# Hitting Times for Finite and Infinite Graphs

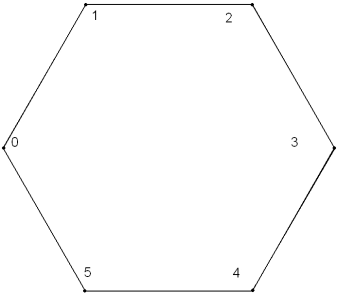
Ari Binder, Williams College
Josh O'Rourke, Reed College

July 31, 2008

Work completed at the Canisius College REU, Summer 2008

*wiki.canisiusmath.net*

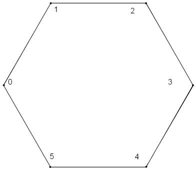
# What is a Random Walk?




- Start at an arbitrary vertex.
- Randomly choose an adjacent destination vertex.
- Move there, and repeat the process.
- I studied mean hitting times on undirected Cayley graphs such as the undirected 6-cycle.

# What is a Cayley Graph?

- A visual representation of a group.
- Vertices represent elements of the group.
- Choose generators; for each generator $h$, start at $e$, connect $e$ to $e + h$ with a directed edge. Then connect $e + h$ to $e + h + h$, and so on.
- The 6-cycle is the Cayley graph of $\mathbb{Z}_6$ on generators 1 and $-1$ (or 5) That is, 6-cycle= $Cay(\mathbb{Z}_6, \{\pm 1\})$.



# What is a Mean Hitting Time?

- Definition: The expected number of steps to reach a given vertex $j$ of a graph $G$ starting from a vertex $i$ of $G$.
- We denote this hitting time as $E_i(T_j)$
- Thus, $E_i(T_j) = \sum_{n=0}^{\infty} n \cdot \mathbb{P}(\textit{walk first reaches } j \textit{ in } n \textit{ steps})$
- But how can we determine these hitting times?

# First, Some Background Stuff

- Definition: A *transition matrix* of an $n$-vertex graph is the $n \times n$ matrix whose $ij$-th entry describes the probability of a random walk moving from state $i$ to state $j$.
- The 6-cycle has the following transition matrix, which we call $P$:

$$P = \begin{bmatrix} 0 & 1/2 & 0 & 0 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 0 & 0 & 1/2 & 0 \end{bmatrix}$$

# Some More Background Stuff

- Definition: We call a graph $G$ *strongly connected* if, for each vertex $v_i$ of $G$ there exist paths from $v_i$ to any other vertex in $G$.
- All Cayley graphs of $\mathbb{Z}_n$ that include 1 or $-1$ as a generator are strongly connected.
- Strong connectivity $\implies$ there exists a stable probability distribution on the vertices of $G$, which we call $\pi$, such that $\pi P = \pi$.
- Definition: Strong connectivity also $\implies$ $P$ is *irreducible*.

# The Fundamental Matrix

- The fundamental matrix $Z$ of an $n$-vertex graph $G$ with irreducible transition matrix $P$ is defined as follows:

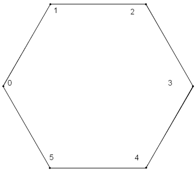
$$Z_{ij} = \sum_{t=0}^{\infty} (P_{ij}^{(t)} - \pi_j)$$

- **Result:** $Z = (I - (P - P_\infty))^{-1} - P_\infty$
- Easily gives us hitting times:
- $E_i(T_j) = \frac{1}{\pi_j}(Z_{jj} - Z_{ij})$
- **Result:** $n$-vertex Cayley graph $\implies \pi$ is uniform, so $\frac{1}{\pi_i} = n \; \forall i$.

# Calculating Hitting Times on the 6-Cycle

- Using above formula, we calculate the $Z$-matrix for 6-cycle:

$$Z = \begin{bmatrix} 35/36 & 5/36 & -13/36 & -19/36 & -13/36 & 5/36 \\ 5/36 & 35/36 & 5/36 & -13/36 & -19/36 & -13/36 \\ -13/36 & 5/36 & 35/36 & 5/36 & -13/36 & -19/36 \\ -19/36 & -13/36 & 5/36 & 35/36 & 5/36 & -13/36 \\ -13/36 & -19/36 & -13/36 & 5/36 & 35/36 & 5/36 \\ 5/36 & -13/36 & -19/36 & -13/36 & 5/36 & 35/36 \end{bmatrix}$$




- $E_0(T_1) = \frac{1}{\pi_1}(Z_{11} - Z_{01}) = 6(\frac{35}{36} - \frac{5}{36}) = 5.$
- $E_0(T_2) = \frac{1}{\pi_2}(Z_{22} - Z_{02}) = 6(\frac{35}{36} + \frac{13}{36}) = 8.$
- $E_0(T_3) = \frac{1}{\pi_1}(Z_{33} - Z_{03}) = 6(\frac{35}{36} + \frac{19}{36}) = 9.$

# Quantifying $E_i(T_j)$ Values Using Only $P$

- $P$ is symmetric, and so can be diagonalized by an orthonormal transformation: $P = U \Lambda U^T$
- This gives $P_{ij} = \sum_{m=1}^{n} \lambda_m(P) u_{im} u_{jm}$
- Defining $P$ exactly in terms of its eigenvectors and eigenvalues leads to the following:
- **Result:**

$$E_i(T_j) = n \sum_{m=2}^{n} (1 - \lambda_m(P))^{-1} u_{jm}(u_{jm} - u_{im})$$

# 6-Cycle Example

$$
\begin{aligned}
E_0(T_1) &= 6\sum_{m=2}^{n}(1-\lambda_m(P))^{-1}u_{1m}(u_{1m}-u_{0m}) \\
&= 6[2\cdot 0(0-1/2) + 2\cdot -1/\sqrt{3}(-1/\sqrt{3}+1/2\sqrt{3}) \\
&\quad + 2/3\cdot 0(0+1/2) + 2/3\cdot 1/\sqrt{3}(1/\sqrt{3}+1/2\sqrt{3}) \\
&\quad + 1/2\cdot 1/\sqrt{6}(1/\sqrt{6}+1/\sqrt{6})] \\
&= 6[2\cdot 0 + 2\cdot -1/\sqrt{3}\cdot -1/2\sqrt{3} + 2/3\cdot 0 \\
&\quad + 2/3\cdot 1/\sqrt{3}\cdot 1/\sqrt{3} + 1/2\cdot 1/\sqrt{6}\cdot 1/\sqrt{6}] \\
&= 6[0+1/3+0+1/3+1/6] \\
&= 5 \\
&= 6(Z_{11}-Z_{01})
\end{aligned}
$$

We can verify that the other hitting times work as well.

# Positive Recurrent Infinite Graphs

# Recurrence vs. Transience

**Recurrent:** The probability of returning to the starting vertex goes to one as time goes to infinity.

**Transient:** There is a non-zero probability of never returning to the starting vertex.

In a strongly connected graph, independent of starting vertex.

# Expected First Return Time

**First Return Time ($T_u^+$):** Given starting vertex $u$, the time a given random walk takes to return to $u$.

**Expected First Return Time ($E_u(T_u^+)$):** Over a large number of random walks starting at $u$, the average first return time.

# Positive Recurrence vs. Null Recurrence

For any vertex $u$ in a transient graph, $E_u(T_u^+) = \infty$.

In a recurrent graph, $E_u(T_u^+)$ can be finite or infinite.

**Positive Recurrent:** $E_u(T_u^+) < \infty$.

**Null Recurrent:** $E_u(T_u^+) = \infty$.

Independent of starting vertex.

# Stationary Measures and Positive Recurrence

**Measure ($\pi$):** A non-negative, real-valued function on the vertices of a graph.

**Transition operator ($P$):** The generalization of the transition matrix to the infinite case.

$P$ acts on measures in the following way:

$$P\pi(u) = \sum_{v \to u} \frac{\pi(v)}{outdeg(v)}$$

If a graph is recurrent, then there exists a measure $\pi$ such that $P\pi = \pi$, unique up to scalar multiples.

The graph is positive recurrent if:

$$\sum_{u \in G} \pi(u) < \infty$$

The graph is null recurrent if:

$$\sum_{u \in G} \pi(u) = \infty$$

# Graphs with indeg = outdeg

**Theorem**

*Let $G$ be a strongly connected, infinite graph with $indeg(u) = outdeg(u)$ for all $u \in G$.*

*$G$ is not positive recurrent.*

$\pi(u) = outdeg(u)$ is a stationary measure and is not summable.

No infinite undirected or Cayley graphs are positive recurrent.

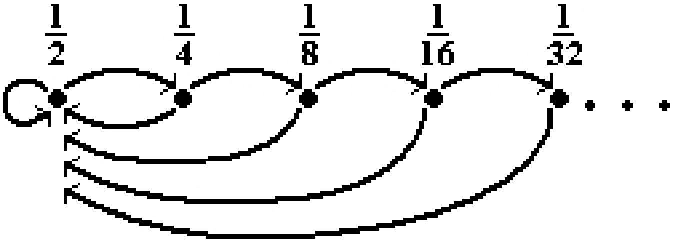
# Stationary Distributions and Expected Return Times

**Distribution:** A measure $\pi$ such that:

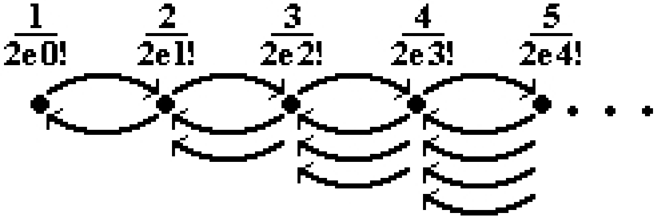
$$\sum_{u \in G} \pi(u) = 1$$

A graph is positive recurrent if and only there exists a distribution $\pi$ such that $P\pi = \pi$. In that case, $E_u(T_u^+) = \frac{1}{\pi(u)}$.

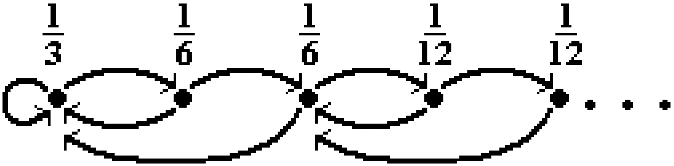
# Some Examples of Positive Recurrent Graphs



A locally finite, positive recurrent graph:



A bounded degree, single-edged, positive recurrent graph:



# References

Aldous, David and Jim Fill. *Reversible Markov Chains and Random Walks on Graphs*. 30 June 2008.
http://stat-www.berkeley.edu/users/aldous/RWG/book.html

Norris, J.R. *Markov Chains*. New York: Cambridge University Press, 1998.

Woess, Wolfgang. *Random Walks on Infinite Graphs and Groups*. Cambridge: Cambridge University Press, 2000.

*wiki.canisiusmath.net*