$

Step II. We get the *first opposite product* by writing the opposite of the result of the *full multiplication*

$$\begin{array}{r|l} & 3x^2 \\ \hline 2x + 1 & 6x^3 + 13x^2 + 13x + 7 \\ & -6x^3 - 3x^2 \\ \hline \end{array}$$

Step III. We get the *first remainder* by oplussing the first opposite product

$$\begin{array}{r|l} & 3x^2 \\ \hline 2x + 1 & 6x^3 + 13x^2 + 13x + 7 \\ & -6x^3 - 3x^2 \\ \hline & \qquad 10x^2 + 13x \end{array}$$

Step IV. We decide if we want to stop or continue the division

- If we decide to *stop* the division,
  - ∗ the *quotient* of the division is $+3x^{+2}$.

* the *remainder* of the division is $+10x^{+2} + 13x$

If we don't care about the *remainder*, we write:

$$\frac{6x^3 + 13x^2 + 13x + 7}{2x + 1} = +3x^{+2} + (...)$$

where we write $+ (...)$ as a reminder that $\dfrac{6x^3 + 13x^2 + 13x + 7}{2x + 1}$ is not exactly equal to $+3x^{+2}$ since there was a *remainder*.

– If we decide to *continue* the division, we start a new cycle

CYCLE 2. Step I. We find the *second monomial* in the quotient by *short division*:

$$\begin{array}{r|l} & \quad 3x^2 \quad + 5x \\ 2x+1 & \; 6x^3 + 13x^2 + 13x + 7 \\ & - 6x^3 \;\; - 3x^2 \\ \hline & \quad\quad 10x^2 + 13x \end{array}$$

Step II. We get the *second opposite product* by writing the opposite of the result of the *full multiplication*

$$\begin{array}{r|l} & \quad 3x^2 \quad + 5x \\ 2x+1 & \; 6x^3 + 13x^2 + 13x + 7 \\ & - 6x^3 \;\; - 3x^2 \\ \hline & \quad\quad 10x^2 + 13x \\ & \quad - 10x^2 \;\; - 5x \\ \hline \end{array}$$

Step III. We get the *second remainder* by oplussing the fir second st opposite product

$$\begin{array}{r|l} & \quad 3x^2 \quad + 5x \\ 2x+1 & \; 6x^3 + 13x^2 + 13x + 7 \\ & - 6x^3 \;\; - 3x^2 \\ \hline & \quad\quad 10x^2 + 13x \\ & \quad - 10x^2 \;\; - 5x \\ \hline & \quad\quad\quad\quad\quad 8x + 7 \end{array}$$

Step IV. We decide if we want to stop or continue the division

– If we decide to *stop* the division,

* the *quotient* of the division is $+3x^{+2} + 5x$.
* the *remainder* of the division is $+8x + 7$

If we don't care about the *remainder*, we write:

$$\frac{6x^3 + 13x^2 + 13x + 7}{2x + 1} = +3x^{+2} + 5x + (...)$$

where we write $+ (...)$ as a reminder that $\dfrac{6x^3 + 13x^2 + 13x + 7}{2x + 1}$ is not exactly equal to $+3x^{+2} + 5x$ since there was a *remainder*.

– If we decide to *continue* the division, we start a new cycle

CYCLE 3. Step I. We find the *third monomial* in the quotient by *short division*:

$$\begin{array}{r|l} & 3x^2 \quad +5x+4 \\ \hline 2x+1 & 6x^3+13x^2+13x+7 \\ & -6x^3 \quad -3x^2 \\ \hline & 10x^2+13x \\ & -10x^2 \quad -5x \\ \hline & 8x+7 \end{array}$$

Step II. We get the *third opposite product* by writing the opposite of the result of the *full multiplication*

$$\begin{array}{r|l} & 3x^2 \quad +5x+4 \\ \hline 2x+1 & 6x^3+13x^2+13x+7 \\ & -6x^3 \quad -3x^2 \\ \hline & 10x^2+13x \\ & -10x^2 \quad -5x \\ \hline & 8x+7 \\ & -8x-4 \\ \hline \end{array}$$

Step III. We get the *third remainder* by oplussing the third opposite product

$$\begin{array}{r|l} & 3x^2 \quad +5x+4 \\ \hline 2x+1 & 6x^3+13x^2+13x+7 \\ & -6x^3 \quad -3x^2 \\ \hline & 10x^2+13x \\ & -10x^2 \quad -5x \\ \hline & 8x+7 \\ & -8x-4 \\ \hline & 3 \end{array}$$

Step IV. We decide if we want to stop or continue the division

– If we decide to *stop* the division,
  * the *quotient* of the division is $+3x^{+2}+5x+4$.
  * the *remainder* of the division is $+3$

  If we don't care about the *remainder*, we write:

$$\frac{6x^3+13x^2+13x+7}{2x+1} = +3x^{+2}+5x+4+(...)$$

where we write $+\ (...)$ as a reminder that $\dfrac{6x^3+13x^2+13x+7}{2x+1}$ is not exactly equal to $+3x^{+2}+5x+4$ since there was a *remainder*.

– If we decide to *continue* the division, we start a new cycle

- When writing the partial remainders, we do not write the monomials beyond those that result from subtracting the *partial product*.

Example 12.

	$+4x$	$+3$		
$-3x^2$ $+5x$ -2)	$-12x^3$	$+11x^2$	$-16x$	$+1$
First *opposite* partial product:	$+12x^3$	$-20x^2$	$+8x$	
First remainder:		$-9x^2$	$-8x$	
Second *opposite* partial product:		$+9x^2$	$-15x$	$+6$
Second remainder:			$-23x$	$+7$

The danger here is that, when we do the next subtraction, we may subtract from 0 rather than from the monomial that was unwritten in the partial remainder.

18.6 Division in Ascending Powers

One major difference between ARITHMETIC and POLYNOMIAL ALGEBRA is that:

- In ARITHMETIC, the *base* in the powers is always *larger* than ONE—in our case it is TEN but, for instance, COMPUTER SCIENCES use TWO, EIGHT and SIXTEEN as well.
- In POLYNOMIAL ALGEBRA, the base in the powers can be *smaller* than ONE as well as *larger* than ONE and, while this has no effect on the procedures we use for *addition*, *subtraction* and *multiplication*, whether x stands for numbers larger than 1 or smaller than 1 makes all the difference in the case of *division*.

This is because division usually does not stop by itself and *we* have to decide when to stop it. But we want to make sure that, after we have replaced the *unspecified numerator* by a *specific numerator*, what we kept of the quotient will give us *most* of what we should get so that we want the size of the successive results to go *diminishing*.

Now, as we already mentioned in Chapter 15, Section 2,

- When x is to be replaced by a numerator that is going to be *large in size*, we will want the Laurent polynomial to be written in *descending order of exponents*.
- When x is to be replaced by a numerator that is going to be *small in size*, we will want the Laurent polynomial to be written in *ascending order of exponents*.

As a result, we need to be able to divide in *ascending* order of exponents as well as in *descending* order of exponents. Fortunately, the procedure is exactly the same.

EXAMPLE 13. In order to compute $\dfrac{-12+23h-h^2-2h^3}{-3+2h}$, we divide $-3+2h$ into $-12+23h-h^2-2h^3$:

	$+4$	$-5h$	$-3h^2$	
$-3\ +2h$)	-12	$+23h$	$-h^2$	$-2h^3$
First *opposite* partial product:	$+12$	$-8h$		
First remainder:		$+15h$	$-h^2$	$-2h^3$
Second *opposite* partial product:		$-15h$	$+10h^2$	
Second remainder:			$+9h^2$	$-2h^3$
Third *opposite* partial product:			$-9h^2$	$+6h^3$
Third remainder:				$+4h^3$

- If we decide to *stop* the division,
  - the *quotient* of the division is $+4-5h-3h^2$.
  - the *remainder* of the division is $+4h^3$. Observe that if we replace the unspecified numerator h by, say, 0.2, then the remainder is equal to $4 \bullet 0.2^3 = 4 \bullet 0.008 = 0.032$ which is indeed small.

  If we don't care about the *remainder*, we write:

  $$\frac{-12+23h-h^2-2h^3}{-3+2h} = +4-5h-3h^2+(...)$$

  where we write $+\ (...)$ as a reminder that $\dfrac{-12+23h-h^2-2h^3}{-3+2h}$ is not exactly equal to $+4-5h-3h^2$ since there was a *remainder*.
- If we were to decide to *continue* the division, we would start a new cycle

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