The Estimate of Some Quantities with Prime Number

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Abstruct

In this paper we would estimate some quantities with prime number by the results obtained from the opimization problem of a certain exponential function. In particular, we would show an estimate for the difference between the consecutive primes. This estimate is a new result in the distribution of the prime numbers.

Keywords; Consecutive primes; Distribution of the prime numbers.

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1. Introduction

Assume that $\overline{\lambda} = (\lambda_1, \lambda_2, \cdots, \lambda_m)$ are real numbers and $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_m \ge 1$. Let $p_1 = 2,\ p_2 = 3, \cdots, p_m, \cdots$ be consecutive primes. We will choose $p_m \ge 5$ arbitrarily and fix it. We define functions $F(\overline{\lambda})$ and $H(\overline{\lambda})$ respectively by

$$F(\overline{\lambda}) = F(\lambda_1,\ \lambda_2, \cdots, \lambda_m) = \prod_{i=1}^{m} \frac{1 - p_i^{-\lambda_i - 1}}{1 - p_i^{-1}},$$

$$H(\overline{\lambda}) = H(\lambda_1,\ \lambda_2, \cdots, \lambda_m) = \frac{\exp\left(\exp\left(e^{-\gamma} \cdot F(\overline{\lambda})\right)\right)}{p_1^{\lambda_1} \cdot p_2^{\lambda_2} \cdots p_m^{\lambda_m}},$$

where $\gamma = 0.577\cdots$ is Euler's constant ([3,4]).

By the theorem 1 of [3], the function $H(\overline{\lambda})$ has the optimum points $\overline{\lambda_0} = (\lambda_1^0,\ \lambda_2^0, \cdots, \lambda_m^0) \in R^m$ in m-dimensional real space R^m. And by the theorem 2 and the theorem 3 in [3], the function value $H(\lambda_1^0,\ \lambda_2^0, \cdots, \lambda_m^0)$ is dependent only on p_m. So we can put

$$C_m = H(\lambda_1^0,\ \lambda_2^0, \cdots, \lambda_m^0) = \frac{\exp\left(\exp\left(e^{-\gamma} \cdot F(\overline{\lambda_0})\right)\right)}{p_1^{\lambda_1^0} \cdot p_2^{\lambda_2^0} \cdots p_m^{\lambda_m^0}}. \tag{1}$$

In this connection, we will put

$$\begin{cases} n_0 = p_1^{\lambda_1^0} p_2^{\lambda_2^0} \cdots p_k^{\lambda_k^0} \cdot p_{k+1}^1 \cdots p_m^1, \quad n_0' = n_0 \cdot p_m^{-1}, \\ \overline{\lambda}_0' = (\lambda_1^0,\ \lambda_2^0, \cdots, \lambda_{m-1}^0) \in R^{m-1}, \\ C_{m-1}' = H(\overline{\lambda}_0') = H(\lambda_1^0,\ \lambda_2^0, \cdots, \lambda_{m-1}^0) \end{cases} \tag{2}$$

and

$$C_{m-1} = \max_{(\lambda_1,\ \lambda_2, \cdots, \lambda_{m-1}) \in R^m} H(\lambda_1,\ \lambda_2, \cdots, \lambda_{m-1}). \tag{3}$$

Then it is clear that $C_{m-1}' \le C_{m-1}$.

Let $\overline{\lambda}' = \left(\lambda_1', \lambda_2', \cdots, \lambda_{m-1}'\right)$ be the optimum points of the function $H\left(\lambda_1, \lambda_2, \cdots, \lambda_{m-1}\right)$ with $(m-1)$-variable in the space R^{m-1}. In general, then we have

$$\lambda_1' > \lambda_2' > \cdots > \lambda_{k-1}' > \lambda_k' = \cdots = \lambda_{m-1}' = 1. \tag{4}$$

Rarely, the last bigger number than 1 in $\left\{\lambda_1', \lambda_2', \cdots, \lambda_{m-1}'\right\}$ could be k. But it is not essential. It is important that for any $i \left(1 \le i \le k-1\right)$

$$p_1^{\lambda_1'+1} = p_2^{\lambda_2'++1} = \cdots = p_{k-1}^{\lambda_{k-1}'+1} = \left(e^{-\gamma} \cdot F\left(\overline{\lambda}'\right)\right) \cdot \exp\left(e^{-\gamma} \cdot F\left(\overline{\lambda}'\right)\right) + 1 \tag{5}$$

holds. We note that it doesn't exceed one in $\left\{\lambda_1^0, \lambda_2^0, \cdots, \lambda_m^0\right\}$.

We also put

$$\begin{cases} n' = p_1^{\lambda_1'} p_2^{\lambda_2'} \cdots p_{k-1}^{\lambda_k'} \cdot p_k^1 \cdots p_{m-1}^1, \\ n_+' = p_1^{\lambda_1'} p_2^{\lambda_2'} \cdots p_{k-1}^{\lambda_k'} \cdot p_k^1 \cdots p_{m-1}^1 \cdot p_m^1 = n' \cdot p_m^1, \\ \overline{\lambda}_+' = \left(\lambda_1', \lambda_2', \cdots, \lambda_{m-1}', 1\right), \ C_m' = H\left(\overline{\lambda}_+'\right). \end{cases} \tag{6}$$

On the other hand, it is well known that

$$\sum_{p \le p_m} \frac{1}{p} = \log\log p_m + b_0 + E_0\left(p_m\right), \tag{7}$$

where

$$b_0 = \gamma + \sum_p \left(\log\left(1 - \frac{1}{p}\right) + \frac{1}{p}\right) = 0.241\cdots \tag{8}$$

([1,2]). And there exists a constant $a > 0$ such that

$$E_0\left(p_m\right) = \mathrm{O}\left(\exp\left(-a\sqrt{\log p_m}\right)\right). \tag{9}$$

In this paper we would estimate some important quantities by the results of [3].

2. The estimate of $F\left(\overline{\lambda}_0\right)$

In this section we will estimate the value $F\left(\overline{\lambda}_0\right)$ for the optimum points $\overline{\lambda}_0 = \left(\lambda_1^0,\ \lambda_2^0, \cdots \lambda_m^0\right) \in R^m$ of the function $H\left(\lambda_1,\ \lambda_2, \cdots, \lambda_m\right)$.

We have

Theorem 1. For the optimum points $\overline{\lambda}_0 = \left(\lambda_1^0,\ \lambda_2^0, \cdots \lambda_m^0\right) \in R^m$ of the function $H\left(\lambda_1,\ \lambda_2, \cdots, \lambda_m\right)$ we have

$$F\left(\overline{\lambda}_0\right) = e^{\gamma} \cdot \log p_m \cdot \left(1 + E_0\left(p_m\right) - \frac{4}{\sqrt{p_m} \cdot \log^{3/2} p_m} + \varepsilon\left(p_m\right)\right), \qquad (10)$$

where $\varepsilon\left(p_m\right) = \mathrm{O}\left(E_0^2\left(p_m\right)\right)$. Hence we also have

$$\left(e^{-\gamma} \cdot F\left(\overline{\lambda}_0\right)\right) \cdot \exp\left(e^{-\gamma} \cdot F\left(\overline{\lambda}_0\right)\right) =$$
$$= p_m \cdot \log p_m \cdot \left(1 + \left(\log p_m + 1\right) \cdot E_0\left(p_m\right) - \frac{4 \cdot \left(\log p_m + 1\right)}{\sqrt{p_m} \cdot \log^{3/2} p_m} + \tilde{\varepsilon}\left(p_m\right)\right), \qquad (11)$$

where $\tilde{\varepsilon}\left(p_m\right) = \mathrm{O}\left(\log^2 p_m \cdot E_0^2\left(p_m\right)\right)$.

Proof. From (5), it is clear that

$$\log F\left(\overline{\lambda}_0\right) = \log\left(\prod_{i=1}^{m} \frac{1 - p_i^{-\lambda_i^0 - 1}}{1 - p_i^{-1}}\right) =$$
$$= \sum_{i=1}^{k} \log\left(1 - \frac{1}{p_i^{\lambda_i^0 + 1}}\right) + \sum_{i=k+1}^{m} \log\left(1 - \frac{1}{p_i^2}\right) + \sum_{i=1}^{m} \log\left(1 - \frac{1}{p_i}\right)^{-1} = \qquad (12)$$
$$= A_1 + A_2 + A_3,$$

where

$$A_1 = \sum_{i=1}^{k} \log\left(1 - \frac{1}{p_i^{\lambda_i^0 + 1}}\right),\ A_2 = \sum_{i=k+1}^{m} \log\left(1 - \frac{1}{p_i^2}\right),\ A_3 = \sum_{i=1}^{m} \log\left(1 - \frac{1}{p_i}\right)^{-1}. \quad (13)$$

First let's see A_1. By Mertens' theorem ([1,2]), preliminarily, we have

$$F\left(\overline{\lambda}_0\right)=\prod_{i=1}^{m}\frac{1-p_i^{-\lambda_i^0-1}}{1-p_i^{-1}}=e^{\gamma}\cdot\log p_m\cdot\left(1+\mathrm{O}\left(\frac{1}{\log^2 p_m}\right)\right). \qquad (14)$$

So we have

$$\left(e^{-\gamma}\cdot F\left(\overline{\lambda}_0\right)\right)\cdot\exp\left(e^{-\gamma}\cdot F\left(\overline{\lambda}_0\right)\right)+1=p_m\cdot\log p_m\cdot\left(1+\mathrm{O}\left(\frac{1}{\log p_m}\right)\right). \qquad (15)$$

Hence From [3], for any $i\ (1\le i\le k)$ we have

$$A_1=\sum_{i=1}^{k}\log\left(1-\frac{1}{p_i^{\lambda_i^0+1}}\right)=k\cdot\log\left(1-\frac{1}{p_i^{\lambda_i^0+1}}\right)=$$
$$=-\frac{k}{p_i^{\lambda_i^0+1}}+\mathrm{O}\left(\frac{k}{p_i^{2\cdot\left(\lambda_i^0+1\right)}}\right)=-\frac{2}{\sqrt{p_m}\cdot\left(\log p_m\right)^{3/2}}\cdot\left(1+\mathrm{O}\left(\frac{\log\log p_m}{\log p_m}\right)\right). \qquad (16)$$

Next let's see A_2. Now we put

$$T\left(x\right)=\sum_{p\le x}\frac{1}{p}=\log\log x+b_0+E_0(x). \qquad (17)$$

Then we have $dT\left(x\right)=\dfrac{dx}{x\cdot\log x}+dE_0(x)$. So we have

$$\sum_{i=k+1}^{m}\frac{1}{p_i^2}=\int_{p_k}^{p_m}\frac{dT\left(t\right)}{t}=\int_{p_k}^{p_m}\frac{1}{t}\cdot\left(\frac{dt}{t\cdot\log t}+dE_0\left(t\right)\right)=$$
$$=\frac{1}{p_k\cdot\log p_k}-\frac{1}{p_m\cdot\log p_m}+\int_{p_k}^{p_m}\frac{dE_0\left(t\right)}{t}= \qquad (18)$$
$$=\frac{2}{\sqrt{p_m}\cdot\left(\log p_m\right)^{3/2}}\cdot\left(1+\mathrm{O}\left(\frac{\log\log p_m}{\log p_m}\right)\right)$$

and

$$A_2=\sum_{i=k+1}^{m}\log\left(1-\frac{1}{p_i^2}\right)=-\sum_{i=k+1}^{m}\frac{1}{p_i^2}+\mathrm{O}\left(\sum_{i=k+1}^{m}\frac{1}{p_i^4}\right)=$$
$$=\frac{-2}{\sqrt{p_m}\cdot\left(\log p_m\right)^{3/2}}\cdot\left(1+\mathrm{O}\left(\frac{\log\log p_m}{\log p_m}\right)\right). \qquad (19)$$

Next let's see A_3. By (8) we have

$$\sum_{p\le p_m}\left(\log\left(1-\frac{1}{p}\right)+\frac{1}{p}\right)=b_0-\gamma-\sum_{p>p_m}\left(\log\left(1-\frac{1}{p}\right)+\frac{1}{p}\right). \qquad (20)$$

And it is clear that

$$\begin{aligned}&\left|\log\left(1-\frac{1}{p}\right)+\frac{1}{p}\right|=\left|-\sum_{j=1}^{\infty}\frac{1}{j\cdot p^j}+\frac{1}{p}\right|=\\&=\left|\frac{1}{2p^2}+\frac{1}{3p^3}+\cdots+\frac{1}{j\cdot p^j}+\cdots\right|\le\\&\le\left|\frac{1}{p^2}+\frac{1}{p^3}+\cdots+\frac{1}{p^j}+\cdots\right|=\frac{1}{p\cdot(p-1)}.\end{aligned} \qquad (21)$$

So we have

$$\begin{aligned}&-\sum_{p>p_m}\left(\log\left(1-\frac{1}{p}\right)+\frac{1}{p}\right)\le\sum_{p>p_m}\frac{1}{p\cdot(p-1)}\le\\&\le\sum_{n>p_m}\frac{1}{n\cdot(n-1)}=\sum_{n>p_m}\left(\frac{1}{n-1}-\frac{1}{n}\right)=\frac{1}{n}=\mathrm{O}\left(\frac{1}{p_m}\right).\end{aligned} \qquad (22)$$

By (7) we have

$$\begin{aligned}A_3&=\sum_{i=1}^{m}\frac{1}{p_i}-\sum_{i=1}^{m}\left(\log\left(1-\frac{1}{p_i}\right)+\frac{1}{p_i}\right)=\\&=\log\log p_m+b_0+E_0(p_m)-\left(b_0-\gamma-\sum_{p>p_m}\left(\log\left(1-\frac{1}{p}\right)+\frac{1}{p}\right)\right)=\\&=\log\log p_m+\gamma+E_0(p_m)+\mathrm{O}\left(\frac{1}{p_m}\right).\end{aligned} \qquad (23)$$

From (16), (19) and (23) we have

$$\begin{aligned}&\log F\left(\bar{\lambda}_0\right)=\log\log p_m+\gamma+E_0(p_m)+\\&\frac{-4}{\sqrt{p_m}\cdot\log^{3/2}p_m}\left(1+\mathrm{O}\left(\frac{\log\log p_m}{\log p_m}\right)\right).\end{aligned} \qquad (24)$$

and hence we have

$$\left(e^{-\gamma}\cdot F\left(\bar{\lambda}_0\right)\right)=\log p_m\cdot\left(1+E_0\left(p_m\right)-\frac{4}{\sqrt{p_m}\log^{3/2}p_m}+\varepsilon\left(p_m\right)\right),$$

where $\varepsilon\left(p_m\right)=\mathrm{O}\left(E_0^2\left(p_m\right)\right)$. Therefore we have

$$\left(e^{-\gamma}\cdot F\left(\bar{\lambda}_0\right)\right)\cdot\exp\left(e^{-\gamma}\cdot F\left(\bar{\lambda}_0\right)\right)=$$
$$=p_m\cdot\log p_m\cdot\left(1+\left(\log p_m+1\right)\cdot E_0\left(p_m\right)-\frac{4\cdot\left(\log p_m+1\right)}{\sqrt{p_m}\log^{3/2}p_m}+\tilde{\varepsilon}\left(p_m\right)\right)\right),$$

where $\tilde{\varepsilon}\left(p_m\right)=\mathrm{O}\left(\log^2 p_m\cdot E_0^2\left(p_m\right)\right)$.

This completes the proof of the theorem 1. □

3. The estimate of $\left(\log C_{m-1}-\log C'_{m-1}\right)$

The aim of this section is to estimate the size of $\left(\log C_{m-1}-\log C'_{m-1}\right)$. This result is used effectively in next section.

We get

Theorem 2. There exists a number m_0 such that for any $m\geq m_0$ we have

$$\log C_{m-1}-\log C'_{m-1}=\frac{p_m-p_{m-1}}{\sqrt{p_{m-1}\cdot\log p_{m-1}}}\cdot\left(\frac{p_m-p_{m-1}}{p_{m-1}}\right)\cdot\left(1+\beta_0\left(p_m\right)\right),\qquad(25)$$

where $\beta_0\left(p_m\right)=\mathrm{O}\left(\dfrac{1}{\log p_m}\right)$. (26)

Proof. From (2) and (3), we have

$$\begin{aligned}&\log C_{m-1}-\log C'_{m-1}=\\&=\left(\exp\left(e^{-\gamma}\cdot F\left(\bar{\lambda}'\right)\right)-\log n'\right)-\left(\exp\left(e^{-\gamma}\cdot F\left(\bar{\lambda}'_0\right)\right)-\log n'_0\right)=\\&=\left(\exp\left(e^{-\gamma}\cdot F\left(\bar{\lambda}'\right)\right)-\exp\left(e^{-\gamma}\cdot F\left(\bar{\lambda}'_0\right)\right)\right)-\left(\log n'-\log n'_0\right)=\\&=R_1-R_2,\end{aligned}\qquad(27)$$

where

$$R_1 = \left(\exp\left(e^{-\gamma}\cdot F\left(\overline{\lambda}'\right)\right) - \exp\left(e^{-\gamma}\cdot F\left(\overline{\lambda}'_0\right)\right)\right), \quad R_2 = \left(\log n' - \log n'_0\right). \qquad (28)$$

Let's see R_1. We can write as

$$\begin{aligned}\left(-R_1\right) &= \left(\exp\left(e^{-\gamma}\cdot F\left(\overline{\lambda}'_0\right)\right) - \exp\left(e^{-\gamma}\cdot F\left(\overline{\lambda}'\right)\right)\right) = \\ &= \exp\left(e^{-\gamma}\cdot F\left(\overline{\lambda}'\right)\right)\cdot\left(\frac{\exp\left(e^{-\gamma}\cdot F\left(\overline{\lambda}'_0\right)\right)}{\exp\left(e^{-\gamma}\cdot F\left(\overline{\lambda}'\right)\right)} - 1\right)\end{aligned} \qquad (29)$$

and here we have

$$\begin{aligned}\log\left(\frac{\exp\left(e^{-\gamma}\cdot F\left(\overline{\lambda}'_0\right)\right)}{\exp\left(e^{-\gamma}\cdot F\left(\overline{\lambda}'\right)\right)}\right) &= \left(e^{-\gamma}\cdot F\left(\overline{\lambda}'_0\right)\right) - \left(e^{-\gamma}\cdot F\left(\overline{\lambda}'\right)\right) = \\ &= \left(e^{-\gamma}\cdot F\left(\overline{\lambda}'\right)\right)\cdot\left(\frac{F\left(\overline{\lambda}'_0\right)}{F\left(\overline{\lambda}'\right)} - 1\right).\end{aligned} \qquad (30)$$

By the Taylor's formula of the function $\log(1+x)$ $(0 < x < 1)$, for any $i\left(1 \le i \le k-1\right)$ we have

$$\begin{aligned}&\log\left(\frac{F\left(\overline{\lambda}'_0\right)}{F\left(\overline{\lambda}'\right)}\right) = \log F\left(\overline{\lambda}'_0\right) - \log F\left(\overline{\lambda}'\right) = \\ &= \log\left(\prod_{i=1}^{m-1}\frac{1-p_i^{-\lambda_i^0-1}}{1-p_i^{-1}}\right) - \log\left(\prod_{i=1}^{m-1}\frac{1-p_i^{-\lambda'_i-1}}{1-p_i^{-1}}\right) = \\ &= (k)\cdot\log\left(\frac{1-p_i^{-\lambda_i^0-1}}{1-p_i^{-1}}\right) - (k-1)\cdot\log\left(\frac{1-p_i^{-\lambda'_i-1}}{1-p_i^{-1}}\right) - \log\left(1+\frac{1}{p_k}\right) = \\ &= (k)\cdot\log\left(\frac{1-p_i^{-\lambda_i^0-1}}{1-p_i^{-\lambda'_i-1}}\right) + \log\left(\frac{1-p_i^{-\lambda'_i-1}}{1-p_k^{-2}}\right) = \\ &= (k)\cdot\log\left(1+\frac{p_i^{\lambda_i^0+1}-p_i^{\lambda'_i+1}}{p_i^{\lambda_i^0+1}\cdot\left(p_i^{\lambda'_i+1}-1\right)}\right) + \log\left(1+\frac{p_i^{\lambda'_i+1}-p_k^2}{p_i^{\lambda'_i+1}\cdot\left(p_k^2-1\right)}\right) = \\ &= (k)\cdot\left(\frac{p_i^{\lambda_i^0+1}-p_i^{\lambda'_i+1}}{p_i^{\lambda_i^0+1}\cdot\left(p_i^{\lambda'_i+1}-1\right)}\right) - \frac{(k)}{2}\cdot\left(\frac{p_i^{\lambda_i^0+1}-p_i^{\lambda'_i+1}}{p_i^{\lambda_i^0+1}\cdot\left(p_i^{\lambda'_i+1}-1\right)}\right)^2 +\end{aligned} \qquad (31)$$

$$+\left(\frac{p_i^{\lambda_i'+1}-p_k^2}{p_i^{\lambda_i'+1}\cdot\left(p_k^2-1\right)}\right)-\frac{1}{2}\cdot\left(\frac{p_i^{\lambda_i'+1}-p_k^2}{p_i^{\lambda_i'+1}\cdot\left(p_k^2-1\right)}\right)^2+\beta_1\left(p_m\right),$$

where $\beta_1\left(p_m\right)=\mathrm{O}\left(k\cdot\left(\frac{p_i^{\lambda_i^0+1}-p_i^{\lambda_i'+1}}{p_i^{\lambda_i^0+1}\cdot\left(p_i^{\lambda_i'+1}-1\right)}\right)^3\right).$ (32)

Hence we have

$$\left(\frac{F\left(\overline{\lambda}_0'\right)}{F\left(\overline{\lambda}'\right)}\right)=1+(k)\cdot\left(\frac{p_i^{\lambda_i^0+1}-p_i^{\lambda_i'+1}}{p_i^{\lambda_i^0+1}\cdot\left(p_i^{\lambda_i'+1}-1\right)}\right)+\frac{k^2}{2}\left(\frac{p_i^{\lambda_i^0+1}-p_i^{\lambda_i'+1}}{p_i^{\lambda_i^0+1}\cdot\left(p_i^{\lambda_i'+1}-1\right)}\right)^2-$$

$$-\frac{(k)}{2}\cdot\left(\frac{p_i^{\lambda_i^0+1}-p_i^{\lambda_i'+1}}{p_i^{\lambda_i^0+1}\cdot\left(p_i^{\lambda_i'+1}-1\right)}\right)^2+\left(\frac{p_i^{\lambda_i'+1}-p_k^2}{p_i^{\lambda_i'+1}\cdot\left(p_k^2-1\right)}\right)+\frac{1}{2}\cdot\left(\frac{p_i^{\lambda_i'+1}-p_k^2}{p_i^{\lambda_i'+1}\cdot\left(p_k^2-1\right)}\right)^2-$$

$$-\frac{1}{2}\cdot\left(\frac{p_i^{\lambda_i'+1}-p_k^2}{p_i^{\lambda_i'+1}\cdot\left(p_k^2-1\right)}\right)^2+\beta_1'\left(p_m\right)=1+(k)\cdot\left(\frac{p_i^{\lambda_i^0+1}-p_i^{\lambda_i'+1}}{p_i^{\lambda_i^0+1}\cdot\left(p_i^{\lambda_i'+1}-1\right)}\right)+ \quad (33)$$

$$+\frac{(k)\cdot(k-1)}{2}\cdot\left(\frac{p_i^{\lambda_i^0+1}-p_i^{\lambda_i'+1}}{p_i^{\lambda_i^0+1}\cdot\left(p_i^{\lambda_i'+1}-1\right)}\right)^2+\left(\frac{p_i^{\lambda_i'+1}-p_k^2}{p_i^{\lambda_i'+1}\cdot\left(p_k^2-1\right)}\right)+\beta_1'\left(p_m\right),$$

where $\beta_1'\left(p_m\right)=\mathrm{O}\left(k^3\cdot\left(\frac{p_i^{\lambda_i^0+1}-p_i^{\lambda_i'+1}}{p_i^{\lambda_i^0+1}\cdot\left(p_i^{\lambda_i'+1}-1\right)}\right)^3\right).$ (34)

From (33), the expression (30) is

$$\log\left(\frac{\exp\left(e^{-\gamma}\cdot F\left(\overline{\lambda}_0'\right)\right)}{\exp\left(e^{-\gamma}\cdot F\left(\overline{\lambda}'\right)\right)}\right)=\left(e^{-\gamma}\cdot F\left(\overline{\lambda}'\right)\right)\cdot\left(\frac{F\left(\overline{\lambda}_0'\right)}{F\left(\overline{\lambda}'\right)}-1\right)=$$

$$=\left(e^{-\gamma}\cdot F\left(\overline{\lambda}'\right)\right)\cdot\left((k)\cdot\left(\frac{p_i^{\lambda_i^0+1}-p_i^{\lambda_i'+1}}{p_i^{\lambda_i^0+1}\cdot\left(p_i^{\lambda_i'+1}-1\right)}\right)+\right.$$

$$\left.+\frac{(k)\cdot(k-1)}{2}\cdot\left(\frac{p_i^{\lambda_i^0+1}-p_i^{\lambda_i'+1}}{p_i^{\lambda_i^0+1}\cdot\left(p_i^{\lambda_i'+1}-1\right)}\right)^2+\left(\frac{p_i^{\lambda_i'+1}-p_k^2}{p_i^{\lambda_i'+1}\cdot\left(p_k^2-1\right)}\right)+\beta_1'\left(p_m\right)\right)$$

(35)

and so we have

$$\left(\frac{\exp\left(e^{-\gamma}\cdot F\left(\overline{\lambda}'_0\right)\right)}{\exp\left(e^{-\gamma}\cdot F\left(\overline{\lambda}'\right)\right)}\right)=1+(k)\cdot\left(e^{-\gamma}\cdot F\left(\overline{\lambda}'\right)\right)\cdot\left(\frac{p_i^{\lambda_i^0+1}-p_i^{\lambda_i'+1}}{p_i^{\lambda_i^0+1}\cdot\left(p_i^{\lambda_i'+1}-1\right)}\right)+$$
$$+\frac{k^2}{2}\cdot\left(e^{-\gamma}\cdot F\left(\overline{\lambda}'\right)\right)^2\cdot\left(\frac{p_i^{\lambda_i^0+1}-p_i^{\lambda_i'+1}}{p_i^{\lambda_i^0+1}\cdot\left(p_i^{\lambda_i'+1}-1\right)}\right)^2+$$
$$+\left(e^{-\gamma}\cdot F\left(\overline{\lambda}'\right)\right)\cdot\left(\frac{p_i^{\lambda_i'+1}-p_k^2}{p_i^{\lambda_i'+1}\cdot\left(p_k^2-1\right)}\right)+\frac{\left(e^{-\gamma}\cdot F\left(\overline{\lambda}'\right)\right)^2}{2}\cdot\left(\frac{p_i^{\lambda_i'+1}-p_k^2}{p_i^{\lambda_i'+1}\cdot\left(p_k^2-1\right)}\right)^2+ \tag{36}$$
$$+\frac{(k)\cdot(k-1)}{2}\cdot\left(e^{-\gamma}\cdot F\left(\overline{\lambda}'\right)\right)\cdot\left(\frac{p_i^{\lambda_i^0+1}-p_i^{\lambda_i'+1}}{p_i^{\lambda_i^0+1}\cdot\left(p_i^{\lambda_i'+1}-1\right)}\right)^2+\beta_1''(p_m),$$

where $\beta_1''(p_m)=\mathrm{O}\left(k^3\cdot\left(e^{-\gamma}\cdot F\left(\overline{\lambda}'\right)\right)^3\cdot\left(\frac{p_i^{\lambda_i^0+1}-p_i^{\lambda_i'+1}}{p_i^{\lambda_i^0+1}\cdot\left(p_i^{\lambda_i'+1}-1\right)}\right)^3\right).$ (37)

Hence we have

$$(-R_1)=\exp\left(e^{-\gamma}\cdot F\left(\overline{\lambda}'\right)\right)\cdot\left(\frac{\exp\left(e^{-\gamma}\cdot F\left(\overline{\lambda}'_0\right)\right)}{\exp\left(e^{-\gamma}\cdot F\left(\overline{\lambda}'\right)\right)}-1\right)=$$
$$=\exp\left(e^{-\gamma}\cdot F\left(\overline{\lambda}'\right)\right)\cdot\left((k)\cdot\left(e^{-\gamma}\cdot F\left(\overline{\lambda}'\right)\right)\cdot\left(\frac{p_i^{\lambda_i^0+1}-p_i^{\lambda_i'+1}}{p_i^{\lambda_i^0+1}\cdot\left(p_i^{\lambda_i'+1}-1\right)}\right)+\right.$$
$$+\frac{k^2}{2}\cdot\left(e^{-\gamma}\cdot F\left(\overline{\lambda}'\right)\right)^2\cdot\left(\frac{p_i^{\lambda_i^0+1}-p_i^{\lambda_i'+1}}{p_i^{\lambda_i^0+1}\cdot\left(p_i^{\lambda_i'+1}-1\right)}\right)^2+\left(e^{-\gamma}\cdot F\left(\overline{\lambda}'\right)\right)\cdot\left(\frac{p_i^{\lambda_i'+1}-p_k^2}{p_i^{\lambda_i'+1}\cdot\left(p_k^2-1\right)}\right)+$$
$$+\frac{1}{2}\cdot\left(e^{-\gamma}\cdot F\left(\overline{\lambda}'\right)\right)^2\cdot\left(\frac{p_i^{\lambda_i'+1}-p_k^2}{p_i^{\lambda_i'+1}\cdot\left(p_k^2-1\right)}\right)^2+$$
$$\left.+\frac{(k)\cdot(k-1)}{2}\cdot\left(e^{-\gamma}\cdot F\left(\overline{\lambda}'\right)\right)\cdot\left(\frac{p_i^{\lambda_i^0+1}-p_i^{\lambda_i'+1}}{p_i^{\lambda_i^0+1}\cdot\left(p_i^{\lambda_i'+1}-1\right)}\right)^2+\beta_1''(p_m)\right).$$

(38)

From (5), since $\left(e^{-\gamma}F\left(\overline{\lambda}'\right)\right)\cdot\exp\left(e^{-\gamma}F\left(\overline{\lambda}'\right)\right)=p_i^{\lambda_i'+1}-1\quad\left(1\le i\le k-1\right)$,

we have

$$\begin{aligned}\left(-R_1\right)=&\left(k\right)\cdot\left(\frac{p_i^{\lambda_i^0+1}-p_i^{\lambda_i'+1}}{p_i^{\lambda_i^0+1}}\right)+\left(\frac{p_i^{\lambda_i'+1}-1}{p_i^{\lambda_i'+1}}\right)\cdot\left(\frac{p_i^{\lambda_i'+1}-p_k^2}{p_k^2-1}\right)+\\&+\frac{k^2}{2}\cdot\frac{\left(e^{-\gamma}\cdot F\left(\overline{\lambda}'\right)\right)}{\left(p_i^{\lambda_i'+1}-1\right)}\cdot\left(\frac{p_i^{\lambda_i^0+1}-p_i^{\lambda_i'+1}}{p_i^{\lambda_i^0+1}}\right)^2+\\&+\frac{1}{2}\cdot\frac{\left(p_i^{\lambda_i'+1}-1\right)}{p_i^{\lambda_i'+1}}\cdot\frac{\left(e^{-\gamma}\cdot F\left(\overline{\lambda}'\right)\right)}{p_i^{\lambda_i'+1}}\cdot\left(\frac{p_i^{\lambda_i'+1}-p_k^2}{p_k^2-1}\right)^2+\\&+\frac{1}{2}\cdot\frac{\left(k\right)\cdot\left(k-1\right)}{\left(p_i^{\lambda_i'+1}-1\right)}\cdot\left(\frac{p_i^{\lambda_i^0+1}-p_i^{\lambda_i'+1}}{p_i^{\lambda_i^0+1}}\right)^2+\beta_1'''\left(p_m\right),\end{aligned}\tag{39}$$

where

$$\begin{aligned}\beta_1'''\left(p_m\right)&=\exp\left(e^{-\gamma}\cdot F\left(\overline{\lambda}'\right)\right)\cdot\beta_1''\left(p_m\right)=\\&=\mathrm{O}\left(k^3\cdot\left(\frac{e^{-\gamma}\cdot F\left(\overline{\lambda}'\right)}{p_i^{\lambda_i'+1}-1}\right)^2\cdot\left(\frac{p_i^{\lambda_i^0+1}-p_i^{\lambda_i'+1}}{p_i^{\lambda_i^0+1}}\right)^3\right).\end{aligned}\tag{40}$$

Next let's see R_2. It is not difficult to see that for any $i\left(1\le i\le k-1\right)$

$$\begin{aligned}\left(-R_2\right)=&\left(\log n_0'-\log n'\right)=\log\left(\frac{p_1^{\lambda_1^0}\cdot p_2^{\lambda_2^0}\cdots p_k^{\lambda_k^0}\,p_{k+1}^1\cdots p_{m-1}^1}{p_1^{\lambda_1'}\,p_2^{\lambda_2'}\,\cdots p_{k-1}^{\lambda_k'}\cdot p_k^1\cdots p_{m-1}^1}\right)=\\&=\left(k\right)\cdot\log\left(\frac{p_i^{\lambda_i^0+1}}{p_i^{\lambda_i'+1}}\right)-\log\left(\frac{p_i^{\lambda_i^0+1}}{p_i^{\lambda_i'+1}}\right)+\log\left(\frac{p_k^{\lambda_k^0+1}}{p_k^2}\right)=\\&=\left(k\right)\cdot\log\left(1+\frac{p_i^{\lambda_i^0+1}-p_i^{\lambda_i'+1}}{p_i^{\lambda_i'+1}}\right)+\log\left(1+\frac{p_i^{\lambda_i'+1}-p_k^2}{p_k^2}\right)=\\&=\left(k\right)\cdot\left(\frac{p_i^{\lambda_i^0+1}-p_i^{\lambda_i'+1}}{p_i^{\lambda_i'+1}}\right)-\left(\frac{k}{2}\right)\cdot\left(\frac{p_i^{\lambda_i^0+1}-p_i^{\lambda_i'+1}}{p_i^{\lambda_i'+1}}\right)^2+\\&+\left(\frac{p_i^{\lambda_i'+1}-p_k^2}{p_k^2}\right)-\frac{1}{2}\cdot\left(\frac{p_i^{\lambda_i'+1}-p_k^2}{p_k^2}\right)^2+\beta_2\left(p_m\right),\end{aligned}\tag{41}$$

where $\beta_2\left(p_m\right) = \mathrm{O}\left(k \cdot \left(\dfrac{p_i^{\lambda_i^0+1} - p_i^{\lambda_i' \ +1}}{p_i^{\lambda_i' \ +1}} \right)^3 \right).$ (42)

Therefore we have

$$
\begin{aligned}
&(-)\cdot\left(\log C_{m-1} - \log C'_{m-1}\right) = -\left(R_1 - R_2\right) = \\
&= (k)\cdot\left(\frac{p_i^{\lambda_i^0+1} - p_i^{\lambda_i'+1}}{p_i^{\lambda_i^0+1}}\right) + \left(\frac{p_i^{\lambda_i'+1} - 1}{p_i^{\lambda_i'+1}}\right)\cdot\left(\frac{p_i^{\lambda_i'+1} - p_k^2}{p_k^2 - 1}\right) + \\
&+ \left(\frac{k^2}{2}\right)\cdot\frac{\left(e^{-\gamma}\cdot F\left(\overline{\lambda}'\right)\right)}{\left(p_i^{\lambda_i'+1} - 1\right)}\cdot\left(\frac{p_i^{\lambda_i^0+1} - p_i^{\lambda_i'+1}}{p_i^{\lambda_i^0+1}}\right)^2 + \\
&+ \frac{1}{2}\cdot\frac{\left(p_i^{\lambda_i'+1} - 1\right)}{p_i^{\lambda_i'+1}}\cdot\frac{\left(e^{-\gamma}\cdot F\left(\overline{\lambda}'\right)\right)}{p_i^{\lambda_i'+1}}\cdot\left(\frac{p_i^{\lambda_i'+1} - p_k^2}{p_k^2 - 1}\right)^2 + \\
&+ \frac{1}{2}\cdot\frac{(k)\cdot(k-1)}{\left(p_i^{\lambda_i'+1} - 1\right)}\cdot\left(\frac{p_i^{\lambda_i^0+1} - p_i^{\lambda_i'+1}}{p_i^{\lambda_i^0+1}}\right)^2 + \beta_1'''\left(p_m\right) - \\
&- (k)\cdot\left(\frac{p_i^{\lambda_i^0+1} - p_{k-1}^{\lambda_i' \ +1}}{p_i^{\lambda_i' \ +1}}\right) + \left(\frac{k}{2}\right)\cdot\left(\frac{p_i^{\lambda_i^0+1} - p_i^{\lambda_i' \ +1}}{p_i^{\lambda_i' \ +1}}\right)^2 - \\
&- \left(\frac{p_i^{\lambda_i' \ +1} - p_k^2}{p_k^2}\right) + \frac{1}{2}\cdot\left(\frac{p_i^{\lambda_i' \ +1} - p_k^2}{p_k^2}\right)^2 - \beta_2\left(p_m\right).
\end{aligned}
\tag{43}
$$

Here the term without k in its coefficients is

$$
\begin{aligned}
&\frac{1}{2}\cdot\frac{\left(p_i^{\lambda_i'+1} - 1\right)}{p_i^{\lambda_i'+1}}\cdot\frac{\left(e^{-\gamma}\cdot F\left(\overline{\lambda}'\right)\right)}{p_i^{\lambda_i'+1}}\cdot\left(\frac{p_i^{\lambda_i'+1} - p_k^2}{p_k^2 - 1}\right)^2 + \\
&+ \frac{\left(p_i^{\lambda_i'+1} - 1\right)}{p_i^{\lambda_i'+1}}\cdot\frac{p_i^{\lambda_i'+1} - p_k^2}{\left(p_k^2 - 1\right)} - \left(\frac{p_i^{\lambda_i' \ +1} - p_k^2}{p_k^2}\right) + \frac{1}{2}\cdot\left(\frac{p_i^{\lambda_i' \ +1} - p_k^2}{p_k^2}\right)^2 = \\
&= \frac{1}{2}\cdot\left(\frac{p_i^{\lambda_i' \ +1} - p_k^2}{p_k^2}\right)^2\cdot\left(\frac{\left(p_i^{\lambda_i'+1} - 1\right)}{p_i^{\lambda_i'+1}}\cdot\frac{\left(e^{-\gamma}\cdot F\left(\overline{\lambda}'\right)\right)}{p_i^{\lambda_i'+1}}\cdot\left(\frac{p_k^2}{p_k^2 - 1}\right)^2 + 1\right) + \\
&+ \left(\frac{p_i^{\lambda_i' \ +1} - p_k^2}{p_k^2}\right)\cdot\left(\frac{\left(p_i^{\lambda_i'+1} - 1\right)}{p_i^{\lambda_i'+1}}\cdot\frac{p_k^2}{\left(p_k^2 - 1\right)} - 1\right) =
\end{aligned}
\tag{44}
$$

$$
=\frac{1}{2}\cdot\left(\frac{p_i^{\lambda_i'+1}-p_k^2}{p_k^2}\right)^2\cdot\left(1+\frac{\left(p_i^{\lambda_i'+1}-1\right)}{p_i^{\lambda_i'+1}}\cdot\frac{\left(e^{-\gamma}\cdot F\left(\overline{\lambda}'\right)\right)}{p_i^{\lambda_i'+1}}\cdot\left(\frac{p_k^2}{p_k^2-1}\right)^2\right)+
$$

$$
+\left(\frac{p_i^{\lambda_i'+1}-p_k^2}{p_k^2}\right)\cdot\left(\frac{\left(p_i^{\lambda_i'+1}-1\right)\cdot p_k^2-p_i^{\lambda_i'+1}\cdot\left(p_k^2-1\right)}{p_i^{\lambda_i'+1}\cdot\left(p_k^2-1\right)}\right)=
$$

$$
=\frac{1}{2}\cdot\left(\frac{p_i^{\lambda_i'+1}-p_k^2}{p_k^2}\right)^2\cdot\left(1+\frac{\left(p_i^{\lambda_i'+1}-1\right)}{p_i^{\lambda_i'+1}}\cdot\frac{\left(e^{-\gamma}\cdot F\left(\overline{\lambda}'\right)\right)}{p_i^{\lambda_i'+1}}\cdot\left(\frac{p_k^2}{p_k^2-1}\right)^2\right)+
$$

$$
+\left(\frac{p_i^{\lambda_i'+1}-p_k^2}{p_k^2}\right)^2\cdot\left(\frac{p_k^2}{p_i^{\lambda_i'+1}\cdot\left(p_k^2-1\right)}\right)=
$$

$$
=\frac{k}{2}\cdot\left(\frac{p_i^{\lambda_i^0+1}-p_i^{\lambda_i'+1}}{p_i^{\lambda_i'+1}}\right)^2\cdot\alpha_1\left(p_m\right),
$$

where

$$
\alpha_1\left(p_m\right)=\frac{1}{k}\cdot\left(\frac{p_i^{\lambda_i'+1}-p_k^2}{p_k^2}\right)^2\cdot\left(\frac{p_i^{\lambda_i^0+1}-p_i^{\lambda_i'+1}}{p_i^{\lambda_i'+1}}\right)^{-2}\times
$$

$$
\times\left(1+\frac{\left(p_i^{\lambda_i'+1}-1\right)}{p_i^{\lambda_i'+1}}\cdot\frac{\left(e^{-\gamma}\cdot F\left(\overline{\lambda}'\right)\right)}{p_i^{\lambda_i'+1}}\cdot\left(\frac{p_k^2}{p_k^2-1}\right)^2+\frac{2\cdot p_k^2}{p_i^{\lambda_i'+1}\cdot\left(p_k^2-1\right)}\right)= \qquad (45)
$$

$$
=\mathrm{O}\left(\sqrt{\frac{\log p_m}{p_m}}\right).
$$

and the term with k in its coefficients is

$$
k\cdot\left(\frac{p_i^{\lambda_i^0+1}-p_i^{\lambda_i'+1}}{p_i^{\lambda_i^0+1}}\right)-k\cdot\left(\frac{p_i^{\lambda_i^0+1}-p_i^{\lambda_i'+1}}{p_i^{\lambda_i'+1}}\right)+\frac{k}{2}\cdot\left(\frac{p_i^{\lambda_i^0+1}-p_i^{\lambda_i'+1}}{p_i^{\lambda_i'+1}}\right)^2=
$$

$$
=k\cdot\left(\frac{p_i^{\lambda_i^0+1}-p_i^{\lambda_i'+1}}{p_i^{\lambda_i^0+1}}-\frac{p_i^{\lambda_i^0+1}-p_i^{\lambda_i'+1}}{p_i^{\lambda_i'+1}}\right)+\frac{k}{2}\cdot\left(\frac{p_i^{\lambda_i^0+1}-p_i^{\lambda_i'+1}}{p_i^{\lambda_i'+1}}\right)^2=
$$

$$
=-k\cdot\frac{\left(p_i^{\lambda_i^0+1}-p_i^{\lambda_i'+1}\right)^2}{p_i^{\lambda_i^0+1}\cdot p_i^{\lambda_i'+1}}+\frac{k}{2}\cdot\left(\frac{p_i^{\lambda_i^0+1}-p_i^{\lambda_i'+1}}{p_i^{\lambda_i'+1}}\right)^2=
$$

$$
=-\frac{k}{2}\cdot\left(\frac{p_i^{\lambda_i^0+1}-p_i^{\lambda_i'+1}}{p_i^{\lambda_i'+1}}\right)^2\cdot\left(2\cdot\frac{p_i^{\lambda_i'+1}}{p_i^{\lambda_i^0+1}}-1\right)=
$$

$$=-\frac{k}{2}\cdot\left(\frac{p_i^{\lambda_i^0+1}-p_i^{\lambda_i'+1}}{p_i^{\lambda_i'+1}}\right)^2\cdot\left(1-2\cdot\frac{p_i^{\lambda_i^0+1}-p_i^{\lambda_i'+1}}{p_i^{\lambda_i^0+1}}\right)=$$
$$=-\frac{k}{2}\cdot\left(\frac{p_i^{\lambda_i^0+1}-p_i^{\lambda_i'+1}}{p_i^{\lambda_i'+1}}\right)^2\cdot\left(1+\alpha_2\left(p_m\right)\right), \tag{46}$$

where $\alpha_2\left(p_m\right)=-2\cdot\frac{p_i^{\lambda_i^0+1}-p_i^{\lambda_i'+1}}{p_i^{\lambda_i^0+1}}=\mathrm{O}\left(\frac{1}{p_m^{\theta}}\right)\ \left(0<\theta<1/2\right).$ (47)

And the term with k^2 in its coefficients is

$$\frac{k^2}{2}\cdot\frac{\left(e^{-\gamma}\cdot F\left(\overline{\lambda}'\right)\right)}{\left(p_i^{\lambda_i'+1}-1\right)}\cdot\left(\frac{p_i^{\lambda_i^0+1}-p_i^{\lambda_i'+1}}{p_i^{\lambda_i^0+1}}\right)^2+\frac{1}{2}\cdot\frac{(k)\cdot(k-1)}{\left(p_i^{\lambda_i'+1}-1\right)}\cdot\left(\frac{p_i^{\lambda_i^0+1}-p_i^{\lambda_i'+1}}{p_i^{\lambda_i^0+1}}\right)^2=$$
$$=\frac{k^2}{2}\cdot\left(\frac{p_i^{\lambda_i^0+1}-p_i^{\lambda_i'+1}}{p_i^{\lambda_i^0+1}}\right)^2\cdot\left(\frac{\left(e^{-\gamma}\cdot F\left(\overline{\lambda}'\right)\right)}{\left(p_i^{\lambda_i'+1}-1\right)}+\frac{k-1}{k\cdot\left(p_i^{\lambda_i'+1}-1\right)}\right)=$$
$$=\frac{k^2}{2}\cdot\left(\frac{p_i^{\lambda_i^0+1}-p_i^{\lambda_i'+1}}{p_i^{\lambda_i'+1}}\right)^2\cdot\left(\frac{p_i^{\lambda_i'+1}}{p_i^{\lambda_i^0+1}}\right)\cdot\left(\frac{\left(e^{-\gamma}\cdot F\left(\overline{\lambda}'\right)\right)}{\left(p_i^{\lambda_i'+1}-1\right)}+\frac{k-1}{k\cdot\left(p_i^{\lambda_i'+1}-1\right)}\right)= \tag{48}$$
$$=\frac{k}{2}\cdot\left(\frac{p_i^{\lambda_i^0+1}-p_i^{\lambda_i'+1}}{p_i^{\lambda_i'+1}}\right)^2\cdot\alpha_3\left(p_m\right),$$

where

$$\alpha_3\left(p_m\right)=k\cdot\left(\frac{p_i^{\lambda_i'+1}}{p_i^{\lambda_i^0+1}}\right)\cdot\left(\frac{\left(e^{-\gamma}\cdot F\left(\overline{\lambda}'\right)\right)}{\left(p_i^{\lambda_i'+1}-1\right)}+\frac{k-1}{k\cdot\left(p_i^{\lambda_i'+1}-1\right)}\right)=$$
$$=\mathrm{O}\left(\sqrt{\frac{\log p_m}{p_m}}\right). \tag{49}$$

And we have

$$\beta_1'''\left(p_m\right)-\beta_2\left(p_m\right)=\left(\frac{k}{2}\right)\cdot\left(\frac{p_i^{\lambda_i^0+1}-p_i^{\lambda_i'+1}}{p_i^{\lambda_i'+1}}\right)^2\cdot\beta_0\left(p_m\right), \tag{50}$$

where $\beta_0\left(p_m\right)=\mathrm{O}\left(\frac{1}{p_m^{\theta}}\right)\left(0<\theta<1/2\right)$. Hence we have

$$\log\left(\frac{C'_{m-1}}{C_{m-1}}\right)=-\frac{k}{2}\cdot\left(\frac{p_i^{\lambda_i^0+1}-p_i^{\lambda'_i+1}}{p_i^{\lambda'_i+1}}\right)^2\cdot\left(1+\alpha_0\left(p_m\right)\right) \tag{51}$$

where

$$\begin{aligned}\alpha_0\left(p_m\right)=\alpha_2\left(p_m\right)-\alpha_1\left(p_m\right)-\alpha_3\left(p_m\right)+\\ +\beta_1'''\left(p_m\right)-\beta_2\left(p_m\right)=\mathrm{O}\left(\frac{1}{p_m^{\theta}}\right).\end{aligned} \tag{52}$$

On the other hand, by (4) and (5) we have

$$p_i^{\lambda_i^0+1}=p_m\cdot\log p_m\cdot\left(1+\varepsilon_1\left(p_m\right)\right),\quad \varepsilon_1\left(p_m\right)=\mathrm{O}\left(\frac{1}{\log p_m}\right) \tag{53}$$

and

$$p_i^{\lambda'_i+1}=p_{m-1}\cdot\log p_{m-1}\cdot\left(1+\varepsilon_2\left(p_{m-1}\right)\right),\quad \varepsilon_2\left(p_{m-1}\right)=\mathrm{O}\left(\frac{1}{\log p_m}\right). \tag{54}$$

and

$$k=2\cdot\sqrt{\frac{p_{m-1}}{\log p_{m-1}}}\left(1+\varepsilon_3\left(p_m\right)\right),\quad \varepsilon_3\left(p_m\right)=\mathrm{O}\left(\frac{\log\log p_m}{\log p_m}\right).$$

Hence

$$\frac{p_i^{\lambda_i^0+1}-p_i^{\lambda'_i+1}}{p_i^{\lambda'_i+1}}=\frac{p_m-p_{m-1}}{p_{m-1}}\cdot\left(1+\varepsilon_4\left(p_m\right)\right), \tag{55}$$

where $\varepsilon_4\left(p_m\right)=\mathrm{O}\left(\frac{1}{\log p_m}\right)$. From this we have

$$\begin{aligned}\log C_{m-1}-\log C'_{m-1}=\frac{k}{2}\cdot\left(\frac{p_i^{\lambda_i^0+1}-p_i^{\lambda'_i+1}}{p_i^{\lambda'_i+1}}\right)^2\cdot\left(1+\alpha_0\left(p_m\right)\right)=\\ =\frac{\left(p_m-p_{m-1}\right)}{\sqrt{p_{m-1}\cdot\log p_{m-1}}}\cdot\left(\frac{p_m-p_{m-1}}{p_{m-1}}\right)\cdot\left(1+\beta_0\left(p_m\right)\right).\end{aligned} \tag{56}$$

where

$$1+\beta_0\left(p_m\right)=\left(1+\varepsilon_1\left(p_m\right)\right)\cdot\left(1+\varepsilon_2\left(p_m\right)\right)^2\cdot\left(1+\alpha_0\left(p_m\right)\right)=$$
$$=1+\mathrm{O}\left(\frac{1}{\log p_m}\right). \tag{57}$$

This is the proof of the theorem 2. □

4. The estimate of $\left(p_m - p_{m-1}\right)$

In this section we will estimate the size of $\left(p_{m+1} - p_m\right)$. Here obtained result on $\left(p_{m+1} - p_m\right)$ is a new result for the distribution of the prime number.

We have

Theorem 3. There exist a number m_0 such that for any $m \geq m_0$ we have

$$\left(p_m - p_{m-1}\right)=\mathrm{O}\left(\sqrt{p_{m-1}}\cdot\log^{5/2} p_{m-1}\right). \tag{58}$$

Proof. It is easy to see that

$$\begin{aligned}
\log C_m - \log C'_m &= \left(\exp\left(e^{-\gamma}\cdot F\left(\overline{\lambda}_0\right)\right)-\log n_0\right)-\left(\exp\left(e^{-\gamma}\cdot F\left(\overline{\lambda}'_+\right)\right)-\log n'_+\right)= \\
&=\left(\exp\left(e^{-\gamma}\cdot F\left(\overline{\lambda}'_0\right)\right)-\exp\left(e^{-\gamma}\cdot F\left(\overline{\lambda}'\right)\right)\right)-\left(\log n'_0-\log n'\right)+ \\
&+\left(\exp\left(e^{-\gamma}\cdot F\left(\overline{\lambda}_0\right)\right)-\exp\left(e^{-\gamma}\cdot F\left(\overline{\lambda}'_0\right)\right)\right)- \\
&-\left(\exp\left(e^{-\gamma}\cdot F\left(\overline{\lambda}'_+\right)\right)-\exp\left(e^{-\gamma}\cdot F\left(\overline{\lambda}'\right)\right)\right)= \\
&=\left(\log C'_{m-1}-\log C_{m-1}\right)+R_0 ,
\end{aligned} \tag{59}$$

where

$$\begin{aligned}
R_0 &=\left(\exp\left(e^{-\gamma}\cdot F\left(\overline{\lambda}_0\right)\right)-\exp\left(e^{-\gamma}\cdot F\left(\overline{\lambda}'_0\right)\right)\right)- \\
&-\left(\exp\left(e^{-\gamma}\cdot F\left(\overline{\lambda}'_+\right)\right)-\exp\left(e^{-\gamma}\cdot F\left(\overline{\lambda}'\right)\right)\right).
\end{aligned} \tag{60}$$

On other hand, by the theorem 2, we have

$$\left(\exp\left(e^{-\gamma}\cdot F\left(\overline{\lambda}_0\right)\right)-\exp\left(e^{-\gamma}\cdot F\left(\overline{\lambda}'_0\right)\right)\right)=$$

$$=\exp\left(e^{-\gamma}\cdot F\left(\overline{\lambda}'_0\right)\right)\cdot\left(\exp\left(e^{-\gamma}\cdot F\left(\overline{\lambda}'_0\right)\cdot\frac{1}{p_m}\right)-1\right)=$$

$$=\frac{1}{p_m}\cdot\left(e^{-\gamma}\cdot F\left(\overline{\lambda}'_0\right)\right)\cdot\exp\left(e^{-\gamma}\cdot F\left(\overline{\lambda}'_0\right)\right)\cdot\left(1+\mathrm{O}\left(\frac{\log p_m}{p_m}\right)\right)=$$

$$=\frac{1}{p_m}\cdot\left(\left(e^{-\gamma}\cdot F\left(\overline{\lambda}_0\right)\right)\cdot\exp\left(e^{-\gamma}\cdot F\left(\overline{\lambda}_0\right)\right)\right)\times$$

$$\times\left(\exp\left(-e^{-\gamma}\cdot F\left(\overline{\lambda}'_0\right)\cdot\frac{1}{p_m}\right)\right)\cdot\left(1-\frac{1}{p_m+1}\right)\cdot\left(1+\mathrm{O}\left(\frac{\log p_m}{p_m}\right)\right)=$$

$$=\frac{1}{p_m}\cdot\left(\left(e^{-\gamma}\cdot F\left(\overline{\lambda}_0\right)\right)\cdot\exp\left(e^{-\gamma}\cdot F\left(\overline{\lambda}_0\right)\right)\right)\cdot\left(1+\mathrm{O}\left(\frac{\log p_m}{p_m}\right)\right)=$$

$$=\log p_m\cdot\left(1+\left(\log p_m+1\right)\cdot E_0\left(p_m\right)-\frac{4\cdot\left(\log p_m+1\right)}{\sqrt{p_m}\log^{3/2}p_m}+\Theta_1\left(p_m\right)\right), \qquad (61)$$

where $\Theta_1\left(p_m\right)=\mathrm{O}\left(\log^2 p_m\cdot E_0^2\left(p_m\right)\right)$.

And we have

$$\exp\left(e^{-\gamma}\cdot F\left(\overline{\lambda}'\right)\right)\cdot\left(\exp\left(e^{-\gamma}\cdot F\left(\overline{\lambda}'\right)\cdot\frac{1}{p_m}\right)-1\right)=$$

$$=\frac{p_{m-1}}{p_m}\cdot\log p_{m-1}\cdot\times\left(1+\left(\log p_{m-1}+1\right)\cdot E_0\left(p_{m-1}\right)-\right. \qquad (62)$$

$$\left.-\frac{4\cdot\left(\log p_{m-1}+1\right)}{\sqrt{p_{m-1}}\log^{3/2}p_{m-1}}+\Theta_2\left(p_{m-1}\right)\right),$$

where $\Theta_2\left(p_{m-1}\right)=\mathrm{O}\left(\log^2 p_{m-1}\cdot E_0^2\left(p_{m-1}\right)\right)$.

From this we obtain

$$R_0 = \log p_m \cdot \left(1 + \left(\log p_m + 1\right) \cdot E_0\left(p_m\right) - \frac{4 \cdot \left(\log p_m + 1\right)}{\sqrt{p_m} \log^{3/2} p_m} + \Theta_1\left(p_m\right)\right) -$$

$$- \frac{p_{m-1} \cdot \log p_{m-1}}{p_m} \cdot \left(1 + \left(\log p_{m-1} + 1\right) \cdot E_0\left(p_{m-1}\right) -\right.$$

$$\left.- \frac{4 \cdot \left(\log p_{m-1} + 1\right)}{\sqrt{p_{m-1}} \log^{3/2} p_{m-1}} + \Theta_2\left(p_{m-1}\right)\right) =$$

$$= \left(\log p_m - \frac{p_{m-1}}{p_m} \cdot \log p_{m-1} \right) + \log p_m \cdot \left(\log p_m + 1\right) \cdot E_0\left(p_m\right) - \qquad (63)$$

$$- \frac{p_{m-1} \cdot \log p_{m-1}}{p_m} \cdot \left(\log p_{m-1} + 1\right) \cdot E_0\left(p_{m-1}\right) +$$

$$+ \frac{4 \cdot \log p_m \cdot \left(\log p_m + 1\right)}{\sqrt{p_m} \log^{3/2} p_m} - \frac{p_{m-1} \cdot \log p_{m-1}}{p_m} \cdot \frac{4 \cdot \left(\log p_{m-1} + 1\right)}{\sqrt{p_{m-1}} \log^{3/2} p_{m-1}} +$$

$$+ \log p_m \cdot \Theta_1\left(p_m\right) - \frac{p_{m-1} \cdot \log p_{m-1}}{p_m} \cdot \Theta_2\left(p_{m-1}\right).$$

Here we have

$$\log p_m - \frac{p_{m-1}}{p_m} \cdot \log p_{m-1} = \log p_m - \left(1 - \frac{p_m - p_{m-1}}{p_m} \right) \cdot \log p_{m-1} =$$

$$= \left(\log p_m - \log p_{m-1}\right) + \left(\frac{p_m - p_{m-1}}{p_m} \right) \cdot \log p_{m-1} = \qquad (64)$$

$$= \log \left(1 + \frac{p_m - p_{m-1}}{p_{m-1}} \right) + \left(\frac{p_m - p_{m-1}}{p_m} \right) \cdot \log p_{m-1} =$$

$$= \frac{p_m - p_{m-1}}{p_{m-1}} - \frac{1}{2} \left(\frac{p_m - p_{m-1}}{p_{m-1}} \right)^2 + \left(\frac{p_m - p_{m-1}}{p_m} \right) \cdot \log p_{m-1} =$$

$$= \frac{p_m - p_{m-1}}{p_{m-1}} \cdot \log p_{m-1} \cdot r_1\left(p_m\right),$$

where $r_1\left(p_m\right) = 1 + \mathrm{O}\left(\frac{1}{\log p_m} \right) = \mathrm{O}(1)$.

By (7) it is true that

$$
\begin{aligned}
&E_0(p_m) = \sum_{i=1}^{m} \frac{1}{p_i} - \log\log p_m - b_0 = \\
&= \left(\sum_{i=1}^{m-1} \frac{1}{p_i} - \log\log p_{m-1} - b_0 \right) - \log\log p_m + \log\log p_{m-1} + \frac{1}{p_m} = \\
&= E_0(p_{m-1}) - \log\left(\frac{\log p_m}{\log p_{m-1}} \right) + \frac{1}{p_m}.
\end{aligned}
\tag{65}
$$

Hence we have

$$
\begin{aligned}
&\log p_m \cdot (\log p_m + 1) \cdot E_0(p_m) - \frac{p_{m-1} \cdot \log p_{m-1}}{p_m} \cdot (\log p_{m-1} + 1) \cdot E_0(p_{m-1}) = \\
&= \log p_m \cdot (\log p_m + 1) \cdot E_0(p_m) - \log p_{m-1} \cdot (\log p_{m-1} + 1) \cdot E_0(p_{m-1}) + \\
&+ \left(\frac{p_m - p_{m-1}}{p_m} \right) \log p_{m-1} \cdot (\log p_{m-1} + 1) \cdot E_0(p_{m-1}) = \\
&= \log p_m \cdot (\log p_m + 1) \cdot \left(E_0(p_m) - E_0(p_{m-1}) \right) + \\
&+ \left(\log p_m \cdot (\log p_m + 1) - \log p_{m-1} \cdot (\log p_{m-1} + 1) \right) \cdot E_0(p_{m-1}) + \\
&+ \left(\frac{p_m - p_{m-1}}{p_m} \right) \log p_{m-1} \cdot (\log p_{m-1} + 1) \cdot E_0(p_{m-1}) = \\
&= \log p_m \cdot (\log p_m + 1) \cdot \left(-\log\left(\frac{\log p_m}{\log p_{m-1}} \right) + \frac{1}{p_m} \right) + \\
&+ \left(\log p_m \cdot (\log p_m + 1) - \log p_{m-1} \cdot (\log p_{m-1} + 1) \right) \cdot E_0(p_{m-1}) + \\
&+ \left(\frac{p_m - p_{m-1}}{p_m} \right) \log p_{m-1} \cdot (\log p_{m-1} + 1) \cdot E_0(p_{m-1}) = \\
&= \left(\frac{p_m - p_{m-1}}{p_{m-1}} \right) \log^2 p_{m-1} \cdot r_2(p_m),
\end{aligned}
\tag{66}
$$

where $r_2(p_m) = \mathrm{O}\left(\frac{1}{p_m - p_{m-1}} + \frac{1}{\log p_m} \right) = \mathrm{O}(1)$.

Next, it is clear that

$$
\begin{aligned}
&\frac{4 \cdot \log p_m \cdot (\log p_m + 1)}{\sqrt{p_m} \log^{3/2} p_m} - \frac{p_{m-1} \cdot \log p_{m-1}}{p_m} \cdot \frac{4 \cdot (\log p_{m-1} + 1)}{\sqrt{p_{m-1}} \log^{3/2} p_{m-1}} = \\
&= \frac{p_m - p_{m-1}}{p_{m-1}} \cdot \log p_{m-1} \cdot r_3(p_m),
\end{aligned}
\tag{67}
$$

and

$$\log p_m \cdot \Theta_1(p_m) - \frac{p_{m-1} \cdot \log p_{m-1}}{p_m} \cdot \Theta_2(p_{m-1}) = \\ = \frac{p_m - p_{m-1}}{p_{m-1}} \cdot \log p_{m-1} \cdot r_4(p_m) \tag{68}$$

where $r_3(p_m) = \mathrm{O}\left(\frac{\log p_m}{\sqrt{p_m}}\right)$ and $r_4(p_m) = \mathrm{O}\left(\frac{1}{\log p_m}\right)$.

Therefore we have

$$R_0 = \frac{p_m - p_{m-1}}{p_{m-1}} \cdot \log^2 p_{m-1} \cdot \delta_0(p_m), \tag{69}$$

where $\delta_0(p_m) = r_2(p_m) + \frac{1}{\log p_m} \cdot \left(r_1(p_m) + r_3(p_m) r_4(p_m)\right) = \mathrm{O}(1)$.

On the other hand, since $C'_m \le C_m$ we have

$$0 < \left(\log C_{m-1} - \log C'_{m-1}\right) < R_0. \tag{70}$$

Thus from the theorem 2, we have

$$\frac{(p_m - p_{m-1})}{\sqrt{p_{m-1} \cdot \log p_{m-1}}} \cdot \left(\frac{p_m - p_{m-1}}{p_{m-1}}\right) \cdot \left(1 + \beta_0(p_m)\right) \le \\ \le \left(\frac{p_m - p_{m-1}}{p_{m-1}}\right) \cdot \log^2 p_{m-1} \cdot \delta_0(p_m). \tag{71}$$

Therefore we have

$$(p_m - p_{m-1}) = \mathrm{O}\left(\sqrt{p_{m-1}} \cdot \log^{5/2} p_{m-1}\right). \tag{72}$$

This is the proof of the theorem 3. □

5. The estimate of $E_0\left(p_m\right)$

In this section we will estimate the size of the error iterm $E_0\left(p_m\right)$ given in the formular (7).

We get

Theorem 4. There exists a number m_0 such that for any $m \ge m_0$ we have

$$E_0\left(p_m\right) = \mathrm{O}\left(\frac{\log^{3/2} p_m}{\sqrt{p_m}}\right). \tag{73}$$

Proof. Since

$$F\left(\overline{\lambda}_0\right) = F\left(\overline{\lambda}'_0\right)\cdot\left(1+\frac{1}{p_m}\right), \tag{74}$$

we have

$$\begin{aligned}&\log p_m\cdot\left(1+E_0\left(p_m\right)-\frac{4}{\sqrt{p_m}\log^{3/2} p_m}+\varepsilon\left(p_m\right)\right)=\\&=\log p_{m-1}\cdot\left(1+E_0\left(p_{m-1}\right)-\frac{4}{\sqrt{p_m}\log^{3/2} p_m}+\varepsilon\left(p_{m-1}\right)\right)\cdot\left(1+\frac{1}{p_m}\right).\end{aligned} \tag{75}$$

From this we have

$$\begin{aligned}&\log p_m\cdot E_0\left(p_m\right)-\log p_{m-1}\cdot\left(1+\frac{1}{p_m}\right)\cdot E_0\left(p_{m-1}\right)=\\&=-\left(\log p_m-\log p_{m-1}-\frac{\log p_{m-1}}{p_m}\right)+\\&+\frac{4}{\sqrt{p_m}\log^{3/2} p_m}\cdot\left(\log p_m-\log p_{m-1}-\frac{\log p_{m-1}}{p_m}\right)+\\&+\log p_m\cdot\varepsilon\left(p_m\right)-\log p_{m-1}\cdot\left(1+\frac{1}{p_m}\right)\cdot\varepsilon\left(p_{m-1}\right).\end{aligned} \tag{76}$$

From (68) the left hand side of (76) is

$$
\begin{aligned}
&\log p_m \cdot E_0\left(p_m\right)-\log p_{m-1}\cdot\left(1+\frac{1}{p_m}\right)\cdot E_0\left(p_{m-1}\right)=\\
&=\log p_m\cdot\left(E_0\left(p_{m-1}\right)-\log\left(\frac{\log p_m}{\log p_{m-1}}\right)+\frac{1}{p_m}\right)-\\
&-\log p_{m-1}\cdot\left(1+\frac{1}{p_m}\right)\cdot E_0\left(p_{m-1}\right)=\\
&\quad=\left(\log p_m-\log p_{m-1}-\frac{\log p_{m-1}}{p_m}\right)\cdot E_0\left(p_{m-1}\right)-\\
&\quad-\log p_m\cdot\left(\log\left(\frac{\log p_m}{\log p_{m-1}}\right)-\frac{1}{p_m}\right).
\end{aligned}
\tag{77}
$$

On the other hand, here we have

$$
\begin{aligned}
&-\log p_m\cdot\left(\log\left(\frac{\log p_m}{\log p_{m-1}}\right)-\frac{1}{p_m}\right)=-\left(\log p_m-\log p_{m-1}-\frac{\log p_{m-1}}{p_m}\right)+\\
&+\left(\log p_m-\log p_{m-1}-\frac{\log p_{m-1}}{p_m}\right)\cdot\left(\frac{\log p_m-\log p_{m-1}}{\log p_{m-1}}\right)+\\
&+\log p_m\cdot\left(\frac{1}{2}\cdot\left(\frac{\log p_m-\log p_{m-1}}{\log p_{m-1}}\right)^2+\Delta\left(p_m\right)\right),
\end{aligned}
\tag{78}
$$

where $\Delta\left(p_m\right)=\mathrm{O}\left(\left(\frac{\log p_m-\log p_{m-1}}{\log p_{m-1}}\right)^3\right).$ (79)

And we have

$$
\begin{aligned}
&\log p_m\cdot\varepsilon\left(p_m\right)-\log p_{m-1}\cdot\left(1+\frac{1}{p_m}\right)\cdot\varepsilon\left(p_{m-1}\right)=\\
&\quad=\mathrm{O}\left(\left(\frac{\log p_m-\log p_{m-1}}{\log p_{m-1}}\right)^2\right).
\end{aligned}
\tag{80}
$$

So from (76) we have

$$\left(\log p_m - \log p_{m-1} - \frac{\log p_{m-1}}{p_m}\right)\cdot E_0\left(p_{m-1}\right) -$$
$$-\left(\log p_m - \log p_{m-1} - \frac{\log p_{m-1}}{p_m}\right)\cdot\left(1 + \frac{\log p_m - \log p_{m-1}}{\log p_{m-1}}\right)$$
$$-\log p_m \cdot \left(\frac{1}{2}\cdot\left(\frac{\log p_m - \log p_{m-1}}{\log p_{m-1}}\right)^2 + \Delta\left(p_m\right)\right) = \tag{81}$$
$$= -\left(\log p_m - \log p_{m-1} - \frac{\log p_{m-1}}{p_m}\right) + \frac{4}{\sqrt{p_m}\log^{3/2} p_m}\times$$
$$\times\left(\log p_m - \log p_{m-1} - \frac{\log p_{m-1}}{p_m}\right) + \mathrm{O}\left(\left(\frac{\log p_m - \log p_{m-1}}{\log p_{m-1}}\right)^2\right).$$

Now if we would exactly write not only the term with $E_0\left(p_m\right)$, but also the term with $E_0^2\left(p_m\right)$ and more $E_0^n\left(p_m\right)$, and would repeat above process, then we would have $\left(\log p_m - \log p_{m-1} - \frac{\log p_{m-1}}{p_m}\right)$ in the every term.

So we will eliminate it from both hand sides of (81). Then since

$$\left(\log p_m - \log p_{m-1}\right)\cdot\left(\log p_m - \log p_{m-1} - \frac{\log p_{m-1}}{p_m}\right)^{-1} =$$
$$= \left(1 - \frac{\log p_{m-1}}{p_m\cdot\left(\log p_m - \log p_{m-1}\right)}\right)^{-1} = \mathrm{O}(1) \tag{82}$$

and

$$\frac{\log p_m - \log p_{m-1}}{\log p_{m-1}} = \frac{1}{\log p_{m-1}}\cdot\log\left(1 + \frac{p_m - p_{m-1}}{p_{m-1}}\right) =$$
$$= \frac{1}{\log p_{m-1}}\cdot\frac{p_m - p_{m-1}}{p_{m-1}}\cdot\left(1 + \mathrm{O}\left(\frac{p_m - p_{m-1}}{p_{m-1}}\right)\right), \tag{83}$$

by the theorem 3, we have

$$E_0(p_m) = \mathrm{O}\left(\frac{\log^{3/2} p_m}{\sqrt{p_m}}\right). \tag{84}$$

This is the proof of the theorem 4. □

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