An Exponential Function and its Optimization Problem

Choe Ryong Gil

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In this paper we will consider an optimization problem on an exponential function with the sum of divisors function. This result is very important at the study of the distribution of the prime numbers. This paper is a continuation of [6].

Assume that $\overline{\lambda} = (\lambda_1, \lambda_2, \cdots, \lambda_m)$ are real numbers and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \geq 1$.

Let $p_1 = 2,\ p_2 = 3, \cdots, p_m, \cdots$ be consecutive primes. We will choose p_m arbitrarily and fix it.

We define functions $F(\overline{\lambda})$ and $H(\overline{\lambda})$ respectively by

$$F(\overline{\lambda}) = F(\lambda_1,\ \lambda_2, \cdots, \lambda_m) = \prod_{i=1}^{m} \frac{1 - p_i^{-\lambda_i - 1}}{1 - p_i^{-1}}, \qquad (1)$$

$$H\left(\overline{\lambda}\right)=H\left(\lambda_1,\ \lambda_2,\cdots,\lambda_m\right)=\frac{\exp\left(\exp\left(e^{-\gamma}\cdot F\left(\overline{\lambda}\right)\right)\right)}{p_1^{\lambda_1}\cdot p_2^{\lambda_2}\cdots p_m^{\lambda_m}}, \qquad (2)$$

where $\gamma=0.577\cdots$ is Euler's constant ([2,5]).

We shall show an existence of the optimum points of the function $H\left(\overline{\lambda}\right)$ in the m-dimensional real space R^m and we will estimate the optimum points.

1. An existence of the optimum points

In this section we will show that the function $H\left(\overline{\lambda}\right)$ has an optimum point in the space R^m. The maximum value theorem of the continuous function is used here. We get

Theorem 1. There exist $\overline{\lambda_0}=\left(\lambda_1^0,\ \lambda_2^0,\cdots,\lambda_m^0\right)\in R^m$ such that for any $\left(\lambda_1,\ \lambda_2,\cdots,\lambda_m\right)\in R^m$ we have $H\left(\overline{\lambda}\right)\le H\left(\overline{\lambda_0}\right)$, that is,

$$H\left(\lambda_1^0,\ \lambda_2^0,\cdots,\lambda_m^0\right)=\max_{(\lambda_1,\ \lambda_2,\cdots,\lambda_m)\in R^m}\frac{\exp\left(\exp\left(e^{-\gamma}\cdot\prod_{i=1}^{m}\frac{1-p_i^{-\lambda_i-1}}{1-p_i^{-1}}\right)\right)}{p_1^{\lambda_1}\cdot p_2^{\lambda_2}\cdots p_m^{\lambda_m}}. \qquad (3)$$

Proof. We put $R_+^1=\left[1,+\infty\right)$ and $R_+^m=R_+^1\times R_+^1\times\cdots\times R_+^1$. Then we have $H\left(\overline{\lambda}\right)>0$ for any $\overline{\lambda}\in R_+^m$. And the function $H\left(\overline{\lambda}\right)$ is continuously differentiable in R_+^m. We set $F_0=\prod_{i=1}^{m}\frac{1}{1-p_i^{-1}}$ and $H_0=\frac{\exp\left(\exp\left(e^{-\gamma}\cdot F_0\right)\right)}{p_1\cdot p_2\cdots p_m}$.

Then both F_0 and H_0 are constants. And we have $F\left(\overline{\lambda}\right)\le F_0$ and $H\left(\overline{\lambda}\right)\le H_0$ for any $\overline{\lambda}\in R_+^m$. This shows that the function $H\left(\overline{\lambda}\right)$ is bouned in R_+^m. So there exists a constant $a>1$ such that the function $H\left(\overline{\lambda}\right)$ is bounded and continuous in a bounded and closed set $\prod=[1,\ a]\times[1,\ a]\times\cdots\times[1,\ a]\subset R_+^m$. Therefore the function $H\left(\overline{\lambda}\right)$ has a maximum value in the set $\prod$, because the set $\prod$ is a compact in the space R^m. Now let $\bar{\lambda}_0 = \left(\lambda_1^0, \lambda_2^0, \cdots, \lambda_m^0\right) \in \prod$ be the optimum points of $H\left(\bar{\lambda}\right)$. Then the points $\bar{\lambda}_0 = \left(\lambda_1^0, \lambda_2^0, \cdots, \lambda_m^0\right)$ are the optimum points of $H\left(\bar{\lambda}\right)$ in the whole space R_+^m. In fact, if it is not then we can take a bigger $a > 1$ again, since for any $\left(\lambda_1, \lambda_2, \cdots, \lambda_m\right) \in R^m$ it holds that

$$0 < H\left(\bar{\lambda}\right) \leq \frac{\exp\left(\exp\left(e^{-\gamma} \cdot F_0\right)\right)}{p_1^{\lambda_1} \cdot p_2^{\lambda_2} \cdots p_m^{\lambda_m}} \tag{4}$$

and

$$\frac{\exp\left(\exp\left(e^{-\gamma} \cdot F_0\right)\right)}{p_1^{\lambda_1} \cdot p_2^{\lambda_2} \cdots p_m^{\lambda_m}} \to 0 \left(\left\|\bar{\lambda}\right\| = \sqrt{\sum_{i=1}^{m} \left|\lambda_i^2\right|} \to \infty \right). \tag{5}$$

These show that there exist optimum points of the function $H\left(\bar{\lambda}\right)$ in the space R_+^m and the maximum value of $H\left(\bar{\lambda}\right)$ is not exceeded the constant H_0.

□

2. The estimate of the optimum points

In this section we will estimate the optimum points $\bar{\lambda}_0 = \left(\lambda_1^0, \lambda_2^0, \cdots, \lambda_m^0\right)$ of the function $H\left(\bar{\lambda}\right)$ obtained from the theorem 1. The optimization problem of the function $H\left(\bar{\lambda}\right)$ with the constraints of the certain inequalities is discussed here. We obtain

Theorem 2. Assume that $p_m \geq 5$. Then for the optimum points $\bar{\lambda}_0 = \left(\lambda_1^0, \lambda_2^0, \cdots, \lambda_m^0\right) \in R^m$ of the function $H\left(\bar{\lambda}\right)$ in the space R^m we have;

① There exist some λ_i^0 in $\left\{\lambda_1^0, \lambda_2^0, \cdots \lambda_m^0\right\}$ such that $\lambda_i^0 = 1$. In particular, we have $\lambda_m^0 = 1$.

② There exist some λ_i^0 in $\{\lambda_1^0, \lambda_2^0, \cdots \lambda_m^0\}$ such that $\lambda_i^0 > 1$. In particular, we have $\lambda_1^0 > 1$.

③ There exists a number k such that

$$\lambda_1^0 > \lambda_2^0 > \cdots > \lambda_k^0 > \lambda_{k+1}^0 = \cdots = \lambda_m^0 = 1. \tag{6}$$

In particular, for any $i\,(1 \le i \le k)$ we have

$$\lambda_i^0 = \left(\frac{\log p_m}{\log p_i} + \frac{\log\log p_m}{\log p_i} - 1 \right) + \mathrm{O}\left(\frac{1}{\log p_i \cdot \log p_m} \right). \tag{7}$$

Proof. Practically, the optimum points $\overline{\lambda}_0 = (\lambda_1^0, \lambda_2^0, \cdots \lambda_m^0)$ of $H(\overline{\lambda})$ in the theorem 1 are given under the constraints with the inequalities $g_i(\overline{\lambda}) = 1 - \lambda_i \le 0 \ (i = 1, 2, \cdots, m)$ in the space R^m. In other words, the $\overline{\lambda}_0 = (\lambda_1^0, \lambda_2^0, \cdots, \lambda_m^0)$ is a solution of the following optimization problem;

$$\begin{cases} -H(\lambda_1, \lambda_2, \cdots \lambda_m) \to \min \\ g_i(\overline{\lambda}) = 1 - \lambda_i \le 0 \ \ (i = 1, 2, \cdots, m). \end{cases} \tag{8}$$

And this problem (8) is equivalent to the problem

$$L(\overline{\lambda}, \overline{\mu}) \overset{D}{=} \left(-H(\overline{\lambda})\right) + \sum_{i=1}^{m} \mu_i \cdot g_i(\overline{\lambda}) \to \min \tag{9}$$

without the constraints, where $\overline{\mu} = (\mu_1, \mu_2, \cdots \mu_m) \in R^m$ are an undetermined multipliers. Since the solution $\overline{\lambda}_0 = (\lambda_1^0, \lambda_2^0, \cdots \lambda_m^0) \in R^m$ of the problem (8) exists, a solution $(\overline{\lambda}_0, \overline{\mu}_0) \in R^{2m}$ of the problem (9) exists. And $(\overline{\lambda}_0, \overline{\mu}_0) \in R^{2m}$ satisfies the equations

$$\begin{cases} \dfrac{\partial H(\overline{\lambda}_0)}{\partial \lambda_i} + \mu_i^0 = 0, \ \ 1 - \lambda_i^0 \le 0, \\ \mu_i^0 \ge 0, \ \ \mu_i^0 \cdot (1 - \lambda_i^0) = 0, \ \ \ (i = 1, 2, \cdots, m) \end{cases}. \tag{10}$$

From this we obtain some results.

First, if $p_m \geq 5$ then there exist some λ_i^0 in $\{\lambda_1^0, \lambda_2^0, \cdots \lambda_m^0\}$ such that $\lambda_i^0 = 1$.

In particular, we have $\lambda_m^0 = 1$.

In fact, if we assume that $\lambda_i^0 > 1$ for any i $(1 \leq i \leq m)$ then we have $\mu_i^0 = 0$ for any i $(1 \leq i \leq m)$ from $\mu_i^0 \cdot (1-\lambda_i^0) = 0$ of (10). So we have $\dfrac{\partial H(\overline{\lambda}_0)}{\partial \lambda_i} = 0$ for any i $(1 \leq i \leq m)$ from $\dfrac{\partial H(\overline{\lambda}_0)}{\partial \lambda_i} + \mu_i^0 = 0$ of (10).

We should calculate the term $\dfrac{\partial H(\overline{\lambda}_0)}{\partial \lambda_i} = 0$. First we have

$$\frac{\partial}{\partial \lambda_i}\left(e^{-\gamma} \cdot \prod_{i=1}^{m} \frac{1-p_i^{-\lambda_i - 1}}{1-p_i^{-1}} \right) = \left(e^{-\gamma} \cdot \prod_{i=1}^{m} \frac{1-p_i^{-\lambda_i - 1}}{1-p_i^{-1}} \right) \cdot \left(\frac{1}{p_i^{\lambda_i + 1} - 1} \right) \cdot \log p_i . \quad (11)$$

So we have

$$\begin{aligned} &\frac{\partial}{\partial \lambda_i}\left(\exp\left(\exp\left(e^{-\gamma} \cdot \prod_{i=1}^{m} \frac{1-p_i^{-\lambda_i - 1}}{1-p_i^{-1}} \right)\right)\right) = \\ &= \exp\left(\exp\left(e^{-\gamma} \cdot \prod_{i=1}^{m} \frac{1-p_i^{-\lambda_i - 1}}{1-p_i^{-1}} \right)\right) \times \exp\left(e^{-\gamma} \cdot \prod_{i=1}^{m} \frac{1-p_i^{-\lambda_i - 1}}{1-p_i^{-1}} \right) \times \\ &\times \left(e^{-\gamma} \cdot \prod_{i=1}^{m} \frac{1-p_i^{-\lambda_i - 1}}{1-p_i^{-1}} \right) \cdot \left(\frac{1}{p_i^{\lambda_i + 1} - 1} \right) \cdot \log p_i . \end{aligned} \quad (12)$$

Next, we have

$$\begin{aligned} &\frac{\partial}{\partial \lambda_i}\left(p_1^{\lambda_1} \cdot p_2^{\lambda_2} \cdots p_m^{\lambda_m} \right) = \left(p_1^{\lambda_1} \cdots \cdot p_{i-1}^{\lambda_{i-1}} \cdot p_{i+1}^{\lambda_{i+1}} \cdots p_m^{\lambda_m} \right) \cdot \frac{\partial}{\partial \lambda_i}\left(p_i^{\lambda_i} \right) = \\ &= \left(p_1^{\lambda_1} \cdots \cdot p_{i-1}^{\lambda_{i-1}} \cdot p_{i+1}^{\lambda_{i+1}} \cdots p_m^{\lambda_m} \right) \cdot \left(p_i^{\lambda_i} \cdot \log p_i \right) = \\ &= \left(p_1^{\lambda_1} \cdot p_2^{\lambda_2} \cdots p_m^{\lambda_m} \right) \cdot \log p_i \end{aligned} \quad (13)$$

Therfore for any i $(1 \leq i \leq m)$ we have

$$\frac{\partial H(\overline{\lambda}_0)}{\partial \lambda_i} = H(\overline{\lambda}_0) \cdot \left[\left(e^{-\gamma} \cdot F(\overline{\lambda}_0) \right) \cdot \exp\left(e^{-\gamma} \cdot F(\overline{\lambda}_0) \right) \cdot \left(\frac{1}{p_i^{\lambda_i + 1} - 1} \right) - 1 \right] = 0 .$$

(14)

Hence we have

$$p_1^{\lambda_1^0+1} = p_2^{\lambda_2^0+1} = \cdots = p_m^{\lambda_m^0+1} = \\ = \left(e^{-\gamma} \cdot F\left(\overline{\lambda}_0\right)\right) \cdot \exp\left(e^{-\gamma} \cdot F\left(\overline{\lambda}_0\right)\right) + 1. \tag{15}$$

In particular, if $\lambda_m^0 > 1$ then we have

$$p_m^{\lambda_m^0+1} = \left(e^{-\gamma} \cdot F\left(\overline{\lambda}_0\right)\right) \cdot \exp\left(e^{-\gamma} \cdot F\left(\overline{\lambda}_0\right)\right) + 1. \tag{16}$$

On the other hand, by the Mertens' theorem [5], it is known that for $p_m \ge 2$

$$\left(e^{-\gamma} \cdot F\left(\overline{\lambda}_0\right)\right) = \log p_m + \varepsilon_0\left(p_m\right), \tag{17}$$

where $\varepsilon_0\left(p_m\right) = \mathrm{O}\left(\frac{1}{\log p_m}\right)$. Therefore we have

$$\begin{aligned} &\left(e^{-\gamma} \cdot F\left(\overline{\lambda}_0\right)\right) \cdot \exp\left(e^{-\gamma} \cdot F\left(\overline{\lambda}_0\right)\right) + 1 = \\ &= \left(\log p_m + \varepsilon_0\left(p_m\right)\right) \cdot \exp\left(\log p_m + \varepsilon_0\left(p_m\right)\right) + 1 = \\ &= \log p_m \cdot \left(1 + \frac{\varepsilon_0\left(p_m\right)}{\log p_m}\right) \cdot p_m \cdot \exp\left(\varepsilon_0\left(p_m\right)\right) + 1 = \\ &= p_m \cdot \log p_m \cdot \left(1 + \frac{\varepsilon_0\left(p_m\right)}{\log p_m}\right) \cdot \exp\left(\varepsilon_0\left(p_m\right)\right) + 1 = \\ &= p_m \cdot \log p_m \cdot \left(\left(1 + \frac{\varepsilon_0\left(p_m\right)}{\log p_m}\right) \cdot \exp\left(\varepsilon_0\left(p_m\right)\right) + \frac{1}{p_m \cdot \log p_m}\right) = \\ &= p_m \cdot \log p_m \cdot \left(1 + \varepsilon\left(p_m\right)\right). \end{aligned} \tag{18}$$

where

$$\varepsilon\left(p_m\right) = \left(1 + \frac{\varepsilon_0\left(p_m\right)}{\log p_m}\right) \cdot \exp\left(\varepsilon_0\left(p_m\right)\right) + \\ + \frac{1}{p_m \cdot \log p_m} - 1 = \mathrm{O}\left(\frac{1}{\log p_m}\right). \tag{19}$$

Put $f\left(p_m\right) = \frac{p_m \cdot \log p_m \cdot \left(1 + \varepsilon\left(p_m\right)\right)}{p_m^2}$ then the function $f\left(p_m\right)$ is monotone decreasing as $p_m \to \infty$ and $f\left(2\right) \le 4.78,\ f\left(3\right) \le 1.79,\ f\left(5\right) \le 0.87$.

Hence we have $\frac{p_m \cdot \log p_m \cdot \left(1+\varepsilon\left(p_m\right)\right)}{p_m^2} < 1$ for any $p_m \geq 5$. From this we have

$$p_m^{\lambda_m^0+1} \leq p_m \cdot \log p_m \cdot \left(1+\varepsilon\left(p_m\right)\right) < p_m^2, \tag{20}$$

but it is contradictive to $\lambda_m^0 > 1$. Therefoer we must obtain $\lambda_m^0 = 1$.

Similarly, if $p_m \cdot \log p_m \cdot \left(1+\varepsilon\left(p_m\right)\right) < p_{m-1}^2$ then we have $\lambda_{m-1}^0 = 1$.

In general, if there is a number $j \ \left(1 \leq j \leq m\right)$ such that

$$p_m \cdot \log p_m \cdot \left(1+\varepsilon\left(p_m\right)\right) < p_j^2, \tag{21}$$

then we have $\lambda_j^0 = 1$. This is the proof of ①.

Second, if $p_m \geq 5$ then there exist some λ_i^0 in $\left\{\lambda_1^0, \lambda_2^0, \cdots, \lambda_m^0\right\}$ such that $\lambda_i^0 > 1$, in particular $\lambda_1^0 > 1$. In fact, if we assume $\lambda_i^0 = 1$ for any $i \ \left(1 \leq i \leq m\right)$ then we have $\frac{\partial H\left(\overline{\lambda}_0\right)}{\partial \lambda_i} = -\mu_i^0 \leq 0$ from $\mu_i^0 \geq 0$ and $\frac{\partial H\left(\overline{\lambda}_0\right)}{\partial \lambda_i} + \mu_i^0 = 0$ of (10). Hence we have

$$\frac{\partial H\left(\overline{\lambda}_0\right)}{\partial \lambda_i} = H\left(\overline{\lambda}_0\right) \cdot \left[\left(e^{-\gamma} \cdot F\left(\overline{\lambda}_0\right)\right) \cdot \exp\left(e^{-\gamma} \cdot F\left(\overline{\lambda}_0\right)\right) \cdot \left(\frac{1}{p_i^{\lambda_i^0+1} - 1}\right) - 1\right] \leq 0 \tag{22}$$

Hence for any $i \ \left(1 \leq i \leq m\right)$ we have

$$p_i^{\lambda_i^0+1} \geq \left(e^{-\gamma} \cdot F\left(\overline{\lambda}_0\right)\right) \cdot \exp\left(e^{-\gamma} \cdot F\left(\overline{\lambda}_0\right)\right) + 1. \tag{23}$$

On the other hand, if $\lambda_i^0 = 1$ for any $i \ \left(i = 1, 2, \cdots, m\right)$ then we have

$$\begin{aligned} F\left(\overline{\lambda}_0\right) &= \prod_{i=1}^{m} \frac{1 - p_i^{-\lambda_i^0 - 1}}{1 - p_i^{-1}} = \prod_{i=1}^{m} \frac{1 - p_i^{-2}}{1 - p_i^{-1}} = \\ &= \prod_{i=1}^{m} \left(1 + \frac{1}{p_i}\right) \to \infty \quad \left(m \to \infty\right). \end{aligned} \tag{24}$$

In particular, if $\lambda_1^0 = 1$ then for $p_m = 5$ we have

$$\begin{aligned} 2^2 &\geq \left(e^{-\gamma} \left(1+\frac{1}{2}\right) \cdot \left(1+\frac{1}{3}\right) \cdot \left(1+\frac{1}{5}\right) \right) \times \\ &\times \exp\left(e^{-\gamma} \left(1+\frac{1}{2}\right) \cdot \left(1+\frac{1}{3}\right) \cdot \left(1+\frac{1}{5}\right) \right) + 1 \geq 5.22 . \end{aligned} \tag{25}$$

But it is constradictive. This is the proof of ②

Finally, if $p_m \geq 5$ then there exist some λ_i^0 in $\{\lambda_1^0, \lambda_2^0, \cdots \lambda_m^0\}$ such that $\lambda_i^0 > 1$ and there exist some λ_i^0 in $\{\lambda_1^0, \lambda_2^0, \cdots, \lambda_m^0\}$ such that $\lambda_i^0 = 1$. So we put

$$\lambda_{k+1}^0 = \lambda_{k+2}^0 = \cdots = \lambda_m^0 = 1 . \tag{26}$$

Then the remaining $\{\lambda_1^0, \lambda_2^0, \cdots, \lambda_k^0\}$ satisfy an equations

$$\begin{aligned} p_1^{\lambda_1^0+1} &= p_2^{\lambda_2^0+1} = \cdots = p_k^{\lambda_k^0+1} = \\ &= \left(e^{-\gamma} F\left(\bar{\lambda}_0\right)\right) \cdot \exp\left(e^{-\gamma} F\left(\bar{\lambda}_0\right)\right) + 1 \end{aligned} \tag{27}$$

since $\lambda_i^0 > 1$ for any $i \ (1 \leq i \leq k)$. From (27) for any $i,\ j \ (1 \leq i < j \leq k)$ we have $p_i^{\lambda_i^0+1} = p_j^{\lambda_j^0+1}$, hence we get

$$\left(\lambda_i^0 + 1\right) \cdot \log p_i = \left(\lambda_j^0 + 1\right) \cdot \log p_j > \left(\lambda_j^0 + 1\right) \cdot \log p_i \tag{28}$$

and $\lambda_i^0 > \lambda_j^0$. Therefore we have

$$\lambda_1^0 > \lambda_2^0 > \cdots > \lambda_k^0 > \lambda_{k+1}^0 = \cdots = \lambda_m^0 = 1 . \tag{29}$$

And from (27) we have

$$\begin{aligned} p_i^{\lambda_i^0+1} &= \left(e^{-\gamma} F\left(\bar{\lambda}_0\right)\right) \cdot \exp\left(e^{-\gamma} F\left(\bar{\lambda}_0\right)\right) + 1 = \\ &= p_m \cdot \log p_m \cdot \left(1 + \varepsilon\left(p_m\right)\right) = \\ &= p_m \log p_m \left(1 + \mathrm{O}\left(\frac{1}{\log p_m}\right)\right). \end{aligned} \tag{30}$$

Therefore we have

$$\lambda_i^0=\left(\frac{\log p_m+\log\log p_m}{\log p_i}-1\right)+\frac{1}{\log p_i}\cdot\log\left(1+\varepsilon\left(p_m\right)\right)=$$
$$=\left(\frac{\log p_m}{\log p_i}+\frac{\log\log p_m}{\log p_i}-1\right)+\mathrm{O}\left(\frac{1}{\log p_i\cdot\log p_m}\right). \tag{31}$$

This is complet proof of the theorem 2. □

The last bigger number k than 1 in the optimum points $\{\lambda_1^0, \lambda_2^0,\cdots,\lambda_m^0\}$ of the function $H(\bar{\lambda})$ is special important. We will here discuss λ_k, p_k and k in detail. In the furture, we assume that $p_m\ge 5$. We have

Theorem 3. For the number k such that $\lambda_1^0>\lambda_k^0>\lambda_{k+1}^0=1$ we have;

① $$\lambda_k^0=1+\mathrm{O}\left(\frac{1}{\log p_m}\right), \tag{32}$$

② $$p_k=\sqrt{p_m\cdot\log p_m}\cdot\left(1+\mathrm{O}\left(\frac{1}{\log p_m}\right)\right), \tag{33}$$

③ $$k=2\sqrt{m}\cdot\left(1+\mathrm{O}\left(\frac{\log\log p_m}{\log p_m}\right)\right). \tag{34}$$

Proof. **First,** The last bigger point λ_k^0 than 1 in the optimum points $\{\lambda_1^0, \lambda_2^0,\cdots,\lambda_m^0\}$ of the function $H(\bar{\lambda})$ is estimated as follows.

Since $\lambda_k^0>1$ we have

$$p_k^2<p_k^{\lambda_k^0+1}=\left(e^{-\gamma}F\left(\bar{\lambda}_0\right)\right)\cdot\exp\left(e^{-\gamma}F\left(\bar{\lambda}_0\right)\right)+1=$$
$$=p_m\cdot\log p_m\cdot\left(1+\varepsilon\left(p_m\right)\right)<p_{k+1}^2. \tag{35}$$

Hence we take the logarithm of the both sides from (35), then we get

$$\left(\lambda_k^0+1\right)=\frac{\log\left(p_m\cdot\log p_m\cdot\left(1+\varepsilon\left(p_m\right)\right)\right)}{\log p_k} \tag{36}$$

and

$$\frac{\log\left(p_m \cdot \log p_m \cdot \left(1+\varepsilon\left(p_m\right)\right)\right)}{\log p_{k+1}} < 2\,. \tag{37}$$

So from (36) and (37), we have

$$\begin{aligned}
\lambda_k^0 - 1 &= \left(\lambda_k^0 + 1\right) - 2 = \frac{\log\left(p_m \cdot \log p_m \cdot \left(1+\varepsilon\left(p_m\right)\right)\right)}{\log p_k} - 2 \le \\
&\le \frac{\log\left(p_m \cdot \log p_m \cdot \left(1+\varepsilon\left(p_m\right)\right)\right)}{\log p_k} - \frac{\log\left(p_m \cdot \log p_m \cdot \left(1+\varepsilon\left(p_m\right)\right)\right)}{\log p_{k+1}} = \\
&= \frac{\log\left(p_m \cdot \log p_m \cdot \left(1+\varepsilon\left(p_m\right)\right)\right)}{\log p_{k+1}} \cdot \left(\frac{\log p_{k+1}}{\log p_k} - 1\right) \le \\
&\le 2 \cdot \left(\frac{\log p_{k+1} - \log p_k}{\log p_k}\right) = \frac{2}{\log p_k} \cdot \left(\log \frac{p_{k+1}}{p_k}\right) = \\
&= \frac{2}{\log p_k} \cdot \log\left(1 + \frac{p_{k+1} - p_k}{p_k}\right) = \\
&= \frac{2}{\log p_k} \cdot \left(\left(\frac{p_{k+1} - p_k}{p_k}\right) + \mathrm{O}\left(\frac{p_{k+1} - p_k}{p_k}\right)^2\right).
\end{aligned} \tag{38}$$

Hence we have

$$\lambda_k^0 = 1 + \varepsilon_1\left(p_k\right), \tag{39}$$

where $\varepsilon_1\left(p_k\right) = \mathrm{O}\left(\frac{p_{k+1} - p_k}{p_k \cdot \log p_k}\right)$.

On the other hand, from [3] it is known that

$$p_{k+1} - p_k = \mathrm{O}\left(p_k^{\theta}\right), \tag{40}$$

where $\theta = \frac{11}{20} - \delta,\ \delta \le \frac{1}{384}$ ([4]) or $\theta = \frac{6}{11} + \varepsilon,\ \varepsilon > 0$ ([3]).

Thus it is easy to see that $p_k \sim p_{k+1}\left(p_k \to \infty\right)$. Hence there is a constant $0 < \alpha = 1 - \theta < 1/2$ such that

$$\lambda_k^0 = 1 + \mathrm{O}\left(\frac{1}{p_k^{\alpha} \cdot \log p_k}\right). \tag{41}$$

And form (35) we have

$$p_k < \sqrt{p_m \cdot \log p_m \cdot \left(1+\varepsilon\left(p_m\right)\right)} < p_{k+1}. \qquad (42)$$

Therefore we have

$$\begin{aligned}\lambda_k^0 &= 1+\varepsilon_1\left(p_k\right) = 1+\mathrm{O}\left(\frac{1}{p_k^{\alpha}\cdot\log p_k}\right) = \\ &= 1+\mathrm{O}\left(\frac{1}{p_k^{\alpha}}\right) = 1+\mathrm{O}\left(\frac{1}{p_{k+1}^{\alpha}}\right) = \\ &= 1+\mathrm{O}\left(\frac{1}{\left(p_m\cdot\log p_m\cdot\left(1+\varepsilon\left(p_m\right)\right)\right)^{\alpha/2}}\right) = \\ &= 1+\mathrm{O}\left(\frac{1}{\left(p_m\cdot\log p_m\right)^{\alpha/2}}\right) = 1+\mathrm{O}\left(\frac{1}{\log p_m}\right).\end{aligned} \qquad (43)$$

This shows that

$$\varepsilon_1\left(p_k\right) = \mathrm{O}\left(\frac{1}{\log p_m}\right) \qquad (44)$$

and $\lambda_k^0 \sim 1 \left(p_m \to \infty\right)$.

Next, we will estimate the k-th prime number p_k.

We can write as

$$p_k = p_k \cdot \frac{\sqrt{p_k^{\lambda_k^0+1}}}{\sqrt{p_k^{\lambda_k^0+1}}} = \sqrt{p_k^{\lambda_k^0+1}} \cdot p_k \cdot p_k^{-\left(\frac{\lambda_k^0+1}{2}\right)} \qquad (45)$$

and from (39) we get

$$\frac{\lambda_k^0+1}{2} = 1+\frac{\varepsilon_1\left(p_k\right)}{2}. \qquad (46)$$

So we have

$$p_k \cdot p_k^{-\left(\frac{\lambda_k^0+1}{2}\right)} = p_k \cdot p_k^{-\left(1+\frac{\varepsilon_1\left(p_k\right)}{2}\right)} = p_k^{-\left(\frac{\varepsilon_1\left(p_k\right)}{2}\right)} = 1+\varepsilon_1'\left(p_k\right), \qquad (47)$$

where $\varepsilon_1'\left(p_k\right) = \log p_k \cdot \mathrm{O}\left(\varepsilon_1\left(p_k\right)\right) = \mathrm{O}\left(\frac{p_{k+1}-p_k}{p_k}\right)$.

From this we get

$$\begin{aligned} p_k &= \sqrt{p_k^{\lambda_k^0+1}} \cdot p_k^{-\left(\frac{\varepsilon_1(p_k)}{2}\right)} = \\ &= \sqrt{p_m \cdot \log p_m \cdot \left(1+\varepsilon\left(p_m\right)\right)} \cdot \left(1+\varepsilon_1'\left(p_k\right)\right) = \\ &= \sqrt{p_m \cdot \log p_m} \cdot \left(1+\varepsilon_2\left(p_m\right)\right), \end{aligned} \tag{48}$$

where $\varepsilon_2\left(p_m\right) = \sqrt{\left(1+\varepsilon\left(p_m\right)\right)} \cdot \left(1+\varepsilon_1'\left(p_k\right)\right) - 1 = \mathrm{O}\left(\frac{1}{\log p_m}\right).$ (49)

In consequence, we have

$$p_k \sim \sqrt{p_m \cdot \log p_m} \ \left(p_m \to \infty\right). \tag{50}$$

From (48) and (49) we have

$$\begin{aligned} \log p_k &= \frac{1}{2} \cdot \left(\log p_m + \log\log p_m\right) + \log\left(1+\varepsilon_2\left(p_m\right)\right) = \\ &= \frac{1}{2} \cdot \log p_m \cdot \left(1 + \frac{\log\log p_m}{\log p_m} + \frac{\log\left(1+\varepsilon_2\left(p_m\right)\right)}{\log p_m}\right) = \\ &= \frac{1}{2} \cdot \log p_m \cdot \left(1+\varepsilon_2'\left(p_m\right)\right), \end{aligned} \tag{51}$$

where

$$\begin{aligned} \varepsilon_2'\left(p_m\right) &= \frac{\log\log p_m}{\log p_m} + \frac{\log\left(1+\varepsilon_2\left(p_m\right)\right)}{\log p_m} = \\ &= \mathrm{O}\left(\frac{\log\log p_m}{\log p_m}\right). \end{aligned} \tag{52}$$

And we have

$$\log p_k \sim \frac{1}{2} \cdot \log p_m \ \left(p_m \to \infty\right). \tag{53}$$

<u>Finally,</u> we will estimate the number k.

Now we recall the function $\pi\left(x\right) = \sum_{p \le x} 1$ ([1,2]). This function $\pi\left(x\right)$ is the number of primes not exceeding the given real number x. By the prime number theorem ([2]), it is well known that

$$\pi(x)=\frac{x}{\log x}\left(1+\delta(x)\right), \tag{54}$$

where $\delta(x)=\mathrm{O}\left(\frac{1}{\log x}\right)$. Thus from (48) and (51) we have

$$\begin{aligned}
k&=\pi(p_k)=\frac{p_k}{\log p_k}\cdot\left(1+\delta(p_k)\right)=\\
&=\frac{2\cdot\sqrt{p_m\cdot\log p_m}\cdot\left(1+\varepsilon_2(p_m)\right)}{\left(\log p_m+\log\log p_m\right)+\log\left(1+\varepsilon_2(p_m)\right)}\cdot\left(1+\delta(p_k)\right)=\\
&=2\cdot\sqrt{\frac{p_m}{\log p_m}}\cdot\frac{\left(1+\varepsilon_2(p_m)\right)}{\left(1+\frac{\log\log p_m}{\log p_m}+\frac{\log\left(1+\varepsilon_2(p_m)\right)}{\log p_m}\right)}\cdot\left(1+\delta(p_k)\right)=\\
&=2\cdot\sqrt{\pi(p_m)}\cdot\left(1+\delta(p_m)\right)^{-1/2}\cdot\left(1+\delta(p_k)\right)\times\\
&\quad\times\frac{\left(1+\varepsilon_2(p_m)\right)}{\left(1+\frac{\log\log p_m}{\log p_m}+\frac{\log\left(1+\varepsilon_2(p_m)\right)}{\log p_m}\right)}=\\
&=2\cdot\sqrt{m}\cdot\left(1+\varepsilon_3(p_m)\right),
\end{aligned} \tag{55}$$

where

$$\begin{aligned}
\left(1+\varepsilon_3(p_m)\right)&=\left(1+\delta(p_m)\right)^{-1/2}\cdot\left(1+\delta(p_k)\right)\times\\
&\quad\times\frac{\left(1+\varepsilon_2(p_m)\right)}{\left(1+\frac{\log\log p_m}{\log p_m}+\frac{\log\left(1+\varepsilon_2(p_m)\right)}{\log p_m}\right)}=\\
&=1+\mathrm{O}\left(\frac{\log\log p_m}{\log p_m}\right)
\end{aligned} \tag{56}$$

From this we have

$$k=2\sqrt{m}\cdot\left(1+\mathrm{O}\left(\frac{\log\log p_m}{\log p_m}\right)\right) \tag{57}$$

and

$$k\sim 2\sqrt{m}\ \left(p_m\to\infty\right). \tag{58}$$

This is the proof of theorem 3. □

Note. In the proof of the theorem 1, we have taken a certain suitable constant $a>1$ determining the region $\prod \subset R_{+}^{m}$ such that there exist the optimum points $\bar{\lambda}_0=\left(\lambda_1^0,\,\lambda_2^0,\cdots\lambda_m^0\right)\in R^m$ of the function $H\left(\bar{\lambda}\right)$.

Let's estimate the size of the constant $a>1$.

In general, since $\lambda_1^0\geq\lambda_2^0\geq\cdots\geq\lambda_m^0\geq 1$, it is sufficient to take a constant $a>1$ such that $1<\lambda_1^0\leq a$. On the other hand, since

$$\begin{aligned} p_1^{\lambda_1^0+1} &= \left(e^{-\gamma}F\left(\bar{\lambda}_0\right)\right)\cdot\exp\left(e^{-\gamma}F\left(\bar{\lambda}_0\right)\right)+1= \\ &= p_m\cdot\log p_m\cdot\left(1+\varepsilon\left(p_m\right)\right), \end{aligned} \tag{59}$$

we get

$$\lambda_1^0=\frac{\log\left(p_m\cdot\log p_m\cdot\varepsilon\left(p_m\right)\right)}{\log p_1}-1.$$

Hence we can take the constant $a>1$ as

$$a=\frac{\log p_m+\log\log p_m}{\log p_1}+1. \tag{60}$$

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Department of Mathematics, University of Sciences, Unjong District, Gwahak 1-dong, Pyongyang, D.P.R.Korea, Email: ryonggilchoe@163.com