

An Estimate for the Error in a Formula on Prime Numbers

Choe Ryong Gil

November 19, 2011

In this paper we will estimate an error in one formula on prime numbers. This result is a new result on the distribution of the prime number. This paper is a continuation of [6] and [7].

1. Introduction

By the theorem 2 of the paper [6], there exist the optimum points $\bar{\lambda}_0 = (\lambda_1^0, \lambda_2^0, \dots, \lambda_m^0) \in R^m$ in m -dimensional real space R^m of the function

$$H(\bar{\lambda}) = H(\lambda_1, \lambda_2, \dots, \lambda_m) = \frac{\exp\left(\exp\left(e^{-\gamma} \cdot F(\bar{\lambda})\right)\right)}{p_1^{\lambda_1} \cdot p_2^{\lambda_2} \cdots p_m^{\lambda_m}}, \quad (1)$$

where

$$F(\bar{\lambda}) = F(\lambda_1, \lambda_2, \dots, \lambda_m) = \prod_{i=1}^m \frac{1 - p_i^{-\lambda_i - 1}}{1 - p_i^{-1}}, \quad (2)$$

and $\gamma = 0.577 \dots$ is Euler's constant ([1,2,5]).

And here we assume that $\bar{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_m)$ are real numbers and $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_m \geq 1$. Let $p_1 = 2, p_2 = 3, p_3 = 5, \dots, p_m, \dots$ be the consecutive primes. We will choose $p_m \geq 5$ arbitrarily and fix it. Then the optimum points $\bar{\lambda}_0 = (\lambda_1^0, \lambda_2^0, \dots, \lambda_m^0) \in R^m$ are estimated as follows. There is a number k ($1 < k < m$) such that for any i ($1 \leq i \leq k$) it holds that

$$\lambda_i^0 = \left(\frac{\log p_m}{\log p_i} + \frac{\log \log p_m}{\log p_i} - 1 \right) + O\left(\frac{1}{\log p_i \cdot \log p_m} \right) \quad (3)$$

and for any i ($k+1 \leq i \leq m$) it holds that $\lambda_i^0 = 1$. So we have

$$\lambda_1^0 > \lambda_2^0 > \dots > \lambda_k^0 > \lambda_{k+1}^0 = \dots = \lambda_m^0 = 1. \quad (4)$$

And for any i ($1 \leq i \leq k$) it holds that

$$p_1^{\lambda_1^0+1} = p_2^{\lambda_2^0+1} = \dots = p_k^{\lambda_k^0+1} = \left(e^{-\gamma} F(\bar{\lambda}_0) \right) \cdot \exp\left(e^{-\gamma} F(\bar{\lambda}_0) \right) + 1. \quad (5)$$

On the other hand, for the number k these hold that

$$\lambda_k^0 = 1 + O\left(\frac{1}{\log p_m} \right), \quad p_k = \sqrt{p_m \cdot \log p_m} \cdot \left(1 + O\left(\frac{1}{\log p_m} \right) \right), \quad (6)$$

$$k = 2\sqrt{m} \cdot \left(1 + O\left(\frac{\log \log p_m}{\log p_m} \right) \right) = 2\sqrt{\frac{p_m}{\log p_m}} \cdot \left(1 + O\left(\frac{\log \log p_m}{\log p_m} \right) \right). \quad (7)$$

This shows that the function value $H(\lambda_1^0, \lambda_2^0, \dots, \lambda_m^0)$ is dependent only on p_m . So we can put

$$C_m = H(\lambda_1^0, \lambda_2^0, \dots, \lambda_m^0) = \frac{\exp\left(\exp\left(e^{-\gamma} \cdot F(\bar{\lambda}_0)\right)\right)}{p_1^{\lambda_1^0} \cdot p_2^{\lambda_2^0} \cdot \dots \cdot p_m^{\lambda_m^0}}. \quad (8)$$

In this connection, we will put

$$\begin{cases} n_0 = p_1^{\lambda_1^0} p_2^{\lambda_2^0} \cdots p_k^{\lambda_k^0} \cdot p_{k+1}^1 \cdots p_m^1, & n'_0 = n_0 \cdot p_m^{-1}, \\ \bar{\lambda}'_0 = (\lambda_1^0, \lambda_2^0, \dots, \lambda_{m-1}^0) \in R^{m-1}, \\ C'_{m-1} = H(\bar{\lambda}'_0) = H(\lambda_1^0, \lambda_2^0, \dots, \lambda_{m-1}^0) \end{cases} \quad (9)$$

and

$$C_{m-1} = \max_{(\lambda_1, \lambda_2, \dots, \lambda_{m-1}) \in R^m} H(\lambda_1, \lambda_2, \dots, \lambda_{m-1}). \quad (10)$$

Then it is clear that $C'_{m-1} \leq C_{m-1}$.

Let $\bar{\lambda}' = (\lambda'_1, \lambda'_2, \dots, \lambda'_{m-1})$ be the optimum points of the function $H(\lambda_1, \lambda_2, \dots, \lambda_{m-1})$ with $(m-1)$ -variable in the space R^{m-1} . In general, then we have

$$\lambda'_1 > \lambda'_2 > \cdots > \lambda'_{k-1} > \lambda'_k = \cdots = \lambda'_{m-1} = 1. \quad (11)$$

Rarely, the last bigger number than 1 in $\{\lambda'_1, \lambda'_2, \dots, \lambda'_{m-1}\}$ could be k . But it is not essential. It is important that for any i ($1 \leq i \leq k-1$)

$$p_1^{\lambda'_i+1} = p_2^{\lambda'_i+1} = \cdots = p_{k-1}^{\lambda'_i+1} = (e^{-\gamma} \cdot F(\bar{\lambda}')) \cdot \exp(e^{-\gamma} \cdot F(\bar{\lambda}')) + 1 \quad (12)$$

holds. We note that it doesn't exceed one in $\{\lambda_1^0, \lambda_2^0, \dots, \lambda_m^0\}$.

We also put

$$\begin{cases} n' = p_1^{\lambda'_1} p_2^{\lambda'_2} \cdots p_{k-1}^{\lambda'_{k-1}} \cdot p_k^1 \cdots p_{m-1}^1, \\ n'_+ = p_1^{\lambda'_1} p_2^{\lambda'_2} \cdots p_{k-1}^{\lambda'_{k-1}} \cdot p_k^1 \cdots p_{m-1}^1 \cdot p_m^1 = n' \cdot p_m^1, \\ \bar{\lambda}'_+ = (\lambda'_1, \lambda'_2, \dots, \lambda'_{m-1}, 1), \quad C'_m = H(\bar{\lambda}'_+). \end{cases} \quad (13)$$

On the other hand, it is well known that

$$\sum_{p \leq p_m} \frac{1}{p} = \log \log p_m + b_0 + E_0(p_m), \quad (14)$$

where

$$b_0 = \gamma + \sum_p \left(\log \left(1 - \frac{1}{p} \right) + \frac{1}{p} \right) = 0.241 \cdots \quad (15)$$

([1,2,5]). And there exists a constant $a > 0$ such that

$$E_0(p_m) = O\left(\exp\left(-a\sqrt{\log p_m}\right)\right). \quad (16)$$

2. The estimate of $F(\bar{\lambda}_0)$

We have

Theorem 1. For the optimum points $\bar{\lambda}_0 = (\lambda_1^0, \lambda_2^0, \dots, \lambda_m^0) \in R^m$ of the function $H(\lambda_1, \lambda_2, \dots, \lambda_m)$ we have

$$F(\bar{\lambda}_0) = e^\gamma \cdot \log p_m \cdot \left(1 + E_0(p_m) - \frac{4}{\sqrt{p_m} \cdot \log^{3/2} p_m} + \varepsilon(p_m)\right), \quad (17)$$

where $\varepsilon(p_m) = O(E_0^2(p_m))$. Hence we also have

$$\begin{aligned} & \left(e^{-\gamma} \cdot F(\bar{\lambda}_0)\right) \cdot \exp\left(e^{-\gamma} \cdot F(\bar{\lambda}_0)\right) = \\ & = p_m \cdot \log p_m \cdot \left(1 + (\log p_m + 1) \cdot E_0(p_m) - \frac{4 \cdot (\log p_m + 1)}{\sqrt{p_m} \cdot \log^{3/2} p_m} + \tilde{\varepsilon}(p_m)\right), \end{aligned} \quad (18)$$

where $\tilde{\varepsilon}(p_m) = O(\log^2 p_m \cdot E_0^2(p_m))$.

Proof. From (2) and (5), it is clear that

$$\begin{aligned} \log F(\bar{\lambda}_0) &= \log \left(\prod_{i=1}^m \frac{1 - p_i^{-\lambda_i^0 - 1}}{1 - p_i^{-1}} \right) = \\ &= \sum_{i=1}^k \log \left(1 - \frac{1}{p_i^{\lambda_i^0 + 1}} \right) + \sum_{i=k+1}^m \log \left(1 - \frac{1}{p_i^2} \right) + \sum_{i=1}^m \log \left(1 - \frac{1}{p_i} \right)^{-1} = \\ &= A_1 + A_2 + A_3, \end{aligned} \quad (19)$$

where

$$A_1 = \sum_{i=1}^k \log \left(1 - \frac{1}{p_i^{\lambda_i^0 + 1}} \right), \quad A_2 = \sum_{i=k+1}^m \log \left(1 - \frac{1}{p_i^2} \right), \quad A_3 = \sum_{i=1}^m \log \left(1 - \frac{1}{p_i} \right)^{-1}. \quad (20)$$

First let's see A_1 . By Mertens' theorem ([1,2]), preliminarily, we have

$$F(\bar{\lambda}_0) = \prod_{i=1}^m \frac{1 - p_i^{-\lambda_i^0 - 1}}{1 - p_i^{-1}} = e^\gamma \cdot \log p_m \cdot \left(1 + O\left(\frac{1}{\log^2 p_m}\right) \right). \quad (21)$$

So we have

$$\left(e^{-\gamma} \cdot F(\bar{\lambda}_0) \right) \cdot \exp\left(e^{-\gamma} \cdot F(\bar{\lambda}_0) \right) + 1 = p_m \cdot \log p_m \cdot \left(1 + O\left(\frac{1}{\log p_m}\right) \right). \quad (22)$$

Hence From (5) and (7) for any i ($1 \leq i \leq k$) we have

$$\begin{aligned} A_1 &= \sum_{i=1}^k \log \left(1 - \frac{1}{p_i^{\lambda_i^0 + 1}} \right) = k \cdot \log \left(1 - \frac{1}{p_i^{\lambda_i^0 + 1}} \right) = \\ &= -\frac{k}{p_i^{\lambda_i^0 + 1}} + O\left(\frac{k}{p_i^{2(\lambda_i^0 + 1)}}\right) = \\ &= -\frac{2}{\sqrt{p_m} \cdot (\log p_m)^{3/2}} \cdot \left(1 + O\left(\frac{\log \log p_m}{\log p_m}\right) \right). \end{aligned} \quad (23)$$

Next let's see A_2 . Now we put

$$T(x) = \sum_{p \leq x} \frac{1}{p} = \log \log x + b_0 + E_0(x). \quad (24)$$

Then we have $dT(x) = \frac{dx}{x \cdot \log x} + dE_0(x)$. So we have

$$\begin{aligned} \sum_{i=k+1}^m \frac{1}{p_i^2} &= \int_{p_k}^{p_m} \frac{dT(t)}{t} = \int_{p_k}^{p_m} \frac{1}{t} \cdot \left(\frac{dt}{t \cdot \log t} + dE_0(t) \right) = \\ &= \frac{1}{p_k \cdot \log p_k} - \frac{1}{p_m \cdot \log p_m} + \int_{p_k}^{p_m} \frac{dE_0(t)}{t} = \\ &= \frac{2}{\sqrt{p_m} \cdot (\log p_m)^{3/2}} \cdot \left(1 + O\left(\frac{\log \log p_m}{\log p_m}\right) \right) \end{aligned} \quad (25)$$

and

$$\begin{aligned}
A_2 &= \sum_{i=k+1}^m \log\left(1 - \frac{1}{p_i^2}\right) = -\sum_{i=k+1}^m \frac{1}{p_i^2} + O\left(\sum_{i=k+1}^m \frac{1}{p_i^4}\right) = \\
&= \frac{-2}{\sqrt{p_m} \cdot (\log p_m)^{3/2}} \cdot \left(1 + O\left(\frac{\log \log p_m}{\log p_m}\right)\right). \tag{26}
\end{aligned}$$

Next let's see A_3 . By (15) we have

$$\sum_{p \leq p_m} \left(\log\left(1 - \frac{1}{p}\right) + \frac{1}{p} \right) = b_0 - \gamma - \sum_{p > p_m} \left(\log\left(1 - \frac{1}{p}\right) + \frac{1}{p} \right). \tag{27}$$

And it is clear that

$$\begin{aligned}
\left| \log\left(1 - \frac{1}{p}\right) + \frac{1}{p} \right| &= \left| -\sum_{j=1}^{\infty} \frac{1}{j \cdot p^j} + \frac{1}{p} \right| = \\
&= \left| \frac{1}{2p^2} + \frac{1}{3p^3} + \dots + \frac{1}{j \cdot p^j} + \dots \right| \leq \\
&\leq \left| \frac{1}{p^2} + \frac{1}{p^3} + \dots + \frac{1}{p^j} + \dots \right| = \frac{1}{p \cdot (p-1)}. \tag{28}
\end{aligned}$$

So we have

$$\begin{aligned}
-\sum_{p > p_m} \left(\log\left(1 - \frac{1}{p}\right) + \frac{1}{p} \right) &\leq \sum_{p > p_m} \frac{1}{p \cdot (p-1)} \leq \\
&\leq \sum_{n > p_m} \frac{1}{n \cdot (n-1)} = \sum_{n > p_m} \left(\frac{1}{n-1} - \frac{1}{n} \right) = \frac{1}{p_m} = O\left(\frac{1}{p_m}\right). \tag{29}
\end{aligned}$$

By (14) we have

$$\begin{aligned}
A_3 &= \sum_{i=1}^m \frac{1}{p_i} - \sum_{i=1}^m \left(\log\left(1 - \frac{1}{p_i}\right) + \frac{1}{p_i} \right) = \\
&= \log \log p_m + b_0 + E_0(p_m) - \left(b_0 - \gamma - \sum_{p > p_m} \left(\log\left(1 - \frac{1}{p}\right) + \frac{1}{p} \right) \right) = \\
&= \log \log p_m + \gamma + E_0(p_m) + O\left(\frac{1}{p_m}\right). \tag{30}
\end{aligned}$$

From (23), (26) and (30) we have

$$\begin{aligned} \log F(\bar{\lambda}_0) &= \log \log p_m + \gamma + E_0(p_m) + \\ &\frac{-4}{\sqrt{p_m} \cdot \log^{3/2} p_m} \left(1 + O\left(\frac{\log \log p_m}{\log p_m}\right) \right). \end{aligned} \quad (31)$$

and hence we have

$$\left(e^{-\gamma} \cdot F(\bar{\lambda}_0) \right) = \log p_m \cdot \left(1 + E_0(p_m) - \frac{4}{\sqrt{p_m} \log^{3/2} p_m} + \varepsilon(p_m) \right),$$

where $\varepsilon(p_m) = O(E_0^2(p_m))$. Therefore we have

$$\begin{aligned} &\left(e^{-\gamma} \cdot F(\bar{\lambda}_0) \right) \cdot \exp\left(e^{-\gamma} \cdot F(\bar{\lambda}_0) \right) = \\ &= p_m \cdot \log p_m \cdot \left(1 + (\log p_m + 1) \cdot E_0(p_m) - \frac{4 \cdot (\log p_m + 1)}{\sqrt{p_m} \log^{3/2} p_m} + \tilde{\varepsilon}(p_m) \right), \end{aligned}$$

where $\tilde{\varepsilon}(p_m) = O(\log^2 p_m \cdot E_0^2(p_m))$.

This completes the proof of the theorem 1. \square

3. The estimate of $(\log C_{m-1} - \log C'_{m-1})$

We get

Theorem 2. There exists a number m_0 such that for any $m \geq m_0$ we have

$$\log C_{m-1} - \log C'_{m-1} = \frac{p_m - p_{m-1}}{\sqrt{p_{m-1}} \cdot \log p_{m-1}} \cdot \left(\frac{p_m - p_{m-1}}{p_{m-1}} \right) \cdot (1 + \tilde{E}(p_m)), \quad (32)$$

$$\text{where } \tilde{E}(p_m) = O\left(\frac{1}{\log p_m}\right). \quad (33)$$

Proof. From (9) and (10), we have

$$\begin{aligned} &\log C_{m-1} - \log C'_{m-1} = \\ &= \left(\exp\left(e^{-\gamma} \cdot F(\bar{\lambda}') \right) - \log n' \right) - \left(\exp\left(e^{-\gamma} \cdot F(\bar{\lambda}_0) \right) - \log n'_0 \right) = \\ &= \left(\exp\left(e^{-\gamma} \cdot F(\bar{\lambda}') \right) - \exp\left(e^{-\gamma} \cdot F(\bar{\lambda}_0) \right) \right) - (\log n' - \log n'_0) = \\ &= R_1 - R_2, \end{aligned} \quad (34)$$

where

$$R_1 = \left(\exp\left(e^{-\gamma} \cdot F(\bar{\lambda}')\right) - \exp\left(e^{-\gamma} \cdot F(\bar{\lambda}'_0)\right) \right), \quad R_2 = (\log n' - \log n'_0). \quad (35)$$

Let's see R_1 . We can write as

$$\begin{aligned} (-R_1) &= \left(\exp\left(e^{-\gamma} \cdot F(\bar{\lambda}'_0)\right) - \exp\left(e^{-\gamma} \cdot F(\bar{\lambda}')\right) \right) = \\ &= \exp\left(e^{-\gamma} \cdot F(\bar{\lambda}')\right) \cdot \left(\frac{\exp\left(e^{-\gamma} \cdot F(\bar{\lambda}'_0)\right)}{\exp\left(e^{-\gamma} \cdot F(\bar{\lambda}')\right)} - 1 \right) \end{aligned} \quad (36)$$

and here we have

$$\begin{aligned} \log\left(\frac{\exp\left(e^{-\gamma} \cdot F(\bar{\lambda}'_0)\right)}{\exp\left(e^{-\gamma} \cdot F(\bar{\lambda}')\right)}\right) &= \left(e^{-\gamma} \cdot F(\bar{\lambda}'_0)\right) - \left(e^{-\gamma} \cdot F(\bar{\lambda}')\right) = \\ &= \left(e^{-\gamma} \cdot F(\bar{\lambda}')\right) \cdot \left(\frac{F(\bar{\lambda}'_0)}{F(\bar{\lambda}')} - 1\right). \end{aligned} \quad (37)$$

By the Taylor's expansion of the function $\log(1+x)$ ($0 < x < 1$), (5) and (12), for any i ($1 \leq i \leq k-1$) we have

$$\begin{aligned} \log\left(\frac{F(\bar{\lambda}'_0)}{F(\bar{\lambda}')}\right) &= \log F(\bar{\lambda}'_0) - \log F(\bar{\lambda}') = \\ &= \log\left(\prod_{i=1}^{m-1} \frac{1-p_i^{-\lambda_i^0-1}}{1-p_i^{-1}}\right) - \log\left(\prod_{i=1}^{m-1} \frac{1-p_i^{-\lambda_i'-1}}{1-p_i^{-1}}\right) = \\ &= (k) \cdot \log\left(\frac{1-p_i^{-\lambda_i^0-1}}{1-p_i^{-1}}\right) - (k-1) \cdot \log\left(\frac{1-p_i^{-\lambda_i'-1}}{1-p_i^{-1}}\right) - \log\left(1 + \frac{1}{p_k}\right) = \\ &= (k) \cdot \log\left(\frac{1-p_i^{-\lambda_i^0-1}}{1-p_i^{-\lambda_i'-1}}\right) + \log\left(\frac{1-p_i^{-\lambda_i'-1}}{1-p_k^{-2}}\right) = \\ &= (k) \cdot \log\left(1 + \frac{p_i^{\lambda_i^0+1} - p_i^{\lambda_i'+1}}{p_i^{\lambda_i^0+1} \cdot (p_i^{\lambda_i'+1} - 1)}\right) + \log\left(1 + \frac{p_i^{\lambda_i'+1} - p_k^2}{p_i^{\lambda_i'+1} \cdot (p_k^2 - 1)}\right) = \\ &= (k) \cdot \left(\frac{p_i^{\lambda_i^0+1} - p_i^{\lambda_i'+1}}{p_i^{\lambda_i^0+1} \cdot (p_i^{\lambda_i'+1} - 1)}\right) - \frac{(k)}{2} \cdot \left(\frac{p_i^{\lambda_i^0+1} - p_i^{\lambda_i'+1}}{p_i^{\lambda_i^0+1} \cdot (p_i^{\lambda_i'+1} - 1)}\right)^2 + \end{aligned} \quad (38)$$

$$+ \left(\frac{p_i^{\lambda'_i+1} - p_k^2}{p_i^{\lambda'_i+1} \cdot (p_k^2 - 1)} \right) - \frac{1}{2} \cdot \left(\frac{p_i^{\lambda'_i+1} - p_k^2}{p_i^{\lambda'_i+1} \cdot (p_k^2 - 1)} \right)^2 + \beta_1(p_m)$$

$$\text{where } \beta_1(p_m) = O \left(k \cdot \left(\frac{p_i^{\lambda'_i+1} - p_i^{\lambda'_i+1}}{p_i^{\lambda'_i+1} \cdot (p_i^{\lambda'_i+1} - 1)} \right)^3 \right). \quad (39)$$

Hence we have

$$\begin{aligned} \left(\frac{F(\bar{\lambda}'_0)}{F(\bar{\lambda}')} \right) &= 1 + (k) \cdot \left(\frac{p_i^{\lambda'_i+1} - p_i^{\lambda'_i+1}}{p_i^{\lambda'_i+1} \cdot (p_i^{\lambda'_i+1} - 1)} \right) + \frac{k^2}{2} \left(\frac{p_i^{\lambda'_i+1} - p_i^{\lambda'_i+1}}{p_i^{\lambda'_i+1} \cdot (p_i^{\lambda'_i+1} - 1)} \right)^2 - \\ &- \frac{(k)}{2} \cdot \left(\frac{p_i^{\lambda'_i+1} - p_i^{\lambda'_i+1}}{p_i^{\lambda'_i+1} \cdot (p_i^{\lambda'_i+1} - 1)} \right)^2 + \left(\frac{p_i^{\lambda'_i+1} - p_k^2}{p_i^{\lambda'_i+1} \cdot (p_k^2 - 1)} \right) + \frac{1}{2} \cdot \left(\frac{p_i^{\lambda'_i+1} - p_k^2}{p_i^{\lambda'_i+1} \cdot (p_k^2 - 1)} \right)^2 - \\ &- \frac{1}{2} \cdot \left(\frac{p_i^{\lambda'_i+1} - p_k^2}{p_i^{\lambda'_i+1} \cdot (p_k^2 - 1)} \right)^2 + \beta'_1(p_m) = \\ &= 1 + (k) \cdot \left(\frac{p_i^{\lambda'_i+1} - p_i^{\lambda'_i+1}}{p_i^{\lambda'_i+1} \cdot (p_i^{\lambda'_i+1} - 1)} \right) + \frac{(k) \cdot (k-1)}{2} \cdot \left(\frac{p_i^{\lambda'_i+1} - p_i^{\lambda'_i+1}}{p_i^{\lambda'_i+1} \cdot (p_i^{\lambda'_i+1} - 1)} \right)^2 + \\ &+ \left(\frac{p_i^{\lambda'_i+1} - p_k^2}{p_i^{\lambda'_i+1} \cdot (p_k^2 - 1)} \right) + \beta'_1(p_m), \end{aligned} \quad (40)$$

$$\text{where } \beta'_1(p_m) = O \left(k^3 \cdot \left(\frac{p_i^{\lambda'_i+1} - p_i^{\lambda'_i+1}}{p_i^{\lambda'_i+1} \cdot (p_i^{\lambda'_i+1} - 1)} \right)^3 \right). \quad (41)$$

From (40), the expression (37) is

$$\begin{aligned} \log \left(\frac{\exp(e^{-\gamma} \cdot F(\bar{\lambda}'_0))}{\exp(e^{-\gamma} \cdot F(\bar{\lambda}'))} \right) &= (e^{-\gamma} \cdot F(\bar{\lambda}')) \cdot \left(\frac{F(\bar{\lambda}'_0)}{F(\bar{\lambda}')} - 1 \right) = \\ &= (e^{-\gamma} \cdot F(\bar{\lambda}')) \cdot \left((k) \cdot \left(\frac{p_i^{\lambda'_i+1} - p_i^{\lambda'_i+1}}{p_i^{\lambda'_i+1} \cdot (p_i^{\lambda'_i+1} - 1)} \right) + \right. \\ &+ \left. \frac{(k) \cdot (k-1)}{2} \cdot \left(\frac{p_i^{\lambda'_i+1} - p_i^{\lambda'_i+1}}{p_i^{\lambda'_i+1} \cdot (p_i^{\lambda'_i+1} - 1)} \right)^2 + \left(\frac{p_i^{\lambda'_i+1} - p_k^2}{p_i^{\lambda'_i+1} \cdot (p_k^2 - 1)} \right) + \beta'_1(p_m) \right) \end{aligned} \quad (42)$$

and so we have

$$\begin{aligned}
& \left(\frac{\exp(e^{-\gamma} \cdot F(\bar{\lambda}'_0))}{\exp(e^{-\gamma} \cdot F(\bar{\lambda}'))} \right) = 1 + (k) \cdot (e^{-\gamma} \cdot F(\bar{\lambda}')) \cdot \left(\frac{p_i^{\lambda'_i+1} - p_i^{\lambda'_i+1}}{p_i^{\lambda'_i+1} \cdot (p_i^{\lambda'_i+1} - 1)} \right) + \\
& + \frac{k^2}{2} \cdot (e^{-\gamma} \cdot F(\bar{\lambda}'))^2 \cdot \left(\frac{p_i^{\lambda'_i+1} - p_i^{\lambda'_i+1}}{p_i^{\lambda'_i+1} \cdot (p_i^{\lambda'_i+1} - 1)} \right)^2 + \\
& + (e^{-\gamma} \cdot F(\bar{\lambda}')) \cdot \left(\frac{p_i^{\lambda'_i+1} - p_k^2}{p_i^{\lambda'_i+1} \cdot (p_k^2 - 1)} \right) + \frac{(e^{-\gamma} \cdot F(\bar{\lambda}'))^2}{2} \cdot \left(\frac{p_i^{\lambda'_i+1} - p_k^2}{p_i^{\lambda'_i+1} \cdot (p_k^2 - 1)} \right)^2 + \\
& + \frac{(k) \cdot (k-1)}{2} \cdot (e^{-\gamma} \cdot F(\bar{\lambda}')) \cdot \left(\frac{p_i^{\lambda'_i+1} - p_i^{\lambda'_i+1}}{p_i^{\lambda'_i+1} \cdot (p_i^{\lambda'_i+1} - 1)} \right)^2 + \beta_1''(p_m),
\end{aligned} \tag{43}$$

$$\text{where } \beta_1''(p_m) = O \left(k^3 \cdot (e^{-\gamma} \cdot F(\bar{\lambda}'))^3 \cdot \left(\frac{p_i^{\lambda'_i+1} - p_i^{\lambda'_i+1}}{p_i^{\lambda'_i+1} \cdot (p_i^{\lambda'_i+1} - 1)} \right)^3 \right). \tag{44}$$

Hence we have

$$\begin{aligned}
(-R_1) &= \exp(e^{-\gamma} \cdot F(\bar{\lambda}')) \cdot \left(\frac{\exp(e^{-\gamma} \cdot F(\bar{\lambda}'_0))}{\exp(e^{-\gamma} \cdot F(\bar{\lambda}'))} - 1 \right) = \\
&= \exp(e^{-\gamma} \cdot F(\bar{\lambda}')) \cdot \left((k) \cdot (e^{-\gamma} \cdot F(\bar{\lambda}')) \cdot \left(\frac{p_i^{\lambda'_i+1} - p_i^{\lambda'_i+1}}{p_i^{\lambda'_i+1} \cdot (p_i^{\lambda'_i+1} - 1)} \right) + \right. \\
&+ \frac{k^2}{2} \cdot (e^{-\gamma} \cdot F(\bar{\lambda}'))^2 \cdot \left(\frac{p_i^{\lambda'_i+1} - p_i^{\lambda'_i+1}}{p_i^{\lambda'_i+1} \cdot (p_i^{\lambda'_i+1} - 1)} \right)^2 + (e^{-\gamma} \cdot F(\bar{\lambda}')) \cdot \left(\frac{p_i^{\lambda'_i+1} - p_k^2}{p_i^{\lambda'_i+1} \cdot (p_k^2 - 1)} \right) + \\
&+ \frac{1}{2} \cdot (e^{-\gamma} \cdot F(\bar{\lambda}'))^2 \cdot \left(\frac{p_i^{\lambda'_i+1} - p_k^2}{p_i^{\lambda'_i+1} \cdot (p_k^2 - 1)} \right)^2 + \\
&\left. + \frac{(k) \cdot (k-1)}{2} \cdot (e^{-\gamma} \cdot F(\bar{\lambda}')) \cdot \left(\frac{p_i^{\lambda'_i+1} - p_i^{\lambda'_i+1}}{p_i^{\lambda'_i+1} \cdot (p_i^{\lambda'_i+1} - 1)} \right)^2 + \beta_1''(p_m) \right).
\end{aligned} \tag{45}$$

From (12), since $(e^{-\gamma} F(\bar{\lambda}')) \cdot \exp(e^{-\gamma} F(\bar{\lambda}')) = p_i^{\lambda_i'+1} - 1$ ($1 \leq i \leq k-1$),

we have

$$\begin{aligned}
(-R_1) &= (k) \cdot \left(\frac{p_i^{\lambda_i^0+1} - p_i^{\lambda_i'+1}}{p_i^{\lambda_i^0+1}} \right) + \left(\frac{p_i^{\lambda_i'+1} - 1}{p_i^{\lambda_i'+1}} \right) \cdot \left(\frac{p_i^{\lambda_i'+1} - p_k^2}{p_k^2 - 1} \right) + \\
&+ \frac{k^2}{2} \cdot \frac{(e^{-\gamma} \cdot F(\bar{\lambda}'))}{(p_i^{\lambda_i'+1} - 1)} \cdot \left(\frac{p_i^{\lambda_i^0+1} - p_i^{\lambda_i'+1}}{p_i^{\lambda_i^0+1}} \right)^2 + \\
&+ \frac{1}{2} \cdot \frac{(p_i^{\lambda_i'+1} - 1)}{p_i^{\lambda_i'+1}} \cdot \frac{(e^{-\gamma} \cdot F(\bar{\lambda}'))}{p_i^{\lambda_i'+1}} \cdot \left(\frac{p_i^{\lambda_i'+1} - p_k^2}{p_k^2 - 1} \right)^2 + \\
&+ \frac{1}{2} \cdot \frac{(k) \cdot (k-1)}{(p_i^{\lambda_i'+1} - 1)} \cdot \left(\frac{p_i^{\lambda_i^0+1} - p_i^{\lambda_i'+1}}{p_i^{\lambda_i^0+1}} \right)^2 + \beta_1''(p_m),
\end{aligned} \tag{46}$$

where

$$\begin{aligned}
\beta_1''(p_m) &= \exp(e^{-\gamma} \cdot F(\bar{\lambda}')) \cdot \beta_1''(p_m) = \\
&= O \left(k^3 \cdot \left(\frac{e^{-\gamma} \cdot F(\bar{\lambda}'))}{p_i^{\lambda_i'+1} - 1} \right)^2 \cdot \left(\frac{p_i^{\lambda_i^0+1} - p_i^{\lambda_i'+1}}{p_i^{\lambda_i^0+1}} \right)^3 \right).
\end{aligned} \tag{47}$$

Next let's see R_2 . It is not difficult to see that for any i ($1 \leq i \leq k-1$)

$$\begin{aligned}
(-R_2) &= (\log n'_0 - \log n') = \log \left(\frac{p_1^{\lambda_1^0} \cdot p_2^{\lambda_2^0} \cdots p_k^{\lambda_k^0} p_{k+1}^1 \cdots p_{m-1}^1}{p_1^{\lambda_1'} p_2^{\lambda_2'} \cdots p_{k-1}^{\lambda_{k-1}'} \cdot p_k^1 \cdots p_{m-1}^1} \right) = \\
&= (k) \cdot \log \left(\frac{p_i^{\lambda_i^0+1}}{p_i^{\lambda_i'+1}} \right) - \log \left(\frac{p_i^{\lambda_i^0+1}}{p_i^{\lambda_i'+1}} \right) + \log \left(\frac{p_k^{\lambda_k^0+1}}{p_k^2} \right) = \\
&= (k) \cdot \log \left(1 + \frac{p_i^{\lambda_i^0+1} - p_i^{\lambda_i'+1}}{p_i^{\lambda_i'+1}} \right) + \log \left(1 + \frac{p_i^{\lambda_i'+1} - p_k^2}{p_k^2} \right) = \\
&= (k) \cdot \left(\frac{p_i^{\lambda_i^0+1} - p_i^{\lambda_i'+1}}{p_i^{\lambda_i'+1}} \right) - \left(\frac{k}{2} \right) \cdot \left(\frac{p_i^{\lambda_i^0+1} - p_i^{\lambda_i'+1}}{p_i^{\lambda_i'+1}} \right)^2 + \\
&+ \left(\frac{p_i^{\lambda_i'+1} - p_k^2}{p_k^2} \right) - \frac{1}{2} \cdot \left(\frac{p_i^{\lambda_i'+1} - p_k^2}{p_k^2} \right)^2 + \beta_2(p_m),
\end{aligned} \tag{48}$$

$$\text{where } \beta_2(p_m) = O\left(k \cdot \left(\frac{p_i^{\lambda_i^0+1} - p_i^{\lambda_i'+1}}{p_i^{\lambda_i'+1}}\right)^3\right). \quad (49)$$

Therefore we have

$$\begin{aligned} (-) \cdot (\log C_{m-1} - \log C'_{m-1}) &= -(R_1 - R_2) = \\ &= (k) \cdot \left(\frac{p_i^{\lambda_i^0+1} - p_i^{\lambda_i'+1}}{p_i^{\lambda_i^0+1}}\right) + \left(\frac{p_i^{\lambda_i'+1} - 1}{p_i^{\lambda_i'+1}}\right) \cdot \left(\frac{p_i^{\lambda_i'+1} - p_k^2}{p_k^2 - 1}\right) + \\ &+ \left(\frac{k^2}{2}\right) \cdot \frac{(e^{-\gamma} \cdot F(\bar{\lambda}'))}{(p_i^{\lambda_i'+1} - 1)} \cdot \left(\frac{p_i^{\lambda_i^0+1} - p_i^{\lambda_i'+1}}{p_i^{\lambda_i^0+1}}\right)^2 + \\ &+ \frac{1}{2} \cdot \frac{(p_i^{\lambda_i'+1} - 1)}{p_i^{\lambda_i'+1}} \cdot \frac{(e^{-\gamma} \cdot F(\bar{\lambda}'))}{p_i^{\lambda_i'+1}} \cdot \left(\frac{p_i^{\lambda_i'+1} - p_k^2}{p_k^2 - 1}\right)^2 + \\ &+ \frac{1}{2} \cdot \frac{(k) \cdot (k-1)}{(p_i^{\lambda_i'+1} - 1)} \cdot \left(\frac{p_i^{\lambda_i^0+1} - p_i^{\lambda_i'+1}}{p_i^{\lambda_i^0+1}}\right)^2 + \beta_1''(p_m) - \\ &- (k) \cdot \left(\frac{p_i^{\lambda_i^0+1} - p_{k-1}^{\lambda_i'+1}}{p_i^{\lambda_i'+1}}\right) + \left(\frac{k}{2}\right) \cdot \left(\frac{p_i^{\lambda_i^0+1} - p_i^{\lambda_i'+1}}{p_i^{\lambda_i'+1}}\right)^2 - \\ &- \left(\frac{p_i^{\lambda_i'+1} - p_k^2}{p_k^2}\right) + \frac{1}{2} \cdot \left(\frac{p_i^{\lambda_i'+1} - p_k^2}{p_k^2}\right)^2 - \beta_2(p_m). \end{aligned} \quad (50)$$

Here the term without k in its coefficients is

$$\begin{aligned} &\frac{1}{2} \cdot \frac{(p_i^{\lambda_i'+1} - 1)}{p_i^{\lambda_i'+1}} \cdot \frac{(e^{-\gamma} \cdot F(\bar{\lambda}'))}{p_i^{\lambda_i'+1}} \cdot \left(\frac{p_i^{\lambda_i'+1} - p_k^2}{p_k^2 - 1}\right)^2 + \\ &+ \frac{(p_i^{\lambda_i'+1} - 1)}{p_i^{\lambda_i'+1}} \cdot \frac{p_i^{\lambda_i'+1} - p_k^2}{(p_k^2 - 1)} - \left(\frac{p_i^{\lambda_i'+1} - p_k^2}{p_k^2}\right) + \frac{1}{2} \cdot \left(\frac{p_i^{\lambda_i'+1} - p_k^2}{p_k^2}\right)^2 = \\ &= \frac{1}{2} \cdot \left(\frac{p_i^{\lambda_i'+1} - p_k^2}{p_k^2}\right)^2 \cdot \left(\frac{(p_i^{\lambda_i'+1} - 1)}{p_i^{\lambda_i'+1}} \cdot \frac{(e^{-\gamma} \cdot F(\bar{\lambda}'))}{p_i^{\lambda_i'+1}} \cdot \left(\frac{p_k^2}{p_k^2 - 1}\right)^2 + 1\right) + \\ &+ \left(\frac{p_i^{\lambda_i'+1} - p_k^2}{p_k^2}\right) \cdot \left(\frac{(p_i^{\lambda_i'+1} - 1)}{p_i^{\lambda_i'+1}} \cdot \frac{p_k^2}{(p_k^2 - 1)} - 1\right) = \end{aligned} \quad (51)$$

$$\begin{aligned}
&= \frac{1}{2} \cdot \left(\frac{p_i^{\lambda'_i+1} - p_k^2}{p_k^2} \right)^2 \cdot \left(1 + \frac{(p_i^{\lambda'_i+1} - 1) \cdot (e^{-\gamma} \cdot F(\bar{\lambda}'))}{p_i^{\lambda'_i+1}} \cdot \left(\frac{p_k^2}{p_k^2 - 1} \right)^2 \right) + \\
&+ \left(\frac{p_i^{\lambda'_i+1} - p_k^2}{p_k^2} \right) \cdot \left(\frac{(p_i^{\lambda'_i+1} - 1) \cdot p_k^2 - p_i^{\lambda'_i+1} \cdot (p_k^2 - 1)}{p_i^{\lambda'_i+1} \cdot (p_k^2 - 1)} \right) = \\
&= \frac{1}{2} \cdot \left(\frac{p_i^{\lambda'_i+1} - p_k^2}{p_k^2} \right)^2 \cdot \left(1 + \frac{(p_i^{\lambda'_i+1} - 1) \cdot (e^{-\gamma} \cdot F(\bar{\lambda}'))}{p_i^{\lambda'_i+1}} \cdot \left(\frac{p_k^2}{p_k^2 - 1} \right)^2 \right) + \\
&+ \left(\frac{p_i^{\lambda'_i+1} - p_k^2}{p_k^2} \right) \cdot \left(\frac{p_k^2}{p_i^{\lambda'_i+1} \cdot (p_k^2 - 1)} \right) = \\
&= \frac{k}{2} \cdot \left(\frac{p_i^{\lambda'_i+1} - p_i^{\lambda'_i+1}}{p_i^{\lambda'_i+1}} \right)^2 \cdot \alpha_1(p_m),
\end{aligned}$$

where

$$\begin{aligned}
\alpha_1(p_m) &= \frac{1}{k} \cdot \left(\frac{p_i^{\lambda'_i+1} - p_k^2}{p_k^2} \right)^2 \cdot \left(\frac{p_i^{\lambda'_i+1} - p_i^{\lambda'_i+1}}{p_i^{\lambda'_i+1}} \right)^{-2} \times \\
&\times \left(1 + \frac{(p_i^{\lambda'_i+1} - 1) \cdot (e^{-\gamma} \cdot F(\bar{\lambda}'))}{p_i^{\lambda'_i+1}} \cdot \left(\frac{p_k^2}{p_k^2 - 1} \right)^2 + \frac{2 \cdot p_k^2}{p_i^{\lambda'_i+1} \cdot (p_k^2 - 1)} \right) = \quad (52) \\
&= O\left(\sqrt{\frac{\log p_m}{p_m}} \right).
\end{aligned}$$

and the term with k in its coefficients is

$$\begin{aligned}
&k \cdot \left(\frac{p_i^{\lambda'_i+1} - p_i^{\lambda'_i+1}}{p_i^{\lambda'_i+1}} \right) - k \cdot \left(\frac{p_i^{\lambda'_i+1} - p_i^{\lambda'_i+1}}{p_i^{\lambda'_i+1}} \right) + \frac{k}{2} \cdot \left(\frac{p_i^{\lambda'_i+1} - p_i^{\lambda'_i+1}}{p_i^{\lambda'_i+1}} \right)^2 = \\
&= k \cdot \left(\frac{p_i^{\lambda'_i+1} - p_i^{\lambda'_i+1}}{p_i^{\lambda'_i+1}} - \frac{p_i^{\lambda'_i+1} - p_i^{\lambda'_i+1}}{p_i^{\lambda'_i+1}} \right) + \frac{k}{2} \cdot \left(\frac{p_i^{\lambda'_i+1} - p_i^{\lambda'_i+1}}{p_i^{\lambda'_i+1}} \right)^2 = \quad (53) \\
&= -k \cdot \frac{(p_i^{\lambda'_i+1} - p_i^{\lambda'_i+1})^2}{p_i^{\lambda'_i+1} \cdot p_i^{\lambda'_i+1}} + \frac{k}{2} \cdot \left(\frac{p_i^{\lambda'_i+1} - p_i^{\lambda'_i+1}}{p_i^{\lambda'_i+1}} \right)^2 = \\
&= -\frac{k}{2} \cdot \left(\frac{p_i^{\lambda'_i+1} - p_i^{\lambda'_i+1}}{p_i^{\lambda'_i+1}} \right)^2 \cdot \left(2 \cdot \frac{p_i^{\lambda'_i+1}}{p_i^{\lambda'_i+1}} - 1 \right) =
\end{aligned}$$

$$\begin{aligned}
&= -\frac{k}{2} \cdot \left(\frac{p_i^{\lambda_i^0+1} - p_i^{\lambda_i'+1}}{p_i^{\lambda_i'+1}} \right)^2 \cdot \left(1 - 2 \cdot \frac{p_i^{\lambda_i^0+1} - p_i^{\lambda_i'+1}}{p_i^{\lambda_i^0+1}} \right) = \\
&= -\frac{k}{2} \cdot \left(\frac{p_i^{\lambda_i^0+1} - p_i^{\lambda_i'+1}}{p_i^{\lambda_i'+1}} \right)^2 \cdot (1 + \alpha_2(p_m)),
\end{aligned}$$

where $\alpha_2(p_m) = -2 \cdot \frac{p_i^{\lambda_i^0+1} - p_i^{\lambda_i'+1}}{p_i^{\lambda_i^0+1}} = O\left(\frac{1}{p_m^\theta}\right)$ ($0 < \theta < 1/2$). (54)

And the term with k^2 in its coefficients is

$$\begin{aligned}
&\frac{k^2}{2} \cdot \frac{(e^{-\gamma} \cdot F(\bar{\lambda}'))}{(p_i^{\lambda_i'+1} - 1)} \cdot \left(\frac{p_i^{\lambda_i^0+1} - p_i^{\lambda_i'+1}}{p_i^{\lambda_i^0+1}} \right)^2 + \frac{1}{2} \cdot \frac{(k) \cdot (k-1)}{(p_i^{\lambda_i'+1} - 1)} \cdot \left(\frac{p_i^{\lambda_i^0+1} - p_i^{\lambda_i'+1}}{p_i^{\lambda_i^0+1}} \right)^2 = \\
&= \frac{k^2}{2} \cdot \left(\frac{p_i^{\lambda_i^0+1} - p_i^{\lambda_i'+1}}{p_i^{\lambda_i^0+1}} \right)^2 \cdot \left(\frac{(e^{-\gamma} \cdot F(\bar{\lambda}'))}{(p_i^{\lambda_i'+1} - 1)} + \frac{k-1}{k \cdot (p_i^{\lambda_i'+1} - 1)} \right) = \\
&= \frac{k^2}{2} \cdot \left(\frac{p_i^{\lambda_i^0+1} - p_i^{\lambda_i'+1}}{p_i^{\lambda_i'+1}} \right)^2 \cdot \left(\frac{p_i^{\lambda_i'+1}}{p_i^{\lambda_i^0+1}} \right) \cdot \left(\frac{(e^{-\gamma} \cdot F(\bar{\lambda}'))}{(p_i^{\lambda_i'+1} - 1)} + \frac{k-1}{k \cdot (p_i^{\lambda_i'+1} - 1)} \right) = \\
&= \frac{k}{2} \cdot \left(\frac{p_i^{\lambda_i^0+1} - p_i^{\lambda_i'+1}}{p_i^{\lambda_i'+1}} \right)^2 \cdot \alpha_3(p_m),
\end{aligned}$$

(55)

where

$$\begin{aligned}
\alpha_3(p_m) &= k \cdot \left(\frac{p_i^{\lambda_i'+1}}{p_i^{\lambda_i^0+1}} \right) \cdot \left(\frac{(e^{-\gamma} \cdot F(\bar{\lambda}'))}{(p_i^{\lambda_i'+1} - 1)} + \frac{k-1}{k \cdot (p_i^{\lambda_i'+1} - 1)} \right) = \\
&= O\left(\sqrt{\frac{\log p_m}{p_m}}\right).
\end{aligned}$$

(56)

And we have

$$\beta_1''(p_m) - \beta_2(p_m) = \left(\frac{k}{2}\right) \cdot \left(\frac{p_i^{\lambda_i^0+1} - p_i^{\lambda_i'+1}}{p_i^{\lambda_i'+1}} \right)^2 \cdot \beta_0(p_m),$$

(57)

where $\beta_0(p_m) = O\left(\frac{1}{p_m^\theta}\right)$ ($0 < \theta < 1/2$). Hence we have

$$\log\left(\frac{C'_{m-1}}{C_{m-1}}\right) = -\frac{k}{2} \cdot \left(\frac{p_i^{\lambda_i^0+1} - p_i^{\lambda_i'+1}}{p_i^{\lambda_i'+1}}\right)^2 \cdot (1 + \alpha_0(p_m)) \quad (58)$$

where

$$\begin{aligned} \alpha_0(p_m) &= \alpha_2(p_m) - \alpha_1(p_m) - \alpha_3(p_m) + \\ &+ \beta_1''(p_m) - \beta_2(p_m) = O\left(\frac{1}{p_m^\theta}\right). \end{aligned} \quad (59)$$

On the other hand, by (11) and (12) we have

$$p_i^{\lambda_i^0+1} = p_m \cdot \log p_m \cdot (1 + \varepsilon_1(p_m)), \quad \varepsilon_1(p_m) = O\left(\frac{1}{\log p_m}\right) \quad (60)$$

and

$$p_i^{\lambda_i'+1} = p_{m-1} \cdot \log p_{m-1} \cdot (1 + \varepsilon_2(p_{m-1})), \quad \varepsilon_2(p_{m-1}) = O\left(\frac{1}{\log p_m}\right). \quad (61)$$

and

$$k = 2 \cdot \sqrt{\frac{p_{m-1}}{\log p_{m-1}}} (1 + \varepsilon_3(p_m)), \quad \varepsilon_3(p_m) = O\left(\frac{\log \log p_m}{\log p_m}\right).$$

Hence

$$\frac{p_i^{\lambda_i^0+1} - p_i^{\lambda_i'+1}}{p_i^{\lambda_i'+1}} = \frac{p_m - p_{m-1}}{p_{m-1}} \cdot (1 + \varepsilon_4(p_m)), \quad (62)$$

where $\varepsilon_4(p_m) = O\left(\frac{1}{\log p_m}\right)$. From this we have

$$\begin{aligned} \log C_{m-1} - \log C'_{m-1} &= \frac{k}{2} \cdot \left(\frac{p_i^{\lambda_i^0+1} - p_i^{\lambda_i'+1}}{p_i^{\lambda_i'+1}}\right)^2 \cdot (1 + \alpha_0(p_m)) = \\ &= \frac{(p_m - p_{m-1})}{\sqrt{p_{m-1} \cdot \log p_{m-1}}} \cdot \left(\frac{p_m - p_{m-1}}{p_{m-1}}\right) \cdot (1 + \beta_0(p_m)). \end{aligned} \quad (63)$$

where

$$\begin{aligned}
1 + \beta_0(p_m) &= (1 + \varepsilon_1(p_m)) \cdot (1 + \varepsilon_2(p_m))^2 \cdot (1 + \alpha_0(p_m)) = \\
&= 1 + O\left(\frac{1}{\log p_m}\right).
\end{aligned} \tag{64}$$

This is the proof of the theorem 2. \square

4. The estimate of $(p_m - p_{m-1})$

We have

Theorem 3. There exist a number m_0 such that for any $m \geq m_0$ we have

$$(p_m - p_{m-1}) = O\left(\sqrt{p_{m-1}} \cdot \log^{5/2} p_{m-1}\right). \tag{65}$$

Proof. It is easy to see that

$$\begin{aligned}
\log C_m - \log C'_m &= \left(\exp\left(e^{-\gamma} \cdot F\left(\bar{\lambda}_0\right)\right) - \log n_0\right) - \left(\exp\left(e^{-\gamma} \cdot F\left(\bar{\lambda}'_+\right)\right) - \log n'_+\right) = \\
&= \left(\exp\left(e^{-\gamma} \cdot F\left(\bar{\lambda}'_0\right)\right) - \exp\left(e^{-\gamma} \cdot F\left(\bar{\lambda}'\right)\right)\right) - (\log n'_0 - \log n') + \\
&+ \left(\exp\left(e^{-\gamma} \cdot F\left(\bar{\lambda}_0\right)\right) - \exp\left(e^{-\gamma} \cdot F\left(\bar{\lambda}'_0\right)\right)\right) - \\
&- \left(\exp\left(e^{-\gamma} \cdot F\left(\bar{\lambda}'_+\right)\right) - \exp\left(e^{-\gamma} \cdot F\left(\bar{\lambda}'\right)\right)\right) = \\
&= (\log C'_{m-1} - \log C_{m-1}) + R_0,
\end{aligned} \tag{66}$$

where

$$\begin{aligned}
R_0 &= \left(\exp\left(e^{-\gamma} \cdot F\left(\bar{\lambda}_0\right)\right) - \exp\left(e^{-\gamma} \cdot F\left(\bar{\lambda}'_0\right)\right)\right) - \\
&- \left(\exp\left(e^{-\gamma} \cdot F\left(\bar{\lambda}'_+\right)\right) - \exp\left(e^{-\gamma} \cdot F\left(\bar{\lambda}'\right)\right)\right).
\end{aligned} \tag{67}$$

On other hand, by (18) in the theorem 1 we have

$$\begin{aligned}
&\left(\exp\left(e^{-\gamma} \cdot F\left(\bar{\lambda}_0\right)\right) - \exp\left(e^{-\gamma} \cdot F\left(\bar{\lambda}'_0\right)\right)\right) = \\
&= \exp\left(e^{-\gamma} \cdot F\left(\bar{\lambda}'_0\right)\right) \cdot \left(\exp\left(e^{-\gamma} \cdot F\left(\bar{\lambda}'_0\right) \cdot \frac{1}{p_m}\right) - 1\right) =
\end{aligned} \tag{68}$$

$$\begin{aligned}
&= \frac{1}{p_m} \cdot \left(e^{-\gamma} \cdot F(\bar{\lambda}'_0) \right) \cdot \exp\left(e^{-\gamma} \cdot F(\bar{\lambda}'_0) \right) \cdot \left(1 + O\left(\frac{\log p_m}{p_m} \right) \right) = \\
&= \frac{1}{p_m} \cdot \left(\left(e^{-\gamma} \cdot F(\bar{\lambda}_0) \right) \cdot \exp\left(e^{-\gamma} \cdot F(\bar{\lambda}_0) \right) \right) \times \\
&\quad \times \left(\exp\left(-e^{-\gamma} \cdot F(\bar{\lambda}'_0) \cdot \frac{1}{p_m} \right) \right) \cdot \left(1 - \frac{1}{p_m + 1} \right) \cdot \left(1 + O\left(\frac{\log p_m}{p_m} \right) \right) = \\
&= \frac{1}{p_m} \cdot \left(\left(e^{-\gamma} \cdot F(\bar{\lambda}_0) \right) \cdot \exp\left(e^{-\gamma} \cdot F(\bar{\lambda}_0) \right) \right) \cdot \left(1 + O\left(\frac{\log p_m}{p_m} \right) \right) = \\
&= \log p_m \cdot \left(1 + (\log p_m + 1) \cdot E_0(p_m) - \frac{4 \cdot (\log p_m + 1)}{\sqrt{p_m} \log^{3/2} p_m} + \Theta_1(p_m) \right),
\end{aligned}$$

where $\Theta_1(p_m) = O(\log^2 p_m \cdot E_0^2(p_m))$.

And we have

$$\begin{aligned}
&\exp\left(e^{-\gamma} \cdot F(\bar{\lambda}') \right) \cdot \left(\exp\left(e^{-\gamma} \cdot F(\bar{\lambda}') \cdot \frac{1}{p_m} \right) - 1 \right) = \\
&= \frac{p_{m-1}}{p_m} \cdot \log p_{m-1} \cdot \left(1 + (\log p_{m-1} + 1) \cdot E_0(p_{m-1}) - \right. \\
&\quad \left. - \frac{4 \cdot (\log p_{m-1} + 1)}{\sqrt{p_{m-1}} \log^{3/2} p_{m-1}} + \Theta_2(p_{m-1}) \right), \tag{69}
\end{aligned}$$

where $\Theta_2(p_{m-1}) = O(\log^2 p_{m-1} \cdot E_0^2(p_{m-1}))$.

From this we obtain

$$\begin{aligned}
R_0 &= \log p_m \cdot \left(1 + (\log p_m + 1) \cdot E_0(p_m) - \frac{4 \cdot (\log p_m + 1)}{\sqrt{p_m} \log^{3/2} p_m} + \Theta_1(p_m) \right) - \\
&\quad - \frac{p_{m-1} \cdot \log p_{m-1}}{p_m} \cdot \left(1 + (\log p_{m-1} + 1) \cdot E_0(p_{m-1}) - \right. \\
&\quad \left. - \frac{4 \cdot (\log p_{m-1} + 1)}{\sqrt{p_{m-1}} \log^{3/2} p_{m-1}} + \Theta_2(p_{m-1}) \right) = \\
&= \left(\log p_m - \frac{p_{m-1}}{p_m} \cdot \log p_{m-1} \right) + \log p_m \cdot (\log p_m + 1) \cdot E_0(p_m) -
\end{aligned} \tag{70}$$

$$\begin{aligned}
& -\frac{p_{m-1} \cdot \log p_{m-1}}{p_m} \cdot (\log p_{m-1} + 1) \cdot E_0(p_{m-1}) + \\
& + \frac{4 \cdot \log p_m \cdot (\log p_m + 1)}{\sqrt{p_m} \log^{3/2} p_m} - \frac{p_{m-1} \cdot \log p_{m-1}}{p_m} \cdot \frac{4 \cdot (\log p_{m-1} + 1)}{\sqrt{p_{m-1}} \log^{3/2} p_{m-1}} + \\
& + \log p_m \cdot \Theta_1(p_m) - \frac{p_{m-1} \cdot \log p_{m-1}}{p_m} \cdot \Theta_2(p_{m-1}).
\end{aligned}$$

Here we have

$$\begin{aligned}
& \log p_m - \frac{p_{m-1}}{p_m} \cdot \log p_{m-1} = \log p_m - \left(1 - \frac{p_m - p_{m-1}}{p_m}\right) \cdot \log p_{m-1} = \\
& = (\log p_m - \log p_{m-1}) + \left(\frac{p_m - p_{m-1}}{p_m}\right) \cdot \log p_{m-1} = \\
& = \log \left(1 + \frac{p_m - p_{m-1}}{p_{m-1}}\right) + \left(\frac{p_m - p_{m-1}}{p_m}\right) \cdot \log p_{m-1} = \tag{71} \\
& = \frac{p_m - p_{m-1}}{p_{m-1}} - \frac{1}{2} \left(\frac{p_m - p_{m-1}}{p_{m-1}}\right)^2 + \left(\frac{p_m - p_{m-1}}{p_m}\right) \cdot \log p_{m-1} = \\
& = \frac{p_m - p_{m-1}}{p_{m-1}} \cdot \log p_{m-1} \cdot r_1(p_m),
\end{aligned}$$

where $r_1(p_m) = 1 + O\left(\frac{1}{\log p_m}\right) = O(1)$.

By (14) it is true that

$$\begin{aligned}
E_0(p_m) &= \sum_{i=1}^m \frac{1}{p_i} - \log \log p_m - b_0 = \\
&= \left(\sum_{i=1}^{m-1} \frac{1}{p_i} - \log \log p_{m-1} - b_0\right) - \log \log p_m + \log \log p_{m-1} + \frac{1}{p_m} = \tag{72} \\
&= E_0(p_{m-1}) - \log \left(\frac{\log p_m}{\log p_{m-1}}\right) + \frac{1}{p_m}.
\end{aligned}$$

Hence we have

$$\begin{aligned}
& \log p_m \cdot (\log p_m + 1) \cdot E_0(p_m) - \frac{p_{m-1} \cdot \log p_{m-1}}{p_m} \cdot (\log p_{m-1} + 1) \cdot E_0(p_{m-1}) = \\
& = \log p_m \cdot (\log p_m + 1) \cdot E_0(p_m) - \log p_{m-1} \cdot (\log p_{m-1} + 1) \cdot E_0(p_{m-1}) + \\
& + \left(\frac{p_m - p_{m-1}}{p_m} \right) \log p_{m-1} \cdot (\log p_{m-1} + 1) \cdot E_0(p_{m-1}) = \\
& = \log p_m \cdot (\log p_m + 1) \cdot (E_0(p_m) - E_0(p_{m-1})) + \\
& + (\log p_m \cdot (\log p_m + 1) - \log p_{m-1} \cdot (\log p_{m-1} + 1)) \cdot E_0(p_{m-1}) + \\
& + \left(\frac{p_m - p_{m-1}}{p_m} \right) \log p_{m-1} \cdot (\log p_{m-1} + 1) \cdot E_0(p_{m-1}) = \\
& = \log p_m \cdot (\log p_m + 1) \cdot \left(-\log \left(\frac{\log p_m}{\log p_{m-1}} \right) + \frac{1}{p_m} \right) + \\
& + (\log p_m \cdot (\log p_m + 1) - \log p_{m-1} \cdot (\log p_{m-1} + 1)) \cdot E_0(p_{m-1}) + \\
& + \left(\frac{p_m - p_{m-1}}{p_m} \right) \log p_{m-1} \cdot (\log p_{m-1} + 1) \cdot E_0(p_{m-1}) = \\
& = \left(\frac{p_m - p_{m-1}}{p_{m-1}} \right) \log^2 p_{m-1} \cdot r_2(p_m), \tag{73}
\end{aligned}$$

where $r_2(p_m) = O\left(\frac{1}{p_m - p_{m-1}} + \frac{1}{\log p_m}\right) = O(1)$.

Next, it is clear that

$$\begin{aligned}
& \frac{4 \cdot \log p_m \cdot (\log p_m + 1)}{\sqrt{p_m} \log^{3/2} p_m} - \frac{p_{m-1} \cdot \log p_{m-1}}{p_m} \cdot \frac{4 \cdot (\log p_{m-1} + 1)}{\sqrt{p_{m-1}} \log^{3/2} p_{m-1}} = \\
& = \frac{p_m - p_{m-1}}{p_{m-1}} \cdot \log p_{m-1} \cdot r_3(p_m), \tag{74}
\end{aligned}$$

and

$$\begin{aligned}
& \log p_m \cdot \Theta_1(p_m) - \frac{p_{m-1} \cdot \log p_{m-1}}{p_m} \cdot \Theta_2(p_{m-1}) = \\
& = \frac{p_m - p_{m-1}}{p_{m-1}} \cdot \log p_{m-1} \cdot r_4(p_m) \tag{75}
\end{aligned}$$

where $r_3(p_m) = O\left(\frac{\log p_m}{\sqrt{p_m}}\right)$ and $r_4(p_m) = O\left(\frac{1}{\log p_m}\right)$.

Therefore we have

$$R_0 = \frac{p_m - p_{m-1}}{p_{m-1}} \cdot \log^2 p_{m-1} \cdot \delta_0(p_m), \quad (76)$$

where $\delta_0(p_m) = r_2(p_m) + \frac{1}{\log p_m} \cdot (r_1(p_m) + r_3(p_m)r_4(p_m)) = O(1)$.

On the other hand, since $C'_m \leq C_m$ we have

$$0 < (\log C_{m-1} - \log C'_{m-1}) < R_0. \quad (77)$$

Thus from the theorem 2, we have

$$\begin{aligned} & \frac{(p_m - p_{m-1})}{\sqrt{p_{m-1}} \cdot \log p_{m-1}} \cdot \left(\frac{p_m - p_{m-1}}{p_{m-1}} \right) \cdot (1 + \beta_0(p_m)) \leq \\ & \leq \left(\frac{p_m - p_{m-1}}{p_{m-1}} \right) \cdot \log^2 p_{m-1} \cdot \delta_0(p_m). \end{aligned} \quad (78)$$

Therefore we have

$$(p_m - p_{m-1}) = O\left(\sqrt{p_{m-1}} \cdot \log^{5/2} p_{m-1}\right). \quad (79)$$

This is the proof of the theorem 3. \square

5. The estimate of $E_0(p_m)$

We get

Theorem 4. There exists a number m_0 such that for any $m \geq m_0$ we have

$$E_0(p_m) = O\left(\frac{\log^2 p_m}{\sqrt{p_m}}\right). \quad (80)$$

Proof. Since

$$F(\bar{\lambda}_0) = F(\bar{\lambda}'_0) \cdot \left(1 + \frac{1}{p_m}\right), \quad (81)$$

we have

$$\begin{aligned}
& \log p_m \cdot \left(1 + E_0(p_m) - \frac{4}{\sqrt{p_m} \log^{3/2} p_m} + \varepsilon(p_m) \right) = \\
& = \log p_{m-1} \cdot \left(1 + E_0(p_{m-1}) - \frac{4}{\sqrt{p_m} \log^{3/2} p_m} + \varepsilon(p_{m-1}) \right) \cdot \left(1 + \frac{1}{p_m} \right).
\end{aligned} \tag{82}$$

From this we have

$$\begin{aligned}
& \log p_m \cdot E_0(p_m) - \log p_{m-1} \cdot \left(1 + \frac{1}{p_m} \right) \cdot E_0(p_{m-1}) = \\
& = - \left(\log p_m - \log p_{m-1} - \frac{\log p_{m-1}}{p_m} \right) + \\
& + \frac{4}{\sqrt{p_m} \log^{3/2} p_m} \cdot \left(\log p_m - \log p_{m-1} - \frac{\log p_{m-1}}{p_m} \right) + \\
& + \log p_m \cdot \varepsilon(p_m) - \log p_{m-1} \cdot \left(1 + \frac{1}{p_m} \right) \cdot \varepsilon(p_{m-1}).
\end{aligned} \tag{83}$$

From (73) the left hand side of (83) is

$$\begin{aligned}
& \log p_m \cdot E_0(p_m) - \log p_{m-1} \cdot \left(1 + \frac{1}{p_m} \right) \cdot E_0(p_{m-1}) = \\
& = \log p_m \cdot \left(E_0(p_{m-1}) - \log \left(\frac{\log p_m}{\log p_{m-1}} \right) + \frac{1}{p_m} \right) - \\
& - \log p_{m-1} \cdot \left(1 + \frac{1}{p_m} \right) \cdot E_0(p_{m-1}) = \\
& = \left(\log p_m - \log p_{m-1} - \frac{\log p_{m-1}}{p_m} \right) \cdot E_0(p_{m-1}) - \\
& - \log p_m \cdot \left(\log \left(\frac{\log p_m}{\log p_{m-1}} \right) - \frac{1}{p_m} \right).
\end{aligned} \tag{84}$$

On the other hand, here we have

$$\begin{aligned}
& -\log p_m \cdot \left(\log \left(\frac{\log p_m}{\log p_{m-1}} \right) - \frac{1}{p_m} \right) = - \left(\log p_m - \log p_{m-1} - \frac{\log p_{m-1}}{p_m} \right) + \\
& + \left(\log p_m - \log p_{m-1} - \frac{\log p_{m-1}}{p_m} \right) \cdot \left(\frac{\log p_m - \log p_{m-1}}{\log p_{m-1}} \right) + \\
& + \log p_m \cdot \left(\frac{1}{2} \cdot \left(\frac{\log p_m - \log p_{m-1}}{\log p_{m-1}} \right)^2 + \Delta(p_m) \right),
\end{aligned} \tag{85}$$

$$\text{where } \Delta(p_m) = O \left(\left(\frac{\log p_m - \log p_{m-1}}{\log p_{m-1}} \right)^3 \right). \tag{86}$$

And we have

$$\begin{aligned}
& \log p_m \cdot \varepsilon(p_m) - \log p_{m-1} \cdot \left(1 + \frac{1}{p_m} \right) \cdot \varepsilon(p_{m-1}) = \\
& = O \left(\left(\frac{\log p_m - \log p_{m-1}}{\log p_{m-1}} \right)^2 \right).
\end{aligned} \tag{87}$$

So from (83) we have

$$\begin{aligned}
& \left(\log p_m - \log p_{m-1} - \frac{\log p_{m-1}}{p_m} \right) \cdot E_0(p_{m-1}) - \\
& - \left(\log p_m - \log p_{m-1} - \frac{\log p_{m-1}}{p_m} \right) \cdot \left(1 + \frac{\log p_m - \log p_{m-1}}{\log p_{m-1}} \right) \\
& - \log p_m \cdot \left(\frac{1}{2} \cdot \left(\frac{\log p_m - \log p_{m-1}}{\log p_{m-1}} \right)^2 + \Delta(p_m) \right) = \\
& = - \left(\log p_m - \log p_{m-1} - \frac{\log p_{m-1}}{p_m} \right) + \frac{4}{\sqrt{p_m} \log^{3/2} p_m} \times \\
& \times \left(\log p_m - \log p_{m-1} - \frac{\log p_{m-1}}{p_m} \right) + O \left(\left(\frac{\log p_m - \log p_{m-1}}{\log p_{m-1}} \right)^2 \right).
\end{aligned} \tag{88}$$

Now if we would exactly write not only the term with $E_0(p_m)$, but also the term with $E_0^2(p_m)$ and more $E_0^n(p_m)$, and would repeat above process, then

we would have $\left(\log p_m - \log p_{m-1} - \frac{\log p_{m-1}}{p_m}\right)$ in the every term.

So we will eliminate it from both hand sides of (88). Then since

$$\begin{aligned} & (\log p_m - \log p_{m-1}) \cdot \left(\log p_m - \log p_{m-1} - \frac{\log p_{m-1}}{p_m}\right)^{-1} = \\ & = \left(1 - \frac{\log p_{m-1}}{p_m \cdot (\log p_m - \log p_{m-1})}\right)^{-1} = O(1) \end{aligned} \quad (89)$$

and

$$\begin{aligned} & \frac{\log p_m - \log p_{m-1}}{\log p_{m-1}} = \frac{1}{\log p_{m-1}} \cdot \log \left(1 + \frac{p_m - p_{m-1}}{p_{m-1}}\right) = \\ & = \frac{1}{\log p_{m-1}} \cdot \frac{p_m - p_{m-1}}{p_{m-1}} \cdot \left(1 + O\left(\frac{p_m - p_{m-1}}{p_{m-1}}\right)\right), \end{aligned} \quad (90)$$

by the theorem 3 we have

$$E_0(p_m) = O\left(\frac{\log^2 p_m}{\sqrt{p_m}}\right). \quad (91)$$

This is the proof of the theorem 4. \square

References

- [1] J. Sandor, D. S. Mitrinovic, B. Crstici, “Handbook of Number theory 1”, Springer, 2006.
- [2] H. L. Montgomery, R. C. Vaugnan, “Multiplicative Number Theory”, Cambridge, 2006.
- [3] S. Lou and Q. Yao, A Chebyshev’s type of prime numbers theorem in a short interval, II. Hardy-Ramanujan J. 15, 1-33, 1992
- [4] C. J. Mozzochi. “On the difference between consecutive primes”, J. Number theory, 24, 181-187, 1986.
- [5] J. B. Rosser, L. Schoenfeld, “ Approximate formulars for some functions

of prime numbers”, Illinois J. Math. 6, 64-94, 1962.

[6] R. G. Choe, The sum of divisors function and the Hardy-Ramanujan’s number, November 12, 2011

[7] R. G. Choe, An exponential function and its optimization problem, November 15, 2011.

See for [6]:

http://commons.wikimedia.org/wiki/File:The_sum_of_divisors_function_and_the_Hardy-Ramanujan%27s_number.pdf

See for [7]:

http://commons.wikimedia.org/wiki/File:An_Exponential_Function_and_itsOptimization_Problem.pdf

*Department of Mathematics, University of Sciences, Unjong District, Gwahak 1-dong, Pyongyang, D.P.R.Korea,
Email: ryonggilchoe@163.com*