The Riemann Hypothesis

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In the paper it is given the proof of famous Riemann Hypothesis.

1. Introduction

Appearing of the zeta-function and the analytical methods in the Number Theory connected with the name of L. Euler (see [18, p.54]). In his works Euler had introduced the zeta-function

$$\zeta(s)=\sum_{n=1}^{\infty}n^{-s} \qquad (1)$$

as a function of real variable s. By using of an identity

$$\zeta(s)=\prod_{p}\left(1-\frac{1}{p^{s}}\right)^{-1},$$

where the product is taken over all prime numbers he gave an analytical proof of the theorem of Euclid on the infiniteness of a set of prime numbers. Euler had given the relationship which is equivalent (see [13]) to the Riemann functional equation. By using of Euler arguments in 1837 L. Dirichlet proved the generalization of Euclid theorem for arithmetic progressions considering L-series.

Great meaning of the zeta-function in the analytical number theory was discovered in 1859 by B. Riemann. In his famous memoir [20] Riemann considered $\zeta(s)$ as a function of complex variable and connected the question on the distribution of prime numbers with the location of complex zeroes of the zeta-function. He proved the functional equation

$$\xi(s)=\xi(1-s);\quad \xi(s)=\frac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

and formulated several hypotheses about the zeta-function. One of them (later RH) was fated to stand a central problem for all of mathematics. The Hypothesis asserts that all of complex zeros of the zeta-function, placed in the strip $0<\operatorname{Re}s<1$, located on the critical line $Re s$=0.5.

D. Hilbert included in 1900 Paris International Congress this Hypothesis into the list of his 23 mathematical problems.

In spite of no decreasing, up to nowadays, attempts of many outstanding mathematicians RH was remaining open. However, they were found several equivalents ([33], [4]) of this Hypothesis and it was arisen an opinion about its insolubility by the methods of mathematical analysis (see [4]).

To make some progress in the direction of this Hypothesis there was developed following brunches in the Number Theory:

1. Investigation of areas free from the zeroes of the zeta-function;
2. Density estimations of zeroes in the critical strip and their applications;
3. Studying of zeroes on the critical line;
4. Studying of a distribution of values of the zeta-function in the critical strip;
5. Computational questions connected with the zeroes and others.

Those directions are classical and in the literature they can be found (see [3,6,12, 16,17,19,22, 24]) historical and other aspects of these problems. We shall consider here, in sketch, only the works of direction 4 and several modern ideas in studying of questions connected with the RH.

Studying of distribution of values of the zeta – function was founded by G. Bohr (see [24, p.279]). In the work [2] the theorem on everywhere denseness of the values of $\zeta(\sigma+it)$, $-\infty<t<\infty, \sigma\in(1/2,1]$ was proven.

The results of S.M. Voronin [25-32] connected with the universality property of the zeta-function founded a new stage of investigations of values of the zeta-function and other functions defined by Dirichlet series. In the works of S.M. Voronin it was studied the distribution of values of some Dirichlet series, and a more general form of D. Hilbert problem on the differential independence of the zeta-function was proven for Dirichlet L-functions. Other generalizations and improvements were considered in the works ([1, 14-16]).

During last several years they begun the studies of some families of Dirichlet series the aim of which were to consider the questions connected with the distribution of zeroes of the zeta and L- functions ([4]). B. Bagchi had considered (see [15 – 16]) the family of Dirichlet series defined as a product

$$F(s;\theta)=\prod_p\left(1-\frac{\chi_p(\theta)}{p^s}\right)^{-1}$$

when Re s >1. He proved that this function can be analytically continued into the strip Re s >1/2 and has not there zeroes for almost all θ, where θ takes values in the topological product of the circles $\left|z_p\right|=1, z_p\in C$, and $\chi_p(\theta)$ is a projection of θ on the circle $\left|z_p\right|=1$. Here the measure is a Haar measure. In the works [1, 14-16] they were investigated the questions connected with the joint universality properties of some Dirichlet series. By using of Ergodic methods the special probability measures were constructed also.

In the work [11] it was gotten the equivalent variant of mentioned above result of B.Bagchi by considering of the function

$$F(s;\theta)=\prod_p\left(1-\frac{e^{2\pi i\theta_p}}{p^s}\right)^{-1},\ 0\le\theta_p\le 1 \qquad (2)$$

in the cube $\Omega=[0,1]\times[0,1]\times\cdots$ with the product of Lebesgue measures.

In the works [33 – 38] they were studied the questions on gaps between consecutive zeroes of the zeta –function on the critical line, on the number of zeroes in the circles with relatively no large radius in the near around of the critical line, and on the repeated zeros.

In the present work we study the distribution of special curves $(\{t\lambda_n\})_{n\ge1}$ (the sign {} means a fractional part and $\lambda_n>0,\lambda_n\to\infty$ as $n\to\infty$) in the subsets of zero measure of the infinite dimensional unite cube on which some series is divergent and the results have not a finite analog. As an application of getting results we prove RH. For this, at first, we shall approximate $\zeta(s)$ in some circle on the right half of the critical strip by a partial products of a view (2) by using of S.M. Voronin`s lemma (see the lemma 1). Further, we extend the gotten approximation to the all right half strip throughout by using of a special structure of a divergence set of some series (see the section 5).

Definition 1. *Let $\sigma:N\to N$ is any one to one mapping of the set of natural numbers. If there exist a natural number m such that $\sigma(n)=n$ for every n >m, we say that σ is a finite permutation. We call the subset $A\subset\Omega$ to be finite – symmetrical if for any element $\theta=(\theta_n)\in A$ and a finite permutation σ we have $\sigma\theta=(\theta_{\sigma(n)})\in A$.*

Let Σ to denote the set of all finite permutations. It is a group which contains any group S_n of *n* – degree permutations as a subgroup (we shall consider every *n* – degree permutation of σ $n=1,2,...$, as a finite permutation in the above sense for which $\sigma(m)=m$ when $m>n$). The set Σ is a countable set and we can ordered its elements and consider it as a sequence.

Theorem. *Let $0<r<1/4$ be a real number. Then there exist a sequence (θ_n) in Ω ($\theta_n\in\Omega, n=1,2,...$) and a sequence ($m_n$) of integers that for every real t*

$$\lim_{n\to\infty}F_n(s+it,\theta_n)=\zeta(s+it)$$

uniformly in the circle $|s-3/4|\le r$; here

$$F_n(s+it,\theta_n)=\prod_{p\le m_n}\left(1-\frac{e^{-2\pi i\vartheta_p^n}}{p^{s+it}}\right)^{-1};\ \theta_n=(\theta_p^n),$$

and the product is taken over the prime numbers, and the components of θ_n are indexed by the prime numbers.

It should be noted that the length of a product depends on *t.*

Corollary. *The Riemann Hypothesis is true, i. e.*

$$\zeta(s) \neq 0$$

when $\sigma > \frac{1}{2}$.

2. Supplementary statements.

The following lemma was proven by S.M. Voronin in [28] (we are formulating it in a little changed form).

Lemma 1. *Let g(s) be an analytical function in the circle $|s| < r < 1/4$, which is continuous and non – vanishing when $|s| \le r$. Then for any $\varepsilon > 0$ and $y > 2$ there exist a finite set of primes M containing all of primes $p \le y$ and an element $\overline{\theta} = (\theta_p)_{p\in M}$ such that:*

1) $0 \le \theta_p \le 1$ *for* $p \in M$ *;*

2) $\theta_p = \theta_p^0$ *are already given numbers for* $p \le y$*;*

3) $\max_{|s|\le r} \left| g(s) - \zeta_M\left(s + 3/4; \overline{\theta}\right)\right| \le \varepsilon$ *;*

here $\zeta_M\left(s+3/4;\overline{\theta}\right)$ *is defined by the equality*

$$\zeta_M\left(s+3/4;\overline{\theta}\right) = \prod_{p\in M}\left(1 - \frac{e^{-2\pi i\theta_p}}{p^{s+3/4}}\right)^{-1}.$$

Proof. The proof of the lemma1 will be conducted by the method of S.M. Voronin's work [28]. The series $u_k(s)$ of this work we define as

$$u_k(s) = \log\left(1 - e^{-2\pi i \vartheta_k} p_k^{-s-3/4}\right).$$

By using of the expansion of the logarithmic function into power series we may write

$$u_k(s) = -e^{-2\pi i\vartheta_k} p_k^{-s-3/4} + v_k(s)\,, \quad (4)$$

where

$$v_k(s) = O(p_k^{2r-3/2})\,.$$

Since $r < 1/4$, we may take such $\delta > 0$ that the inequality $2\delta + 2r - 3/2 < -1$ would satisfied. Then definition of $u_k(s)$ and (4) with the last inequality show that the series

$$\sum_{n=1}^{\infty} \eta_n(s); \eta_n(s) = -e^{-2\pi i\vartheta_n} p_n^{-s-3/4} \quad (5)$$

differs from the series $\sum u_n(s)$ by an absolutely convergent series. We must show the convergence of some permutation of the series $\sum u_k(s)$ uniformly to $\varphi(s)$ for any $\varphi(s) \in H_2^{(\gamma r)}$ ($0 < \gamma < 1$) (the definition of the Hardy space $H_2^{(\gamma r)}$ was given in [24, p.323]). Further, we consider (5) following by [28] and note that

$$\sum_{k=1}^{\infty} \|\eta_k(s)\|^2 < \infty .$$

We have

$$(\eta_k(s), \varphi(s)) = -\operatorname{Re} \int_{|s| \le R} e^{-2\pi i \vartheta_k} p_k^{-(s+3/4)} \overline{\varphi(s)} d\sigma dt = \operatorname{Re}\left[-e^{-2\pi i \vartheta_k} \Delta(\log p_k)\right],$$

where

$$\Delta(x) = \iint_{|s| \le R} e^{-x(s+3/4)} \overline{\varphi(s)} d\sigma dt .$$

As it was showen in [28], $\Delta(x)$ can be expanded into power series

$$\Delta(x) = \pi R^2 e^{-3x/4} \sum_{m=0}^{\infty} \frac{\beta_m}{m!} (xR)^m$$

by using of expansion of the function $\varphi(s)$. Define the entire function

$$F(u) = \sum_{m=0}^{\infty} \frac{\beta_m}{m!} u^m ; |\beta_m| \le 1 .$$

Repeating the reasoning of the work [28] we show that for any $\delta > 0$ there exist a sequence $u_1, u_2, \ldots \to \infty$ satisfying the following inequality

$$|\Delta(x_j)| > c e^{-x_j (3/4 + R + 2\delta R)} ; x_j = u_j / R .$$

Furthermore, we get the inequality

$$\max_{|x - x_j| \le 1} |\Delta(x)| > e^{-(1-\delta) x_j}$$

(see [24, p.244]) for every *j=1,2,…*. Now we take $\vartheta_k = k/4$ when $p_k > y$ (for *k* with $p_k \le y$ the numbers ϑ_k are any) and can separate from the series (5) sub series diverging to $+\infty$ and $-\infty$ correspondingly. From the results of the work [24, p.243] we deduce that

$$|\Delta(x)| \le \pi R^2 e^{-x/2} ;$$

so $|(\eta_k(s), \varphi(s))| \to 0$ as $k \to \infty$. Then some permutation of the series

$$\sum_{n=1}^{\infty} (\eta_k(s), \varphi(s))$$

converges conditionally. So (see [24, p.339]) there exist a permutation of the series $\sum_{p_n > y} u_n(s)$ converging to the $\varphi(s) - \sum_{p_n \le y} u_n(s)$ in regard to the norm of the space $H_2^{(\gamma r)}$. From this by a known way (see [22, p. 345]) we get the convergence in the usual sense uniformly in any compact sub domain of the circle $|s| < r.$ Taking sufficiently large partial sums of this series we get a suitable result. Lemma 1 is proved.

Lemma 2. *Let the series of analytical functions*

$$\sum_{n=1}^{\infty} f_n(s)$$

be given in the one-connected domain G of the complex s - plane and absolutely converges almost everywhere in the G in Lebesgue meaning and the function

$$\Phi(\sigma,t) = \sum_{n=1}^{\infty} |f_n(s)|$$

is a summable function in the domain G. Then the given series uniformly converges in any compact sub domain of the G; particularly the sum of this series will be an analytical function in the G.

Proof. It is enough to show that the theorem is true for any rectangle C in the domain G. Let C be a rectangle in the G and C' is another rectangle lying directly in the interior of the C, moreover, the sides of them are parallel to the axis. We can suppose that on contour the series is convergent almost everywhere in correspondence with the theorem of G.Fubini (see. [7, p.208]). We deduce from the theorem of Lebesgue on a bounded convergence (see. [21, p.293]):

$$\frac{1}{2\pi i}\int_C \frac{\Phi_0(s)}{s-\xi}ds = \sum_{n=1}^{\infty} \frac{1}{2\pi i}\int_C \frac{f_n(s)}{s-\xi}ds\,,$$

where the integrals are taken in Lebesgue meaning and $\Phi_0(s) = \Phi_0(\sigma,t)$ is a sum of given series at the points of convergence. Because on the right hand side of the equality the integrals exist in the Riemann meaning, we get (by applying Cauchy's formula)

$$\Phi_1(\xi) = \frac{1}{2\pi i}\int_C \frac{\Phi_0(s)}{s-\xi}ds = \sum_{n=1}^{\infty} f_n(\xi)\,,$$

where $\Phi_1(\xi) = \Phi_0(\xi)$ almost everywhere and ξ is any point on or in the contour. Further, the series in the C' is bounded by following inequality

$$|f_n(\xi)| \le \frac{1}{2\pi}\int_C \frac{|f_n(s)|}{|s-\xi|}|ds| \le \frac{1}{2\pi\delta}\int_C |f_n(s)||ds|\,,$$

if δ is a minimal distance between sides of the C and C'. The series

$$\sum_{n=1}^{\infty}\int_C |f_n(s)||ds|$$

converges in agree with the theorem of Lebesgue on a monotone convergence (see [21, p.290]).

Therefore, the series $\sum_{n=1}^{\infty} f_n(\xi)$ converges uniformly in the C'. The lemma 2 is proved.

3. Basic auxiliary results.

Let $\omega \in \Omega$, $\Sigma(\omega) = \{\sigma\omega \mid \sigma \in \Sigma\}$ and $\Sigma'(\omega)$ means the closed set of all limit points of the sequence $\Sigma(\omega)$. For real *t* we denote $\{t\Lambda\} = (\{t\lambda_n\})$, where $\Lambda = (\lambda_n)$. Below we denote by μ a product of linear Lebesgue measures of *m* defined in the segment [0,1]: $\mu = m \times m \times \cdots$.

Lemma 3. *Let $A \subset \Omega$ is a finite-symmetrical subset of zero measure and $\Lambda = (\lambda_n)$ is a unbounded monotonically increasing sequence of positive numbers any subset of components of which is linearly independent over the field of rational numbers. Let $B \supset A$ is any open set with $\mu(B) < \varepsilon$ and*

$$E_0 = \{0 \le t \le 1 \mid (\{t\Lambda\}) \in A \wedge \Sigma'(\{t\Lambda\}) \subset B\}$$

Then we have $m(E_0) \le 6c\varepsilon$, where c is an absolute constant and m means the linear Lebesgue measure.

Proof. Let ε is any small positive number. Since the numbers λ_n are linearly independent, we for any finite permutation σ have $(\{t_1\lambda_n\}) \ne (\{t_2\lambda_{\sigma(n)}\})$ when $t_1 \ne t_2$. Really, in the other case we could have the equality $\{t_1\lambda_m\} = \{t_2\lambda_m\}$ for some sufficiently large natural *m,* i.e. $(t_1 - t_2)\lambda_m = k, \quad k \in Z$. Further, by writing the same equality for other integer $r > m$ we have the relation

$$\frac{k_1}{\lambda_r} - \frac{k}{\lambda_m} = \frac{k_1\lambda_m - k\lambda_r}{\lambda_r\lambda_m} = 0,$$

which contradicts the linear independence of the numbers λ_n. So, for any pair of different numbers t_1 and t_2 $(\{t_1\lambda_n\}) \notin \left\{(\{t_2\lambda_{\sigma(n)}\}) \mid \sigma \in \Sigma\right\}$. We can find a family of open spheres (in the Tichonov topology) such that each of them do not contain any other (the sphere being contained in the other one may be omitted) and

$$A \subset B \subset \bigcup_{j=1}^{\infty} B_j, \sum \mu(B_j) < 1.5\varepsilon.$$

Now we take the permutation $\sigma \in \Sigma$ defined by the equalities $\sigma(1) = n_1, \ldots, \sigma(k) = n_k$ with the natural numbers n_j set by following way. At first we take N such that

$$\mu(B'_N) < 2\varepsilon_1,$$

where the B'_N is a projection of the sphere B_1 into the subspace of first N axes and $\mu(B_1)=\varepsilon_1$. We cover the B'_N by cubes with the rib δ and summarized measure not exceeding $3\varepsilon_1$. Let us to write $k=N$ and define the numbers $n_1,\ldots,n_k$ by using of following inequalities

$$\lambda_{n_1}>1,\ \lambda_{n_2}^{-1}<\frac{1}{4}\delta\lambda_{n_1}^{-1},\ \lambda_{n_3}^{-1}<\frac{1}{9}\delta\lambda_{n_2}^{-1},\ldots,\lambda_{n_k}^{-1}<\frac{1}{k^2}\delta\lambda_{n_{k-1}}^{-1},\ \delta<1. \qquad (6)$$

Now we take any cube with the rib δ and center $(\alpha_m)_{1\le m\le k}$. Then the point $(\{t\lambda_{n_m}\})$ would lie in this cube if

$$\left|\{t\lambda_{n_m}\}-\alpha_m\right|\le\delta/2 .$$

From the definition of the fractional part we may write for some integral *r* taking *m=1:*

$$\frac{r+\alpha_1-\delta/2}{\lambda_{n_1}}\le t\le\frac{r+\alpha_1+\delta/2}{\lambda_{n_1}}. \qquad (7)$$

The measure of a set of such *t* does not exceed the value $\delta\lambda_{n_1}^{-1}$. The number of such intervals corresponding to the different values of $r=[t\lambda_{n_1}]\le\lambda_{n_1}$ does not exceed $[\lambda_{n_1}]+2\le\lambda_{n_1}+2$. The total measure of those intervals is

$$\le(\lambda_{n_1}+2)\delta\lambda_{n_1}^{-1}\le(1+2\lambda_{n_1}^{-1})\delta .$$

Now we examine one of the intervals (6) and taking *m=2* write

$$\frac{s+\alpha_2-\delta/2}{\lambda_{n_2}}\le t\le\frac{s+\alpha_2+\delta/2}{\lambda_{n_2}}, \qquad (8)$$

with $s=[t\lambda_{n_2}]\le\lambda_{n_2}$. Since we take the conditions (6) and (8) simultaneously, we must estimate the total measures of intervals (8) having nonempty intersections with the intervals (7) by using of the conditions (6). The number of intervals with the length having nonempty intersection with one of the intervals of the view (7) does not exceed the value

$$[\delta\lambda_{n_1}^{-1}\lambda_{n_2}]+2\le\delta\lambda_{n_1}^{-1}\lambda_{n_2}+2 .$$

Therefore, the measure of a set of such *t* for all of which the conditions (7) and (8) are satisfied simultaneously does not exceed

$$(\lambda_{n_1}+2)(2+\delta\lambda_{n_1}^{-1}\lambda_{n_2})\delta\lambda_{n_2}^{-1} .$$

One may continue those reasoning by taking all of conditions of the form

$$\frac{l+\alpha-\delta/2}{\lambda_{n_m}}\le t\le\frac{l+\alpha+\delta/2}{\lambda_{n_m}},\quad m=1,\ldots,k .$$

Then we find the following estimation for the measure $m(\delta)$ of a set of such *t* for which the points $(\{t\lambda_{n_m}\})$ lie into the cubes with the rib of δ:

$$m(\delta) \leq (2+\lambda_{n_1})(2+\delta\lambda_{n_1}^{-1}\lambda_{n_2})\cdots(2+\delta\lambda_{n_{k-1}}^{-1}\lambda_{n_k})\delta\lambda_{n_k}^{-1} \leq \delta^k \prod_{m=1}^{\infty}(1+2m^{-2}).$$

Therefore, by summing over all of such cubes we get the upper bound for the measure of a set of such t for which $(\{t\lambda_{n_m}\}) \in B_1$ the value $\leq 3c\varepsilon_1$, $c>0$.

Note that the sequence $\Lambda = (\lambda_n)$ defined above depends on δ. We shall fix for every of defined above spheres B_k some sequence Λ_k by using of conditions (6). Considering all of such spheres we denote $\Sigma_0 = \{\Lambda_k \mid k = 1,2,...\}$. Since the set A is finite symmetrical, the measure of interested us values of t can be estimated by using of any sequence Λ_k because as it was noted above the sets $\Sigma(\{t\Lambda\})$ for various values of t have empty intersection.

Further, for any point t of the E_0 the set $\Sigma(\{t\Lambda\})$ has a non-empty intersection only with finite number of spheres B_k. Really, if else some limit point (which is contained by the open set B) of $\Sigma(\Lambda)$ belong say to B_s. Let d is a distance from θ to the bound of B_s. Then for infinitely many indexes n_k beginning from some k all of spheres B_{n_k} would belong into the sphere with radius < $d/2$, and the center θ. So, for sufficiently large k the all of such spheres would belong into B_s which is contradiction. Consequently, the set E_0 can be represented as a union of subsets E_k, $k=1,2,\ldots,$ where

$$E_k = \{t \in E_0 \mid \Sigma(\{t\Lambda\}) \cap \bigcup_{m>k} B_m = \varnothing\}.$$

Then,

$$E_k = \{t \in E_0 \mid \Sigma(\{t\Lambda\}) \subset \bigcup\nolimits_{k \leq m} B_k\},\ E_0 = \bigcup\nolimits_{k=1}^{\infty} E_k;\ E_k \subset E_{k+1} (k \geq 1).$$

So, we have

$$m(E_0) \leq \limsup\nolimits_{\Lambda\in\Sigma_0} m(E(\Lambda)) \leq \sum_k \limsup\nolimits_{\Lambda\in\Sigma_0} m(E^{(k)}(\Lambda)) \leq$$

$$\leq 3c(\varepsilon_1 + \varepsilon_2 + \cdots) = 3c\varepsilon,$$

where $E(\Lambda) = \{t \in E_0 \mid (\{t\Lambda\}) \in B\}$ and $E^{(k)}(\Lambda) = \{t \in E_0 \mid \{t\Lambda\} \in B_k\}$. The proof of the lemma 3 is completed.

4. Local approximation

Lemma 4. *There exist a sequence of points (θ_K) ($\theta_K \in \Omega$) and natural numbers (m_k) such that $\theta_K \to 0$ and*

$$\lim_{k\to\infty} F_k\left(s+\frac{3}{4}, \theta_k\right) = \zeta\left(\frac{3}{4}+s\right)$$

in the circle $|s| \leq r$, $0 < r < 1/4$ uniformly by s (F_k was defined above).

Proof. Let $y > 2$ is a whole positive number which will be precisely defined below. We suppose

$$y_0 = y,\ y_1 = 2y_0, \ldots, y_m = 2y_{m-1} = 2^m y_0, \ldots .$$

From the lemma 2 it follows that for the given ε and the whole number $y > 2$ there exists a set M_1 of primes and a point $\theta_1 = (\theta_p^0)_{p \in M_1}$ such that M_1 contains all the primes $p \le y$ with $\theta_p^0 = 0$ and (to be brief we write $s_1 = s + 3/4$)

$$\max_{|s| \le r} \left|\zeta(s_1) - \eta_1(s_1)\right| \le \varepsilon\,;\ \eta_1(s_1) = \prod_{p \in M_1} \left(1 - \frac{e^{-2\pi i \theta_p^0}}{p^{s_1}}\right)^{-1} .$$

Now we denote

$$F_1(s_1;\theta) = \prod_{p \le m_1} \left(1 - e^{-2\pi i \theta_p} p^{-s_1}\right)^{-1}$$

and

$$h_1(s_1;\theta) = F_1(s_1;\theta) \prod_{p \in M_1} \left(1 - e^{-2\pi i \theta_p^0} p^{-s_1}\right) - 1, ;$$

here $\theta_p = \theta_p^0$ when $p \in M_1$ and $m_1 = \max\limits_{m \in M_1} m$. Let n to denote natural numbers the canonical factorizations of which contain only primes $p,\ p \notin M_1,\ p \le m_1$, and

$$a_n(\theta) = e^{2\pi i \sum \alpha_p \theta_p}; n = \prod p^{\alpha_p} .$$

If $r + \delta < 1/4$ we have

$$\int_{\Omega_1} \left(\iint_{|s| \le r + \delta} |h_1(s_1;\theta)|^2 \, d\sigma dt \right) d\theta \le \iint_{|s| \le r+\delta} \left(\int_{\Omega_1} |h_1(s_1;\theta)|^2 \, d\theta \right) d\sigma dt \le$$

$$\le \pi(r+\delta)^2 \max_{|s| \le r+\delta} \int_{\Omega_1} \left| \sum_{n > y} n^{-s_1} a_n(\theta) \right|^2 d\theta \le \frac{4\pi(r+\delta)^2}{1 - 4r - 4\delta} y^{-1/2 + 2r + 2\delta};$$

here Ω_1 indicates the projection of Ω into the subspace of coordinate axes θ_p, $p \notin M_1$.Then from the inequality gotten above it follows an existence of a point $\theta_1' = (\theta_p)_{p \notin M_1}$ such that

$$\iint_{|s| \le r+\delta} |h_1(s_1;\theta_1')|^2 \, d\sigma dt \le \frac{4\pi(r+\delta)^2}{1 - 4r - 4\delta} y^{2\delta + 2r - 1/2}$$

or

$$\max_{|s| \le r} |h_1(s_1;\theta_1')| \le \sqrt{2}\delta^{-1} \left(\frac{1}{2\pi} \iint_{|s| \le r+\delta} |h_1(s_1;\theta_1'|^2 \, d\sigma dt \right)^{1/2} \le c(\delta) y^{\delta + r - 1/4}$$

(see [22, p. 345]) with a constant $c(\delta)>0$. So, taking $\theta_1 = (\theta_0, \theta_1')$, $\theta_0 = (\theta_p^0)_{p \in M_1}$ we shall have

$$\max_{|s|\le r}\left\{\left|\zeta\left(\frac{3}{4}+s\right)-F_1\left(\frac{3}{4}+s;\theta_1\right)\right|\right\}\le \max_{|s|\le r}\left\{\left|\zeta\left(\frac{3}{4}+s\right)-\eta_1\left(\frac{3}{4}+s\right)\right|+\left|\eta_1\left(\frac{3}{4}+s\right)\right|\cdot\left|h_1(s_1;\theta_1')\right|\right\}\le$$

$$\le \varepsilon+(A+1)c(\delta)y_0^{r+\delta-1/4}\le 2\varepsilon;$$

if $y=y_0$ would taken satisfying the condition

$$c(\delta)y_0^{r+\delta-1/4}(A+1)\le\varepsilon;\ A=\max_{|s|\le r}\left|\zeta\left(\frac{3}{4}+s\right)\right|.$$

We replace now ε by $\varepsilon/2$. There exist a set of primes M_2 containing all of the prime numbers $p\le 2y_0=y_1$ and satisfying, by the lemma 1, the inequality

$$\max_{|s|\le r}\left|\zeta\left(\frac{3}{4}+s\right)-\eta_2\left(\frac{3}{4}+s\right)\right|\le\frac{\varepsilon}{2}$$

with

$$\eta_2(s_1)=\prod_{p\in M_2}\left(1-\frac{e^{-2\pi i\theta_p^{(1)}}}{p^{s_1}}\right)^{-1},$$

and $\theta_p^{(1)}=0$ if $p\le y_1$. By like way we find $\theta_2'\in\Omega_2$ (Ω_2 is a projection of Ω into the subspace of coordinate axes θ_p, $p\notin M_2$) such that

$$\max_{|s|\le r}\left|\zeta\left(\frac{3}{4}+s\right)-F_2\left(\frac{3}{4}+s;\theta_2\right)\right|\le 2^{1+(r+\delta-1/4)}\varepsilon;\quad \theta_2=(\theta_1,\theta_2').$$

Really,

$$\left|F_2\left(\frac{3}{4}+s\right)-\eta_2\left(\frac{3}{4}+s\right)\right|=\left|\eta_2\left(\frac{3}{4}+s\right)\right|\left|h_2(s_1;\theta_2')\right|.$$

Now by taking of the mean value we get

$$\max_{|s|\le r}\left|h_2(s_1;\theta_2')\right|\le\sqrt{2}\delta^{-1}\left(\frac{1}{2\pi}\iint_{|s|\le r+\delta}\left|h_2(s_1;\theta_2'\right|^2 d\sigma dt\right)^{1/2}\le c(\delta)(2y_0)^{\delta+r-1/4}.$$

Therefore,

$$\max_{|s|\le r}\left|\zeta\left(\frac{3}{4}+s\right)-F_2\left(\frac{3}{4}+s;\theta_2\right)\right|\le\frac{\varepsilon}{2}+2^{\delta+r-1/4}\varepsilon\le 2^{1+(r+\delta-1/4)}\varepsilon;\quad\theta_2=(\theta_1,\theta_2').$$

By repeating this calculus we for every $\kappa>1$ find $\theta_\kappa=(\theta_k,\theta_{k+1}')\in\Omega$, $\theta_k=(\theta_p^k)_{p\in M_{k+1}}$ such that $\theta_p^{(k)}=0$ when $p\le y_\kappa$ and

$$\max_{|s|\le r}\left|\zeta\left(\frac{3}{4}+s\right)-F_{k+1}\left(\frac{3}{4}+s;\theta_{k+1}\right)\right|\le 2^{1+k(r+\sigma-\frac{1}{4})}\varepsilon,$$

where

$$F_{k+1}(s_1;\theta)=\prod_{p\le m_{k+1}}\left(1-e^{-2\pi i\theta_p}p^{-s_1}\right)^{-1};m_{k+1}=\max_{m\in M_{k+1}}m.$$

Consequently, uniformly by s, $|s|\le r$ we have

$$\lim_{k\to\infty}F_k\left(\frac{3}{4}+s,\,\theta_k\right)=\zeta\left(\frac{3}{4}+s\right).$$

Lemma 4 is proved.

5. Proof of the theorem.

Now we consider the integral

$$B_k=\int_\Omega\left(\iint_{|s|\le r}\left|F_{M_{k+1}}\left(\frac{3}{4}+s;\theta_{k+1}+\theta\right)-F_{M_k}\left(\frac{3}{4}+s;\,\theta_k+\theta\right)\right|d\sigma d\tau\right)d\theta,$$

where κ = 0, 1,…, and (for $\kappa=0$ we let $F_{M_0}(3/4+s,\theta_0+\theta)=0$). By applying the Schwartz inequality and changing the order of the integration we find as above:

$$B_k^2\le 4\pi r^2\iint_{|s|\le r}d\sigma d\tau\int_\Omega\left|\prod_{p\le 2^{k-1}y_0}\left(1-p^{-\frac{3}{4}-s}\cdot e^{2\pi i(\theta_p+\theta_p^k)}\right)^{-1}\right|^2\prod_{p\le 2^{k-1}y_0}d\theta_p\cdot\sum_{n>2^{k-1}y_0}n^{2r+2\delta-\frac{3}{2}}\le$$

$$\le c_\delta\left(2^{k-1}y_0\right)^{2r+2\delta-\frac{1}{2}};c_\delta>0.$$

Since $2r+2\delta-\dfrac{1}{2}<0,$ from this estimation it follows the convergence of the series below almost everywhere (for all $\theta\in\Omega_0$, where Ω_0 is a subset of full measure and the set $A=\Omega\setminus\Omega_0$ is finite symmetrical) by θ

$$\sum_{k=1}^{\infty}\iint_{|s|\le r}\left|F_k\left(\frac{3}{4}+s,\,\theta_k+\theta\right)-F_{k-1}\left(\frac{3}{4}+s,\,\theta_{k-1}+\theta\right)\right|d\sigma d\tau;\,s=\sigma+i\tau.\quad(9)$$

By the theorem of Yegorov (see [7, p. 166]) the series above is converging almost uniformly in the outside of some subset $\Omega_1',\,\mu(\Omega_1')=0$. We can suppose the set $A\cup\Omega_1'$ to be finite symmetrical (if else one can take all permutations of all its elements). We can find some countable family of spheres B_r with the total measure does not exceeding ε and the union of which contains the set $A\cup\Omega_1'$. For any natural number *n* we define the set $\Sigma_n'(t\Lambda)$ as a set of all limit points of the sequence $\Sigma_n(\omega)=\{\sigma\omega\mid\sigma\in\Sigma\wedge\sigma(1)=1\wedge\cdots\wedge\sigma(n)=n\}$. Let $\Lambda=(\lambda_n)$, $\lambda_n=(1/2\pi)\log p_n$, where p_n denote the *n*- th prime number and $B^{(n)}=\{t\mid\{t\Lambda\}\in A\wedge\Sigma_n'(\{t\Lambda\})\subset\bigcup_r B_r\}$, $n=1,2,\ldots$. We have $B^{(n)}\subset B^{(n+1)}$. Therefore, if we let $B=\bigcup_n B^{(n)}$, we get $m(B)\le\sup m(B^{(n)})$. The set $\Sigma_n'(\{t\Lambda\})$ is closed. It is clear that if we would restrict the sequences $\{t\Lambda\}$ by taking only the components $\{t\lambda_n\}$ with indexes greater than n and denote by $\{t\Lambda\}'$ the restricted sequence, the set $\Sigma'(\{t\Lambda\}')$ were also a closed set. Now we consider the products $[0,1]^n \times \{\{t\Lambda\}'\}$ for every t (the exterior parentheses, in the difference from the interior ones, sign a set of one element). We have

$$\{t\Lambda\} \in [0,1]^n \times \{\{t\Lambda\}'\} \subset A\,,$$

because if the series (9) above is divergent for given $\{t\Lambda\}$, it is divergent for every point of the set $[0,1]^n \times \{\{t\Lambda\}'\}$ also. So, the taken open set contains the all of such points.

The example below shows that from this fact it does not follow the equality $A=\Omega$. Let $I=[0,1]$; $U=[0;1/2]$; $V=[1/2;1]$, and

$$X_0 = U\times U\times\cdots,\ X_1 = V\times U\times\cdots,\ X_2 = I\times V\times U\times\cdots,\ldots,\ X_{s+1} = I^s\times V\times U\times\cdots,\ldots.$$

It is clear that $\mu(X_s)=0$ for the all s. Then $\mu(X)=0$, where

$$X=\bigcup_{s=0}^{\infty} X_s\,.$$

As it is seen from the construction of X the equality $X=[0,1]^s\times X$ is satisfied for every s.

Since the set $[0,1]^n \times \{\{t\Lambda\}'\}$ is closed, there exists only finite set R of natural numbers such that $[0,1]^n \times \{\{t\Lambda\}'\} \subset \bigcup_{r\in R} B_r$. Consider the set of restricted points θ' of the spheres B_r. Let $B_r' = \{\theta' \mid \theta\in B_r\}$. Then the intersection of them being an open set contains the point $\{t\Lambda\}'$. So, we have

$$[0,1]^n \times \{\{t\Lambda\}'\} \subset [0,1]^n \times \bigcap_{r\in R} B_r' \subset \bigcup_{r\in R} B_r \qquad (10)$$

for every considered t. The analogical relation is true if we would exchange the point $\{t\Lambda\}$ by any limit point ω of the sequence $\Sigma(\{t\Lambda\})$, because $\omega\in B_r$. If by B' we denote the union of all open sets of the view $\bigcap_{r\in R} B_r'$, we get the relation

$$\{t\Lambda\} \in [0,1]^n \times \{\{t\Lambda\}'\} \subset A \subset [0,1]^n \times B' \subset \bigcup_r B_r$$

for each considered values of t, or

$$\{\omega\} \in [0,1]^n \times \{\omega'\}\} \subset A \subset [0,1]^n \times B' \subset \bigcup_r B_r$$

for any limit point of ω. From this it follows that $\mu(B')\le\varepsilon$. The set B' is an open set and $\Sigma'(\{t\Lambda\}')\subset B'$. Now we can apply the lemma 4 and get the bound $m(B^{(n)})\le 6c\varepsilon$. So, we have $m(B)\le 6c\varepsilon$.

Consequently, by taking $n=y_k$, $k=1, 2, 3,\ldots$ we find such a limit point $\omega_k \in \Omega\setminus\bigcup_r B_r$ of the sequence $\Sigma_n(\{t\Lambda\})$ for which the series

$$\sum_{l=1}^{\infty} \iint_{|s|\le r} \left| F_l\left(\frac{3}{4}+s,\, \theta_l+\omega_k\right) - F_{l-1}\left(\frac{3}{4}+s,\, \theta_{l-1}+\omega_k\right)\right| d\sigma d\tau$$

is converging for all values of $t \notin B$. Since the set $\Omega \setminus \bigcup_r B_r$ is closed then the limit point $\overline{\omega} = (\{t\Lambda\})$ of the sequence (ω_k) will belong into $\Omega \setminus \bigcup_r B_r$, because the series (9) is uniformly convergent in the set $\Omega \setminus \bigcup_r B_r$. So the series below is convergent

$$\sum_{l=1}^{\infty} \iint_{|s|\le r} \left| F_l\left(\frac{3}{4}+s,\, \theta_l+\omega_k\right) - F_{l-1}\left(\frac{3}{4}+s,\, \theta_{l-1}+\omega_k\right)\right| d\sigma d\tau$$

for all values of $t \notin B$. Consequently, this series is convergent for all values of *t* with exception of their set of a measure not exceeding $6c\varepsilon$. Since ε is any, the latest result shows the convergence of the series (9) for almost all *t* such that $0 \le t \le 1$. It is clear that the last condition is not a main one and the result is true for almost all real *t.* Then by the lemma 2 for
any given $\delta_0 < 1$ the sequence

$$F_k\left(\frac{3}{4}+s,\, \theta_k + i\{t\Lambda\}\right), \qquad (11)$$

for the all such *t* converges in the circle $|s| \le r\delta_0$ $(\delta_0 < 1)$ uniformly to some analytical function $f(s_1;\, t)$:

$$\lim_{k\to\infty} F_k\left(\frac{3}{4}+s+it,\, \theta_k\right) = f(s_1;t)\,.$$

In spite of the getting result we cannot use *t* as a variable because left and right sides (the right side is defined as a limit of the sequence (11)) can have different arguments. Therefore, we cannot use the principle of analytical continuation. To complete the proof of the theorem, we take any large positive number *T.* Since the set of taken values of *t* is everywhere dense in the interval $[T, -T]$, the union of the circles $C(t)=\{3/4+it+s\colon |s|\le r\delta_0\}$ contains the rectangle

$$K: 3/4 - r\delta_0^2 \le \mathrm{Re}(s+3/4) \le 3/4 + r\delta_0^2, -T \le \mathrm{Im}(s+3/4) \le T\,,$$

in which, as it was shown above, the conditions of the lemma 2 are satisfied for the series

$$F_{M_1}(s+3/4;\vartheta_1) + \left(F_{M_2}(s+3/4;\vartheta_2) - F_{M_1}(s+3/4;\vartheta_1)\right) + \cdots.$$

Therefore it defines some analytical function $F(s)$ in this rectangle.

For applying of the principle of analytical continuation we must take an one-connected open domain, where both of the functions $\log\zeta(s)$ and $\log F(s)$ are regular. Let $\rho_1,\ldots,\rho_L$ are all possible zeroes of the zeta-function in the rectangle *K* on the contour of which the zeta-function has not any zeroes. We take the cross cuts over the segments $1/2 \le \mathrm{Re}\, s \le \mathrm{Re}\,\rho_l$, $\mathrm{Im}\, s = \mathrm{Im}\,\rho_l, l = 1,\ldots,L$. In the open domain of the considering rectangle the functions $\log\zeta(s)$

and $\log F(s)$ are regular. From the lemma 4 it follows that the left side of the (11) converges absolutely and uniformly to the $\zeta(s)$ when *t=0*. Therefore, the equality $\zeta(s) = F(s)$ is satisfied in the open domain defined above by the principle of analytical continuation. Now we get the equality $\zeta(s) = F(s)$ in the all rectangle (without the cross cut), because both of those functions are regular. The proof of the theorem is completed.

6. Proof of the corollary.

The deduction of the corollary comes out from the theorem of Rouch'e (see [23,p.137]). It is enough to show that for any $0 < r < 1/4$ in the circle C={ s | $|s - 3/4 - it| = r$ }, on which any possible zero of $\zeta(s)$ does not exist, we have $\zeta(s) \neq 0$. Let

$$m = \min_{s \in C} |\zeta(s)|.$$

By the theorem had proven above we can find such *n=n(t)* for which in and on the contour *C* the following inequality holds

$$|F_n(s; \theta_n) - \zeta(s)| \leq 0.25m.$$

Then on the *C* the inequality

$$|F_n(s; \theta_n) - \zeta(s)| \leq |\zeta(s)|$$

is satisfied. By the theorem of Rouch'e the functions $\zeta(s)$ and $F_n(s;\theta_n)$ have the same number of zeroes. in the *C*. But $F_n(s;\theta_n)$ has not any zeroes in the circle *C*. Therefore, $\zeta(s)$ also has not zeroes in the *C*. Since *t* is any, we deduce from this that the strip *-r<Re s-3/4<r* (for any *0<r<1/4*)is free from the zeroes of the function $\zeta(s)$. The proof of the corollary is completed.

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